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# Examining the Hermite–Hadamard Inequalities for *k*-Fractional Operators Using the Green Function

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**Abstract:** For $k$-Riemann–Liouville fractional integral operators, the Hermite–Hadamard inequality is already well-known in the literature. In this regard, this paper presents the Hermite–Hadamard inequalities for $k-$Riemann–Liouville fractional integral operators by using a novel method based on Green's function. Additionally, applying these identities to the convex and monotone functions, new Hermite–Hadamard type inequalities are established. Furthermore, a different form of the Hermite–Hadamard inequality is also obtained by using this novel approach. In conclusion, we believe that the approach presented in this paper will inspire more research in this area.

**Keywords:** Hermite–Hadamard inequality; convex functions; green functions; $k$-Riemann–Liouville fractional operators

**MSC:** 26D15; 26D10; 26A51; 26A33; 41A55

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## 1. Introduction

Convex functions are different from other function classes in that they have many applications in the fields of mathematics, statistics, optimization theory, and applied sciences; and their definition has a geometric interpretation. Additionally, it is one of the fundamental components of inequality theory and has evolved into the main motivating element behind several inequalities. Although there are many areas of mathematical analysis and statistics where convex functions can be applied, the inequality theory has shown to be the most significant one. In this regard, a number of traditional and analytical inequalities, particularly Hermite–Hadamard-, Ostrowski-, Simpson-, Fejér-, and Hardy-type inequalities, have been established [1–3].

The definition of the convex function is:

**Definition 1.** *A function $\psi : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\psi(\xi\varkappa_1 + (1-\xi)\varkappa_2) \leq \xi\psi(\varkappa_1) + (1-\xi)\psi(\varkappa_2)$$

*holds for all $\varkappa_1, \varkappa_2 \in I$ and $\xi \in [0,1]$.*

The Hermite–Hadamard inequality, which is the main result of convex functions' widespread application and excellent geometrical interpretation, has received a lot of attention in fundamental mathematics. Recent years have seen a rapid development in the theory of inequality [4–6]. Important inequalities, such as the Hermite–Hadamard inequality, are one of the most important reasons for this development. It is worth reflecting on the fact that the theories of inequality and convexity are closely related to one another. In recent years, several new extensions, generalizations, and definitions of novel convexity have been given, and parallel developments in the theory of convexity inequality, particularly integral inequalities theory, have been emphasized. The Hermite–Hadamard inequality is formally expressed as follows:

Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function of the interval $I$ of real numbers and $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$.

$$\psi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \psi(\phi) d\phi \leq \frac{\psi(\varkappa_1) + \psi(\varkappa_2)}{2}. \tag{1}$$

The inequality in (1) will hold in reverse directions if $\psi$ is a concave function. The Hermite–Hadamard inequality, which is based on geometry, gives an upper and lower estimate for the integral mean of any convex function defined in a closed and limited domain, which includes the endpoints and midpoint of the domain of the function. Due to the significance of this inequality, several variations of the Hermite–Hadamard inequality have been examined in the literature for various classes of convexity, including harmonically convex, exponentially convex, $s$-convex, $h$-convex, and co-ordinate convex functions [7–10].

Inequalities involving fractional integrals are a special focus of the calculus of non-integer order, widely known as fractional calculus. This subject deals with the generalization of integrals and derivative operators. Several definitions are used for fractional integral operators, such as Hadamard integral, the $k$-Riemann–Liouville fractional integral, Caputo–Fabrizio fractional integral, Riemann–Liouville fractional integral, and conformable fractional integral [11–14]. By adding new parameters to such fractional integral operators, one can generalize the fractional operators, yielding to the following inequalities: Ostrowski, Grüss, Minkowski, Hermite–Hadamard, and others [15–17]. Such generalizations inspire future research to present more novel ideas with unified fractional operators and obtain inequalities involving such generalized fractional operators. In many different branches of research, inequalities relating to fractional integral operators have many practical applications. The theory of fractional calculus is also essential in the solution of many other special function problems, including those involving the solution of integral-differentiable equations, differential equations, and integral equations.

To obtain some remarks and corollaries, it is important for us to remember the preliminary formulae and notations of some well-known Riemann–Liouville and $k$-Riemann–Liouville fractional integral operators.

Several varieties of fractional integrals have been described in the literature; the most traditional are the Riemann–Liouville fractional integrals, which are defined as follows:

**Definition 2** ([18])**.** *Let $\psi \in L_1[\varkappa_1, \varkappa_2]$. The Riemann–Liouville integrals $J^{\alpha}_{\varkappa_1^+}\psi$ and $J^{\alpha}_{\varkappa_2^-}\psi$ of order $\alpha > 0$ with $\varkappa_1 \geq 0$ are defined by*

$$J^{\alpha}_{\varkappa_1^+}\psi(\phi) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa_1}^{\phi} (\phi - \xi)^{\alpha-1} \psi(\xi) d\xi, \ \ \phi > \varkappa_1$$

*and*

$$J^{\alpha}_{\varkappa_2^-}\psi(\phi) = \frac{1}{\Gamma(\alpha)} \int_{\phi}^{\varkappa_2} (\xi - \phi)^{\alpha-1} \psi(\xi) d\xi, \ \ \phi < \varkappa_2$$

*respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$. Here is $J^{0}_{\varkappa_1^+}\psi(\phi) = J^{0}_{\varkappa_2^-}\psi(\phi) = \psi(\phi)$.*

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [6], Sarıkaya et al. proved the following Hadamard-type inequalities for fractional integrals as follows:

**Theorem 1.** *Let $\psi : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a positive function with $0 \leq \varkappa_1 < \varkappa_2$ and $\psi \in L_1[\varkappa_1, \varkappa_2]$. If $\psi$ is a convex function on $[\varkappa_1, \varkappa_2]$, then the following inequalities for fractional integrals hold:*

$$\psi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\varkappa_2 - \varkappa_1)^{\alpha}} \left[J^{\alpha}_{\varkappa_1^+}\psi(\varkappa_2) + J^{\alpha}_{\varkappa_2^-}\psi(\varkappa_1)\right] \leq \frac{\psi(\varkappa_1) + \psi(\varkappa_2)}{2}$$

*with* $\alpha > 0$.

In [19], the $k$-Gamma function and its properties were introduced by Diaz et al. as follows:

**Definition 3.** *For* $k > 0$*, the* $k$*-Gamma function* $\Gamma_k$ *is given by*

$$\Gamma_k(\phi) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{\phi}{k}-1}}{(\phi)_{n,k}}, \ \ \phi \in \mathbb{C}\backslash k\mathbb{Z}^-.$$

**Definition 4.** *Let* $\phi \in \mathbb{C}$*,* $Re(\phi) > 0$*. Then, the* $k$*-Gamma function is defined by the following integral form:*

$$\Gamma_k(\phi) = \int_0^\infty \xi^{\phi-1} e^{-\frac{\xi^k}{k}} d\xi.$$

**Proposition 1.** *The* $k$*-Gamma function* $\Gamma_k(\phi)$ *satisfies the following properties:*

1. $\Gamma_k(\phi + k) = \phi\Gamma_k(\phi)$.
2. $(\phi)_{n,k} = \frac{\Gamma_k(\phi+nk)}{\Gamma_k(\phi)}$.
3. $\Gamma_k(k) = 1$.

**Theorem 2** ([14])**.** *The* $k$*-Riemann–Liouville integrals* $I^{\lambda}_{\varkappa_1^+,k}\psi$ *and* $I^{\lambda}_{\varkappa_2^-,k}\psi$ *of order* $\lambda > 0$ *with* $\varkappa_1 \geq 0$ *are defined by*

$$I^{\lambda}_{\varkappa_1^+,k}\psi(\phi) = \frac{1}{k\Gamma_k(\lambda)} \int_{\varkappa_1}^{\phi} (\phi - \xi)^{\frac{\lambda}{k}-1} \psi(\xi) d\xi, \ \ \phi > \varkappa_1$$

*and*

$$I^{\lambda}_{\varkappa_2^-,k}\psi(\phi) = \frac{1}{k\Gamma_k(\lambda)} \int_{\phi}^{\varkappa_2} (\xi - \phi)^{\frac{\lambda}{k}-1} \psi(\xi) d\xi, \ \ \phi < \varkappa_2.$$

The following Hadamard-type inequalities for $k$-fractional integrals were established by Farid et al. in [20].

**Theorem 3.** *Let* $\psi : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ *be a positive function with* $0 \leq \varkappa_1 < \varkappa_2$ *and* $\psi \in L_1[\varkappa_1, \varkappa_2]$*. If* $\psi$ *is a convex function of* $[\varkappa_1, \varkappa_2]$*, then the following inequalities for* $k$*-fractional integrals hold:*

$$\psi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{\Gamma_k(\lambda + k)}{2(\varkappa_2 - \varkappa_1)^{\frac{\alpha}{k}}} \left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) + J^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right] \leq \frac{\psi(\varkappa_1) + \psi(\varkappa_2)}{2}$$

*with* $\lambda, k > 0$.

A different form of Hadamard's inequality is given in the following theorem:

**Theorem 4** ([21,22])**.** *Let* $\psi : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ *be positive mapping with* $0 \leq \varkappa_1 < \varkappa_2$ *and* $\psi \in L_1[\varkappa_1, \varkappa_2]$*. If* $\psi$ *is a convex function of* $[\varkappa_1, \varkappa_2]$*, then*

$$\begin{aligned}
&\psi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \\
\leq\ & \frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda + k)}{(\varkappa_2 - \varkappa_1)^{\frac{\lambda}{k}}} \left[I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2) + I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1)\right] \\
\leq\ & \frac{\psi(\varkappa_1) + \psi(\varkappa_2)}{2}
\end{aligned}$$

*with* $\lambda, k > 0$.

The fact that $k-$fractional integrals generalize certain varieties of fractional integrals, such as the Riemann–Liouville fractional integral, is their most important component. One may check contemporary publications and books for further information [17,23–28]. As a result, in recent years, these fractional operators have been investigated and utilized to expand inequalities of the Hadamard, Grüss, Minkowski, Chebychev, and Pólya–Szegö kinds.

This article aims to present a novel approach to obtain the Hermite–Hadamard inequalities using the $k$-Riemann–Liouville fractional operator. By using the Green function in this approach, we are able to get several identities involving the $k$-Riemann–Liouville fractional integral operators. Additionally, we get new Hermite–Hadamard-type inequalities by applying these identities to the convex and monotone functions. Finally, using this novel approach, a different form of the Hermite–Hadamard inequality is obtained.

## 2. Main Results

In [29], Mehmood et al. established the following Lemma, which will be used to prove our main results:

**Lemma 1.** *Let $\varkappa_1 < \varkappa_2$ and G be the Green function defined on $[\varkappa_1, \varkappa_2] \times [\varkappa_1, \varkappa_2]$ by*

$$G(\lambda,\mu) = \begin{cases} \varkappa_1 - \mu, & \varkappa_1 \le \mu \le \lambda \\ \varkappa_1 - \lambda, & \lambda \le \mu \le \varkappa_2. \end{cases}$$

*Then, any $\psi \in C^2[\varkappa_1, \varkappa_2]$ can be expressed as*

$$\psi(\xi) = \psi(\varkappa_1) + (\xi - \varkappa_1)\psi'(\varkappa_2) + \int_{\varkappa_1}^{\varkappa_2} G(\xi,\mu)\psi''(\mu)d\mu. \tag{2}$$

**Proof.** The above equation can be easily obtained by employing the methods of integration by parts in $\int_{\varkappa_1}^{\varkappa_2} G(\phi,\mu)\psi''(\mu)d\mu$. So, the details of the proof are left to interested readers. □

The following theorem gives the Hermite–Hadamard inequality for $k$-fractional operators. The Hermite–Hadamard inequality has been proved by many researchers for different operators and many new inequalities have thus been obtained. Additionally, many important inequalities have also been established in the theory of inequality using Green's functions (see [25,26,30–33]).

**Theorem 5.** *Let $\psi \in C^2[\varkappa_1, \varkappa_2]$. If $\psi$ is a convex function of $[\varkappa_1, \varkappa_2]$, then we have the following inequalities:*

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \le \frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) + I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right] \le \frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2},$$

*which is the well-known Hermite–Hadamard inequality for k-fractional operators with $\lambda, k > 0$.*

**Proof.** Substituting $\xi = \frac{\varkappa_1+\varkappa_2}{2}$ in identity (2), we have

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) = \psi(\varkappa_1) + \left(\frac{\varkappa_2-\varkappa_1}{2}\right)\psi'(\varkappa_2) + \int_{\varkappa_1}^{\varkappa_2} G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)\psi''(\mu)d\mu. \tag{3}$$

Also, using identity (2), the following calculations are performed:

$$\begin{aligned}
& I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) \\
= & \frac{1}{k\Gamma_k(\lambda)}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}\psi(\xi)d\xi \\
= & \frac{1}{k\Gamma_k(\lambda)}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}\left\{\psi(\varkappa_1)+(\xi-\varkappa_1)\psi'(\varkappa_2)+\int_{\varkappa_1}^{\varkappa_2}G(\xi,\mu)\psi''(\mu)d\mu\right\}d\xi \\
= & \frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}d\xi+\psi'(\varkappa_2)\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}(\xi-\varkappa_1)d\xi\right. \\
& \left.+\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) = & \frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}{\frac{\lambda}{k}}+\psi'(\varkappa_2)\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)}\right. \\
& \left.+\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.
\end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned}
I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1) = & \frac{1}{k\Gamma_k(\lambda)}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}\psi(\xi)d\xi \\
= & \frac{1}{k\Gamma_k(\lambda)}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}\left\{\psi(\varkappa_1)+(\xi-\varkappa_1)\psi'(\varkappa_2)+\int_{\varkappa_1}^{\varkappa_2}G(\xi,\mu)\psi''(\mu)d\mu\right\}d\xi \\
= & \frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}d\xi+\psi'(\varkappa_2)\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}(\xi-\varkappa_1)d\xi\right. \\
& \left.+\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1) = & \frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}{\frac{\lambda}{k}}+\psi'(\varkappa_2)\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right. \\
& \left.+\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.
\end{aligned} \tag{5}$$

Now, adding (4) and (5), and multiplying the result by $\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}$, we have the following result:

$$\begin{aligned}
& \frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right] \\
= & \psi(\varkappa_1)+\psi'(\varkappa_2)\frac{\varkappa_2-\varkappa_1}{2}+\frac{\lambda}{2k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right. \\
& \left.+\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right].
\end{aligned} \tag{6}$$

Subtracting (6) from (3), we have

$$
\begin{aligned}
&\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right] \qquad (7)\\
=&\int_{\varkappa_1}^{\varkappa_2}G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)\psi''(\mu)d\mu-\frac{\lambda}{2k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right.\\
&\left.+\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right]\\
=&\int_{\varkappa_1}^{\varkappa_2}\left[G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)-\frac{\lambda}{2k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right.\right.\\
&\left.\left.+\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right\}\right]\psi''(\mu)d\mu.
\end{aligned}
$$

According to the Green function's definition,

$$
G(\xi,\mu)=\begin{cases}\varkappa_1-\mu, & \varkappa_1\le\mu\le\xi\\ \varkappa_1-\xi, & \xi\le\mu\le\varkappa_2,\end{cases}
$$

we obtain

$$
\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi=\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)} \qquad (8)
$$

and

$$
\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi=\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)}+\frac{(\varkappa_1-\mu)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}{\frac{\lambda}{k}}. \qquad (9)
$$

Substituting identity (8) and (9) into (7), we obtain

$$
\begin{aligned}
&\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\\
=&\int_{\varkappa_1}^{\varkappa_2}\left[G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)-\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}+\frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}\right. \qquad (10)\\
&\left.-\frac{\varkappa_1-\mu}{2}-\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]\psi''(\mu)d\mu.
\end{aligned}
$$

So, we take

$$
\begin{aligned}
\mathcal{F}(\mu)=&\;G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)-\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}+\frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}\\
&-\frac{\varkappa_1-\mu}{2}-\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}.
\end{aligned}
$$

Additionally, note that

$$
G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)=\begin{cases}\varkappa_1-\mu, & \varkappa_1\le\mu\le\frac{\varkappa_1+\varkappa_2}{2}\\ \frac{\varkappa_1-\varkappa_2}{2}, & \frac{\varkappa_1+\varkappa_2}{2}\le\mu\le\varkappa_2.\end{cases} \qquad (11)
$$

In the above identity (11), if we choose $\varkappa_1 \le \mu \le \frac{\varkappa_1+\varkappa_2}{2}$, then we obtain

$$\mathcal{F}(\mu) = \frac{\varkappa_1 - \mu}{2} - \frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} + \frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)} - \frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}},$$

$$\mathcal{F}'(\mu) = -\frac{1}{2} + \frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}}}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} - \frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}}}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} \le 0.$$

This demonstrates that $\mathcal{F}$ is decreasing. As a result, for all $\varkappa_1 \le \mu \le \frac{\varkappa_1+\varkappa_2}{2}$, $\mathcal{F}(\mu) \le 0$ from $\mathcal{F}(\varkappa_1) = 0$.

On the other hand, if $\frac{\varkappa_1+\varkappa_2}{2} \le \mu \le \varkappa_2$, then

$$\mathcal{F}(\mu) = \frac{\mu - \varkappa_2}{2} - \frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} + \frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)} - \frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}.$$

Therefore,

$$\mathcal{F}'(\mu) = \frac{1}{2} + \frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}}}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} - \frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}}}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}},$$

$$\mathcal{F}''(\mu) = -\frac{\lambda(\varkappa_2-\mu)^{\frac{\lambda}{k}-1}}{2k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} - \frac{\lambda(\mu-\varkappa_1)^{\frac{\lambda}{k}-1}}{2k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} \le 0,$$

which demonstrates that $\mathcal{F}'$ is decreasing and $\mathcal{F}'(\varkappa_2) = 0$ and so $\mathcal{F}'(\mu) \ge 0$. Consequently, $\mathcal{F}$ is increasing and $\mathcal{F}(\varkappa_2) = 0$. Hence, $\mathcal{F}(\mu) \le 0$ for all $\frac{\varkappa_1+\varkappa_2}{2} \le \mu \le \varkappa_2$. Moreover, $\psi''(\mu) \ge 0$ because $\psi$ is convex.

Taking into account the two situations mentioned above, we may conclude that $\mathcal{F}(\mu) \le 0$ for all $\mu \in [\varkappa_1, \varkappa_2]$.

The first inequality is derived from (10), as follows:

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \le \frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) + I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right].$$

For the right-hand side of the Hermite–Hadamard inequality, we recall

$$\psi(\xi) = \psi(\varkappa_1) + (\xi - \varkappa_1)\psi'(\varkappa_2) + \int_{\varkappa_1}^{\varkappa_2} G(\xi,\mu)\psi''(\mu)d\mu.$$

Let $\xi = \varkappa_2$. From the above identity, we have

$$\begin{aligned} \psi(\varkappa_2) &= \psi(\varkappa_1) + (\varkappa_2-\varkappa_1)\psi'(\varkappa_2) + \int_{\varkappa_1}^{\varkappa_2} G(\varkappa_2,\mu)\psi''(\mu)d\mu \\ \frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2} &= \psi(\varkappa_1) + \frac{\varkappa_2-\varkappa_1}{2}\psi'(\varkappa_2) + \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2} G(\varkappa_2,\mu)\psi''(\mu)d\mu \qquad (12) \end{aligned}$$

If we subtract (6) from (12), then we have

$$
\begin{aligned}
&\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right] \qquad (13)\\
=\;&\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}G(\varkappa_2,\mu)\psi''(\mu)d\mu-\frac{\lambda}{2k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right.\\
&\left.+\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right]\\
=\;&\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[G(\varkappa_2,\mu)-\frac{\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right.\right.\\
&\left.\left.+\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right\}\right]\psi''(\mu)d\mu\\
=\;&\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[G(\varkappa_2,\mu)-\frac{\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)}\right.\right.\\
&\left.\left.+\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)}+\frac{(\varkappa_1-\mu)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}{\frac{\lambda}{k}}\right\}\right]\psi''(\mu)d\mu.
\end{aligned}
$$

When we set

$$
\Im(\mu)=G(\varkappa_2,\mu)-\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}+\frac{\varkappa_2-\varkappa_1}{\frac{\lambda}{k}+1}-\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}-\varkappa_1+\mu,
$$

then, for $\varkappa_1\leq\mu\leq\varkappa_2$, we obtain

$$
\Im(\mu)=\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}-(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}.
$$

If $\varkappa_1\leq\mu\leq\frac{\varkappa_1+\varkappa_2}{2}$, then we get

$$
\Im'(\mu)=\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}}-(\mu-\varkappa_1)^{\frac{\lambda}{k}}}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\geq 0
$$

which proves that $\Im$ is increasing. $\Im(\varkappa_1)=0$, and so we obtain $\Im(\mu)\geq 0$.

Similarly, if we take $\frac{\varkappa_1+\varkappa_2}{2}\leq\mu\leq\varkappa_2$, we have

$$
\Im'(\mu)=\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}}-(\mu-\varkappa_1)^{\frac{\lambda}{k}}}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\leq 0.
$$

This suggests that $\Im$ is a decreasing function and $\Im(\varkappa_2)=0$, and consequently $\Im(\mu)\geq 0$. Moreover, $\psi''(\mu)\geq 0$ because $\psi$ is convex.

Taking into account the two situations mentioned above, we may conclude that $\mathcal{F}(\mu)\geq 0$ for all $\mu\in[\varkappa_1,\varkappa_2]$.

The second inequality is obtained from (10), as follows:

$$
\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}\geq\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right].
$$

That completes the proof. As a result, Hermite–Hadamard inequality for the $k$-fractional integral operator is proven again. □

**Theorem 6.** *Let $\psi \in C^2[\varkappa_1, \varkappa_2]$ and $\lambda, k > 0$. As a result, the following arguments are true:*

**1** *If we choose an increasing function of $|\psi''|$, then we have*

$$\left|\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) + I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \tag{14}$$
$$\le \frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2 - \frac{\lambda}{k} + 2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left\{\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right| + |\psi''(\varkappa_2)|\right\}.$$

**2** *If we choose a decreasing function of $|\psi''|$, then we have*

$$\left|\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) + I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \tag{15}$$
$$\le \frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2 - \frac{\lambda}{k} + 2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left\{|\psi''(\varkappa_1)| + \left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\right\}.$$

**3** *If $|\psi''|$ is a convex function of $[\varkappa_1, \varkappa_2]$, then*

$$\left|\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) + I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \tag{16}$$
$$\le \frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2 - \frac{\lambda}{k} + 2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left[\max\left\{|\psi''(\varkappa_1)|, \left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\right\}\right.$$
$$\left. + \max\left\{\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|, |\psi''(\varkappa_2)|\right\}\right].$$

**Proof.** In order to obtain inequality (14), if we use identity (10), we obtain

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2) + I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]$$
$$= \int_{\varkappa_1}^{\varkappa_2}\left[G\left(\frac{\varkappa_1+\varkappa_2}{2}, \mu\right) - \frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} + \frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}\right.$$
$$\left. - \frac{\varkappa_1-\mu}{2} - \frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]\psi''(\mu)d\mu$$
$$= \int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\left[\varkappa_1 - \mu - \frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} + \frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}\right.$$
$$\left. - \frac{\varkappa_1-\mu}{2} - \frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]\psi''(\mu)d\mu$$
$$+ \int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\left[\frac{\varkappa_1-\varkappa_2}{2} - \frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}} + \frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}\right.$$
$$\left. - \frac{\varkappa_1-\mu}{2} - \frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]\psi''(\mu)d\mu.$$

Taking absolute values and using triangle inequality on the above identity, utilizing simple calculations, we obtain

$$\begin{aligned}
&\left|\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \\
\leq\ &\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\left[\frac{\varkappa_1-\mu}{2}-\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right. \\
&\left.+\frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}-\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]d\mu \\
&+|\psi''(\varkappa_2)|\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\left[\frac{\mu-\varkappa_2}{2}-\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right. \\
&\left.+\frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}-\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]d\mu \\
=\ &\frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2-\frac{\lambda}{k}+2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left\{\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|+|\psi''(\varkappa_2)|\right\}.
\end{aligned}$$

So, the inequality of (14) is established. It can be easily determined using the same procedure for inequality (15). Also, to obtain the inequality of (16), we utilize the fact that the convex function $\psi$ is bounded above by $\max\{|\psi(\varkappa_1)|,|\psi(\varkappa_2)|\}$ since it is defined on the interval $[\varkappa_1,\varkappa_2]$. As a result, we obtain the inequality (16) from (10) as follows:

$$\begin{aligned}
&\left|\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \\
\leq\ &\frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2-\frac{\lambda}{k}+2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)} \\
&\times\max\left[|\psi''(\varkappa_1)|+\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|,\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|+|\psi''(\varkappa_2)|\right] \\
=\ &\frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2-\frac{\lambda}{k}+2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left[\max\left\{|\psi''(\varkappa_1)|,\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\right\}\right. \\
&\left.+\max\left\{\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|,|\psi''(\varkappa_2)|\right\}\right].
\end{aligned}$$

□

**Remark 1.** *In Theorem 6, if we choose $k=1$, then we obtain Theorem 7 in [25].*

**Theorem 7.** *Let $\psi\in C^2[\varkappa_1,\varkappa_2]$ and $\lambda,k>0$. Then, the following arguments are true:*

**1** *If we choose an increasing function of $|\psi''|$, then we have*

$$\left|\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \tag{17}$$
$$\leq \frac{\lambda(\varkappa_2-\varkappa_1)^2}{2k\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}|\psi''(\varkappa_2)|.$$

**2** *If we choose an decreasing function of $|\psi''|$, then we have*

$$\left|\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \tag{18}$$
$$\leq \frac{\lambda(\varkappa_2-\varkappa_1)^2}{2k\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}|\psi''(\varkappa_1)|.$$

**3** *If $|\psi''|$ is a convex function on $[b_1,b_2]$, then*

$$\left|\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \tag{19}$$
$$\leq \frac{\lambda(\varkappa_2-\varkappa_1)^2}{2k\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\max\{|\psi''(\varkappa_1)|,|\psi''(\varkappa_2)|\}.$$

**Proof.** We use the following identity, which we established from (13), to demonstrate the inequality (17):

$$\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]$$
$$= \int_{\varkappa_1}^{\varkappa_2}\left\{\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}-(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right\}\psi''(\mu)d\mu.$$

Taking absolute values on both sides of the above identity and using triangle inequality and $|\psi''|$ as an increasing function, we obtain

$$\left|\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right|$$
$$\leq |\psi''(\varkappa_2)|\int_{\varkappa_1}^{\varkappa_2}\left\{\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}-(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right\}d\mu$$
$$= \frac{\lambda(\varkappa_2-\varkappa_1)^2}{2k\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}|\psi''(\varkappa_2)|.$$

Thus, the inequality of (17) is established. The inequality (18) can be determined in a similar way. Finally, for inequality (19), we make use of (13) and the fact that every convex function $\psi$ defined on the interval $[\varkappa_1,\varkappa_2]$ is bound above by $\max\{|\psi(\varkappa_1)|,|\psi(\varkappa_2)|\}$ to have

$$
\begin{aligned}
&\left|\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \\
\leq\ & \max\{|\psi''(\varkappa_1)|,|\psi''(\varkappa_2)|\}\int_{\varkappa_1}^{\varkappa_2}\left\{\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}-(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right\}d\mu \\
\leq\ & \frac{\lambda(\varkappa_2-\varkappa_1)^2}{2k\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\max\{|\psi''(\varkappa_1)|,|\psi''(\varkappa_2)|\}
\end{aligned}
$$

which is the required result. □

**Remark 2.** *In Theorem 7, if we take $k=1$, then we have Theorem 5 in [25].*

**Theorem 8.** *Let $\psi\in C^2[\varkappa_1,\varkappa_2]$ and $\lambda,k>0$. If $|\psi''|$ is a convex function of $[\varkappa_1,\varkappa_2]$, then we have the following inequalities:*

$$
\begin{aligned}
&\left|\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]-\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2-\frac{\lambda}{k}+2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\{|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\}.
\end{aligned}
$$

**Proof.** Using identity (10), we have

$$
\begin{aligned}
&\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]-\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \\
=\ & \int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\left[\frac{\mu-\varkappa_1}{2}+\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}-\frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}\right. \\
&\left.+\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]\psi''(\mu)d\mu \\
&+\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\left[\frac{\varkappa_2-\mu}{2}+\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}-\frac{\varkappa_2-\varkappa_1}{2\left(\frac{\lambda}{k}+1\right)}\right. \\
&\left.+\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{2\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right]\psi''(\mu)d\mu.
\end{aligned}
$$

Setting $\mu=(1-\xi)\varkappa_1+\xi\varkappa_2$ where $d\mu=(\varkappa_2-\varkappa_1)d\xi$, after some calculation, then we have

$$
\begin{aligned}
&\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]-\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \qquad (20)\\
=\ & \frac{(\varkappa_2-\varkappa_1)^2}{2\left(\frac{\lambda}{k}+1\right)}\left\{\int_0^{\frac{1}{2}}\left[(1-\xi)^{\frac{\lambda}{k}+1}-1+\xi\left(\frac{\lambda}{k}+1\right)+\xi^{\frac{\lambda}{k}+1}\right]\psi''((1-\xi)\varkappa_1+\xi\varkappa_2)d\xi\right. \\
&\left.+\int_{\frac{1}{2}}^{1}\left[(1-\xi)^{\frac{\lambda}{k}+1}+(1-\xi)\frac{\lambda}{k}-\xi+\xi^{\frac{\lambda}{k}+1}\right]\psi''((1-\xi)\varkappa_1+\xi\varkappa_2)d\xi\right\}.
\end{aligned}
$$

In Equation (20), taking the absolute value on both sides and using the convexity of $|\psi''|$, we get

$$\begin{aligned}
&\left|\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]-\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2\left(\frac{\lambda}{k}+1\right)}\left\{\int_0^{\frac{1}{2}}\left[(1-\xi)^{\frac{\lambda}{k}+2}-(1-\xi)+\xi(1-\xi)\left(\frac{\lambda}{k}+1\right)+(1-\xi)\xi^{\frac{\lambda}{k}+1}\right]|\psi''(\varkappa_1)|d\xi\right. \\
&+\int_0^{\frac{1}{2}}\left[\xi(1-\xi)^{\frac{\lambda}{k}+1}-\xi+\xi^2\left(\frac{\lambda}{k}+1\right)+\xi^{\frac{\lambda}{k}+2}\right]|\psi''(\varkappa_2)|d\xi \\
&+\int_{\frac{1}{2}}^1\left[(1-\xi)^{\frac{\lambda}{k}+2}+(1-\xi)^2\frac{\lambda}{k}-\xi(1-\xi)+(1-\xi)\xi^{\frac{\lambda}{k}+1}\right]|\psi''(\varkappa_1)|d\xi \\
&\left.+\int_{\frac{1}{2}}^1\left[\xi(1-\xi)^{\frac{\lambda}{k}+1}+\xi(1-\xi)\frac{\lambda}{k}-\xi^2+\xi^{\frac{\lambda}{k}+2}\right]|\psi''(\varkappa_2)|d\xi\right\}.
\end{aligned}$$

If the necessary simple calculations are made, the desired result is obtained. That is:

$$\begin{aligned}
&\left|\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]-\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2\left(\left(\frac{\lambda}{k}\right)^2-\frac{\lambda}{k}+2\right)}{16\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\{|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\}.
\end{aligned}$$

□

**Remark 3.** *By setting $k=1$ in Theorem 8, then we find the result presented (Theorem 9) in [25].*

**Theorem 9.** *Let $\psi\in C^2[\varkappa_1,\varkappa_2]$ and $\lambda,k>0$. If $|\psi''|$ is a convex function of $[\varkappa_1,\varkappa_2]$, then we obtain the following inequalities:*

$$\begin{aligned}
&\left|\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2\frac{\lambda}{k}}{4\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\{|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\}.
\end{aligned}$$

**Proof.** We begin by recalling the identity given in (13) as follows:

$$\begin{aligned}
&\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right] \\
=\ & \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[G(\varkappa_2,\mu)-\frac{\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\frac{(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)}\right.\right. \\
&\left.\left.+\frac{(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)}+\frac{(\varkappa_1-\mu)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}{\frac{\lambda}{k}}\right\}\right]\psi''(\mu)d\mu \\
=\ & \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left\{\frac{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}+1}-(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}-(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\right\}\psi''(\mu)d\mu.
\end{aligned}$$

By taking $\mu=(1-\xi)\varkappa_1+\xi\varkappa_2$ and $d\mu=(\varkappa_2-\varkappa_1)d\xi$ with $\xi\in[0,1]$, we get

$$\begin{aligned}
&\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\\
=\ &\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left\{-\frac{(\varkappa_2-\varkappa_1)^2(1-\xi)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)}+\frac{(\varkappa_2-\varkappa_1)^2}{\left(\frac{\lambda}{k}+1\right)}-\frac{(\varkappa_2-\varkappa_1)^2\xi^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)}\right\}\psi''((1-\xi)\varkappa_1+\xi\varkappa_2)d\xi.
\end{aligned}$$

If we take the absolute value on both sides and use the convexity of $|\psi''|$, then

$$\begin{aligned}
&\left|\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\varkappa_1^+,k}\psi(\varkappa_2)+I^{\lambda}_{\varkappa_2^-,k}\psi(\varkappa_1)\right]\right|\\
\leq\ &\frac{(\varkappa_2-\varkappa_1)^2}{2\left(\frac{\lambda}{k}+1\right)}\left\{\int_0^1\left[-(1-\xi)^{\frac{\lambda}{k}+2}+(1-\xi)-(1-\xi)\xi^{\frac{\lambda}{k}+1}\right]|\psi''(\varkappa_1)|d\xi\right.\\
&\left.+\int_0^1\left[-\xi(1-\xi)^{\frac{\lambda}{k}+1}-\xi-\xi^{\frac{\lambda}{k}+2}\right]|\psi''(\varkappa_2)|d\xi\right\}\\
=\ &\frac{(\varkappa_2-\varkappa_1)^2\frac{\lambda}{k}}{4\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\{|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\}
\end{aligned}$$

which is our required inequality. Thus, the proof is completed. □

**Remark 4.** *Letting $k=1$ in Theorem 9 gives Theorem 11 in [25].*

**Theorem 10.** *Let $\psi\in C^2[\varkappa_1,\varkappa_2]$ and $\lambda,k>0$. If $\psi$ is a convex function of $[\varkappa_1,\varkappa_2]$, then we have the following inequalities:*

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\leq\frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2)+I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1)\right]\leq\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}. \tag{21}$$

**Proof.** First of all, from Definition 2 and utilizing identity (2), we can do the following calculations

$$\begin{aligned}
&I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2)\\
=\ &\frac{1}{k\Gamma_k(\lambda)}\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}\psi(\xi)d\xi\\
=\ &\frac{1}{k\Gamma_k(\lambda)}\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}\left\{\psi(\varkappa_1)+(\xi-\varkappa_1)\psi'(\varkappa_2)+\int_{\varkappa_1}^{\varkappa_2}G(\xi,\mu)\psi''(\mu)d\mu\right\}d\xi\\
=\ &\frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}d\xi+\psi'(\varkappa_2)\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}(\xi-\varkappa_1)d\xi\right.\\
&\left.+\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.
\end{aligned}$$

Therefore,

$$I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2) \tag{22}$$
$$= \frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\frac{\left(\frac{\varkappa_2-\varkappa_1}{2}\right)^{\frac{\lambda}{k}}}{\frac{\lambda}{k}}+\psi'(\varkappa_2)\frac{\left(\frac{\varkappa_2-\varkappa_1}{2}\right)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}\left(\frac{\lambda}{k}+1\right)}\right.$$
$$\left.+\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.$$

In the same way,

$$\begin{aligned}
&I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1)\\
=\;& \frac{1}{k\Gamma_k(\lambda)}\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}\psi(\xi)d\xi\\
=\;& \frac{1}{k\Gamma_k(\lambda)}\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}\left\{\psi(\varkappa_1)+(\xi-\varkappa_1)\psi'(\varkappa_2)+\int_{\varkappa_1}^{\varkappa_2}G(\xi,\mu)\psi''(\mu)d\mu\right\}d\xi\\
=\;& \frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}d\xi+\psi'(\varkappa_2)\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}}d\xi\right.\\
&\left.+\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.
\end{aligned}$$

Therefore,

$$I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1) \tag{23}$$
$$= \frac{1}{k\Gamma_k(\lambda)}\left\{\psi(\varkappa_1)\frac{\left(\frac{\varkappa_2-\varkappa_1}{2}\right)^{\frac{\lambda}{k}}}{\frac{\lambda}{k}}+\psi'(\varkappa_2)\frac{\left(\frac{\varkappa_2-\varkappa_1}{2}\right)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right.$$
$$\left.+\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right\}.$$

Adding (22) and (23), and multiplying the result by $\frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}$, we obtain the following result:

$$\frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2)+I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1)\right] \tag{24}$$
$$= \psi(\varkappa_1)+\frac{\varkappa_2-\varkappa_1}{2}\psi'(\varkappa_2)+\frac{2^{\frac{\lambda}{k}-1}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right.$$
$$\left.+\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right].$$

Subtracting (24) from (3), we get

$$\begin{aligned}
&\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2)+I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1)\right]\\
=\;& \int_{\varkappa_1}^{\varkappa_2} G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)\psi''(\mu)d\mu-\frac{2^{\frac{\lambda}{k}-1}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right.\\
&\left.+\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right]\\
=\;& \int_{\varkappa_1}^{\varkappa_2}\left[G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)-\frac{2^{\frac{\lambda}{k}-1}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right.\right.\\
&\left.\left.+\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right\}\right]\psi''(\mu)d\mu.
\end{aligned}$$

So, we take

$$\begin{aligned}
\mathcal{L}(\mu) &= G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)-\frac{2^{\frac{\lambda}{k}-1}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right.\\
&\left.+\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right\}.
\end{aligned} \tag{25}$$

According to the Green function's definition, we can write:

$$G(\xi,\mu)=\begin{cases}\varkappa_1-\mu, & \varkappa_1\le\mu\le\xi\\ \varkappa_1-\xi, & \xi\le\mu\le\varkappa_2,\end{cases}$$

and

$$G\left(\frac{\varkappa_1+\varkappa_2}{2},\mu\right)=\begin{cases}\varkappa_1-\mu, & \varkappa_1\le\mu\le\frac{\varkappa_1+\varkappa_2}{2}\\ \frac{\varkappa_1-\varkappa_2}{2}, & \frac{\varkappa_1+\varkappa_2}{2}\le\mu\le\varkappa_2.\end{cases}$$

Hence, if we choose $\varkappa_1\le\mu\le\frac{\varkappa_1+\varkappa_2}{2}$ in (25), then we obtain

$$\begin{aligned}
\mathcal{L}(\mu) &= (\varkappa_1-\mu)-\frac{2^{\frac{\lambda}{k}-1}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}(\varkappa_1-\mu)d\xi\right.\\
&\left.+\int_{\varkappa_1}^{\mu}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}(\varkappa_1-\xi)d\xi+\int_{\mu}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}(\varkappa_1-\mu)d\xi\right\}\\
&= -\frac{2^{\frac{\lambda}{k}-1}(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\le 0.
\end{aligned} \tag{26}$$

On the other hand, if we choose $\frac{\varkappa_1+\varkappa_2}{2}\le\mu\le\varkappa_2$ in Equation (25), then we have

$$\begin{aligned}
\mathcal{L}(\mu) &= \left(\frac{\varkappa_1-\varkappa_2}{2}\right)-\frac{2^{\frac{\lambda}{k}-1}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\mu}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}(\varkappa_1-\xi)d\xi\right.\\
&\left.+\int_{\mu}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}(\varkappa_1-\mu)d\xi+\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}(\varkappa_1-\xi)d\xi\right\}\\
&= -\frac{2^{\frac{\lambda}{k}-1}(\varkappa_2-\mu)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\le 0.
\end{aligned} \tag{27}$$

Since $\psi$ is a convex function of $[\varkappa_1, \varkappa_2],$; therefore, $\psi''(\mu) \geq 0$ and by using (26) and (27) in (25), we get

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2) + I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1)\right],$$

which is the left half inequality of (21).

Next, we prove the right half inequality of (21). For this purpose, we take $\xi = \varkappa_2$ in Equation (2), and we have

$$\begin{aligned} \psi(\varkappa_2) &= \psi(\varkappa_1) + (\varkappa_2-\varkappa_1)\psi'(\varkappa_2) + \int_{\varkappa_1}^{\varkappa_2} G(\varkappa_2,\mu)\psi''(\mu)d\mu \\ \frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2} &= \psi(\varkappa_1) + \frac{\varkappa_2-\varkappa_1}{2}\psi'(\varkappa_2) + \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2} G(\varkappa_2,\mu)\psi''(\mu)d\mu. \end{aligned} \tag{28}$$

If we subtract (24) from (28), then we have

$$\begin{aligned} &\frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2} - \frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda+k)}{(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2) + I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1)\right] \qquad (29) \\ = &\ \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2} G(\varkappa_2,\mu)\psi''(\mu)d\mu - \frac{2^{\frac{\lambda}{k}-1}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left[\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right. \\ &\left. + \int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\int_{\varkappa_1}^{\varkappa_2}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)\psi''(\mu)d\mu d\xi\right] \\ = &\ \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[G(\varkappa_2,\mu) - \frac{2^{\frac{\lambda}{k}}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right.\right. \\ &\left.\left. + \int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right\}\right]\psi''(\mu)d\mu \end{aligned}$$

When we set

$$\begin{aligned} \pounds(\mu) &= G(\varkappa_2,\mu) - \frac{2^{\frac{\lambda}{k}}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right. \\ &\left. + \int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right\} \\ &= (\varkappa_1-\mu) - \frac{2^{\frac{\lambda}{k}}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right. \\ &\left. + \int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}G(\xi,\mu)d\xi\right\}, \end{aligned}$$

then for $\varkappa_1 \leq \mu \leq \frac{\varkappa_1+\varkappa_2}{2}$, we obtain

$$\begin{aligned} \pounds(\mu) &= (\varkappa_1-\mu) - \frac{2^{\frac{\lambda}{k}}\lambda}{k(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}\left\{\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}(\varkappa_2-\xi)^{\frac{\lambda}{k}-1}(\varkappa_1-\mu)d\xi\right. \\ &\left. + \int_{\varkappa_1}^{\mu}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}(\varkappa_1-\xi)d\xi + \int_{\mu}^{\frac{\varkappa_1+\varkappa_2}{2}}(\xi-\varkappa_1)^{\frac{\lambda}{k}-1}(\varkappa_1-\mu)d\xi\right\} \\ &= -\varkappa_1 + \mu - \frac{2^{\frac{\lambda}{k}}(\mu-\varkappa_1)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2-\varkappa_1)^{\frac{\lambda}{k}}}. \end{aligned}$$

Therefore

$$\pounds'(\mu) = 1 - \frac{2^{\frac{\lambda}{k}}(\mu - \varkappa_1)^{\frac{\lambda}{k}}}{(\varkappa_2 - \varkappa_1)^{\frac{\lambda}{k}}} \geq 0$$

which proves that $\pounds(\mu)$ is increasing. $\pounds(\varkappa_1) = 0$, and so we have $\pounds(\mu) \geq 0$.

Similarly, if we take $\frac{\varkappa_1+\varkappa_2}{2} \leq \mu \leq \varkappa_2$, we get

$$\begin{aligned} \pounds(\mu) &= (\varkappa_1 - \mu) - \frac{2^{\frac{\lambda}{k}}\lambda}{k(\varkappa_2 - \varkappa_1)^{\frac{\lambda}{k}}} \left\{ \int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\mu} (\varkappa_2 - \xi)^{\frac{\lambda}{k}-1}(\varkappa_1 - \xi)d\xi \right. \\ &\quad \left. + \int_{\mu}^{\varkappa_2} (\varkappa_2 - \xi)^{\frac{\lambda}{k}-1}(\varkappa_1 - \mu)d\xi + \int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}} (\xi - \varkappa_1)^{\frac{\lambda}{k}-1}(\varkappa_1 - \xi)d\xi \right\} \\ &= \varkappa_2 - \mu - \frac{2^{\frac{\lambda}{k}}(\varkappa_2 - \mu)^{\frac{\lambda}{k}+1}}{\left(\frac{\lambda}{k}+1\right)(\varkappa_2 - \varkappa_1)^{\frac{\lambda}{k}}}. \end{aligned}$$

Hence

$$\pounds'(\mu) = -1 + \frac{2^{\frac{\lambda}{k}}(\varkappa_2 - \mu)^{\frac{\lambda}{k}}}{(\varkappa_2 - \varkappa_1)^{\frac{\lambda}{k}}} \leq 0.$$

This suggests that $\pounds(\mu)$ is a decreasing function and $\pounds(\varkappa_2) = 0$, and consequently $\pounds(\mu) \geq 0$.

Taking into account the two situations mentioned above, we may conclude that $\pounds(\mu) \geq 0$ for all $\mu \in [\varkappa_1, \varkappa_2]$. Also, $\psi''(\mu) \geq 0$ because $\psi$ is convex.

The right half inequality of (21) is obtained from (29), as follows:

$$\frac{\psi(\varkappa_1) + \psi(\varkappa_2)}{2} \geq \frac{2^{\frac{\lambda}{k}-1}\Gamma_k(\lambda + k)}{(\varkappa_2 - \varkappa_1)^{\frac{\lambda}{k}}} \left[ I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^+,k}\psi(\varkappa_2) + I^{\lambda}_{\left(\frac{\varkappa_1+\varkappa_2}{2}\right)^-,k}\psi(\varkappa_1) \right].$$

Finally, we arrive at the required result. As a result, it is demonstrated that the Hermite–Hadamard inequality for the *k*-fractional integral operator is a special case. □

## 3. Conclusions

In this article, we presented a new method to prove the Hermite–Hadamard inequality using the *k*-Riemann–Liouville fractional integral operators, based on a Green's function and obtained some new identities for convex and monotone functions. Also, using this new method, a different form of the Hermite–Hadamard inequality was obtained. In particular, we found that utilizing this new approach and the other Green's functions—$G_2$, $G_3$ and $G_4$ in [29]–different types of integral inequalities can be obtained. In addition to these identities, researchers can also obtain new inequalities for the *q*-th power of different convexities by using the Hölder and Power-mean inequalities or others (In particular, Theorem 10 can be used). Using this method, new and different identities can be obtained for concave functions. We believe that the new consequences and methods presented in this work will encourage researchers to investigate a more interesting sequel in this field.

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