fractal and fractional

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Lie Symmetries and Third- and Fifth-Order Time-Fractional Polynomial Evolution Equations

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Abstract: This paper is concerned with a class of ten time-fractional polynomial evolution equations. The one-parameter Lie point symmetries of these equations are found and the symmetry reductions are provided. These reduced equations are transformed into nonlinear ordinary differential equations, which are challenging to solve by conventional methods. We search for power series solutions and demonstrate the convergence properties of such a solution.

Keywords: time fractional; Lie symmetry; Erdélyi–Kober

Citation: Kubayi, J.T.; Jamal, S. Lie Symmetries and Third- and Fifth-Order Time-Fractional Polynomial Evolution Equations. *Fractal Fract.* **2023**, *7*, 125. https://doi.org/10.3390/fractalfract7020125

Academic Editor: Riccardo Caponetto

Received: 7 January 2023
Revised: 25 January 2023
Accepted: 28 January 2023
Published: 29 January 2023

1. Introduction

Decades ago, Fujimoto-Watanabe [1] derived a complete list of the third-order polynomial evolution equations that admit nontrivial Lie–Bäcklund symmetries. Recursion operators map symmetries to symmetries so that certain integrable evolution equations admit infinitely many symmetries [2]. If the recursion operator is hereditary [3], the infinite series of symmetries commute with each other (see [1], Equation (2.7), p. 2). Most of these equations possess a hereditary recursion operator so that the Lie algebras of their Lie–Bäcklund symmetries are infinitely dimensional and commutative. From the third-order equations, all except the seventh equation admit a recursion operator (see Remark 1 in [1], p. 3).

Further, two fifth-order equations were also presented. In this work, we consider the time-fractional version of this class of equations. They are the following eight equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_x^3 u_{xxx} + a u_x^4, \tag{1}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_x^3 u_{xxx} + a u_x^3, \tag{2}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^3 u_{xxx} + 3u^2 u_x u_{xx} + a(u^3 u_{xx} + u^2 u_x^2) + \frac{2}{9} a^2 u^3 u_x, \tag{3}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^3 u_{xxx} + 3u^2 u_x u_{xx} + 4a u^3 u_x, \tag{4}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^3 u_{xxx} + 3u^2 u_x u_{xx} + 3a u^2 u_x, \tag{5}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^3 u_{xxx} + a u^3 u_x, \tag{6}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^3 u_{xxx} + \frac{3}{2} u^2 u_x u_{xx} + a(u^3 u_{xx} + u^2 u_x^2) + \frac{2}{9} a^2 u^3 u_x, \tag{7}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^3 u_{xxx} + \frac{3}{2} u^2 u_x u_{xx} + a u^2 u_x, \tag{8}$$

where $a > 0$ is a constant, and also two fifth-order equations that do not belong to the above hierarchies of equations

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^5 u_{xxxxx} + 5u^4(u_x u_{xxxx} + 2u_{xx}u_{xxx}), \tag{9}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u^5 u_{xxxxx} + 5u^4(u_x u_{xxxx} + \frac{1}{2}u_{xx}u_{xxx}) + \frac{15}{4}u^3 u_x^2 u_{xxx}. \tag{10}$$

Moreover, it is possible to construct chains of differential substitutions that connect the Fujimoto-Watanabe equations with the KdV, Sawada–Kotera, and Kaup equations [4]. These equations find applications in different areas such as mathematical physics, but they are primarily studied from the perspective of waves and ocean science [5].

Lie symmetry analysis has been widely applied to investigate nonlinear differential equations arising in both mathematics and physics [6,7], particularly for constructing their exact solutions. A Lie symmetry group of a system of differential equations is a group of transformations. The group of transformations relies on continuous parameters and maps any solution to another solution of the system. Lie group analysis is a systematic and direct method for deriving new exact and explicit solutions. The above equations were considered in [5,8] from the classical integer derivative perspective. Fractional derivatives are of superior interest in recent literature, see for example [9–13]. FDEs are often considered superior to classical integer-order equations since the latter experience memory effects and FDEs allow for the study of intermediate evolutionary behaviour at fractional time. Fractional differential equations, or FDEs, may most commonly contain Riemann and Liouville or Caputo derivatives. Symmetry methods have been extended to FDEs [14–17].

The plan of the paper is as follows. In Section 2, we define the preliminary mathematical notation and definitions required for this study. Thereafter, in Section 3, we list the symmetries for each of the ten equations under study. Section 4 contains the series solutions and reductions of the equations. A demonstration for testing of convergence of the series is given, and finally, in Section 5, we conclude.

2. Fractional Calculus and Symmetries

In this section, we present the mathematical framework required in subsequent sections of this paper. In existence, there are several different definitions of fractional derivatives. Time-fractional derivatives are commonly discussed in terms of Caputo, Grünwald–Letnikov, or Riemann–Liouville derivatives [18–20]. In this work, we limit ourselves to the latter—that is, we shall introduce the linear operators of differentiation in the framework of Riemann–Liouville fractional calculus, followed by the procedure for determining point symmetries of time-fractional PDEs.

$$G := \frac{\partial^\alpha u}{\partial t^\alpha} - \kappa(x, t, u, u_x, u_{xx}, \ldots) = 0. \tag{11}$$

Here, $0 < \alpha < 1$ is the parameter describing the order of the fractional time derivative.

The Riemann–Liouville fractional derivative is defined by

$$D_t^\alpha u(t,x) = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\theta,x)}{(t-\theta)^{\alpha+1-n}} d\theta, & n-1 \leq \alpha \leq n, n \in \mathbb{N} \end{cases} \tag{12}$$

where $\Gamma(z)$ is the Euler gamma function.

Suppose that (1) is invariant under the one-parameter Lie group of point transformations

$$\begin{aligned}
\bar{t} &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\
\bar{x} &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\
\bar{u} &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\
\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \tfrac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_\alpha^0(x,t,u) + O(\epsilon^2), \\
\frac{\partial \bar{u}}{\partial \bar{x}} &= \tfrac{\partial u}{\partial x} + \epsilon\eta^x(x,t,u) + O(\epsilon^2), \\
\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \tfrac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(x,t,u) + O(\epsilon^2), \\
&\vdots
\end{aligned} \tag{13}$$

where ϵ is an infinitesimal parameter and, for example, the individual terms in the above expression are

$$\begin{aligned}
\eta^x &= \eta_x + \eta_u u_x - (\xi_x + \xi_u u_x)u_x - (\tau_x + \tau_u u_x)u_t, \\
\eta^{xx} &= \eta_{xx} + 2\,\eta_{xu} u_x - 2\,\xi_{xu} {u_x}^2 - \xi_{xx} u_x - 2\,\tau_{ux} u_t u_x - \tau_{xx} u_t - \xi_{uu} {u_x}^3 \\
&\quad -\tau_{uu} {u_x}^2 u_t + \eta_{uu} {u_x}^2 - 3\,\xi_u u_x u_{xx} - \tau_u u_t u_{xx} + \eta_u u_{xx} - 2\,\xi_x u_{xx} \\
&\quad -2\,\tau_u u_x u_{tx} - 2\,\tau_x u_{tx}.
\end{aligned} \tag{14}$$

Using the generalised Leibniz rule [21–23] and a generalisation of the chain rule, we have that [15]

$$\begin{aligned}
\eta_\alpha^0 &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) \\
&\quad + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n}{\partial t^n}\eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}.
\end{aligned} \tag{15}$$

The D_t^α is the total fractional derivative operator, and

$$\begin{aligned}
\mu &= \textstyle\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\
&\quad \times (-u)^r \tfrac{\partial^m}{\partial t^m}\left(u^{k-r}\right)\tfrac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.
\end{aligned} \tag{16}$$

It is important to remark that by convention in the literature, $\eta(x,t,u)$ is taken to be linear in the variable u so that μ vanishes. We adopt this idea in the work hereafter as well.

Let the generator

$$X = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{17}$$

span the associated Lie algebra—that is,

$$\tau(x,t,u) = \frac{dt^*}{d\epsilon}\Big|_{\epsilon=0}, \qquad \xi(x,t,u) = \frac{dx^*}{d\epsilon}\Big|_{\epsilon=0}, \qquad \eta(x,t,u) = \frac{du^*}{d\epsilon}\Big|_{\epsilon=0}.$$

The infinitesimal criterion for invariance is given by $XG = 0, \quad \text{when} \quad G = 0$, where X is extended to all derivatives appearing in the equation through an appropriate prolongation. Moreover, it is essential that the transformation (13) leaves the lower limit of the fractional derivative invariant $\frac{\partial^\alpha u}{\partial t^\alpha}$, which translates into the additional constraint condition

$$\tau(x,t,u)\Big|_{t=0} = 0. \tag{18}$$

We further require the following two definitions. The definition of the Erdélyi–Kober fractional integral operator given by

$$\left(K_{\beta}^{l,m} g\right)(z) = \begin{cases} g(z), & m = 0, \\ \frac{1}{\Gamma(m)} \int_1^{\infty} (u-1)^{m-1} u^{-(l+m)} g(z u^{\frac{1}{\beta}}) du, & m > 0, \end{cases} \tag{19}$$

and the Erdélyi–Kober fractional differential operator is

$$P_{\beta}^{q,r} w = \prod_{j=0}^{n-1} \left(q + j - \frac{1}{\beta} z \frac{\partial}{\partial t}\right) \left(K_{\beta}^{q+r,n-r} w\right)(z), \tag{20}$$

At this stage, we also recall the formula [24]

$$\frac{d^{\alpha} x^{\beta}}{dx^{\alpha}} = \frac{x^{-\alpha+\beta} \Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)}, \qquad \beta > -1, \tag{21}$$

which is useful in the reduction of the FDE.

In the next two sections, we shall study the main equations of the paper. The definitions and formulas discussed above will be used to investigate each of the cases.

3. Symmetry Analysis

The application of the theory of Section 2 shows that we have the following symmetries (see Table 1) corresponding to each of the above ten Equations (1)–(10), corresponding to Cases 1–10. Detailed calculations of these symmetries are omitted due to their volume.

Table 1. Lie point symmetries of Equations (1)–(10).

Case	Symmetries	Dimension
1	$X_1 = \partial_x,\ X_2 = 3t\partial_t - u\alpha\partial_u,\ X_u = \partial_u,$	The Lie algebra spanned by point symmetries is 3 dimensional.
2	$X_1,\ X_3 = -3t\partial_t + 3u\alpha\partial u + x\alpha\partial_x,\ X_u,$	The Lie algebra spanned by point symmetries is 3 dimensional.
3	$X_1,\ X_2,\ X_a = e^{-\frac{ax}{3}} u\partial_u - \frac{3}{a} e^{-\frac{ax}{3}} \partial_x,$	The Lie algebra spanned by point symmetries is 3 dimensional.
4	$X_1,\ X_2,$	The Lie algebra spanned by point symmetries is 2 dimensional.
5,8	$X_1,\ X_4 = -3t\partial_t + 2u\alpha\partial u + x\alpha\partial_x,$	The Lie algebra spanned by point symmetries is 2 dimensional.
6	$X_1,\ X_2,\ X_5 = u\cos(\sqrt{a}x)\sqrt{a}\partial_u + \sin(\sqrt{a}x)\partial_x,$ $X_6 = u\sin(\sqrt{a}x)\sqrt{a}\partial_u - \cos(\sqrt{a}x)\partial_x,$	The Lie algebra spanned by point symmetries is 4 dimensional.
7	$X_1,\ X_2,\ X_{aa} = 2e^{-\frac{2ax}{3}} u\partial_u - \frac{3}{a} e^{-\frac{2ax}{3}} \partial_x,$	The Lie algebra spanned by point symmetries is 3 dimensional.
9, 10	$X_1,\ X_7 = 5t\partial_t - \alpha u\partial_u,\ X_8 = u\partial u + x\partial_x, X_9 = 2ux\partial_u + x^2\partial_x,$	The Lie algebra spanned by point symmetries is 4 dimensional.

The Lie brackets are given in Tables 2–8. As for Cases 1, 3, and 7, the algebra is solvable. Case 4 is abelian, nilpotent, and solvable—all commutators vanish. Cases 2, 5, and 8 are indecomposable and solvable. Cases 6, 9, and 10 are decomposable.

Table 2. Lie brackets for Case 1.

[,]	X_1	X_2	X_u
X_1	0	0	0
X_2	0	0	αX_u
X_u	0	$-\alpha X_u$	0

Table 3. Lie brackets for Case 2.

[,]	X_1	X_3	X_u
X_1	0	αX_1	0
X_3	$-\alpha X_1$	0	$-3\alpha X_u$
X_u	0	$3\alpha X_u$	0

Table 4. Lie brackets for Case 3.

[,]	X_1	X_2	X_a
X_1	0	0	$-\frac{a}{3}X_a$
X_2	0	0	0
X_a	$\frac{a}{3}X_a$	0	0

Table 5. Lie brackets for Cases 5 and 8.

[,]	X_1	X_4
X_1	0	αX_1
X_4	$-\alpha X_1$	0

Table 6. Lie brackets for Case 6.

[,]	X_1	X_2	X_5	X_6
X_1	0	0	$-\sqrt{a}X_6$	$\sqrt{a}X_5$
X_2	0	0	0	0
X_5	$\sqrt{a}X_6$	0	0	$\sqrt{a}X_1$
X_6	$-\sqrt{a}X_5$	0	$-\sqrt{a}X_1$	0

Table 7. Lie brackets for Case 7.

[,]	X_1	X_2	X_{aa}
X_1	0	0	$-\frac{2a}{3}X_{aa}$
X_2	0	0	0
X_{aa}	$\frac{2a}{3}X_{aa}$	0	0

Table 8. Lie brackets for Cases 9 and 10.

[,]	X_1	X_7	X_8	X_9
X_1	0	0	X_1	$2\,X_8$
X_7	0	0	0	0
X_8	$-X_1$	0	0	X_9
X_9	$-2\,X_8$	0	$-X_9$	0

4. Reductions and Power Series Solutions

In this section, we consider several transformed equations via the symmetries listed above. The solutions are found with power series or, alternatively, the equation is reduced with the use of Erdélyi–Kober operators. We consider Cases 1, 3, 4, 6, 7, and Cases 9 and 10, which admit the symmetries X_2 and where the last two cases admit X_7. The symmetry X_2 produces the invariants

$$u(x,t) = w(x)t^{-\frac{\alpha}{3}}, \tag{22}$$

while X_7 produces the invariants

$$u(x,t) = w(x)t^{-\frac{\alpha}{5}}. \tag{23}$$

In each case, these invariants will provide us with a fractional-order ODE, whereupon the fractional terms are manipulated with the application of (21) to obtain an integer-order ODE. The latter ODE is solved using the power series method [25]. The convergence and uniqueness of the solution can then be determined via the implicit functional theorem. Case 1 is performed in detail. Cases 2, 5, and 8 are best reduced with Erdélyi–Kober operators given the symmetries they admit.

4.1. Case 1

Consider Equation (1), a reduction using (22) followed by application of (21) generates the following ODE to solve, viz.

$$w(x)\Gamma\left(1-\frac{\alpha}{3}\right)\left(\Gamma\left(1-\frac{4\,\alpha}{3}\right)\right)^{-1} - \left(\frac{\mathrm{d}}{\mathrm{d}x}w(x)\right)^{3}\frac{\mathrm{d}^3}{\mathrm{d}x^3}w(x) - a\left(\frac{\mathrm{d}}{\mathrm{d}x}w(x)\right)^{4} = 0. \tag{24}$$

This ODE is very difficult to solve using most techniques. We show that power series are highly effective. Thus, the power series

$$w(x) = \sum_{r_1=0}^{\infty} a_{r_1}x^{r_1}, \tag{25}$$

is substituted into (24). We find that a_0, a_1, a_2 are arbitrary and that a solution may be expressed as

$$\begin{aligned} w(x) \quad = \quad & a_0 + a_1x + a_2x^2 + \\ & x^3\frac{a_0\Gamma\left(1-\frac{\alpha}{3}\right) - aa_1^4\Gamma\left(1-\frac{4\alpha}{3}\right)}{6a_1^3\Gamma\left(1-\frac{4\alpha}{3}\right)} + \\ & x^4\frac{-8aa_2a_1^2\Gamma\left(1-\frac{4\alpha}{3}\right) - \frac{6a_2\left(a_0\Gamma\left(1-\frac{\alpha}{3}\right)-aa_1^4\Gamma\left(1-\frac{4\alpha}{3}\right)\right)}{a_1^2} + \Gamma\left(1-\frac{\alpha}{3}\right)}{24a_1^2\Gamma\left(1-\frac{4\alpha}{3}\right)} \\ & + \ldots \end{aligned} \tag{26}$$

with graphical solution expressed in Figure 1.

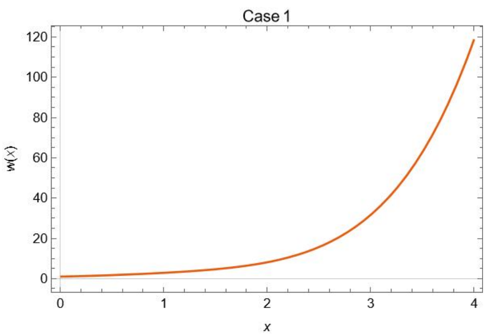

Figure 1. We let $a_0 = a_1 = a_2 = a = 1, \alpha = \frac{1}{2}$, for the graphical solution.

Testing for Convergence

A natural question that arises is whether or not the above series solution converges. The following illustrates how to test for convergence. Suppose we consider (24) with the power series (25) substituted into it; then, we have that

$$\sum_{r_1=0}^{\infty}\Bigg(a_{r_1}\Gamma\Big(1-\frac{\alpha}{3}\Big)\Gamma\Big(1-\frac{4\,\alpha}{3}\Big)^{-1}- \\ \sum_{r_2=3}^{r_1+6}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3} r_4 a_{r_4}(r_3-r_4)a_{r_3-r_4}(r_2-r_3)a_{r_2-r_3}(r_1+6-r_2)(r_1+-r_2)(r_1+-r_2)a_{r_1+6-r_2}- \\ a\sum_{r_2=3}^{r_1+4}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3} r_4 a_{r_4}(r_3-r_4)a_{r_3-r_4}(r_2-r_3)a_{r_2-r_3}(r_1+4-r_2)a_{r_1+4-r_2}\Bigg)x^{r_1}=0. \tag{27}$$

Hence, by formal calculations we have that, in general, the coefficients in the above sum are given by

$$a_{r_1+3}=\frac{\frac{a_{r_1}\Gamma(1-\frac{\alpha}{3})}{\Gamma(1-\frac{\alpha}{3})}}{a_1^3(r_1+3)(r_1+2)(r_1+1)} \\ -\frac{\sum_{r_2=4}^{r_1+6}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3} r_4 a_{r_4}(r_3-r_4)a_{r_3-r_4}(r_2-r_3)}{a_1^3(n_1+3)(r_1+2)(r_1+1)}\times \\ a_{r_2-r_3}(r_1+6-r_2)(r_1+5-r_2)(r_1+4-r_2)a_{r_1+6-r_2} \\ -\frac{a\sum_{r_2=2}^{r_1+4}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3} r_4 a_{r_4}(r_3-r_4)a_{r_3-r_4}(r_2-r_3)a_{r_2-r_3}(r_1+4-r_2)a_{r_1+4-r_2}}{a_1^3(r_1+3)(r_1+2)(r_1+1)}, \tag{28}$$

for $r_1 \geq 0$, such that (26) reads as

$$w(x)=a_0+a_1x+a_2x^2+\sum_{r_1=0}^{\infty}a_{r_1+3}x^{r_1+3}. \tag{29}$$

Next, we prove the convergence of the power series solution (29). From (28), we obtain

$$|a_{r_1+3}| \leq M\Bigg(|a_{r_1}|+\sum_{r_2=4}^{r_1+6}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}|a_{r_4}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_1+6-r_2}| \\ +\sum_{r_2=2}^{r_1+4}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}|a_{r_4}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_1+4-r_2}|\Bigg), \tag{30}$$

where $r_1=0,1,2...$, and $M=max\{\frac{\Gamma(1-\frac{\alpha}{3})}{\Gamma(1-\frac{\alpha}{3})},\frac{|a|}{a_1^3}\}$.

Suppose we have the power series $\mu=R(x)=\Sigma_{r_1=0}^{\infty}p_{r_1}x^{r_1}$ where

$$p_k=|a_k|,\ k=0,1,2 \tag{31}$$

and

$$p_{r_1+3} = M\Bigg(p_{r_1}+\sum_{r_2=4}^{r_1+6}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}p_{r_4}p_{r_3-r_4}p_{r_2-r_3}p_{r_1+6-r_2} \\ +\sum_{r_2=2}^{r_1+4}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}p_{r_4}p_{r_3-r_4}p_{r_2-r_3}p_{r_1+4-r_2}\Bigg). \tag{32}$$

Hence,

$$|a_{r_1}| \le p_{r_1},\ r_1 = 0, 1, 2... \tag{33}$$

Next, we prove that μ is convergent in a neighbourhood of a point. Note that μ is a majorant series of Equation (29) and can be written as follows:

$$\begin{aligned} R(x) &= p_0 + p_1 x + p_2 x^2 + \sum_{r_1=0}^{\infty} p_{r_1+3} x^{r_1+3} \\ &= p_0 + p_1 x + p_2 x^2 + \\ &\quad M \sum_{r_1=0}^{\infty} \left(p_{r_1} + \sum_{r_2=4}^{r_1+6} \sum_{r_3=1}^{r_2} \sum_{r_4=1}^{r_3} p_{r_4} p_{r_3-r_4} p_{r_2-r_3} p_{r_1+6-r_2} \right. \\ &\quad + \left. \sum_{r_2=2}^{r_1+4} \sum_{r_3=1}^{r_2} \sum_{r_4=1}^{r_3} p_{r_4} p_{r_3-r_4} p_{r_2-r_3} p_{r_1+4-r_2} \right) x^{r_1+3} \\ &= p_0 + p_1 x + p_2 x^2 \\ &\quad + M[R^4 + R^2 \rho(x) + R\sigma(x) + \nu(x)], \end{aligned} \tag{34}$$

where $\nu(x) = \theta(x) - p_0^4$ and $\theta(x), \rho(x)$, and $\sigma(x)$ are polynomials with each term having a degree of x of at least one. Hence, let

$$\begin{aligned} F(x, \mu) &= \mu - p_0 - p_1 x - p_2 x^2 \\ &\quad - M[R^4 + R^2 \rho(x) + R\sigma(x) + \nu(x)], \end{aligned} \tag{35}$$

be the implicit function equation, where we obtain that $F(0, p_0) = 0$ and $F_\mu(0, r_0) = 1 - 4Mp_0^3 \neq 0$. By virtue of the implicit function theorem [26], $\mu = R(x)$ is analytic and convergent in a neighbourhood of the point $(0, p_0)$ in the plane and with a positive radius. Then, the power series solution (29) is convergent in the neighbourhood of a point $(0, p_0)$.

Therefore, (29) can be written as

$$\begin{aligned} w(x) &= a_0 + a_1 x + a_2 x^2 + \\ &\quad \Sigma_{r_1=0}^{\infty} \left(\frac{\frac{a_{r_1} \Gamma(1-\frac{\alpha}{3})}{\Gamma(1-\frac{\alpha}{3})}}{a_1^3 (r_1+3)(r_1+2)(r_1+1)} \right. \\ &\quad - \frac{\sum_{r_2=4}^{r_1+6} \sum_{r_3=1}^{r_2} \sum_{r_4=1}^{r_3} r_4 a_{r_4} (r_3 - r_4) a_{r_3-r_4} (r_2 - r_3)}{a_1^3 (n_1+3)(r_1+2)(r_1+1)} \times \\ &\quad a_{r_2-r_3} (r_1 + 6 - r_2)(r_1 + 5 - r_2)(r_1 + 4 - r_2) a_{r_1+6-r_2} \\ &\quad - \left. \frac{a \sum_{r_2=2}^{r_1+4} \sum_{r_3=1}^{r_2} \sum_{r_4=1}^{r_3} r_4 a_{r_4} (r_3 - r_4) a_{r_3-r_4} (r_2 - r_3) a_{r_2-r_3} (r_1 + 4 - r_2) a_{r_1+4-r_2}}{a_1^3 (r_1+3)(r_1+2)(r_1+1)} \right) x^{r_1+3}. \end{aligned} \tag{36}$$

Thus, the exact power series solution of Equation (1) can be written as

$$u(x,t) = \left(a_0 + a_1 x + a_2 x^2 + \Sigma_{r_1=0}^{\infty} a_{r_1+3} x^{r_1+3} \right) t^{-\frac{\alpha}{3}} \tag{37}$$

where $a_0 \neq 0, a_1, a_2$, are arbitrary constants and the rest of the constants are to be determined by (28). This produces the solution (26).

Similarly, and very tediously, we can construct convergent power series solutions to all reduced equations in this paper. The solutions can then be transformed back into original variables, given the invertible transformations stated for each reduction. Due to how lengthy the above test for convergence is, we omit the convergence details for all other cases.

4.2. Case 2

As mentioned above, this case is best reduced by the elegant Erdélyi–Kober operators instead of the how we treated Case 1. Suppose we take X_3; then, we have the invariants $z = xt^{\frac{\alpha}{3}}$ and $u(x,t) = w(x,t)t^{-\alpha}$. Now, we transform the LHS of (2) using the following transformation:

$$u_t^\alpha = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} w(s^{-\frac{\alpha}{3}}x)s^{-\alpha}ds\right]. \tag{38}$$

Let $s = t/v$; then, $ds = -\frac{-t}{v^2}dv$ so that the above becomes

$$\begin{aligned} u_t^\alpha =& \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_\infty^1 \left(t-\frac{t}{v}\right)^{n-\alpha-1} w\left(\left(\frac{t}{v}\right)^{-\frac{\alpha}{3}}x\right)\left(\frac{t}{v}\right)^{-\alpha}\left(-\frac{t}{v^2}\right)dv\right] \\ =& \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_\infty^1 \left(t-\frac{t}{v}\right)^{n-\alpha-1} w\left(\left(\frac{t}{v}\right)^{-\frac{\alpha}{3}}x\right)\left(\frac{t}{v}\right)^{-\alpha}\left(-\frac{t}{v^2}\right)dv\right] \\ =& \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-2\alpha}}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1} w\left(zv^{\frac{\alpha}{3}}\right)(v)^{-(n-2\alpha-1)}dv\right]. \end{aligned}$$

Then, by the definition of the Erdélyi–Kober fractional integral operator (19), $\beta = \frac{3}{\alpha}$ and $m = n-\alpha$, and from the powers of v we have $-(n-2\alpha+1) \implies -[(n-\alpha)-1-\alpha] \implies -[m+l]$, where $l = -1-\alpha$.

So, the above becomes

$$u_t^\alpha = \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-2\alpha}}{\Gamma(n-\alpha)}\left(K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}w\right)(z)\right]. \tag{39}$$

From $z = xt^{-\frac{\alpha}{3}}$, by the chain rule, we obtain

$$t\frac{\partial}{\partial t}\phi(z) = -\frac{\alpha}{3}z\frac{\partial}{\partial t}. \tag{40}$$

Therefore, the RHS of (39) is

$$\frac{\partial^n}{\partial t^n}\left[t^{n-2\alpha}\left(K_{\frac{3}{\alpha}}^{1-\alpha,n-\alpha}w\right)(z)\right] = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}t^{n-2\alpha}\left(K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}w\right)(z)\right], \tag{41}$$

which by product rule gives

$$\frac{\partial^{n-1}}{\partial t^{n-1}}\left[(n-2\alpha)t^{n-2\alpha-1}\left(K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}w\right)(z) + t^{n-2\alpha-1}t\left(K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}w\right)'(z)\right]. \tag{42}$$

Now, we will have

$$\frac{\partial^{n-1}}{\partial t^{n-1}}\left[(n-2\alpha)t^{n-2\alpha-1}\left(n-2\alpha-\frac{\alpha}{3}z\frac{\partial}{\partial t}\right)\left(K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}w\right)(z)\right].$$

Then, by repeating this $n-1$ times, we obtain

$$\frac{\partial^n}{\partial t^n}\left[t^{n-2\alpha}\left(K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}w\right)(z)\right] = t^{-2\alpha}\prod_{j=0}^{n-1}\left(1+j-2\alpha-\frac{\alpha}{3}z\frac{\partial}{\partial t}\right)K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}. \tag{43}$$

By the definition of the fractional operator (20) and by comparing the subscripts in the $K_{\beta}^{q+r,n-r}$ term, we have that $r = \alpha$ and $q = -1 - 2\alpha$, and (43) can be written as

$$\frac{\partial^n}{\partial t^n}\left[t^{n-2\alpha}\left(K_{\frac{3}{\alpha}}^{-1-\alpha,n-\alpha}w\right)(z)\right] = t^{-2\alpha}\left(P_{\frac{3}{\alpha}}^{-1-2\alpha,\alpha}w\right)(z). \quad (44)$$

Then, from (39), we will have

$$u_t^{\alpha} = t^{-2\alpha}\left(P_{\frac{3}{\alpha}}^{-1-2\alpha,\alpha}w\right)(z). \quad (45)$$

Hence, Equation (2) transforms to

$$\left(P_{\frac{3}{\alpha}}^{-1-2\alpha,\alpha}w\right)(z) = \left(w'(z)\right)^3\left(w'''(z) + a\right).$$

4.3. Case 3

Similar to Case 1, consider a reduction using (22) followed by application of (21) to generate the following ODE

$$w(x)\Gamma\left(1-\frac{\alpha}{3}\right)\left(\Gamma\left(1-\frac{4\,\alpha}{3}\right)\right)^{-1} - (w(x))^3\frac{d^3}{dx^3}w(x) - 3\,(w(x))^2\left(\frac{d^2}{dx^2}w(x)\right)\frac{d}{dx}w(x)$$
$$-a\left((w(x))^3\frac{d^2}{dx^2}w(x) + (w(x))^2\left(\frac{d}{dx}w(x)\right)^2\right) - \frac{2\,a^2(w(x))^3\frac{d}{dx}w(x)}{9} = 0,$$

which we solve with a power series $w(x) = \sum_{r_1=0}^{\infty} a_{r_1}x^{r_1}$ to obtain a_0, a_1, a_2 as arbitrary and where

$$\begin{aligned} w(x) = \; & a_0 + a_1x + a_2x^2 + \\ & \frac{x^3}{54a_0^3\Gamma\left(1-\frac{4\alpha}{3}\right)}\Bigg(-2a^2a_1a_0^3\Gamma\left(1-\frac{4\alpha}{3}\right) - 18aa_2a_0^3\Gamma\left(1-\frac{4\alpha}{3}\right) \\ & -9aa_1^2a_0^2\Gamma\left(1-\frac{4\alpha}{3}\right) - 54a_1a_2a_0^2\Gamma\left(1-\frac{4\alpha}{3}\right) + 9a_0\Gamma\left(1-\frac{\alpha}{3}\right)\Bigg) \\ & + \ldots \end{aligned} \quad (46)$$

with graphical solution expressed in Figure 2.

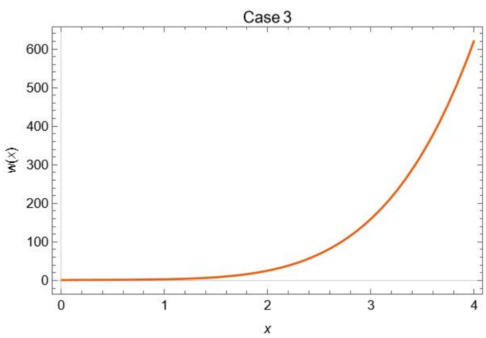

Figure 2. We let $a_0 = 1, a_1 = 1, a_2 = 1, \alpha = \frac{1}{2}, a = 1$.

4.4. Case 4

In this case again, consider a reduction using (22) followed by application of (21) to generate the following ODE

$$w(x)\Gamma\left(1-\frac{\alpha}{3}\right)\left(\Gamma\left(1-\frac{4\,\alpha}{3}\right)\right)^{-1}-(w(x))^3\frac{\mathrm{d}^3}{\mathrm{d}x^3}w(x)-3\,(w(x))^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}w(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}w(x)$$
$$-4\,a(w(x))^3\frac{\mathrm{d}}{\mathrm{d}x}w(x)=0,$$

which we solve with a power series $w(x)=\sum_{r_1=0}^{\infty}a_{r_1}x^{r_1}$ to obtain a_0,a_1,a_2 as arbitrary and where

$$\begin{aligned} w(x) &= a_0+a_1x+a_2x^2+ \\ &\quad \frac{x^3}{6a_0^2\Gamma\left(1-\frac{4\alpha}{3}\right)}\left(-4aa_1a_0^2\Gamma\left(1-\frac{4\alpha}{3}\right)\right. \\ &\quad \left.-9aa_1^2a_0^2\Gamma\left(1-\frac{4\alpha}{3}\right)-54a_1a_2a_0^2\Gamma\left(1-\frac{4\alpha}{3}\right)+9a_0\Gamma\left(1-\frac{\alpha}{3}\right)\right) \\ &\quad +\ldots \end{aligned} \tag{47}$$

with graphical solution expressed in Figure 3.

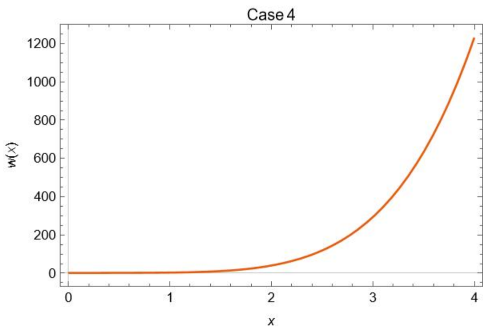

Figure 3. We let $a_0=1,a_1=1,a_2=1,\alpha=\frac{1}{2},a=1$.

4.5. Case 5

This case admits the symmetry X_4, which gives the invariants $z=xt^{\frac{\alpha}{3}}$ and $w(z)=ut^{\frac{2\alpha}{3}}$. This case was considered in [27] with $a=1$, where it is shown that

$$u_t^{\alpha}=t^{\frac{-5\alpha}{3}}\left(P_{\frac{3}{\alpha}}^{-1-\frac{5\alpha}{3},\alpha}w\right)(z). \tag{48}$$

Hence, for our case, using the above, we obtain that (5) transforms to

$$\left(P_{\frac{3}{\alpha}}^{-1-2\alpha,\alpha}w\right)(z)=3w^2(z)\left((a+w''(z))w'(z)-\frac{1}{3}w(z)w'''(z)\right).$$

4.6. Case 6

Similar to Case 1, consider a reduction using (22) followed by application of (21) to generate the following ODE

$$w(x)\Gamma\left(1-\frac{\alpha}{3}\right)\left(\Gamma\left(1-\frac{4\,\alpha}{3}\right)\right)^{-1}-(w(x))^3\frac{\mathrm{d}^3}{\mathrm{d}x^3}w(x)-a(w(x))^3\frac{\mathrm{d}}{\mathrm{d}x}w(x)=0,$$

which we solve with a power series $w(x) = \sum_{r_1=0}^{\infty} a_{r_1} x^{r_1}$ to obtain a_0, a_1, a_2 as arbitrary and where

$$\begin{aligned} w(x) &= a_0 + a_1 x + a_2 x^2 + \\ & \frac{x^3}{6a_1^3\Gamma\left(1-\frac{4\alpha}{3}\right)}\left(a_0\Gamma\left(1-\frac{\alpha}{3}\right) - a a_0 a_1 \Gamma\left(1-\frac{4\alpha}{3}\right)\right) \\ & + \dots \end{aligned} \tag{49}$$

with graphical solution expressed in Figure 4.

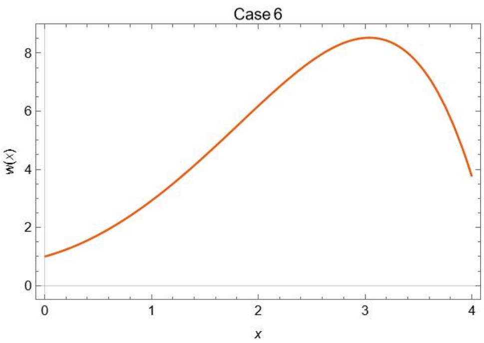

Figure 4. We let $a_0 = 1, a_1 = 1, a_2 = 1, \alpha = \frac{1}{2}, a = 1$.

4.7. Case 7

Similar to Case 1, consider a reduction using (22) followed by application of (21) to generate the following ODE

$$\begin{aligned} & w(x)\Gamma\left(1-\tfrac{\alpha}{3}\right)\left(\Gamma\left(1-\tfrac{4\alpha}{3}\right)\right)^{-1} - (w(x))^3 \tfrac{d^3}{dx^3}w(x) - \frac{3\,(w(x))^2\left(\frac{d^2}{dx^2}w(x)\right)\frac{d}{dx}w(x)}{2} \\ & -a\left((w(x))^3\tfrac{d^2}{dx^2}w(x) + (w(x))^2\left(\tfrac{d}{dx}w(x)\right)^2\right) - \frac{2\,a^2(w(x))^3\,\frac{d}{dx}w(x)}{9} = 0, \end{aligned}$$

which we solve with a power series $w(x) = \sum_{r_1=0}^{\infty} a_{r_1} x^{r_1}$ to obtain a_0, a_1, a_2 as arbitrary and where

$$\begin{aligned} w(x) &= a_0 + a_1 x + a_2 x^2 + \\ & \frac{x^3}{54a_0^3\Gamma\left(1-\frac{4\alpha}{3}\right)}\Bigg(-2a^2 a_1 a_0^3 \Gamma\left(1-\frac{4\alpha}{3}\right) - 18 a a_2 a_0^3 \Gamma\left(1-\frac{4\alpha}{3}\right) \\ & -9 a a_1^2 a_0^2 \Gamma\left(1-\frac{4\alpha}{3}\right) - 27 a_1 a_2 a_0^2 \Gamma\left(1-\frac{4\alpha}{3}\right) + 9 a_0 \Gamma\left(1-\frac{\alpha}{3}\right)\Bigg) \\ & + \dots \end{aligned} \tag{50}$$

with graphical solution expressed in Figure 5.

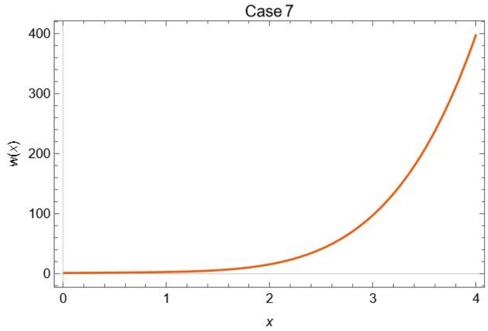

Figure 5. We let $a_0 = 1, a_1 = 1, a_2 = 1, \alpha = \frac{1}{2}, a = 1$.

4.8. Case 8

This case shares the symmetry X_4 with Case 5; hence, the reduction is the same for the LHS of the equation, but the RHS differs, so that (8) transforms to

$$\left(P_{\frac{3}{\alpha}}^{-1-2\alpha,\alpha} w\right)(z) = w^2(z)\left((a + \frac{3}{2}w''(z))w'(z) + w(z)w'''(z)\right).$$

4.9. Case 9

Similar to Case 1, consider a reduction using (22) followed by application of (21) to generate the following ODE

$$w(x)\Gamma\left(1-\tfrac{\alpha}{5}\right)\left(\Gamma\left(1-\tfrac{6\alpha}{5}\right)\right)^{-1} - (w(x))^5 \tfrac{d^5}{dx^5}w(x) - 5\,(w(x))^4\left(\left(\tfrac{d^4}{dx^4}w(x)\right)\tfrac{d}{dx}w(x)\right.$$
$$\left. +2\left(\tfrac{d^2}{dx^2}w(x)\right)\tfrac{d^3}{dx^3}w(x)\right) = 0,$$

which we solve with a power series $w(x) = \sum_{r_1=0}^{\infty} a_{r_1}x^{r_1}$ to obtain a_0, a_1, a_2, a_3, a_4 as arbitrary and where

$$\begin{aligned} w(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \\ &\quad \frac{x^5}{120a_0^4\Gamma(1-\frac{6\alpha}{5})}\left(-120a_2a_3a_0^3\Gamma\left(1-\frac{6\alpha}{5}\right)\right. \\ &\quad \left.-120a_1a_4a_0^3\Gamma\left(1-\frac{6\alpha}{5}\right) + \Gamma\left(1-\frac{\alpha}{5}\right)\right) \\ &\quad +\ldots \end{aligned} \tag{51}$$

with graphical solution expressed in Figure 6.

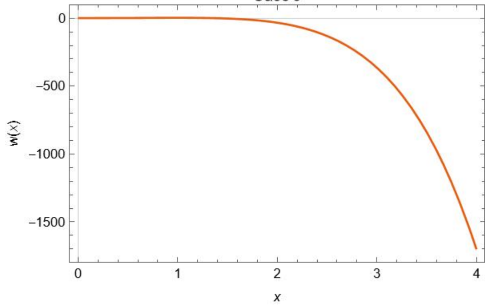

Figure 6. We let $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, \alpha = \frac{1}{2}, a = 1$.

4.10. Case 10

Finally, consider a reduction using (22) followed by application of (21) to generate the following ODE

$$w(x)\Gamma\left(1-\tfrac{\alpha}{5}\right)\left(\Gamma\left(1-\tfrac{6\alpha}{5}\right)\right)^{-1}-(w(x))^5\tfrac{d^5}{dx^5}w(x)$$
$$-5\,(w(x))^4\left(\left(\tfrac{d^4}{dx^4}w(x)\right)\tfrac{d}{dx}w(x)+1/2\left(\tfrac{d^2}{dx^2}w(x)\right)\tfrac{d^3}{dx^3}w(x)\right)$$
$$-\frac{15\,(w(x))^3\left(\frac{d}{dx}w(x)\right)^2\frac{d^3}{dx^3}w(x)}{4}=0,$$

which we solve with a power series $w(x)=\sum_{r_1=0}^{\infty}a_{r_1}x^{r_1}$ to obtain a_0,a_1,a_2,a_3,a_4 as arbitrary and where

$$\begin{aligned} w(x) &= a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+ \\ &\quad \frac{x^5}{240a_0^5\Gamma\left(1-\frac{6\alpha}{5}\right)}\left(-60a_2a_3a_0^4\Gamma\left(1-\frac{6\alpha}{5}\right)\right. \\ &\quad -240a_1a_4a_0^4\Gamma\left(1-\frac{6\alpha}{5}\right)-45a_1^2a_3a_0^3\Gamma\left(1-\frac{6\alpha}{5}\right) \\ &\quad \left.+2a_0\Gamma\left(1-\frac{\alpha}{5}\right)\right)+\dots \end{aligned} \tag{52}$$

with graphical solution expressed in Figure 7.

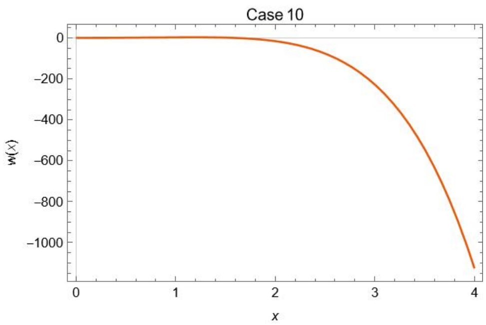

Figure 7. We let $a_0=1,a_1=1,a_2=1,a_3=1,a_4=1,\alpha=\frac{1}{2},a=1$.

5. Concluding Remarks

It has remained a topic of debate as to how to find the solutions of highly nonlinear equations. There are many challenges associated with finding reductions and solutions of FDEs in particular. We have shown that Lie symmetries combined with power series methods are extremely effective in the analysis. The purpose of this study is to show how power series may be applied to Lie symmetry reductions. We have restricted our attention to FDEs with the Riemann–Liouville derivative; however, reductions from Caputo fractional derivatives will work in practice. In this regard, power series may be used to address problems in solving reduced equations obtained from other methods as well.

Author Contributions: J.T.K. conducted formal analysis, software applications and wrote the first draft. S.J. conceptualised the study and reviewed and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: National Research Foundation of South Africa.

Data Availability Statement: This study did not report any data.

Conflicts of Interest: The authors declare no conflict of interest.

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