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Investigation of a Coupled System of Hilfer–Hadamard Fractional Differential Equations with Nonlocal Coupled Hadamard Fractional Integral Boundary Conditions

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Abstract: We investigate the existence criteria for solutions of a nonlinear coupled system of Hilfer–Hadamard fractional differential equations of different orders complemented with nonlocal coupled Hadamard fractional integral boundary conditions. The desired results are accomplished with the aid of standard fixed-point theorems. We emphasize that the fixed point approach is one of the effective methods to establish the existence results for boundary value problems. Examples illustrating the obtained results are constructed.

Keywords: Hadamard fractional derivative; Hilfer–Hadamard fractional derivative; nonlocal integral boundary conditions; existence and uniqueness; fixed point

MSC: 26A33; 34A08; 34A34; 34B10

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1. Introduction

During the past few decades, fractional calculus has evolved as an important and popular area of research in mathematical analysis. It has been mainly due to the extensive use of its tools in developing innovative mathematical models associated with several phenomena occurring in science, engineering, mechanics, economics, etc. For examples and details, see [1–4]. Among a variety of fractional derivatives introduced so far, there has been shown a great interest in the study of the Riemann–Liouville and Caputo fractional derivatives and their applications. In [5], Hilfer introduced a fractional derivative (known as Hilfer fractional derivative), which reduces to the Riemann–Liouville and Caputo fractional derivatives for the extreme values of the parameter involved in its definition. One can find details about this derivative in [6,7]. For some interesting results on Hilfer-type initial and boundary value problems, for example, see [8–13]. In a recent work [14], some existence and Ulam–Hyers stability results for a fully coupled system of nonlinear sequential Hilfer fractional differential equations and integro-multistrip-multipoint boundary conditions were obtained. A boundary value problem for hybrid generalized Hilfer fractional differential equations was studied in [15].

A fractional derivative involving a logarithmic function with an arbitrary exponent in its kernel, introduced by Hadamard [16] in 1892, is known as Hadamard fractional derivative. Later, some variants of this derivative such as Caputo–Hadamard, and Hilfer–Hadamard fractional derivatives were studied in [17–21]. It is imperative to mention that Hadamard and Caputo–Hadamard fractional derivatives appear as special cases of the Hilfer–Hadamard fractional derivative with parameter β for $\beta = 0$ and $\beta = 1$, respectively. In [22], the authors discussed the Hyers–Ulam stability for Hilfer–Hadamard-type coupled fractional differential equations. A two-point boundary value problem for a system of nonlinear Hilfer–Hadamard sequential fractional differential equations was studied in [23]. The authors in [24] derived some existence results for nonlocal mixed Hilfer–Hadamard type fractional boundary value problems. Some existence results for a Hilfer–Hadamard fractional differential equation equipped with nonlocal integro-multipoint boundary conditions were derived in [25]. The authors studied a coupled Hilfer and Hadamard fractional differential system in generalized Banach spaces [26].

Inspired by the recent works presented in [23,24], we introduce and investigate a nonlinear nonlocal coupled boundary value problem involving Hilfer–Hadamard derivatives and Hadamard fractional integral operators of different orders given by

$$\begin{cases} {}^{HH}\mathcal{D}_{1^+}^{\alpha_1,\beta_1} u(t) = \varrho_1(t,u(t),v(t)), \ 1<\alpha_1\le 2, \ t\in\mathcal{E} := [1,T], \\ {}^{HH}\mathcal{D}_{1^+}^{\alpha_2,\beta_2} v(t) = \varrho_2(t,u(t),v(t)), \ 2<\alpha_2\le 3, \ t\in\mathcal{E} := [1,T], \end{cases} \tag{1}$$

$$\begin{cases} u(1)=0, \qquad u(T)=\lambda_1 \, {}^{H}I_{1^+}^{\delta_1} v(\eta_1), \\ v(1)=0, \qquad v(\eta_2)=0, \qquad v(T)=\lambda_2 \, {}^{H}I_{1^+}^{\delta_2} u(\eta_3), \qquad 1<\eta_1,\eta_2,\eta_3<T, \end{cases} \tag{2}$$

where ${}^{HH}\mathcal{D}_{1^+}^{\alpha_j,\beta_j}$ denotes the Hilfer-Hadamarad fractional derivative operator of order α_j and type β_j; $j=1,2$, ${}^{H}I_{1^+}^{\xi}$ is the Hadamard fractional integral operator of order $\xi\in\{\delta_1,\delta_2\}$, $\varrho_1,\varrho_2 : \mathcal{E}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $\delta_1,\delta_2>0$ and $\lambda_1,\lambda_2\in\mathbb{R}^+$.

Here we emphasize that the problem investigated in the present study is novel in the configuration of Hilfer–Hadamard fractional differential equations of different orders and coupled Hadamard-type fractional integral boundary conditions. We apply the fixed point approach to obtain the existence and uniqueness results for the problem (1) and (2). Our strategy is to convert the given problem into an equivalent fixed point problem and then use Leray–Schauder alternative and Banach's fixed point theorem to prove the existence and uniqueness results for the given problem, respectively. Our results are new and contribute to the literature on boundary value problems involving coupled systems of fractional differential equations.

The rest of the paper is organized as follows. Some preliminary concepts of fractional calculus related to our work are outlined in Section 2. An auxiliary lemma dealing with the linear variant of the problem (1) and (2) is proved in Section 3, while the main results together with illustrative examples are presented in Section 4. Some concluding remarks are given in Section 5.

2. Preliminaries

Definition 1 ([27])**.** *The Hadamard fractional integral of order $p>0$ for a continuous function $f:[a,\infty)\to\mathbb{R}$ is given by*

$${}^{H}I_{a^+}^{p} f(t) = \frac{1}{\Gamma(p)}\int_a^t \left(\log\frac{t}{s}\right)^{p-1}\frac{f(s)}{s}ds, \quad t>a,$$

where $\log(.)=\log_e(.)$.

Definition 2 ([27])**.** *The Hadamard fractional derivative of order $p>0$ for a function $f:[a,\infty)\to\mathbb{R}$ is defined by*

$${}^{H}D_{a^+}^{p} f(t) = \delta^n\left({}^{H}I_{a^+}^{n-p} f\right)(t), \quad n=[p]+1,$$

where $\delta^n = t^n\frac{d^n}{dt^n}$ and $[p]$ denotes the integer part of the real number p.

Lemma 1 ([27])**.** *If $p,q>0$ and $0<a<b<\infty$, then*

(i) $\left({}^{H}I_{a^+}^{p}\left(\log\frac{t}{a}\right)^{q-1}\right)(x) = \frac{\Gamma(q)}{\Gamma(q+p)}\left(\log\frac{x}{a}\right)^{q+p-1}$;

(ii) $\left({}^{H}D^{p}_{a^+}\left(\log\frac{t}{a}\right)^{q-1}\right)(x)=\frac{\Gamma(q)}{\Gamma(q-p)}\left(\log\frac{x}{a}\right)^{q-p-1}$.

In particular, for $q=1$, we have

$$\left({}^{H}D^{p}_{a^+}\right)(1)=\frac{1}{\Gamma(1-p)}\left(\log\frac{x}{a}\right)^{-p}\neq 0,\quad 0<p<1.$$

Definition 3 ([7])**.** *For $n-1<p<n$ and $0\leq q\leq 1$, the Hilfer-Hadamarad fractional derivative of order p and type q for $f\in L^1(a,b)$ is defined as*

$$\begin{aligned}\left({}^{HH}\mathcal{D}^{p,q}_{a^+}f\right)(t)&=\left({}^{H}I^{q(n-p)}_{a^+}\delta^{n}\,{}^{H}I^{(n-p)(1-q)}_{a^+}f\right)(t)\\&=\left({}^{H}I^{q(n-p)}_{a^+}\delta^{n}\,{}^{H}I^{(n-\gamma)}_{a^+}f\right)(t)\\&=\left({}^{H}I^{q(n-p)}_{a^+}\ {}^{H}D^{\gamma}_{a^+}f\right)(t),\quad \gamma=p+nq-pq,\end{aligned}$$

where ${}^{H}I^{(\cdot)}_{a^+}$and ${}^{H}D^{(\cdot)}$ are given in Definitions 1 and 2, respectively.

Theorem 1 ([28])**.** *If $g\in L^1(a,b)$, $0<a<b<\infty$, and $\left({}^{H}I^{n-\gamma}_{a^+}g\right)(t)\in AC^{n}_{\delta}[a,b]$, then*

$$\begin{aligned}{}^{H}I^{p}_{a^+}\left({}^{HH}\mathcal{D}^{p,q}_{a^+}g\right)(t)&={}^{H}I^{\gamma}_{a^+}\left({}^{HH}\mathcal{D}^{\gamma}_{a^+}g\right)(t)\\&=g(t)-\sum_{j=0}^{n-1}\frac{(\delta^{(n-j-1)}({}^{H}I^{n-\gamma}_{a^+}g))(a)}{\Gamma(\gamma-j)}\left(\log\frac{t}{a}\right)^{\gamma-j-1},\end{aligned}$$

where $p>0, 0\leq q\leq 1$ and $\gamma=p+nq-pq, n=[p]+1$. Observe that $\Gamma(\gamma-j)$ exists for all $j=1,2,\ldots,n-1$ and $\gamma\in[p,n]$.

3. An Auxiliary Lemma

Before presenting the main results, we prove an auxiliary lemma for the linear variant of the problem (1) and (2).

Lemma 2. *Let $h_1,h_2\in C(\mathcal{E},\mathbb{R})$ and*

$$\begin{aligned}\Delta=&\frac{\lambda_1\lambda_2\Gamma(\gamma_1)\Gamma(\gamma_2-1)}{\Gamma(\gamma_1+\delta_2)\Gamma(\delta_1+\gamma_2-1)}(\log\eta_1)^{\delta_1+\gamma_2-2}(\log\eta_3)^{\delta_2+\gamma_1-1}\left[\log\eta_2-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\right]\\&+(\log T)^{\gamma_1+\gamma_2-3}\log\left(\frac{T}{\eta_2}\right)\neq 0,\end{aligned}\tag{3}$$

$$\overline{\Delta}=(\log T)^{\gamma_1+\gamma_2-3}-\frac{\lambda_1\lambda_2\Gamma(\gamma_1)\Gamma(\gamma_2-1)}{\Gamma(\gamma_1+\delta_2)\Gamma(\delta_1+\gamma_2-1)}(\log\eta_1)^{\delta_1+\gamma_2-2}(\log\eta_3)^{\delta_2+\gamma_1-1}\neq 0.\tag{4}$$

Then the solution of the following linear Hilfer–Hadamard coupled boundary value problem:

$$\begin{cases}{}^{HH}\mathcal{D}^{\alpha_1,\beta}_{1^+}u(t)=h_1(t),\ 1<\alpha_1\leq 2,\\ {}^{HH}\mathcal{D}^{\alpha_2,\beta}_{1^+}v(t)=h_2(t),\ 2<\alpha_2\leq 3,\\ u(1)=0,\ u(T)=\lambda_1\ {}^{H}I^{\delta_1}_{1^+}v(\eta_1),\\ v(1)=0, v(\eta_2)=0,\ v(T)=\lambda_2\ {}^{H}I^{\delta_1}_{1^+}u(\eta_3),\ 1<\eta_1,\eta_2,\eta_3<T,\end{cases}\tag{5}$$

is given by

$$\begin{aligned}u(t)=&\ {}^{H}I^{\alpha_1}_{1^+}h_1(t)+\frac{(\log t)^{\gamma_1-1}}{\Delta}\Bigg\{(\log T)^{\gamma_2-2}\log\left(\frac{T}{\eta_2}\right)\left[\lambda_1\ {}^{H}I^{\delta_1+\alpha_2}_{1^+}h_2(\eta_1)-{}^{H}I^{\alpha_1}_{1^+}h_1(T)\right]\\&-\frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\gamma_2+\delta_1-1)}\left[\log\eta_2-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\right]\left[\lambda_2\ {}^{H}I^{\delta_2+\alpha_1}_{1^+}h_1(\eta_3)\right.\end{aligned}$$

$$- \,{}^{H}I_{1^+}^{\alpha_2} h_2(T)\Big] - \frac{\lambda_1 \Gamma(\gamma_2 - 1)(\log \eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\gamma_2 + \delta_1 - 1)} \Big(\frac{\log T}{\log \eta_2}\Big)^{\gamma_2 - 2} \times$$
$$\Big[\log T - \frac{\gamma_2 - 1}{\gamma_2 + \delta_1 - 1} \log \eta_1\Big] \,{}^{H}I_{1^+}^{\alpha_2} h_2(\eta_2)\bigg\}, \tag{6}$$

and

$$v(t) = \,{}^{H}I_{1^+}^{\alpha_2} h_2(t) - \Big(\frac{\log t}{\log \eta_2}\Big)^{\gamma_2 - 2} \,{}^{H}I_{1^+}^{\alpha_2} h_2(\eta_2) + \frac{(\log t)^{\gamma_2 - 2}}{\Delta} \log\Big(\frac{t}{\eta_2}\Big) \times$$
$$\bigg\{(\log T)^{\gamma_1 - 1}\Big[\lambda_2 \,{}^{H}I_{1^+}^{\delta_2 + \alpha_1} h_1(\eta_3) - \,{}^{H}I_{1^+}^{\alpha_2} h_2(T)\Big] + \frac{\lambda_2 \Gamma(\gamma_1)(\log \eta_3)^{\gamma_1+\delta_2-1}}{\Gamma(\gamma_1 + \delta_2)} \times$$
$$\Big[\lambda_1 \,{}^{H}I_{1^+}^{\delta_1 + \alpha_2} h_2(\eta_1) - \,{}^{H}I_{1^+}^{\alpha_1} h_1(T)\Big] + \frac{\overline{\Delta}}{(\log \eta_2)^{\gamma_2 - 2}} \,{}^{H}I_{1^+}^{\alpha_2} h_2(\eta_2)\bigg\}. \tag{7}$$

Proof. Applying the Hadamard integral operators ${}^{H}I_{1^+}^{\alpha_1}$ and ${}^{H}I_{1^+}^{\alpha_2}$ to the first and second equations in (5), respectively, and using Theorem 1, we get

$$u(t) = \,{}^{H}I_{1^+}^{\alpha_1} h_1(t) + c_0(\log t)^{\gamma_1 - 1} + c_1(\log t)^{\gamma_1 - 2}, \ \gamma_1 = \alpha_1 + (2 - \alpha_1)\beta_1, \tag{8}$$
$$v(t) = \,{}^{H}I_{1^+}^{\alpha_2} h_2(t) + d_0(\log t)^{\gamma_2 - 1} + d_1(\log t)^{\gamma_2 - 2} + d_2(\log t)^{\gamma_2 - 3}, \ \gamma_2 = \alpha_2 + (3 - \alpha_2)\beta_2, \tag{9}$$

where c_0, c_1, d_0, d_1 and d_2 are unknown arbitrary constants. Combining (8) and (9) with $u(1) = 0$ and $v(1) = 0$, we get $c_1 = 0$, $d_2 = 0$ since $\gamma_1 \in [\alpha_1, 2]$ and $\gamma_2 \in [\alpha_2, 3]$. Consequently, Equations (8) and (9) become

$$u(t) = \,{}^{H}I_{1^+}^{\alpha_1} h_1(t) + c_0(\log t)^{\gamma_1 - 1}, \tag{10}$$
$$v(t) = \,{}^{H}I_{1^+}^{\alpha_2} h_2(t) + d_0(\log t)^{\gamma_2 - 1} + d_1(\log t)^{\gamma_2 - 2}. \tag{11}$$

Using the condition $v(\eta_2) = 0$ in (11), we get

$$d_1 = -\frac{1}{(\log \eta_2)^{\gamma_2 - 2}} \Big\{ \,{}^{H}I_{1^+}^{\alpha_2} h_2(\eta_2) + d_0(\log \eta_2)^{\gamma_2 - 1}\Big\}. \tag{12}$$

Inserting the value of d_1 in (11), we have

$$v(t) = \,{}^{H}I_{1^+}^{\alpha_2} h_2(t) + d_0(\log t)^{\gamma_2 - 2} \log\Big(\frac{t}{\eta_2}\Big) - \Big(\frac{\log t}{\log \eta_2}\Big)^{\gamma_2 - 2} \,{}^{H}I_{1^+}^{\alpha_2} h_2(\eta_2). \tag{13}$$

Now, using (10) and (13) in the conditions: $u(T) = \lambda_1 \,{}^{H}I_{1^+}^{\delta_1} v(\eta_1)$ and $v(T) = \lambda_2 \,{}^{H}I_{1^+}^{\delta_2} u(\eta_3)$, we find that

$$c_0 A_1 + d_0 A_2 = J_1,$$
$$c_0 B_1 + d_0 B_2 = J_2, \tag{14}$$

where

$$A_1 = (\log T)^{\gamma_1 - 1}, \ A_2 = \lambda_1 \frac{\Gamma(\gamma_2 - 1)(\log \eta_1)^{\delta_1 + \gamma_2 - 2}}{\Gamma(\delta_1 + \gamma_2 - 1)} \Big\{\log \eta_2 - \frac{\gamma_2 - 1}{\delta_1 + \gamma_2 - 1} \log \eta_1\Big\},$$
$$J_1 = \lambda_1 \,{}^{H}I_{1^+}^{\delta_1 + \alpha_2} h_2(\eta_1) - \lambda_1 \frac{\Gamma(\gamma_2 - 1)}{\Gamma(\delta_1 + \gamma_2 - 1)} \frac{(\log \eta_1)^{\delta_1 + \gamma_2 - 2}}{(\log \eta_2)^{\gamma_2 - 2}} \,{}^{H}I_{1^+}^{\alpha_2} h_2(\eta_2) - \,{}^{H}I_{1^+}^{\alpha_1} h_1(T),$$
$$B_1 = -\frac{\lambda_2 \Gamma(\gamma_1)}{\Gamma(\delta_2 + \gamma_1)} (\log \eta_3)^{\delta_2 + \gamma_1 - 1}, \ B_2 = (\log T)^{\gamma_2 - 2} \log\Big(\frac{T}{\eta_2}\Big),$$
$$J_2 = \lambda_2 \,{}^{H}I_{1^+}^{\delta_2 + \alpha_1} h_1(\eta_3) + \Big(\frac{\log T}{\log \eta_2}\Big)^{\gamma_2 - 2} \,{}^{H}I_{1^+}^{\alpha_2} h_2(\eta_2) - \,{}^{H}I_{1^+}^{\alpha_2} h_2(T).$$

Solving the system (14) for c_0 and d_0, we get

$$\begin{aligned}
c_0 =& \frac{1}{\Delta}\Bigg\{(\log T)^{\gamma_2-2}\log\left(\frac{T}{\eta_2}\right)\left[\lambda_1\, {}^{H}I_{1^+}^{\delta_1+\alpha_2}h_2(\eta_1) - {}^{H}I_{1^+}^{\alpha_1}h_1(T)\right] \\
&-\lambda_1\frac{\Gamma(\gamma_2-1)}{\Gamma(\delta_1+\gamma_2-1)}(\log\eta_1)^{\delta_1+\gamma_2-2}\left[\log\eta_2 - \frac{\gamma_2-1}{\delta_1+\gamma_2-1}\log\eta_1\right]\times \\
&\left[\lambda_2\, {}^{H}I_{1^+}^{\alpha_1+\delta_2}h_1(\eta_3) - {}^{H}I_{1^+}^{\alpha_2}h_2(T)\right] \\
&-\frac{\lambda_1\Gamma(\gamma_2-1)}{\Gamma(\delta_1+\gamma_2-1)}(\log\eta_1)^{\delta_1+\gamma_2-2}\left(\frac{\log T}{\log\eta_2}\right)^{\gamma_2-2}\times \\
&\left[\log T - \frac{\gamma_2-1}{\delta_1+\gamma_2-1}\log\eta_1\right]\, {}^{H}I_{1^+}^{\alpha_2}h_2(\eta_2)\Bigg\},
\end{aligned} \tag{15}$$

$$\begin{aligned}
d_0 =& \frac{1}{\Delta}\Bigg\{(\log T)^{\gamma_1-1}\left[\lambda_2\, {}^{H}I_{1^+}^{\delta_2+\alpha_1}h_1(\eta_3) - {}^{H}I_{1^+}^{\alpha_2}h_2(T)\right] \\
&+\frac{\lambda_2\Gamma(\gamma_1)}{\Gamma(\delta_2+\gamma_1)}(\log\eta_3)^{\delta_2+\gamma_1-1}\left[\lambda_1\, {}^{H}I_{1^+}^{\delta_1+\alpha_2}h_2(\eta_1) - {}^{H}I_{1^+}^{\alpha_1}h_1(T)\right] \\
&+\frac{\overline{\Delta}}{(\log\eta_2)^{\gamma_2-2}}\, {}^{H}I_{1^+}^{\alpha_2}h_2(\eta_2)\Bigg\},
\end{aligned} \tag{16}$$

where Δ, $\overline{\Delta}$ are defined in (3) and (4), respectively. Substituting the value of c_0 from (15) into (10), and the values of d_0 and d_1, respectively, from (16) and (12) into (11), we get the solution (6) and (7). The converse of this lemma can be established by direct computation. □

4. Main Results

For a Banach space $U = C(\mathcal{E}, \mathbb{R})$ equipped with the norm $\|u\| = \sup_{t\in\mathcal{E}}|u(t)|$, the product space $(U\times U, \|.\|_{U\times U})$ endowed with the norm $\|(u,v)\|_{U\times U} = \|u\| + \|v\|$ for $(u,v)\in U\times U$ is also a Banach space.

In view of Lemma 2, we introduce an operator $\Upsilon : U\times U \to U\times U$ as follows:

$$\Upsilon(u,v)(t) := (\Upsilon_1(u,v)(t), \Upsilon_2(u,v)(t)), \tag{17}$$

where

$$\begin{aligned}
\Upsilon_1(u,v)(t) =& \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{\varrho_1(s,u(s),v(s))}{s}ds + \frac{(\log t)^{\gamma_1-1}}{\Delta}\times \\
&\Bigg\{(\log T)^{\gamma_2-2}\log\left(\frac{T}{\eta_2}\right)\Bigg[\frac{\lambda_1}{\Gamma(\delta_1+\alpha_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\delta_1+\alpha_2-1}\frac{\varrho_2(s,u(s),v(s))}{s}ds \\
&-\frac{1}{\Gamma(\alpha_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_1-1}\frac{\varrho_1(s,u(s),v(s))}{s}ds\Bigg] \\
&-\frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\gamma_2+\delta_1-1)}\left[\log\eta_2 - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\right]\times \\
&\Bigg[\frac{\lambda_2}{\Gamma(\delta_2+\alpha_1)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{s}\right)^{\delta_3+\alpha_1-1}\frac{\varrho_1(s,u(s),v(s))}{s}ds \\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_2-1}\frac{\varrho_2(s,u(s),v(s))}{s}ds\Bigg] \\
&-\frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\alpha_2)\Gamma(\gamma_2+\delta_1-1)}\left(\frac{\log T}{\log\eta_2}\right)^{\gamma_2-2}\left[\log T - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\right]
\end{aligned}$$

$$\times \int_1^{\eta_2} \Big(\log \frac{\eta_2}{s} \Big)^{\alpha_2 - 1} \frac{\varrho_2(s, u(s), v(s))}{s} ds \Big\},$$

and

$$\begin{aligned}
&\Upsilon_2(u,v)(t) \\
&= \frac{1}{\Gamma(\alpha_2)} \int_1^t \Big(\log \frac{t}{s} \Big)^{\alpha_2 - 1} \frac{\varrho_2(s, u(s), v(s))}{s} ds \\
&- \frac{1}{\Gamma(\alpha_2)} \Big(\frac{\log t}{\log \eta_2} \Big)^{\gamma_2 - 1} \int_1^{\eta_2} \Big(\log \frac{\eta_2}{s} \Big)^{\alpha_2 - 1} \frac{\varrho_2(s, u(s), v(s))}{s} ds \\
&+ \frac{(\log t)^{\gamma_2 - 2}}{\Delta} \log \Big(\frac{t}{\eta_2} \Big) \Big\{ (\log T)^{\gamma_1 - 1} \Big[\frac{\lambda_2}{\Gamma(\delta_2 + \alpha_1)} \int_1^{\eta_3} \Big(\log \frac{\eta_3}{s} \Big)^{\delta_2 + \alpha_1 - 1} \frac{\varrho_1(s, u(s), v(s))}{s} ds \\
&- \frac{1}{\Gamma(\alpha_2)} \int_1^T \Big(\log \frac{T}{s} \Big)^{\alpha_2 - 1} \frac{\varrho_2(s, u(s), v(s))}{s} ds \Big] + \frac{\lambda_2 \Gamma(\gamma_2)(\log \eta_3)^{\gamma_1 + \delta_2 - 1}}{\Gamma(\gamma_1 + \delta_2)} \times \\
&\Big[\frac{\lambda_1}{\Gamma(\delta_1 + \alpha_2)} \int_1^{\eta_1} \Big(\log \frac{\eta_1}{s} \Big)^{\delta_1 + \alpha_2 - 1} \frac{\varrho_2(s, u(s), v(s))}{s} ds \\
&- \frac{1}{\Gamma(\alpha_1)} \int_1^T \Big(\log \frac{T}{s} \Big)^{\alpha_1 - 1} \frac{\varrho_1(s, u(s), v(s))}{s} ds \Big] \\
&+ \frac{\overline{\Delta}}{\Gamma(\alpha_2)(\log \eta_2)^{\gamma_2 - 2}} \int_1^{\eta_2} \Big(\log \frac{\eta_2}{s} \Big)^{\alpha_2 - 1} \frac{\varrho_2(s, u(s), v(s))}{s} ds \Big\}.
\end{aligned}$$

In the sequel, we need the following assumptions.

(H_1) For real constants $\kappa_i, \overline{\kappa}_i \geq 0$ $(i = 1, 2)$ and $\kappa_0 > 0$, $\overline{\kappa}_0 > 0$, we have

$$|\varrho_1(t, u, v)| \leq \kappa_0 + \kappa_1 |u| + \kappa_2 |v|, \;\; |\varrho_2(t, u, v)| \leq \overline{\kappa}_0 + \overline{\kappa}_1 |u| + \overline{\kappa}_2 |v|, \;\; \forall u, v \in \mathbb{R}.$$

(H_2) For positive constants l_1 and l_2, we have

$$\begin{aligned}
|\varrho_1(t, \nu_1, \mu_1) - \varrho_1(t, \nu_2, \mu_2)| &\leq l_1(|\nu_1 - \nu_2| + |\mu_1 - \mu_2|), \\
|\varrho_2(t, \nu_1, \mu_1) - \varrho_2(t, \nu_2, \mu_2)| &\leq l_2(|\nu_1 - \nu_{21}| + |\mu_1 - \mu_2|), \forall t \in \mathcal{E}, \; \nu_i, \; \mu_i \in \mathbb{R}, \; i = 1, 2.
\end{aligned}$$

Moreover, we set the notation:

$$\begin{aligned}
\Omega_1 &= \frac{(\log T)^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{(\log T)^{\alpha_1 + \gamma_1 + \gamma_2 - 3}}{|\Delta| \Gamma(\alpha_1 + 1)} \log \Big(\frac{T}{\eta_2} \Big) \\
&+ \lambda_1 \lambda_2 \frac{\Gamma(\gamma_2 - 1)(\log \eta_1)^{\gamma_2 + \delta_1} (\log \eta_3)^{\delta_2 + \alpha_1} (\log T)^{\gamma_1 - 1}}{|\Delta| \Gamma(\gamma_2 + \delta_1 + 1) \Gamma(\delta_2 + \alpha_1 + 1)} \Big| \log \eta_2 - \frac{\gamma_2 - 1}{\gamma_2 + \delta_1 - 1} \log \eta_1 \Big|,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\Omega_2 &= \lambda_1 \frac{(\log \eta_1)^{\alpha_2 + \delta_1} (\log T)^{\gamma_1 + \gamma_2 - 3}}{|\Delta| \Gamma(\delta_1 + \alpha_2 + 1)} \log \Big(\frac{T}{\eta_2} \Big) \\
&+ \lambda_1 \frac{\Gamma(\gamma_2 - 1)(\log \eta_1)^{\gamma_2 + \delta_2} (\log T)^{\gamma_1 + \alpha_2 - 1}}{|\Delta| \Gamma(\gamma_2 + \delta_1 - 1) \Gamma(\alpha_2 + 1)} \Big| \log \eta_2 - \frac{\gamma_2 - 1}{\gamma_2 + \delta_1 - 1} \log \eta_1 \Big| \\
&+ \lambda_1 \frac{\Gamma(\gamma_2 - 1)(\log \eta_1)^{\gamma_2 + \delta_2} (\log T)^{\gamma_1 - 1} (\log \eta_2)^{\alpha_2}}{|\Delta| \Gamma(\gamma_2 + \delta_1 - 1) \Gamma(\alpha_2 + 1)} \Big| \log T - \frac{\gamma_2 - 1}{\gamma_2 + \delta_1 - 1} \log \eta_1 \Big|,
\end{aligned} \tag{19}$$

$$\overline{\Omega}_1 = \log \Big(\frac{T}{\eta_2} \Big) \Big[\frac{\lambda_2 (\log T)^{\gamma_1 + \gamma_2 - 3} (\log \eta_3)^{\delta_2 + \alpha_1}}{|\Delta| \Gamma(\delta_2 + \alpha_1 + 1)} + \lambda_2 \frac{\Gamma(\gamma_1)(\log \eta_3)^{\gamma_1 + \delta_2 - 1} (\log T)^{\gamma_2 + \alpha_1 - 2}}{|\Delta| \Gamma(\gamma_1 + \delta_1) \Gamma(\alpha_1 + 1)} \Big], \tag{20}$$

$$\overline{\Omega}_2 = \frac{(\log T)^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \frac{(\log T)^{\gamma_2 - 2}}{\Gamma(\alpha_2 + 1)(\log \eta_2)^{\gamma_2 - \alpha_2 - 2}} + \log \Big(\frac{T}{\eta_2} \Big) \Big[\frac{(\log T)^{\gamma_1 + \gamma_2 + \alpha_2 - 3}}{|\Delta| \Gamma(\alpha_2 + 1)}$$

$$+\quad \lambda_1\lambda_2\frac{\Gamma(\gamma_1)(\log\eta_1)^{\delta_1+\alpha_2}(\log\eta_3)^{\gamma_1+\delta_2-1}(\log T)^{\gamma_2-2}}{|\Delta|\Gamma(\gamma_1+\delta_2)\Gamma(\delta_1+\alpha_2+1)}+\frac{|\overline{\Delta}|(\log T)^{\gamma_2-1}(\log\eta_2)^{\alpha_2}}{|\Delta|\Gamma(\alpha_2+1)}\Big], \tag{21}$$

$$\Phi = \min\Big\{1-[(\Omega_1+\overline{\Omega}_1)\kappa_1+(\Omega_2+\overline{\Omega}_2)\overline{\kappa}_1],1-[(\Omega_1+\overline{\Omega}_1)\kappa_1+(\Omega_2+\overline{\Omega}_2)\overline{\kappa}_1]\Big\}. \tag{22}$$

We apply the following known result to prove the existence of solutions for the problem (1) and (2).

Lemma 3 (Leray-Schauder alternative [29]). *Let $\Theta(\chi)=\{x\in\mathcal{D}:x=\kappa\chi(x)$ for some $0<\kappa<1\}$, where $\chi:\mathcal{D}\to\mathcal{D}$ is a completely continuous operator. Then either the set $\Theta(\chi)$ is unbounded or there exists at least one fixed point for the operator χ.*

Theorem 2. *Let Δ, $\overline{\Delta}$, $\Phi\neq 0$, where Δ, $\overline{\Delta}$ and Φ are defined in (3), (4) and (22), respectively. If the assumption (H_1) holds, and that*

$$(\Omega_1+\Omega_2)\kappa_1+(\overline{\Omega}_1+\overline{\Omega}_2)\overline{\kappa}_1<1, \tag{23}$$

and

$$(\Omega_1+\Omega_2)\kappa_2+(\overline{\Omega}_1+\overline{\Omega}_2)\overline{\kappa}_2<1. \tag{24}$$

Then there exists at least one solution for the problem (1) and (2) on $\mathcal{E}$.

Proof. Let us first establish that the operator $\Upsilon:\ U\times U\to U\times U$ defined by (17) is completely continuous. Obviously, continuity of the operator Υ (in terms of Υ_1 and Υ_2) follows from that of ϱ_1 and ϱ_2. Next we show that the operator Υ is uniformly bounded. For that, let $\mathcal{M}\subset U\times U$ be a bounded set. Then we can find positive constants N_1 and N_2 satisfying $|\varrho_1(t,u(t),v(t)|\leq N_1$ and $|\varrho_2(t,u(t),v(t))|\leq N_2$, $\forall(u,v)\in\mathcal{M}$. Consequently, we obtain

$$\begin{aligned}
&\|\Upsilon_1(u,v)\|\\
&=\sup_{t\in\mathcal{E}}|\Upsilon_1(u,v)(t)|\\
&\leq\frac{1}{\Gamma(\alpha_1)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\frac{|\varrho_1(s,u(s),v(s))|}{s}ds\\
&+\frac{(\log t)^{\gamma_1-1}}{|\Delta|}\Bigg\{(\log T)^{\gamma_2-2}\log\Big(\frac{T}{\eta_2}\Big)\Big[\frac{\lambda_1}{\Gamma(\delta_1+\alpha_2)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\delta_1+\alpha_2-1}\frac{|\varrho_2(s,u(s),v(s))|}{s}ds\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^T\Big(\log\frac{T}{s}\Big)^{\alpha_1-1}\frac{|\varrho_1(s,u(s),v(s))|}{s}ds\Big]\\
&+\frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\gamma_2+\delta_1-1)}\Big|\log\eta_2-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\times\\
&\Big[\frac{\lambda_2}{\Gamma(\delta_2+\alpha_1)}\int_1^{\eta_3}\Big(\log\frac{\eta_3}{s}\Big)^{\delta_3+\alpha_1-1}\frac{|\varrho_1(s,u(s),v(s))|}{s}ds\\
&+\frac{1}{\Gamma(\alpha_2)}\int_1^T\Big(\log\frac{T}{s}\Big)^{\alpha_2-1}\frac{|\varrho_2(s,u(s),v(s))|}{s}ds\Big]\\
&+\frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\alpha_2)\Gamma(\gamma_2+\delta_1-1)}\Big(\frac{\log T}{\log\eta_2}\Big)^{\gamma_2-2}\Big|\log T-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\times\\
&\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\alpha_2-1}\frac{|\varrho_2(s,u(s),v(s))|}{s}ds\Bigg\},\\
&\leq N_1\Big\{\frac{(\log T)^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{(\log T)^{\alpha_1+\gamma_1+\gamma_2-3}}{|\Delta|\Gamma(\alpha_1+1)}\log\Big(\frac{T}{\eta_2}\Big)
\end{aligned}$$

$$+\lambda_1\lambda_2\frac{\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1}(\log\eta_3)^{\delta_2+\alpha_1}(\log T)^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_2+\delta_1+1)\Gamma(\delta_2+\alpha_1+1)}\Big|\log\eta_1-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\Big\}$$
$$+N_2\Big\{\lambda_1\frac{(\log\eta_1)^{\alpha_2+\delta_1}(\log T)^{\gamma_1+\gamma_2-3}}{|\Delta|\Gamma(\delta_1+\alpha_2+1)}\log\Big(\frac{T}{\eta_2}\Big)$$
$$+\lambda_1\frac{\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_2}(\log T)^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_2+\delta_1-1)\Gamma(\alpha_2+1)}\Big|\log\eta_2-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|$$
$$+\lambda_1\frac{\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_2}(\log T)^{\gamma_1-1}(\log\eta_2)^{\alpha_2}}{|\Delta|\Gamma(\gamma_2+\delta_1-1)\Gamma(\alpha_2+1)}\Big|\log T-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\Big\},$$

which, in view of the notation (18) and (19) yields

$$\|\Upsilon_1(u,v)\|\leq\Omega_1N_1+\Omega_2N_2. \tag{25}$$

Likewise, using the notation (20) and (21), we have

$$\|\Upsilon_2(u,v)\|\leq\overline{\Omega}_1N_1+\overline{\Omega}_2N_2. \tag{26}$$

Then it follows from (25) and (26) that

$$\|\Upsilon(u,v)\| \quad\leq\quad (\Omega_1+\overline{\Omega}_1)N_1+(\Omega_2+\overline{\Omega}_2)N_2, \tag{27}$$

which shows that the operator Υ is uniformly bounded.

In order to establish that Υ is equicontinuous, we take $t_1,\ t_2\in\mathcal{E}$ with $t_1<t_2$. Then, we find that

$$|\Upsilon_1(u,v)(t_2)-\Upsilon_1(u,v)(t_1)|$$
$$\leq\frac{N_1}{\Gamma(\alpha_1)}\Big[\int_1^{t_1}\Big|\Big(\log\frac{t_2}{s}\Big)^{\alpha_1-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha_1-1}\Big|\frac{ds}{s}$$
$$+\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha_1-1}\frac{ds}{s}\Big]+\frac{N_1\Big|(\log t_2)^{\gamma_1-1}-(\log t_1)^{\gamma_1-1}\Big|}{|\Delta|}\times$$
$$\Big\{\frac{(\log T)^{\gamma_2-2}}{\Gamma(\alpha_1)}\log\Big(\frac{T}{\eta_2}\Big)\int_1^T\Big(\log\frac{T}{s}\Big)^{\alpha_1-1}\frac{ds}{s}+\frac{\lambda_1\lambda_2\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\delta_2+\alpha_1)\Gamma(\gamma_2+\delta_1-1)}\Big|\log\eta_2$$
$$-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\int_1^{\eta_3}\Big(\log\frac{\eta_3}{s}\Big)^{\delta_3+\alpha_1-1}\frac{ds}{s}\Big\}$$
$$+\frac{N_2\Big|(\log t_2)^{\gamma_1-1}-(\log t_1)^{\gamma_1-1}\Big|}{|\Delta|}\Big\{\frac{\lambda_1(\log T)^{\gamma_2-2}}{\Gamma(\delta_1+\alpha_2)}\log\Big(\frac{T}{\eta_2}\Big)\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\delta_1+\alpha_2-1}\frac{ds}{s}$$
$$+\frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\alpha_2)\Gamma(\gamma_2+\delta_1-1)}\Big|\log\eta_2-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\int_1^T\Big(\log\frac{T}{s}\Big)^{\alpha_2-1}\frac{ds}{s}$$
$$+\frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\alpha_2)\Gamma(\gamma_2+\delta_1-1)}\Big(\frac{\log T}{\log\eta_2}\Big)^{\gamma_2-2}\Big|\log\ T$$
$$-\frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\alpha_2-1}\frac{ds}{s}\Big\},\to 0\text{ as }t_2\to t_1,$$

independent of $(u,v)\in\mathcal{M}$. Likewise, it can be shown that $|\Upsilon_2(u,v)(t_2)-\Upsilon_2(u,v)(t_1)|\to 0$ as $t_2\to t_1$ independent of $(u,v)\in\mathcal{M}$. Thus, the equicontinuity of Υ_1 and Υ_2 implies that the operator Υ is equicontinuous. Hence, the operator Υ is compact by Arzela-Ascoli's theorem.

Finally, we establish the boundedness of the set $\Theta(\Upsilon)=\{(u,v)\in U\times U:(u,v)=\kappa\Upsilon(u,v)\ ;\ 0\leq\kappa\leq 1\}$. Let $(u,v)\in\Theta(\Upsilon)$. Then $(u,v)=\kappa\Upsilon(u,v)$, which implies that $u(t)=\kappa\Upsilon_1(u,v)(t),\ v(t)=\kappa\Upsilon_2(u,v)(t)$ for any $t\in\mathcal{E}$.

By the assumption (H_1), we get

$$\begin{aligned}
|u(t)| \leq & \frac{1}{\Gamma(\alpha_1)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1-1} \Big[\kappa_0 + \kappa_1|u| + \kappa_2|v|\Big] \frac{ds}{s} \\
& + \frac{(\log t)^{\gamma_1-1}}{|\Delta|} \Bigg\{ (\log T)^{\gamma_2-2} \log\left(\frac{T}{\eta_2}\right) \Big[\frac{\lambda_1}{\Gamma(\delta_1+\alpha_2)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\delta_1+\alpha_2-1} \Big[\overline{\kappa}_0 + \overline{\kappa}_1|u| \\
& + \overline{\kappa}_2|v|\Big] \frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_1-1} \Big[\kappa_0 + \kappa_1|u| + \kappa_2|v|\Big] \frac{ds}{s} \Big] \\
& + \frac{\lambda_1 \Gamma(\gamma_2-1)(\log \eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\gamma_2+\delta_1-1)} \left| \log \eta_2 - \frac{\gamma_2-1}{\gamma_2+\delta_1-1} \log \eta_1 \right| \times \\
& \Big[\frac{\lambda_2}{\Gamma(\delta_2+\alpha_1)} \int_1^{\eta_3} \left(\log \frac{\eta_3}{s}\right)^{\delta_3+\alpha_1-1} \Big[\kappa_0 + \kappa_1|u| + \kappa_2|v|\Big] \frac{ds}{s} \\
& + \frac{1}{\Gamma(\alpha_2)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_2-1} \Big[\overline{\kappa}_0 + \overline{\kappa}_1|u| + \overline{\kappa}_2|v|\Big] \frac{ds}{s} \Big] \\
& + \frac{\lambda_1 \Gamma(\gamma_2-1)(\log \eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\alpha_2)\Gamma(\gamma_2+\delta_1-1)} \left(\frac{\log T}{\log \eta_2}\right)^{\gamma_2-2} \left| \log T - \frac{\gamma_2-1}{\gamma_2+\delta_1-1} \log \eta_1 \right| \\
& \times \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\alpha_2-1} \Big[\overline{\kappa}_0 + \overline{\kappa}_1|u| + \overline{\kappa}_2|v|\Big] \frac{ds}{s} \Bigg\} \\
& \leq \Omega_1[\kappa_0 + \kappa_1|u| + \kappa_2|v|] + \Omega_2[\overline{\kappa}_0 + \overline{\kappa}_1|u| + \overline{\kappa}_2|v|],
\end{aligned}$$

which implies that

$$\|u\| = \sup_{t \in \mathcal{E}} |u(t)| \leq \Omega_1\kappa_0 + \Omega_2\overline{\kappa}_0 + (\Omega_1\kappa_1 + \Omega_2\overline{\kappa}_1)||u|| + (\Omega_1\kappa_2 + \Omega_2\overline{\kappa}_2)||v||. \tag{28}$$

Similarly, one can find that

$$\|v\| \leq \overline{\Omega}_1\kappa_0 + \overline{\Omega}_2\overline{\kappa}_0 + (\overline{\Omega}_1\kappa_1 + \overline{\Omega}_2\overline{\kappa}_1)\|u\| + (\overline{\Omega}_1\kappa_2 + \overline{\Omega}_2\overline{\kappa}_2)\|v\|. \tag{29}$$

From (28) and (29), we obtain

$$\begin{aligned}
\|u\| + \|v\| \quad \leq \quad & (\Omega_1 + \overline{\Omega}_1)\kappa_0 + (\Omega_2 + \overline{\Omega}_2)\overline{\kappa}_0 + [(\Omega_1 + \overline{\Omega}_1)\kappa_1 + (\Omega_2 + \overline{\Omega}_2)\overline{\kappa}_1]||u|| \\
+ \quad & [(\Omega_1 + \overline{\Omega}_1)\kappa_2 + (\Omega_2 + \overline{\Omega}_2)\overline{\kappa}_2]\|v\|,
\end{aligned} \tag{30}$$

which, by $\|(u,v)\| = \|u\| + \|v\|$, yields

$$\|(u,v)\| \leq \frac{1}{\Phi}[(\Omega_1 + \overline{\Omega}_1)\kappa_0 + (\Omega_2 + \overline{\Omega}_2)\overline{\kappa}_0].$$

In consequence, $\Theta(\Upsilon)$ is bounded. Thus, the conclusion of Lemma 3 applies and hence the operator Υ has at least one fixed point, which is indeed a solution of the problem (1) and (2). □

In the following result, the existence of a unique solution for the problem (1) and (2) will be proved by means of a fixed point theorem due to Banach.

Theorem 3. *If the assumption (H_2) is satisfied and that*

$$\left(\Omega_1 + \overline{\Omega}_1\right)l_1 + \left(\Omega_2 + \overline{\Omega}_2\right)l_2 < 1, \tag{31}$$

where Ω_i and $\overline{\Omega}_i$, $(i = 1,2)$ are given in (18)–(21), then the problem (1) and (2) has a unique solution on $\mathcal{E}$.

Proof. Letting $K_1 = \sup_{t\in\mathcal{E}} |\varrho_1(t,0,0)| < \infty$ and $K_2 = \sup_{t\in\mathcal{E}} |\varrho_2(t,0,0)| < \infty$, it follows by the assumption (H_1) that

$$|\varrho_1(t,u,v)| \le l_1(\|u\| + \|v\|) + K_1 \le l_1\|(u,v)\| + K_1,$$

and

$$|\varrho_2(t,u,v)| \le l_2\|(u,v)\| + K_2.$$

Firstly, we show that $\Upsilon\mathcal{B}_\rho \subset \mathcal{B}_\rho$, where $\mathcal{B}_\rho = \{(u,v) \in U \times U : \|(u,v)\| \le \rho\}$, with

$$\rho > \frac{(\Omega_1 + \overline{\Omega}_1)K_1 + (\Omega_2 + \overline{\Omega}_2)K_2}{1 - (\Omega_1 + \overline{\Omega}_1)l_1 - (\Omega_2 + \overline{\Omega}_2)l_2}. \tag{32}$$

For $(u,v) \in \mathcal{B}_\rho$, we have

$$\begin{aligned}
\|\Upsilon_1(u,v)\| &= \sup_{t\in\mathcal{E}} |\Upsilon_1(u,v)(t)| \\
&\le \frac{1}{\Gamma(\alpha_1)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\frac{|\varrho_1(s,u(s),v(s))|}{s}ds \\
&+ \frac{(\log t)^{\gamma_1-1}}{|\Delta|}\Bigg\{(\log T)^{\gamma_2-2}\log\Big(\frac{T}{\eta_2}\Big)\times \\
&\Big[\frac{\lambda_1}{\Gamma(\delta_1+\alpha_2)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\delta_1+\alpha_2-1}\frac{|\varrho_2(s,u(s),v(s))|}{s}ds \\
&+ \frac{1}{\Gamma(\alpha_1)}\int_1^T\Big(\log\frac{T}{s}\Big)^{\alpha_1-1}\frac{|\varrho_1(s,u(s),v(s))|}{s}ds\Big] \\
&+ \frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\gamma_2+\delta_1-1)}\Big|\log\eta_2 - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\times \\
&\Big[\frac{\lambda_2}{\Gamma(\delta_2+\alpha_1)}\int_1^{\eta_3}\Big(\log\frac{\eta_3}{s}\Big)^{\delta_3+\alpha_1-1}\frac{|\varrho_1(s,u(s),v(s))|}{s}ds \\
&+ \frac{1}{\Gamma(\alpha_2)}\int_1^T\Big(\log\frac{T}{s}\Big)^{\alpha_2-1}\frac{|\varrho_2(s,u(s),v(s))|}{s}ds\Big] \\
&+ \frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\alpha_2)\Gamma(\gamma_2+\delta_1-1)}\Big(\frac{\log T}{\log\eta_2}\Big)^{\gamma_2-2}\Big|\log T - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\times \\
&\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\alpha_2-1}\frac{|\varrho_2(s,u(s),v(s))|}{s}ds\Bigg\},
\end{aligned}$$

which yields

$$\begin{aligned}
\|\Upsilon_1(u,v)\| \le (l_1\rho + K_1)\Big\{&\frac{(\log T)^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{(\log T)^{\alpha_1+\gamma_1+\gamma_2-3}}{|\Delta|\Gamma(\alpha_1+1)}\log\Big(\frac{T}{\eta_2}\Big) \\
&+ \lambda_1\lambda_2\frac{\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1}(\log\eta_3)^{\delta_2+\alpha_1}(\log T)^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_2+\delta_1+1)\Gamma(\delta_2+\alpha_1+1)}\Big|\log\eta_1 \\
&- \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\Big\} + (l_2\rho + K_2)\Big\{\lambda_1\frac{(\log\eta_1)^{\alpha_2+\delta_1}(\log T)^{\gamma_1+\gamma_2-3}}{|\Delta|\Gamma(\delta_1+\alpha_2+1)}\log\Big(\frac{T}{\eta_2}\Big) \\
&+ \lambda_1\frac{\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_2}(\log T)^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_2+\delta_1-1)\Gamma(\alpha_2+1)}\Big|\log\eta_2 - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big| \\
&+ \lambda_1\frac{\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_2}(\log T)^{\gamma_1-1}(\log\eta_2)^{\alpha_2}}{|\Delta|\Gamma(\gamma_2+\delta_1-1)\Gamma(\alpha_2+1)}\Big|\log T - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\Big|\Big\}.
\end{aligned}$$

Using the notation (18)–(21), we get

$$\|\Upsilon_1(u,v)\| \leq (l_1\Omega_1 + l_2\Omega_2)\rho + \Omega_1 K_1 + \Omega_2 K_2. \tag{33}$$

Likewise, we can find that

$$\|\Upsilon_2(u,v)\| \leq (l_1\overline{\Omega}_1 + l_2\overline{\Omega}_2)\rho + \overline{\Omega}_1 K_1 + \overline{\Omega}_2 K_2. \tag{34}$$

Then it follows from (32)–(34) that

$$\|\Upsilon(u,v)\| = \|\Upsilon_1(u,v)\| + \|\Upsilon_2(u,v)\| \leqslant \rho.$$

Therefore, $\Upsilon\mathcal{B}_\rho \subset \mathcal{B}_\rho$ as $(u,v) \in \mathcal{B}_\rho$ is an arbitrary element.

In order to verify that the operator Υ is a contraction, let $\nu_i,\ \mu_i \in B_\rho,\ i = 1,2$. Then, we get

$$\begin{aligned}
&\|\Upsilon_1(\nu_1,\mu_1) - \Upsilon_1(\nu_2,\mu_2)\| \\
&\leq \frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1} |\varrho_1(s,\nu_1(s),\mu_1(s)) - \varrho_1(s,\nu_2(s),\mu_2(s))|\frac{ds}{s} \\
&+ \frac{(\log t)^{\gamma_1-1}}{|\Delta|}\Bigg\{(\log T)^{\gamma_2-2}\log\left(\frac{T}{\eta_2}\right)\times \\
&\Bigg[\frac{\lambda_1}{\Gamma(\delta_1+\alpha_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\delta_1+\alpha_2-1}|\varrho_2(s,\nu_1(s),\mu_1(s)) - \varrho_2(s,\nu_2(s),\mu_2(s))|\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_1-1}|\varrho_1(s,\nu_1(s),\mu_1(s)) - \varrho_1(s,\nu_2(s),\mu_2(s))|\frac{ds}{s}\Bigg] \\
&+ \frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\gamma_2+\delta_1-1)}\left|\log\eta_2 - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\right|\times \\
&\Bigg[\frac{1}{\Gamma(\alpha_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_2-1}|\varrho_2(s,\nu_1(s),\mu_1(s)) - \varrho_2(s,\nu_2(s),\mu_2(s))|\frac{ds}{s} \\
&+ \frac{\lambda_2}{\Gamma(\delta_2+\alpha_1)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{s}\right)^{\delta_3+\alpha_1-1}|\varrho_1(s,\nu_1(s),\mu_1(s)) - \varrho_1(s,\nu_2(s),\mu_2(s))|\frac{ds}{s}\Bigg] \\
&+ \frac{\lambda_1\Gamma(\gamma_2-1)(\log\eta_1)^{\gamma_2+\delta_1-2}}{\Gamma(\alpha_2)\Gamma(\gamma_2+\delta_1-1)}\left(\frac{\log T}{\log\eta_2}\right)^{\gamma_2-2}\left|\log T - \frac{\gamma_2-1}{\gamma_2+\delta_1-1}\log\eta_1\right|\times \\
&\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\alpha_2-1}|\varrho_2(s,\nu_1(s),\mu_1(s)) - \varrho_2(s,\nu_2(s),\mu_2(s))|\frac{ds}{s}\Bigg\},
\end{aligned}$$

which, by (H_2), yields

$$\|(\Upsilon_1(\nu_1,\mu_1) - \Upsilon_1(\nu_2,\mu_2)\| \leq (\Omega_1 l_1 + \Omega_2 l_2)\Big[\|\nu_1-\nu_2\| + \|\mu_1-\mu_2\|\Big]. \tag{35}$$

In a similar manner, we can find that

$$\|\Upsilon_2(\nu_1,\mu_1) - \Upsilon_2(\nu_2,\mu_2)\| \leq (\overline{\Omega}_1 l_1 + \overline{\Omega}_2 l_2)\Big[\|\nu_1-\nu_2\| + \|\mu_1-\mu_2\|\Big]. \tag{36}$$

Consequently, it follows from (35) and (36) that

$$\begin{aligned}
\|\Upsilon(\nu_1,\mu_1) - \Upsilon(\nu_2,\mu_2)\| &= \|\Upsilon_1(\nu_1,\mu_1) - \Upsilon_1(\nu_2,\mu_2)\| + \|\Upsilon_2(\nu_1,\mu_1) - \Upsilon_2(\nu_2,\mu_2)\| \\
&\leq \Big[(\Omega_1+\overline{\Omega}_1)l_1 + (\Omega_2+\overline{\Omega}_2)l_2\Big]\Big(\|\nu_1-\nu_2\| + \|\mu_1-\mu_2\|\Big),
\end{aligned}$$

which, by the condition (31), implies that Υ is a contraction. Therefore, the operator Υ has a unique fixed point as an application of Banach fixed point theorem. Hence, there exists a unique solution for the problem (1) and (2) on $\mathcal{E}$. □

Examples

Consider the following coupled Hilfer–Hadamard fractional differential system:

$$\begin{cases} {}^{HH}\mathcal{D}_{1^+}^{\frac{1}{4},\frac{1}{4}} u(t) = \varrho_1(t, u(t), v(t)),\ t \in [1, 10], \\ {}^{HH}\mathcal{D}_{1^+}^{\frac{5}{2},\frac{1}{2}} v(t) = \varrho_2(t, u(t), v(t)), \end{cases} \tag{37}$$

supplemented with nonlocal coupled Hadamard integral boundary conditions:

$$\begin{cases} u(1) = 0, \qquad u(10) = 3\ {}^{H}I_{1^+}^{\frac{1}{3}} v(6), \\ v(1) = 0, \qquad v(\frac{4}{3}) = 0, \qquad v(10) = 2\ {}^{H}I_{1^+}^{\frac{3}{4}} u(5). \end{cases} \tag{38}$$

Here $\alpha_1 = 1/4,\ \beta_1 = 1/4, \alpha_2 = 5/2,\ \beta_2 = 1/2, T = 10,\ \delta_1 = 1/3,\ \delta_2 = 3/4, \eta_1 = 6$, $\eta_2 = 4/3, \eta_3 = 5,\ \lambda_1 = 3$, and $\lambda_2 = 2$. With the given data, it is found that $\gamma_1 = 11/16$, $\gamma_2 = 11/4, \Delta_1 = -19.38876,\ \overline{\Delta}_2 = -16.87,\ \Omega_1 = 1.95131,\ \Omega_2 = 2.88353, \overline{\Omega}_1 = 1.34903$, $\overline{\Omega}_2 = 7.54898$.

(1) For illustrating Theorem 2, we take

$$\begin{aligned} \varrho_1(t, u(t), v(t)) &= \sqrt{2t+1} + \frac{|u(t)|}{25(1+|u(t)|)} + \frac{\cos v(t)}{5t+10}, \\ \varrho_2(t, u(t), v(t)) &= e^{-2t} + \frac{\tan^{-1} u(t)}{30t} + \frac{1}{45}\sin v(t). \end{aligned} \tag{39}$$

Clearly the condition (H_1) is satisfied with $\kappa_0 = \sqrt{3},\ \kappa_1 = 1/25,\ \kappa_2 = 1/15, \overline{\kappa}_0 = 1/e^2$, $\overline{\kappa}_1 = 1/30,\ \overline{\kappa}_2 = 1/45$. Moreover, we have

$$\begin{aligned} &\frac{1}{25}(\Omega_1 + \Omega_2) + \frac{1}{30}(\overline{\Omega}_1 + \overline{\Omega}_2) \approx 0.635578 < 1, \\ &\frac{1}{15}(\Omega_1 + \Omega_2) + \frac{1}{45}(\overline{\Omega}_1 + \overline{\Omega}_2) \approx 0.617112 < 1. \end{aligned}$$

Thus, the hypothesis of Theorem 2 holds true. Therefore, the conclusion of Theorem 2 applies and hence the problem (37) and (38) with ϱ_1 and ϱ_2 given by (39) has at least one solution on $[1, 10]$.

(2) We illustrate Theorem 3 by considering

$$\begin{aligned} \varrho_1(t, u(t), v(t)) &= \frac{1}{t^2+4} + \frac{1}{10\sqrt{2t+7}}(\sin u(t) + |v(t)|), \\ \varrho_2(t, u(t), v(t)) &= e^{-2t} + \frac{1}{5(t+4)}\Big(\tan^{-1} u(t) + \cos v(t)\Big). \end{aligned} \tag{40}$$

In a straightforward manner, we find that (H_2) is satisfied with $l_1 = 1/30$ and $l_2 = 1/25$, and

$$\frac{1}{30}(\Omega_1 + \overline{\Omega}_1) + \frac{1}{25}(\Omega_2 + \overline{\Omega}_2) \approx 0.527312 < 1.$$

As the hypothesis of Theorem 3 holds true, therefore, it follows by its conclusion that the problem (37) and (38) with ϱ_1 and ϱ_2 given by (40) has a unique solution on $[1, 10]$.

5. Conclusions

We have presented the existing criteria for solutions of a coupled system of nonlinear Hilfer–Hadamard fractional differential equations of different orders subject to nonlocal coupled Hadamard integral boundary conditions. The methods of modern analysis are successfully applied to establish the desired results. It is imperative to mention that the work presented in the given configuration is new and enriches the literature on the topic. Moreover, our results reduce to the ones for system (1) with boundary conditions of the form: $u(1) = 0, u(T) = 0, v(1) = 0, v(\eta_2) = 0, v(T) = 0, 1 < \eta_2 < T$, when we choose $\lambda_1 = 0$ and $\lambda_2 = 0$, which are indeed new. In the future, we plan to extend the present work to a coupled system of nonlinear Hilfer–Hadamard sequential fractional differential equations of different orders subject to integro-multipoint boundary conditions. We will also study the multivalued analog of the problem investigated in this paper.

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