fractal and fractional

Article

Fractional Study of the Non-Linear Burgers' Equations via a Semi-Analytical Technique

Naveed Iqbal [1,*], Muhammad Tajammal Chughtai [2] and Roman Ullah [3]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
[2] Department of Electrical Engineering, College of Engineering, University of Hail, Hail 55427, Saudi Arabia
[3] Department of General Studies, Higher Colleges of Technology, Dubai Women Campus, Dubai 16062, United Arab Emirates
* Correspondence: n.iqbal@uoh.edu.sa

Abstract: Most complex physical phenomena are described by non-linear Burgers' equations, which help us understand them better. This article uses the transformation and the fractional Taylor's formula to find approximate solutions for non-linear fractional-order partial differential equations. Solving non-linear Burgers' equations with the right starting data shows that the method utilized is correct and can be utilized. Based on the limit of the idea, a rapid convergence McLaurin series is used to obtain close series solutions for both models with less work and more accuracy. To see how time-Caputo fractional derivatives affect how the results of the above models behave, in three dimension figures are drawn. The results showed that the proposed method is an easy, flexible, and helpful way to solve and understand a wide range of non-linear physical models.

Keywords: fractional non-linear Burgers' equations; caputo operator; residual power series transform method; analytical solution

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1. Introduction

Fractional differential equations have become more well known in recent years because they are useful in many scientific and engineering fields. For example, a fractional derivative can be used to describe the non-linear oscillations of an earthquake, and the fractional of the fluid traffic dynamic model can solve the insufficiency that comes from assuming that traffic flows continuously [1,2]. Fractional differential problems are also used to model a wide range of mathematical biology, chemical processes and applied sciences models [3–6]. Different physical, biological, and chemical phenomena can be defined by non-linear partial differential equations (NPDEs). The main goal of current research is to find precise traveling wave solutions for equations like these. Scientists can better understand the complicated physical phenomena and dynamic processes that NPDEs show [7–9] when they become clear and exact answers. In the last 40 years, many important methods for obtaining accurate answers to NPDEs [10,11] have been proposed [12–14].

Bateman came up with the Burgers' equation in 1915 [15], later changing it to the Burgers' equation [16]. The Burgers' equation is used extensively in engineering and science, especially when solving problems with non-linear equations. More and more important and interesting things are being conducted with Burgers' equation by mathematicians and researchers. It has been known for a long time that this equation can be used to model things such as dynamics, heat conduction, acoustic waves, turbulence, and many more [17–22]. Most of the time, you need to use special methods to solve this kind of non-linear PDE because it cannot be solved analytically. Some researchers and scientists have applied analytical techniques such as the Variational iteration method, the Adomian decomposition method, the Homotopy analysis method, the Homotopy perturbation method and the differential transform method [23–27] to solve these kinds of problems.

A numerical analytic method for solving many forms of ordinary, partial, integrodifferential equations and fractional fuzzy differential equations is known as the residual power series (RPS) approach. Since it offers closed-form solutions of well-known functions, it is an efficient optimization strategy. The RFPS technique solves fuzzy FDEs, various FDEs, and integral equations having fractional order, such as the Newell–Whitehead–Segel equation of fractional order [28], coupled fractional resonant Schrödinger equations [29], time fractional Kundu–Eckhaus and massive Thirring equations [30], time fractional Fokker–Planck models [31], fractional partial differential equations [32], singular initial value problems [33], dractional Fredholm integro-differential equations having order 2b [34], and several classes of fractional fuzzy differential equations [35,36]. The Elzaki transform (ET) is a powerful tool for resolving numerous complex models that appear in various branches of the natural sciences. When analytical methods are combined with the ET operator, non-linear problems can be solved more quickly and precisely.

The main goal of this work is to examine the approximate and analytical solutions for linear and non-linear systems using the Elzaki residual power series (ERPS) method, which was introduced and developed in [37]. The ET and RPS approaches are combined in the ERPS approach, which provides both approximate and accurate solutions as quickly fractional power series (FPS) solutions. The proposed problem is converted to Elzaki space, and the solutions in the form of algebraic equations are created. Lastly, the inverse Elzaki is applied to the proposed problem results. In contrast to the FRPS approach, which depends on the fractional derivative and consumes time to compute the various fractional derivatives in determining the solutions, the unknown coefficients in a modified Elzaki expansion can be identified by employing the limit notion. The ERPS approach takes less time and provides higher accuracy with minor computational requirements.

This work is arranged as follows: Section 2 revisits some valid basic results concerning the Elzaki transformation and fractional power series representations. The proposed technique for obtaining the solutions for the fractional partial differential equation is in Section 3. In Section 4, the simplicity and applicability of RPSM are studied by solving fractional order Burgers' equations. Last, concluding remarks on our findings are drawn in Section 5.

2. Preliminaries

In this section, we go over some basic concepts and definitions of the Caputo derivative (CD), ET, and theorems that will be useful in this paper.

Definition 1. *Assume that the existence axioms of the ET are satisfied by $\Im(\vartheta,\tau)$. Then, the ET of $\Im(\vartheta,\tau)$ is formulated as [38]:*

$$\Xi[\Im(\vartheta,\tau)] = \aleph(\vartheta,\omega) = \omega\int_0^\infty \Im(\vartheta,\tau)e^{-\frac{\tau}{\omega}}d\tau,\ \ \tau\geq 0,\ \ e_1\leq\omega\leq e_2.$$

Definition 2. *The FD of $\Im(\vartheta,\tau)$ of order ϱ in the CD sense is defined as follows [38]:*

$$D_\tau^\varrho\Im(\vartheta,\tau) = \mathcal{J}_\varrho^{r-\varrho}\Im^{(n)}(\vartheta,\tau),\ \ \tau\geq 0,\ \ n-1<\varrho\leq n,$$

where $\mathcal{J}_\tau^{n-\varrho}$ is the $R-L$ integral of $\Im(\vartheta,\tau)$.

Theorem 1. *Assume that the multiple fractional power series (MFPS) representation for the function $\Xi[\Im(\vartheta,\tau)] = \aleph(\vartheta,s)$ is given by [38]*

$$\aleph(\vartheta,\omega) = \sum_{n=0}^\infty \Theta_n(\vartheta)\omega^{n\varrho+2},$$

then we have

$$\Theta_n(\vartheta) = D_\tau^{n\varrho}\Im(\vartheta, 0),$$

where, $D_\tau^{n\varrho} = D_\tau^{\varrho} \cdot D_\tau^{\varrho} \dots D_\tau^{\varrho}(n-$ *times). The conditions for the convergence of the MFPS is determined in the following theorem.*

Theorem 2. *Let* $\Xi[\Im(\vartheta, \tau)] = \aleph(\vartheta, \omega)$ *can be represented as the new form of MFPS explained in Theorem 1. If* $\left|\frac{1}{\omega^2}\Xi\left[D_\tau^{(n+1)\varrho}\Im(\vartheta, \tau)\right]\right| \leq \mathcal{Z}$, *then the remainder* $\mathcal{R}_n(\vartheta, \omega)$ *of MFPS satisfies the following inequality [38]:*

$$\left|\mathcal{R}_j(\vartheta, \omega)\right| \leq \mathcal{Z}\omega^{(n+1)\varrho+2}.$$

3. General Implementation of Elzaki Residual Power Series Method

Consider the general fractional partial differential equation

$$D_\tau^{\varrho} \nu(\vartheta, \tau) = cD_\vartheta^2 \nu(\vartheta, \tau) + a\nu(\vartheta, \tau) - b\nu^4(\vartheta, \tau), \tag{1}$$

with the initial condition

$$\nu(\vartheta, \tau) = f_0(\vartheta). \tag{2}$$

First, we use the Elzaki transformation to (1),

$$\Xi\left[D_\tau^{\varrho}\nu(\vartheta, \tau)\right] = c\Xi\left[D_\vartheta^2\nu(\vartheta, \tau)\right] + a\Xi[\nu(\vartheta, \tau)^2] - b\Xi\left[\nu^4(\vartheta, \tau)\right]. \tag{3}$$

By the fact that $\Xi\left[D_\tau^{\varrho}w(\vartheta, \tau)\right] = s^a\Xi[w(\vartheta, \tau)] - s^{a-1}\nu(\vartheta, 0)$ and using the initial condition (2), we rewrite (3) as

$$V(\vartheta, s) = s^2 f_0(\vartheta) + s^{\varrho}cD_\vartheta^2 V(\vartheta, s) + s^{\varrho}aV(\vartheta, s) - s^{\varrho}b\Xi\left[\left[\Xi^{-1}[V(\vartheta, s)]^2\right]^{\varrho}\right], \tag{4}$$

where $V(\vartheta, s) = \Xi[w(\vartheta, \tau)]$.

Second, we define the transform term $V(\vartheta, s)$ as the following expression

$$V(\vartheta, s) = \sum_{n=0}^{\infty} s^{n\varrho+1} f_s(\vartheta). \tag{5}$$

The series form of kth-truncated of Equation (5)

$$V_k(\vartheta, s) = \sum_{n=0}^{k} s^{n\varrho+1} f_s(\vartheta) = s^2 f_o(\vartheta) + \sum_{n=1}^{k} s^{n\varrho+1} f_k(\vartheta). \tag{6}$$

The Elzaki residual function to (5) is

$$\Xi Res_k(\vartheta, s) = V_k(\vartheta, s) - sf_0(\vartheta) - s^{\varrho}cD_s^2 V_k(\vartheta, s) - s^{\varrho}aV_k(\vartheta, s) + s^{\varrho}b\Xi\left[\left[\Xi^{-1}[V_k(\vartheta, s)]\right]^q\right]. \tag{7}$$

Third, we use a few properties that come up in the standard RPSM [25,26] to point out certain facts:
$\Xi\,\mathrm{Res}(\vartheta, s) = 0$ and $\lim_{k\to\infty} \Xi\,\mathrm{Res}\, s_k(\vartheta, s) = \Xi\,\mathrm{Res}(\vartheta, s)$ for each $s > 0$.
$\lim_{s\to\infty} s\Xi\,\mathrm{Res}(\vartheta, s) = 0 \Rightarrow \lim_{s\to\infty} s\Xi\,\mathrm{Res}(\vartheta, s) = 0$.
$\lim_{s\to\infty} s^{k\varrho+1}\Xi\,\mathrm{Res}(\vartheta, s) = \lim_{s\to\infty} s^{k\varrho+1}\Xi\,\mathrm{Res}_k(\vartheta, s) = 0, 0 < \varrho \leq 1, k = 1, 2, 3, \dots$

So, to find the co-efficient functions $f_n(\vartheta)$, we solve the following scheme successively:

$$\lim_{s\to\infty}\left(s^{ka+1}\Xi Res_k(\vartheta, s)\right) = 0, \;\; 0 < \varrho \leq 1, \;\; k = 1, 2, 3, \dots$$

Finally, we use the inverse Elzaki to $V_k(\vartheta, s)$, to achieved the kth approximated supportive result $\nu_k(\vartheta, \tau)$.

4. Numerical Examples

Example 1. *Consider the fractional two-dimensional Burger's equation*

$$D_\tau^\varrho \nu(\vartheta,\zeta,\tau) = \nu(\vartheta,\zeta)\nu_\vartheta(\vartheta,\zeta) + \nu_{\vartheta\vartheta}(\vartheta,\zeta) + \nu_{\zeta\zeta}(\vartheta,\zeta), \quad 0 < \varrho \leq 1 \tag{8}$$

with the initial condition (IC)

$$\nu(\vartheta,\zeta,0) = \vartheta + \zeta = f_0(\vartheta) \tag{9}$$

The exact solution of (8) $(\varrho = 1)$ is

$$\nu(\vartheta,\zeta,\tau) = \frac{\vartheta+\zeta}{1-\tau} \tag{10}$$

Using ET to (8) and applying the IC (9), we get

$$\begin{aligned} V(\vartheta,s) = {} & s^2(\vartheta+\zeta) + s^\varrho \Xi_\tau[[\Xi_\tau^{-1}\{V_\vartheta(\vartheta,\zeta,s)\}][\Xi_\tau^{-1}\{V_\vartheta(\vartheta,\zeta,s)\}]] \\ & + s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{V_{\vartheta\vartheta}(\vartheta,\zeta,s)\}] + s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{V_{yy}(\vartheta,\zeta,s)\}]. \end{aligned} \tag{11}$$

The k-th truncated term series of (11) is

$$V_k(\vartheta,\zeta,s) = s^2(\vartheta+\zeta) + \sum_{n=1}^{k} s^{n\varrho+1} f_n(\vartheta), \tag{12}$$

and the k-th Elzaki residual function is

$$\begin{aligned} \Xi_\tau Res_k(\vartheta,\zeta,s) = {} & V_k(\vartheta,\zeta,s) - s^2(\vartheta+\zeta) - s^\varrho \Xi_\tau[[\Xi_\tau^{-1}\{V(\vartheta,\zeta,s)\}][\Xi_\tau^{-1}\{\frac{\partial}{\partial\vartheta} V_\vartheta(\vartheta,\zeta,s)\}]] \\ & - s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{\frac{\partial^2}{\partial\vartheta^2} V(\vartheta,\zeta,s)\}] - s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{\frac{\partial^2}{\partial\zeta^2} V_{yy}(\vartheta,\zeta,s)\}]. \end{aligned} \tag{13}$$

Now, to calculate $f_k(\vartheta,\zeta)$, $k = 1,2,3,\cdots$, we put the kth-truncate series (12) into the kth-Elzaki residual term (13), multiply the solution equation by $s^{k\varrho+1}$, and then the relation is solved recursively $\lim_{s\to\infty}[s^{k\varrho+1} Res_k(\vartheta,\zeta,s)] = 0$, $k = 1,2,3,\cdots$ *for f_k. The following are the some terms $f_k(\vartheta,\zeta)$*

$$\begin{aligned} f_1(\vartheta,\zeta) &= (\vartheta+\zeta) \\ f_2(\vartheta,\zeta) &= 2(\vartheta+\zeta) \\ f_3(\vartheta,\zeta) &= (\vartheta+\zeta)\Big[4 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\Big] \\ f_4(\vartheta,\zeta) &= 2(\vartheta+\zeta)\Big[4 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2} + \frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}\Big] \\ &\vdots \end{aligned} \tag{14}$$

Putting the value of $f_n(\vartheta,\zeta)$, $(n \geq 1)$ in Equation (12), we have

$$\begin{aligned} V(\vartheta,\zeta,s) = {} & s^2\vartheta + \zeta + s^{\varrho+1}(\vartheta+\zeta) + s^{2\varrho+1}2(\vartheta+\zeta) + s^{3\varrho+1}(\vartheta+\zeta)\Big[4 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\Big] \\ & + s^{4\varrho+1}(\vartheta+\zeta)\Big[4 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2} + \frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}\Big] + \cdots. \end{aligned} \tag{15}$$

$$V(\vartheta,\zeta,s) = (\vartheta+\zeta)\bigg[s^2 + s^{\varrho+1} + 2s^{2\varrho+1} + s^{3\varrho+1}\Big[4 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\Big] + s^{4\varrho+1}\Big[4 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2} + \frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}\Big] + \cdots\bigg]. \tag{16}$$

Applying inverse Elzaki transform, we get

$$\nu(\vartheta,\zeta,\tau) = (\vartheta+\zeta)\Bigg[1 + \frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + \frac{2\tau^{\varrho}}{\Gamma(2\varrho+1)} + \frac{[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}]}{\Gamma(3\varrho+1)}\tau^{3\varrho} + \frac{[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2} + \frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}]}{\Gamma(4\varrho+1)}\tau^{4\varrho} + \cdots\Bigg]. \tag{17}$$

Now, if we substitute $\varrho = 1$ in Equation (17), it gives

$$\nu(\vartheta,\zeta,\tau) = (\vartheta+\zeta)\Big[1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \cdots\Big] \tag{18}$$

The result of Equation (18) closed contact with the Maclaurin series of

$$\nu(\vartheta,\zeta,\tau) = \frac{\vartheta+\zeta}{1-\tau}. \tag{19}$$

In Figure 1, the exact and analytical solution of $\nu(\vartheta,\zeta,\tau)$ at $\varrho = 1$ and 0.8 of Example 1. In Figure 2, the analytical solutions of $\nu(\vartheta,\zeta,\tau)$ at $\varrho = 0.6$ and 0.4 and similarly in Figure 3, the different fractional order of ϱ of $\nu(\vartheta,\zeta,\tau)$ of Example 1, which show that the close contact with the exact solution.

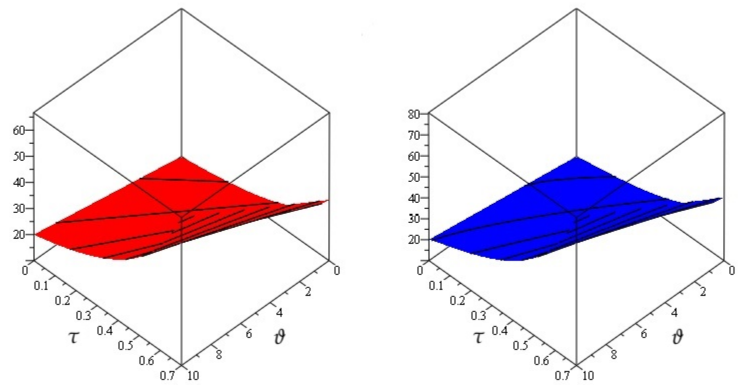

Figure 1. Exact and analytical solution of $\nu(\vartheta,\zeta,\tau)$ at $\varrho = 1$ and 0.8 of Example 1.

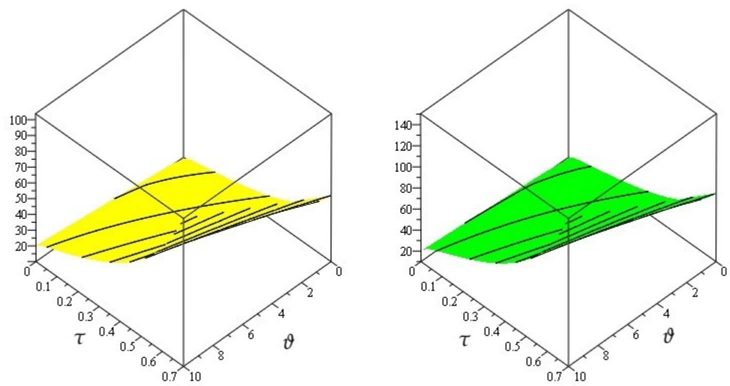

Figure 2. The analytical solutions of $\nu(\vartheta,\zeta,\tau)$ at $\varrho = 0.6$ and 0.4 of Example 1.

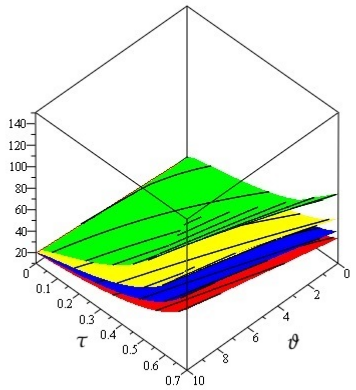

Figure 3. The different fractional-order of ϱ of $\nu(\vartheta,\zeta,\tau)$ of Example 1.

Example 2. *Consider the fractional two-dimensional Burger's equation*

$$D_\tau^\varrho \nu = \nu\nu_\vartheta + \nu_{\vartheta\vartheta} + \nu_{yy} + \nu_{zz}, \quad 0 < \varrho \le 1 \tag{20}$$

with the IC

$$\nu(\vartheta,\zeta,z,0) = \vartheta + \zeta + z = f_0(\vartheta) \tag{21}$$

The exact solution of (20) $(\varrho = 1)$ is

$$\nu(\vartheta,\zeta,z,\tau) = \frac{\vartheta+\zeta+z}{1-\tau} \tag{22}$$

Using Elzaki transformation to (20) and applying the IC (21), we obtain

$$\begin{aligned} V(\vartheta,\zeta,z,s) &= s^2(\vartheta+\zeta+z) + s^\varrho \Xi_\tau[[\Xi_\tau^{-1}\{V_\vartheta(\vartheta,\zeta,z,s)\}][\Xi_\tau^{-1}\{V_\vartheta(\vartheta,\zeta,z,s)\}]] \\ &+ s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{V_{\vartheta\vartheta}(\vartheta,\zeta,z,s)\}] + s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{V_{yy}(\vartheta,\zeta,z,s)\}] + s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{V_{zz}(\vartheta,\zeta,z,s)\}]. \end{aligned} \tag{23}$$

The k-th truncated term series of (23) is

$$V_k(\vartheta,\zeta,z,s) = s^2(\vartheta+\zeta+z) + \sum_{n=1}^{k} s^{n\varrho+1} f_n(\vartheta,\zeta,z). \tag{24}$$

and the k-th Elzaki residual function is

$$\begin{aligned} \Xi_\tau Res_k(\vartheta,\zeta,z,s) &= V_k(\vartheta,\zeta,z,s) - s^2(\vartheta+\zeta+z) - s^\varrho \Xi_\tau[[\Xi_\tau^{-1}\{V(\vartheta,\zeta,z,s)\}][\Xi_\tau^{-1}\{\frac{\partial}{\partial\vartheta}V_k(\vartheta,\zeta,z,s)\}]] \\ &- s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{\frac{\partial^2}{\partial\vartheta^2}V_k(\vartheta,\zeta,z,s)\}] - s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{\frac{\partial^2}{\partial\zeta^2}V_k(\vartheta,\zeta,z,s)\}] + s^\varrho \Xi_\tau[\Xi_\tau^{-1}\{\frac{\partial^2}{\partial z^2}V_k(\vartheta,\zeta,z,s)\}]. \end{aligned} \tag{25}$$

Now, to calculate $f_k(\vartheta,\zeta)$, $k = 1,2,3,\cdots$, we put the kth-truncate series (24) into the kth-Elzaki residual term (25), multiply the solution equation by $s^{k\varrho+1}$, and then the relation is solved recursively $\lim_{s\to\infty}[s^{k\varrho+1}Res_k(\vartheta,\zeta,s)] = 0$, $k = 1,2,3,\cdots$ *for f_k. The following are the some terms $f_k(\vartheta,\zeta)$*

$$
\begin{aligned}
f_1(\vartheta,\zeta,z) &= (\vartheta+\zeta+z)\\
f_2(\vartheta,\zeta,z) &= 2(\vartheta+\zeta+z)\\
f_3(\vartheta,\zeta,z) &= (\vartheta+\zeta+z)[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}]\\
f_4(\vartheta,\zeta,z) &= 2(\vartheta+\zeta+z)\left[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}+\frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}\right]\\
&\vdots
\end{aligned}
\tag{26}
$$

put the value of $f_n(\vartheta,\zeta,z)$, $(n \geq 1)$ in Equation (24), we obtain

$$
\begin{aligned}
V(\vartheta,\zeta,z,s) &= s^2\vartheta+\zeta+z+s^{\varrho+1}(\vartheta+\zeta+z)+s^{2\varrho+1}2(\vartheta+\zeta+z)+s^{3\varrho+1}(\vartheta+\zeta+z)\left[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\right]\\
&+s^{4\varrho+1}2(\vartheta+\zeta+z)\left[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}+\frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}\right]+\cdots.
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
V(\vartheta,\zeta,z,s) = (\vartheta+\zeta+z)\Bigg[&s^2+s^{\varrho+1}+2s^{2\varrho+1}+s^{3\varrho+1}\left[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\right]\\
&+s^{4\varrho+1}\left[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}+\frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}\right]+\cdots\Bigg].
\end{aligned}
\tag{28}
$$

Applying inverse Elzaki transform, we obtain

$$
\begin{aligned}
v(\vartheta,\zeta,z,\tau) = (\vartheta+\zeta+z)\Bigg[&1+\frac{\tau^{\varrho}}{\Gamma(\varrho+1)}+\frac{2\tau^{\varrho}}{\Gamma(2\varrho+1)}+\frac{[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}]}{\Gamma(3\varrho+1)}\tau^{3\varrho}\\
&+\frac{[4+\frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}+\frac{2\Gamma(1+3\varrho)}{\Gamma(1+\varrho)\Gamma(1+2\varrho)}]}{\Gamma(4\varrho+1)}\tau^{4\varrho}+\cdots\Bigg].
\end{aligned}
\tag{29}
$$

Now, if we substitute $\varrho = 1$ in Equation (29), it gives

$$
v(\vartheta,\zeta,z,\tau) = (\vartheta+\zeta+z)[1+\tau+\tau^2+\tau^3+\tau^4+\cdots] \tag{30}
$$

The solution of Equation (30) agrees with the Maclaurin series is

$$
v(\vartheta,\zeta,z,\tau) = \frac{\vartheta+\zeta+z}{1-\tau}. \tag{31}
$$

In Figure 4, the exact and analytical solution of $v(\vartheta,\zeta,\tau)$ at $\varrho = 1$ and 0.8 of Example 2. In Figure 5, the analytical solutions of $v(\vartheta,\zeta,\tau)$ at $\varrho = 0.6$ and 0.4 and similarly in Figure 6, the different fractional order of ϱ of $v(\vartheta,\zeta,\tau)$ of Example 2, which show that the close contact with the exact solution.

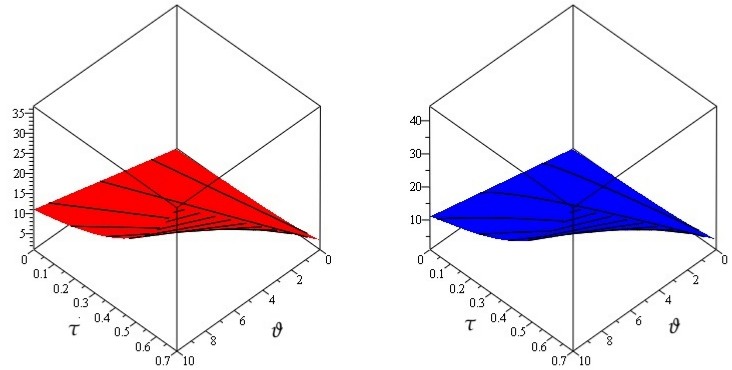

Figure 4. Exact and analytical solution of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 1$ and 0.8 of Example 2.

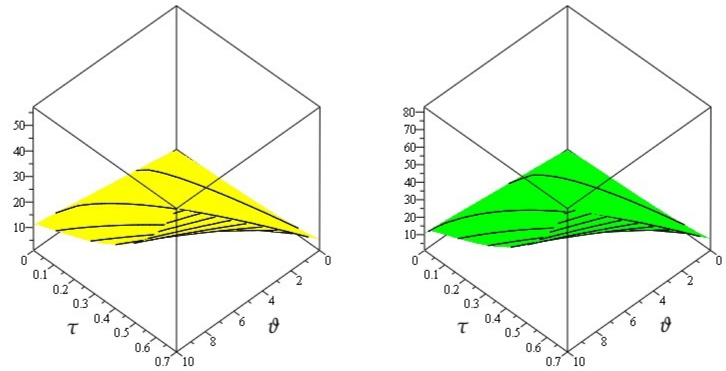

Figure 5. The analytical solutions of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 0.6$ and 0.4 of Example 2.

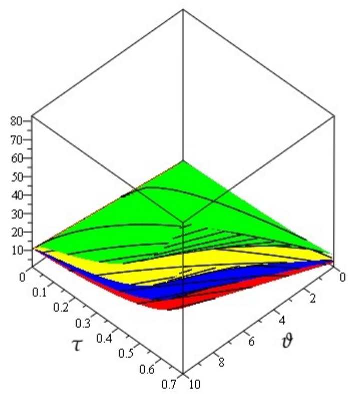

Figure 6. The different fractional-order of ϱ of $\nu(\vartheta, \zeta, z, \tau)$ of Example 2.

Example 3. *Consider the fractional one dimensional system Burger's equation*

$$\begin{aligned} D_\tau^\varrho \nu &= 2\nu\nu_\vartheta - \nu_{\vartheta\vartheta} + (\nu\psi)_\vartheta, \\ D_\tau^\varrho \psi &= -2\psi\psi_\vartheta - \psi_{\vartheta\vartheta} - (\nu\psi)_\vartheta, \quad 0 < \varrho \leq 1 \end{aligned} \tag{32}$$

with the ICs

$$\nu(\vartheta, 0) = \sin(\vartheta), \;\; \psi(\vartheta, 0) = -\sin(\vartheta). \tag{33}$$

Using Elzaki transformation to (32) and the ICs given in (32), we obtain

$$\begin{aligned}
V(\vartheta,s) &= s^2\sin(\vartheta) + 2s^{\varrho}\Xi_\tau[\Xi_\tau^{-1}\{V(\vartheta,s)\frac{\partial}{\partial\vartheta}V(\vartheta,s)\}] \\
&\quad + s^{\varrho}\Xi_\tau([\Xi_\tau^{-1}V(\vartheta,s)][\Xi_\tau^{-1}V(\vartheta,s)])_\vartheta - s^{\varrho}\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}V(\vartheta,s)]\}, \\
\Psi(\vartheta,s) &= s^2 - \sin(\vartheta) - 2s^{\varrho}\Xi_\tau[\Xi_\tau^{-1}\{\Psi(\vartheta,s)\frac{\partial}{\partial\vartheta}\Psi(\vartheta,s)\}] \\
&\quad - s^{\varrho}\Xi_\tau([\Xi_\tau^{-1}V(\vartheta,s)][\Xi_\tau^{-1}\Psi(\vartheta,s)])_\vartheta - s^{\varrho}\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}\Psi(\vartheta,s)]\}.
\end{aligned} \tag{34}$$

The k-th truncated term series of Equation (34) is

$$\begin{aligned}
V_k(\vartheta,s) &= s^2\sin(\vartheta) + \sum_{n=0}^{\infty} s^{n\varrho+1} f_n(\vartheta), \\
\Psi_k(\vartheta,s) &= -s^2\sin(\vartheta)s + \sum_{n=0}^{\infty} s^{n\varrho+1} g_n(\vartheta).
\end{aligned} \tag{35}$$

and the k-th Elzaki residual function is

$$\begin{aligned}
\Xi_\tau Res_k(\vartheta,s) &= V_k(\vartheta,s) - \frac{\sin(\vartheta)}{s} - 2s^{\varrho}\Xi_\tau[\Xi_\tau^{-1}\{V_k(\vartheta,s)\frac{\partial}{\partial\vartheta}V_k(\vartheta,s)\}] \\
&\quad - s^{\varrho}\Xi_\tau([\Xi_\tau^{-1}V_k(\vartheta,s)][\Xi_\tau^{-1}\Psi_k(\vartheta,s)])_\vartheta + s^{\varrho}\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}V_k(\vartheta,s)]\}, \\
\Xi_\tau Res_k(\vartheta,s) &= \Psi(\vartheta,s) - \frac{-\sin(\vartheta)}{s} + 2s^{\varrho}\Xi_\tau[\Xi_\tau^{-1}\{\Psi_k(\vartheta,s)\frac{\partial}{\partial\vartheta}\Psi_k(\vartheta,s)\}] \\
&\quad - s^{\varrho}\Xi_\tau([\Xi_\tau^{-1}V_k(\vartheta,s)][\Xi_\tau^{-1}\Psi_k(\vartheta,s)])_\vartheta - s^{\varrho}\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}\Psi_k(\vartheta,s)]\}.
\end{aligned} \tag{36}$$

Now, to determine $f_k(\vartheta)$ and $g_k(\vartheta)$, $k = 1,2,3,\cdots$, we substitute the kth-truncated series (35) into the kth-Elzaki residual function (36), multiply the resulting equation by $s^{k\varrho+1}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\varrho+1}Res_k(\vartheta,s)] = 0$, $k = 1,2,3,\cdots$ for f_k and g_k. Some terms of the sequences $f_k(\vartheta)$ and $g_k(\vartheta)$

$$\begin{aligned}
f_1(\vartheta) &= \sin(\vartheta), \\
g_1(\vartheta) &= -\sin(\vartheta). \\
f_2(\vartheta) &= \sin(\vartheta), \\
g_2(\vartheta) &= -\sin(\vartheta). \\
f_3(\vartheta) &= \sin(\vartheta), \\
g_3(\vartheta) &= -\sin(\vartheta). \\
f_4(\vartheta) &= \sin(\vartheta), \\
g_4(\vartheta) &= -\sin(\vartheta). \\
&\vdots
\end{aligned} \tag{37}$$

Putting the values of $f_n(\vartheta)$ and $g_n(\vartheta)$ for $(n \geq 1)$ in Equation (35), we obtain

$$\begin{aligned}
V(\vartheta,s) &= s^2\sin(\vartheta) + s^{\varrho+1}\sin(\vartheta) + s^{2\varrho+1}\sin(\vartheta) + s^{3\varrho+1}\sin(\vartheta) + \cdots, \\
\Psi(\vartheta,s) &= -s^2\sin(\vartheta) - s^{\varrho+1}\sin(\vartheta) - s^{2\varrho+1}\sin(\vartheta) - s^{3\varrho+1}\sin(\vartheta) + \cdots.
\end{aligned} \tag{38}$$

$$\begin{aligned}
V(\vartheta,s) &= \sin(\vartheta)\left[s + s^{\varrho+1} + s^{2\varrho+1} + s^{3\varrho+1} + \cdots\right], \\
\Psi(\vartheta,s) &= -\sin(\vartheta)\left[s + s^{\varrho+1} + s^{2\varrho+1} + s^{3\varrho+1} + \cdots\right].
\end{aligned} \tag{39}$$

Applying the inverse Elzaki transform to Equation (39), we obtain

$$\begin{aligned}\nu(\vartheta,\tau) &= \sin(\vartheta)\left[1+\frac{\tau^{\varrho}}{\Gamma(\varrho+1)}+\frac{\tau^{2\varrho}}{\Gamma(2\varrho+1)}+\frac{\tau^{3\varrho}}{\Gamma(3\varrho+1)}+\cdots\right],\\ \psi(\vartheta,\tau) &= -\sin(\vartheta)\left[1+\frac{\tau^{\varrho}}{\Gamma(\varrho+1)}+\frac{\tau^{2\varrho}}{\Gamma(2\varrho+1)}+\frac{\tau^{3\varrho}}{\Gamma(3\varrho+1)}+\cdots\right].\end{aligned} \tag{40}$$

Putting $\varrho = 1$, we obtain the solution of Equation (32) in closed form

$$\begin{aligned}\nu(\vartheta,\tau) &= e^{\tau}\sin(\vartheta),\\ \psi(\vartheta,\tau) &= -e^{\tau}\sin(\vartheta).\end{aligned} \tag{41}$$

Figure 7 shows the exact and analytical solution of $\nu(\vartheta,\zeta,z,\tau)$ at $\varrho = 1$ and 0.8 of Example 3. Figure 8 shows the analytical solutions of $\nu(\vartheta,\zeta,z,\tau)$ at $\varrho = 0.6$ and 0.4. Similarly, Figure 9, shows the different fractional-order of ϱ of $\nu(\vartheta\zeta,z,\tau)$ of Example 3, which show the close contact with the exact solution. Figures 10–12 show the close relation with respect to $\nu(\vartheta,\zeta,z,\tau)$ of Example 3.

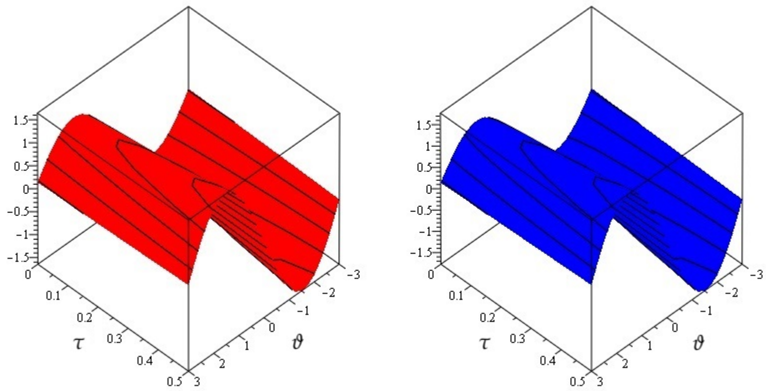

Figure 7. Exact and analytical solution of $\nu(\vartheta,\zeta,z,\tau)$ at $\varrho = 1$ and 0.8 of Example 3.

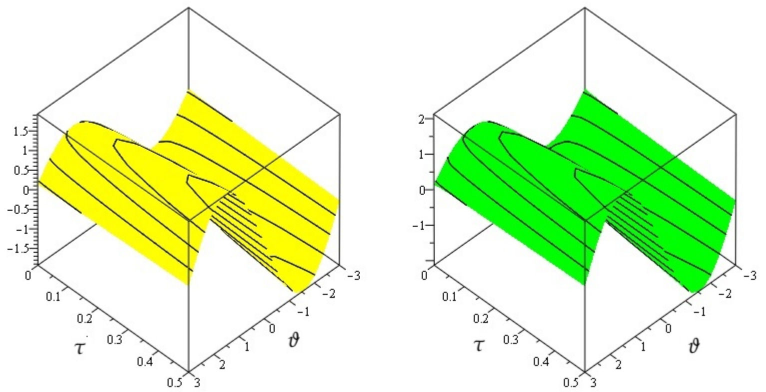

Figure 8. The analytical solutions of $\nu(\vartheta,\zeta,z,\tau)$ at $\varrho = 0.6$ and 0.4 of Example 3.

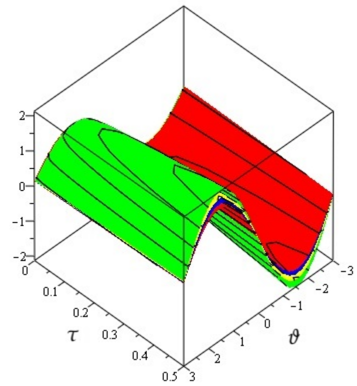

Figure 9. The different fractional-order of ϱ of $\nu(\vartheta, \zeta, z, \tau)$ of Example 3.

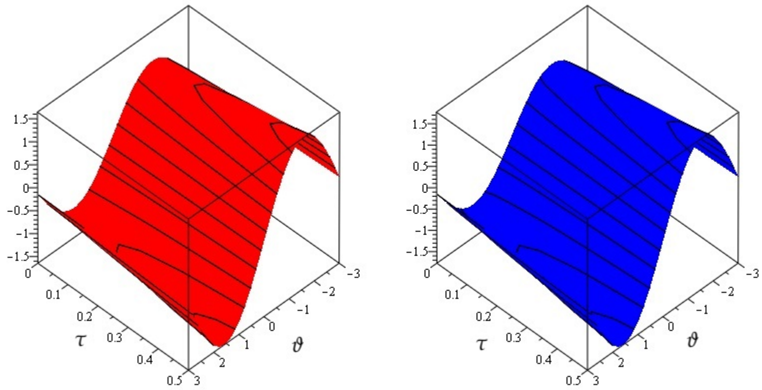

Figure 10. Exact and analytical solution of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 1$ and 0.8 of Example 3.

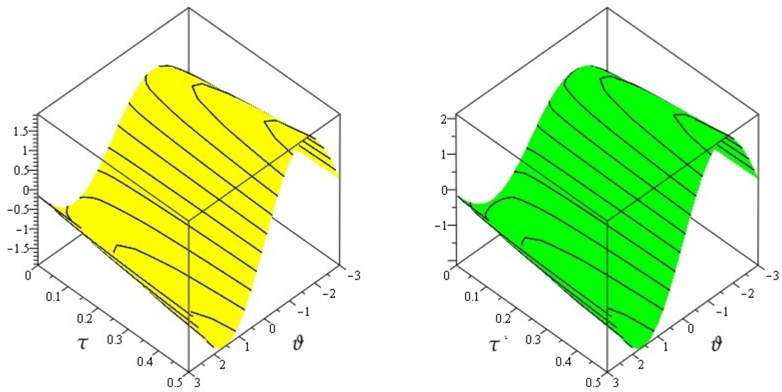

Figure 11. The analytical solutions of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 0.6$ and 0.4 of Example 3.

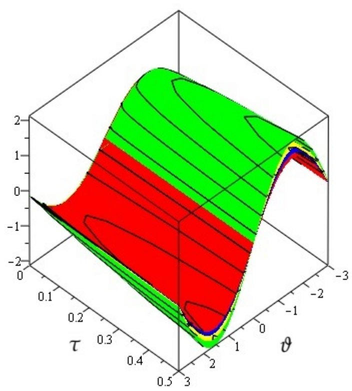

Figure 12. The different fractional-order of ϱ of $\nu(\vartheta, \zeta, z, \tau)$ of Example 3.

Example 4. *Consider the fractional two dimensional system of Burger's equation*

$$
\begin{aligned}
D_\tau^\varrho \nu &= \nu_{\vartheta\vartheta} + \nu_{\zeta\zeta} - \nu\nu_\vartheta - \psi\nu_\zeta, \\
D_\tau^\varrho \psi &= \psi_{\vartheta\vartheta} + \nu_{\zeta\zeta} - \nu\psi_\vartheta - \psi\psi_\zeta, \ \ 0 < \varrho \leq 1
\end{aligned} \tag{42}
$$

with the ICs

$$
\nu(\vartheta,\zeta,0) = \vartheta + \zeta, \ \ \psi(\vartheta,\zeta,0) = \vartheta - \zeta. \tag{43}
$$

Applying Elzaki transform to Equation (42) and applying the ICs in Equation (43), we obtain

$$
\begin{aligned}
V(\vartheta,\zeta,s) &= s^2\vartheta + \zeta + s^\varrho \xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}V(\vartheta,\zeta,s)]\} + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\zeta^2}V(\vartheta,\zeta,s)]\} \\
&\quad - s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[V(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\vartheta}V(\vartheta,\zeta,s)]\} - s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\Psi(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\zeta}V(\vartheta,\zeta,s)]\}, \\
\Psi(\vartheta,\zeta,s) &= s^2\vartheta + \zeta + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}\Psi(\vartheta,\zeta,s)]\} + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\zeta^2}\Psi(\vartheta,\zeta,s)]\} \\
&\quad - s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[V(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\vartheta}\Psi(\vartheta,\zeta,s)]\} - s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\Psi(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\zeta}\Psi(\vartheta,\zeta,s)]\},
\end{aligned} \tag{44}
$$

The k-th truncated functions series of Equation (44) is

$$
\begin{aligned}
V_k(\vartheta,\zeta,s) &= \frac{\vartheta+\zeta}{s} + \sum_{n=0}^{\infty} \frac{f_n(\vartheta,\zeta)}{s^{n\varrho+1}}, \\
\Psi_k(\vartheta,\zeta,s) &= \frac{\vartheta-\zeta}{s} + \sum_{n=0}^{\infty} \frac{g_n(\vartheta,\zeta)}{s^{n\varrho+1}}.
\end{aligned} \tag{45}
$$

and k-th Elzaki residual function is

$$
\begin{aligned}
\Xi_\tau Res_k(\vartheta,\zeta,s) &= V_k(\vartheta,\zeta,s) - s^2\vartheta + \zeta + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}V_k(\vartheta,\zeta,s)]\} - s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\zeta^2}V_k(\vartheta,\zeta,s)]\} \\
&\quad + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[V_k(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\vartheta}V_k(\vartheta,\zeta,s)]\} + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\Psi_k(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\zeta}V_k(\vartheta,\zeta,s)]\}, \\
\Xi_\tau Res_k(\vartheta,\zeta,s) &= \Psi(\vartheta,\zeta,s) - s^2\vartheta + \zeta + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\vartheta^2}\Psi_k(\vartheta,\zeta,s)]\} - s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\frac{\partial^2}{\partial\zeta^2}\Psi_k(\vartheta,\zeta,s)]\} \\
&\quad + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[V_k(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\vartheta}\Psi_k(\vartheta,\zeta,s)]\} + s^\varrho \Xi_\tau\{\Xi_\tau^{-1}[\Psi_k(\vartheta,\zeta,s)]\Xi_\tau^{-1}[\frac{\partial}{\partial\zeta}\Psi_k(\vartheta,\zeta,s)]\},
\end{aligned} \tag{46}
$$

Now, to determine $f_k(\vartheta,\zeta)$ and $g_k(\vartheta,\zeta)$,$k = 1,2,3,\cdots$, we substitute the kth-truncated series (45) into the kth-Elzaki residual function (46), multiply the resulting equation by $s^{k\varrho+1}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\varrho+1}Res_k(\vartheta,s)] = 0$, $k = 1,2,3,\cdots$ for f_k and g_k. The first some terms of the sequences $f_k(\vartheta,\zeta)$ and $g_k(\vartheta,\zeta)$

$$
\begin{aligned}
f_1(\vartheta,\zeta) &= -2\vartheta, \\
g_1(\vartheta,\zeta) &= -2\zeta. \\
f_2(\vartheta,\zeta) &= 4(\vartheta+\zeta), \\
g_2(\vartheta,\zeta) &= 4(\vartheta-\zeta). \\
f_3(\vartheta,\zeta) &= -16\vartheta - 4\vartheta\frac{\Gamma(2\varrho+1)}{\Gamma(\varrho+1)^2}, \\
g_3(\vartheta,\zeta) &= -16\zeta - 4\zeta\frac{\Gamma(2\varrho+1)}{\Gamma(\varrho+1)^2}. \\
&\ \ \vdots
\end{aligned} \tag{47}
$$

substituting the value of $f_n(\vartheta,\zeta)$ and $g_n(\vartheta,\zeta)$, for $n \geq 1$ in Equation (45), we have

$$
\begin{aligned}
U(\vartheta,\zeta,s) &= s^2\vartheta + \zeta - 2s^{\varrho+1}\vartheta + 4s^{2\varrho+1}(\vartheta+\zeta) + s^{\varrho+1}\left(-16\vartheta - 4\vartheta\frac{\Gamma(2\varrho+1)}{\Gamma(\varrho+1)^2}\right) + \cdots, \\
\Psi(\vartheta,\zeta,s) &= s^2\vartheta - \zeta - 2s^{\varrho+1}\zeta + 4s^{\varrho+1}(\vartheta+\zeta) + s^{\varrho+1}\left(-16\zeta - 4\zeta\frac{\Gamma(2\varrho+1)}{\Gamma(\varrho+1)^2}\right) + \cdots.
\end{aligned} \tag{48}
$$

Applying inverse Elzaki transform to Equation (48), we have

$$
\begin{aligned}
\nu(\vartheta,\zeta,\tau) &= \vartheta + \zeta + \frac{-2\vartheta}{\Gamma(\varrho+1)}\tau^{\varrho} + \frac{4(\vartheta+\zeta)}{\Gamma(2\varrho+1)}\tau^{2\varrho} + \frac{-16\vartheta - 4\vartheta\frac{\Gamma(2\varrho+1)}{\Gamma(\varrho+1)^2}}{\Gamma(3\varrho+1)}\tau^{3\varrho} + \cdots, \\
\psi(\vartheta,\zeta,\tau) &= \vartheta - \zeta + \frac{-2\zeta}{\Gamma(\varrho+1)}\tau^{\varrho} + \frac{4(\vartheta+\zeta)}{\Gamma(2\varrho+1)}\tau^{2\varrho} + \frac{-16\zeta - 4\zeta\frac{\Gamma(2\varrho+1)}{\Gamma(\varrho+1)^2}}{\Gamma(3\varrho+1)}\tau^{3\varrho} + \cdots.
\end{aligned} \tag{49}
$$

Putting $\varrho = 1$, we obtain the solution of Equation (42) in closed form

$$
\begin{aligned}
\nu(\vartheta,\zeta,\tau) &= \vartheta + \zeta - 2\vartheta\tau + 2(\vartheta+\zeta)\tau^2 - 4\vartheta\tau^3 + 4(\vartheta+\zeta)\tau^4 - 8\vartheta\tau^5 + \cdots, \\
\psi(\vartheta,\zeta,\tau) &= \vartheta - \zeta - 2\zeta\tau + 2(\vartheta+\zeta)\tau^2 - 4\zeta\tau^3 + 4(\vartheta-\zeta)\tau^4 - 8\vartheta\tau^5 + \cdots. \\
\nu(\vartheta,\zeta,\tau) &= \frac{\vartheta + \zeta - 2\vartheta\tau}{1 - 2\tau^2}, \\
\psi(\vartheta,\zeta,\tau) &= \frac{\vartheta - \zeta - 2\zeta\tau}{1 - 2\tau^2}.
\end{aligned} \tag{50}
$$

Figure 13 shows the exact and analytical solution of $\nu(\vartheta,\zeta,z,\tau)$ at $\varrho = 1$ and 0.8 of Example 4. Figure 14 shows the analytical solutions of $\nu(\vartheta,\zeta,z,\tau)$ at $\varrho = 0.6$ and 0.4. Figure 15 similarly shows the different fractional-order of ϱ of $\nu(\vartheta,\zeta,z,\tau)$ of Example 4, which show the close contact with the exact solution. Figures 16–18 show the close relation with respect to $\nu(\vartheta,\zeta,z,\tau)$ of Example 4.

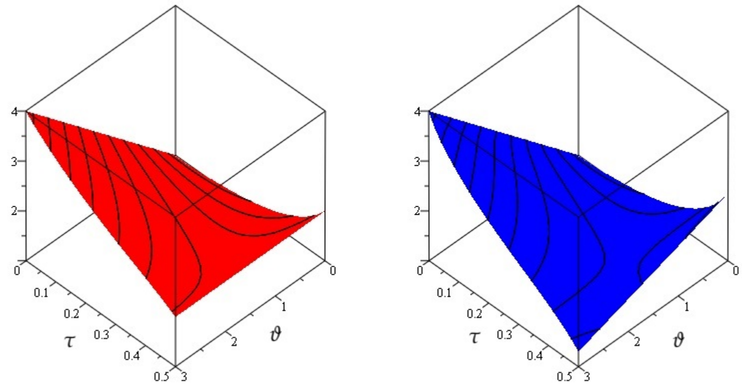

Figure 13. Exact and analytical solution of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 1$ and 0.8 of Example 4.

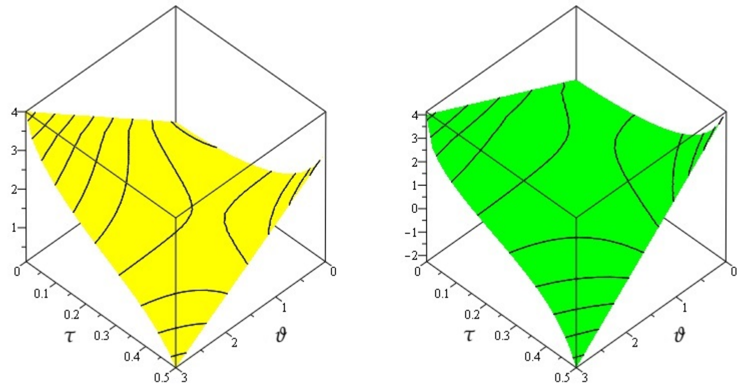

Figure 14. The analytical solutions of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 0.6$ and 0.4 of Example 4.

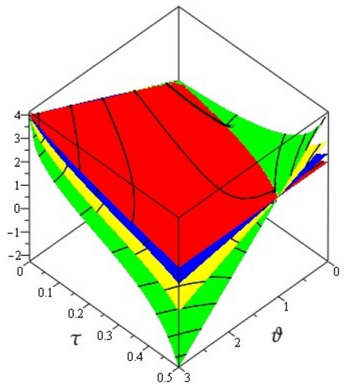

Figure 15. The different fractional-order of ϱ of $\nu(\vartheta, \zeta, z, \tau)$ of Example 4.

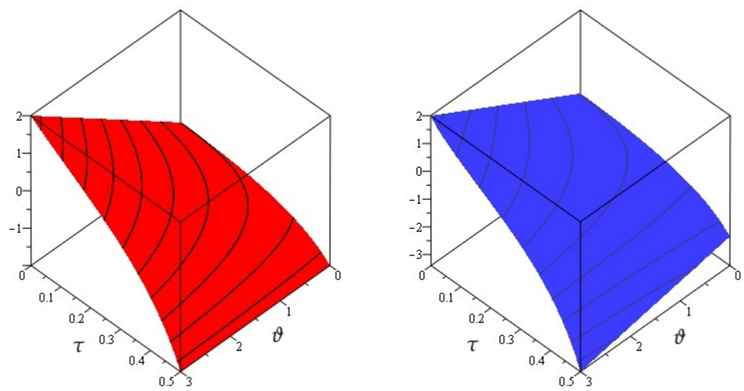

Figure 16. Exact and analytical solution of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 1$ and 0.8 of Example 4.

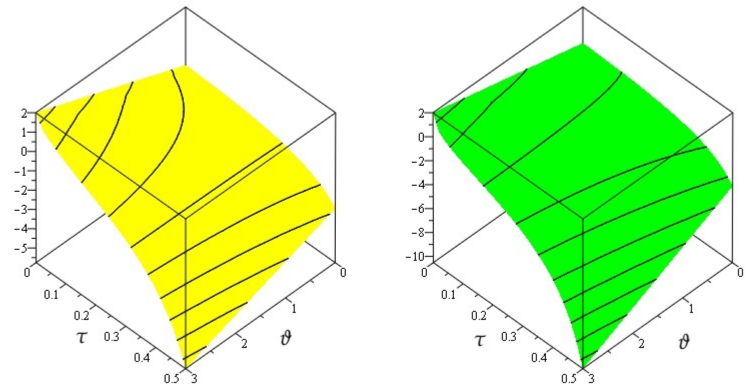

Figure 17. The analytical solutions of $\nu(\vartheta, \zeta, z, \tau)$ at $\varrho = 0.6$ and 0.4 of Example 4.

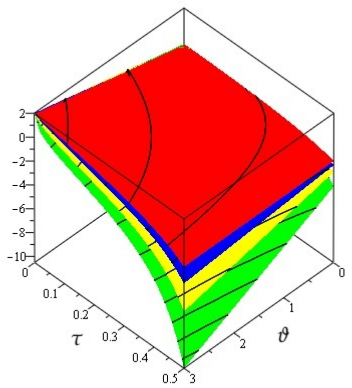

Figure 18. The different fractional-order of ϱ of $\nu(\vartheta, \zeta, z, \tau)$ of Example 4.

5. Conclusions

In this paper, the analytical solutions are constructed and analyzed for Burgers' equations with suitable initial conditions utilizing the RPSTM under time-Caputo differentiability. The fractional RPSTM is modified in the current method by tying it to the Yang transform operator. The advantage of use the RPSTM is that it gives more accurate convergence McLaurin series and needs only a small size of calculation without involving the perturbation, discretization, or any other physical restrictive condition. Two well-known physical applications are tested in order to show the viability and superiority of the proposed technique. Graphics and numerical simulation are used to discuss the resulting approximations. The resulting results are contrasted with those of other widely used, published methodologies. In light of these findings, it is clear that the RPSTM is a simple and practical method for handling the wide spectrum of non-linear fractional partial differential equations that might emerge in engineering and scientific situations. Future research can use the RPSTM to discover both precise and approximative answers to fractional partial differential equations. As a result, physical models and dynamical models can be handled by the RPSTM application.

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References

1. Al-Sawalha, M.M.; Agarwal, R.P.; Shah, R.; Ababneh, O.Y.; Weera, W. A reliable way to deal with fractional-order equations that describe the unsteady flow of a polytropic gas. *Mathematics* **2022**, *10*, 2293. [CrossRef]
2. Said, L.A.; Madian, A.H.; Radwan, A.G.; Soliman, A.M. Fractional order oscillator with independent control of phase and frequency. In Proceedings of the 2014 2nd International Conference on Electronic Design (ICED), Penang, Malaysia, 19–21 August 2014; IEEE: Piscataway, NJ, USA, 2014; pp. 224–229.
3. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 1140–1153. [CrossRef]
4. Sabatier, J.A.T.M.J.; Agrawal, O.P.; Machado, J.T. *Advances in Fractional Calculus*; Springer: Dordrecht, The Netherlands, 2007; Volume 4.
5. Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus: Models and Numerical Methods*; World Scientific: Singapore, 2012; Volume 3.
6. Debnath, L. Recent applications of fractional calculus to science and engineering. *Int. J. Math. Math. Sci.* **2003**, *2003*, 3413–3442. [CrossRef]
7. Al-Sawalha, M.M.; Alshehry, A.S.; Nonlaopon, K.; Shah, R.; Ababneh, O.Y. Fractional view analysis of delay differential equations via numerical method. *AIMS Math.* **2022**, *7*, 20510–20523. [CrossRef]
8. Mukhtar, S.; Shah, R.; Noor, S. The Numerical Investigation of a Fractional-Order Multi-Dimensional Model of Navier-Stokes Equation via Novel Techniques. *Symmetry* **2022**, *14*, 1102. [CrossRef]
9. Al-Sawalha, M.M.; Alshehry, A.S.; Nonlaopon, K.; Shah, R.; Ababneh, O.Y. Approximate analytical solution of time-fractional vibration equation via reliable numerical algorithm. *AIMS Math.* **2022**, *7*, 19739–19757. [CrossRef]
10. Al-Sawalha, M.M.; Khan, A.; Ababneh, O.Y.; Botmart, T. Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives. *AIMS Math.* **2022**, *7*, 18334–18359. [CrossRef]
11. Kai, Y.; Chen, S.; Zhang, K.; Yin, Z. Exact solutions and dynamic properties of a non-linear fourth-order time-fractional partial differential equation. *Waves Random Complex Media* **2022** , 1–12. [CrossRef]
12. Cao, Y.; Nikan, O.; Avazzadeh, Z. A localized meshless technique for solving 2D nonlinear integro-differential equation with multi-term kernels. *Appl. Numer. Math.* **2023**, *183*, 140–156. [CrossRef]
13. Jassim, H.K.; Shareef, M.A. On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator. *J. Math. Comput. Sci.* **2021**, *23*, 58–66. [CrossRef]
14. Oderinu, R.A.; Owolabi, J.A.; Taiwo, M. Approximate solutions of linear time-fractional differential equations. *J. Math. Comput. Sci.* **2022**, *2023*, 60–72. [CrossRef]
15. Bateman, H. Some recent researches on the motion of fluids. *Mon. Weather Rev.* **1915**, *43*, 163–170. [CrossRef]
16. Burgers, J.M. A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.* **1948**, 1, 171–199.
17. Cole, J.D. On a quasilinear parabolic equations occurring in aerodynamics. *Quart. Appl. Math.* **1951**, 9, 225–236. [CrossRef]
18. Wazwaz, A.M. *Partial Differential Equations: Methods and Applications*; CRC Press: Boca Raton, FL, USA, 2002; p. 474.
19. Naghipour, A.; Manafian, J. Application of the laplace adomian decomposition and implicit methods for solving Burgers' equation. *TWMS J. Pure Appl. Math.* **2015**, *6*, 68–77.
20. Rashidi, M.M.; Erfani, E. New analytical method for solving Burgers and nonlinear heat transfer equations and comparison with HAM. *Comput. Phys. Comm.* **2009**, *180*, 1539–1544. [CrossRef]
21. Moslem, W.M.; Sabry, R. Zakharov-Kuznetsov-Burgers' equation for dust ion acoustic waves. *Chaos Solitons Fractals* **2008**, 36, 628–634. [CrossRef]
22. Basto, M.; Semiao, V.; Calheiros, F. Dynamics and synchronization of numerical solutions of the Burgers' equation. *J. Comput. Appl. Math.* **2009**, *231*, 793–806. [CrossRef]
23. Adomian, G. *Solving Frontier Problems in Physics: The Decomposition Method*; Kluwer Academic: Boston, MA, USA, 1994.
24. He, J.H. Variational iteration method-a kind of nonlinear analytical technique: Some examples. *Int. J. Non Linear Mech.* **1999**, *34*, 699–708. [CrossRef]
25. He, J.H. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Engrg.* **1999**, *178*, 257–262. [CrossRef]
26. Liao, S.J. *Beyond Perturbation: Introduction to the Homotopy Analysis Method*; Chapman and Hall/CRC Press: Boca Raton, FL, USA, 2003.
27. Zhou, J.K. *Differential Transform and Its Application for Electrical Circuits*; Huazhong University Press: Wuhan, China, 1986. (In Chinese)
28. Saadeh, R.; Alaroud, M.; Al-Smadi, M.; Ahmad, R.R.; Din, U.K.S. Application of fractional residual power series algorithm to solve Newell–Whitehead–Segel equation of fractional order. *Symmetry* **2019**, *11*, 1431. [CrossRef]

29. Freihet, A.; Hasan, S.; Alaroud, M.; Al-Smadi, M.; Ahmad, R.R.; Din, U.K.S. Toward computational algorithm for timefractional Fokker–Planck models. *Adv. Mech. Eng.* **2019**, *11*, 1–11. [CrossRef]
30. Al-Smadi, M.; Arqub, O.A.; Hadid, S. Approximate solutions of nonlinear fractional Kundu–Eckhaus and coupled fractional massive Thirring equations emerging in quantum field theory using conformable residual power series method. *Phys. Scripta* **2020**, *95*, 105205. [CrossRef]
31. Al-Smadi, M.; Arqub, O.A.; Momani, S. Numerical computations of coupled fractional resonant Schrödinger equations arising in quantum mechanics under conformable fractional derivative sense. *Phys. Scripta* **2020**, *95*, 075218. [CrossRef]
32. Al-Smadi, M.; Arqub, O.A.; Hadid, S. An attractive analytical technique for coupled system of fractional partial differential equations in shallow water waves with conformable derivative. *Commun. Theor. Phys.* **2020**, *72*, 085001. [CrossRef]
33. Komashynska, I.; Al-Smadi, M.; Arqub, O.A.; Momani, S. An efficient analytical method for solving singular initial value problems of nonlinear systems. *Appl. Math. Inform. Sci.* **2016**, *10*, 647-–656. [CrossRef]
34. Alaroud, M.; Al-smadi, M.; Ahmad, R.R.; Din, U.K.S. Numerical computation of fractional Fredholm integrodifferential equation of order 2b arising in natural sciences. *J. Phys. Conf. Ser.* **2019**, *1212*, 012022. [CrossRef]
35. Alaroud, M.; Al-Smadi, M.; Ahmad, R.R.; Din, U.K.S. An Analytical Numerical Method for Solving Fuzzy Fractional Volterra Integro-Differential Equations. *Symmetry* **2019**, *11*, 205. [CrossRef]
36. Alaroud, M.; Al-Smadi, M.; Ahmad, R.R.; Din, U.K.S. Computational optimization of residual power series algorithm for certain classes of fuzzy fractional differential equations. *Int. J. Diff. Equ.* **2018**, *2018*, 8686502. [CrossRef]
37. Eriqat, T.; El-Ajou, A.; Oqielat, M.N.; Al-Zhour, Z.; Momani, S. A New Attractive Analytic Approach for Solutions of Linear and Nonlinear Neutral Fractional Pantograph Equations. *Chaos Solitons Fract.* **2020**, *138*, 109957. [CrossRef]
38. Zhang, J.; Chen, X.; Li, L.; Zhou, C. Elzaki Transform Residual Power Series Method for the Fractional Population Diffusion Equations. *Eng. Lett.* **2021**, *29*, 1561–1572.