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Globally Existing Solutions to the Problem of Dirichlet for the Fractional 3*D* Poisson Equation

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Abstract: A general approach to solving the Dirichlet problem, both for bounded 3*D* domains and for their unbounded complements, in terms of the fractional (3*D*) Poisson equation, is presented. Lauren Schwartz class solutions are sought for tempered distributions. The solutions found are represented by a formula that contains the volume Riesz potential and the one-layer potential, the latter depending on the boundary data. Infinite regularity of fractional harmonic functions, analogous to the infinite smoothness of the classical harmonic functions, is also proved in the respective domain, no matter what the boundary conditions are. Other properties of the solutions, that are presumably of interest to mathematical physics, are also investigated. In particular, an intrinsic decay property, valid far from the common boundary, is shown.

Keywords: fractional laplacian; Riesz potentials; integral equations; unbounded domains; explicit solutions; regularity

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1. Introduction

The topic of the fractional Laplacian has recently accumulated increasing interest (e.g., [1], wherein a large number of relevant results were cited [2–5]). As discussed in [1], the leading motivation was inspired by possible applications. Dealing with such Laplacian operators (with the appropriate boundary value problem) stems from essential questions concerning the Brownian motion phenomena. From the accepted view point, Brownian particle behavior, close to a surface barrier $\partial\Omega$ (given a bounded, say 3*D*, domain Ω) generally admits some kind of anomalous stage, called subordination (see the explanations and references in [1]). The particle motion is guided by a Levy process. The well-known Dirichlet and Neumann problems, for the fractional Laplace (Poisson) equation, are adequate mathematical models. The progress in the fractional Laplacian topics is due to the results of many authors (see [1]) over the last two decades. There are, in addition, interesting recent results in related fields, e.g., [6,7].

However, the Dirichlet problem for the fractional Poisson equation has been relatively weakly explored until now (see [1]). (The same applies for the Neumann problem). Regarding the Dirichlet problem, several results from the last decade should be noted, as these built the framework of the contemporary state [2–5,8]. In [2], nonzero Dirichlet boundary data was imposed on $\partial\Omega$, for the Poisson equation with non-homogeneous fractional Laplacian. There was a difference with our main result. It was seen from the observation that the fractional harmonic function $u(x)$ (i.e., $(\Delta)^{\alpha/2}u = 0$ in Ω) regarding the Laplacian, introduced below, was not generally fractionally harmonic, regarding the non-homogeneous fractional Laplacian. Nonzero (local) boundary data was considered in [5], but for the well-posedness Dirichlet problem a finite dimensional linear condition was assumed for the equation and boundary data $\{f, \varphi\}$. Similar results were obtained in [3,4], actually under zero boundary conditions, in the case of stopped α, stable motion, which were not related to what was found in our study. It is worth mentioning the work of [3,5], which represents an important upgrade of the ideas of HŐrmander and early Vishik and Eskin, based on pseudo-differential operators of fractional degree. Note that, in [8], a fractional Poisson–Boltzmann equation was analyzed under zero boundary conditions (as in [4]). Nevertheless, the analysis of the Dirichlet problem in the exterior of bounded, say $3D$, domains has not been undertaken up to the present, for the fractional Laplace (and Poisson) equation. Similar remarks can be made concerning the (infinite) interior regularity of the fractional harmonic functions (satisfying the equation $(-\Delta)^{\alpha/2}u = 0$, for a given domain $\mathcal{D} \subset \mathbb{R}^3$. Our interest in the case of unbounded exterior $\mathbb{R}^3 \setminus (\Omega \cup \partial\Omega)$ of a bounded domain Ω was inspired by the possibility of anomalous electric potential distributions in heterogeneous material systems possessing some kind of quasi-vacuum sub-phases. Via the anomalous behavior of the Brownian motion, we observed, in addition, the generally realistic process for particles coming from the exterior into the bounded zone. We dealt here with $3D$ bounded domains $\Omega \subset \mathbb{R}^3$ and their complements $C\Omega := \mathbb{R}^3 \setminus (\Omega \cup \Gamma)$, with $\Gamma = \partial\Omega$, the closed ($2D$) boundary surface of Ω, assumed to be of second order (C^2) regularity.

For the Laplacian degree $\alpha/2$ $(-\Delta)^{\alpha/2}$, assuming $1 < \alpha \leq 2$ and the action $(-\Delta)^{\alpha/2}u$ is defined by its Fourier transform $|\xi|^\alpha \hat{u}(\xi)$, for $u \in S' = S'(\mathbb{R}^3)$, the class of the Schwartz tempered distributions (e.g., [9]) gives rise to the following: $|\xi|$ is the length of the vector $\xi \in \mathbb{R}^3$ and $\hat{u}(\xi) = \mathbb{F}[u](\xi)$ is the Fourier image of u. Concerning the Fourier transformation, we proceed with the convention $\hat{\phi}(\xi) = \int_{\mathbb{R}^3} \exp(-i\langle x, \xi\rangle)\phi(x)dx$ and $\phi(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \exp(i\langle x, \xi\rangle)\hat{\phi}(x)dx$, $\phi \in S = S(\mathbb{R}^3)$, the Schwartz class of the fast decreasing (infinitely smooth) functions ([9]), $\langle x, \xi\rangle$ is the scalar product of the vectors x, ξ. Thus, we have $(-\Delta)^{\alpha/2}u(x) := \mathbb{F}^{-1}[|\xi|^\alpha \hat{u}(\xi)](x)$, $u \in S'$, x varying in $\mathbb{R}^3$, where $\mathbb{F}^{-1}$ is the inverse map of $\mathbb{F}$ (The symbol $(-\Delta)^{\alpha/2}$ for the fractional Laplacian is the one introduced in [1]).

Our approach to the problem of Dirichlet is based on exploring a simple, but effective, idea to deal with the global Laplacian, i.e., $(-\Delta)^{\alpha/2}u = \mathbb{F}^{-1}[|.|^\alpha \mathbb{F}]$, on such distributions u from S' that the action product $(-\Delta)^{\alpha/2}u$ coincides on Ω with a prescribed function (distribution) f, and possesses traces $(u|_\Gamma)$ on Γ (with prescribed values $\varphi(x)$ of $u|_\Gamma$, $x \in \Gamma$). Additionally, we remark that the way we looked for globally-defined solutions (given a boundary value problem) was first suggested by the $1D$ case of the Poisson equation $(-\Delta)^{\alpha/2}u = f$, with $\Delta = \dfrac{d^2}{dx^2}$, considered in unbounded intervals $l_0 < x < \infty$, $l_0 \in \mathbb{R}^1$. (some details regarding this case are provided in the Appendix A).

Slightly formalized, the above idea gives rise to the following. Given a function $f(x)$, $x \in \Omega$, say bounded, i.e., $f \in L^\infty(\Omega)$, and boundary data $\varphi(x)$, $x \in \Gamma$ for instance $\varphi \in L_2(\Gamma)$, the point is to solve the (extended on $\mathbb{R}^3$) equation $(-\Delta)^{\alpha/2}u = F^0$, with $F^0 \in S' : F^0|_\Omega = f$, by a suitable distribution $u \in S'$, satisfying the condition $u|_\Gamma = \varphi$ (As seen below, the proper choice for F^0 is $F^0|_{C\Omega} = 0$). If we assume we have found $u \in S'$, we could get a globally-existing (i.e., defined on $\mathbb{R}^3$) solution of the problem under consideration, reformulated now in the form:

$$(a)\ (-\Delta)^{\alpha/2}u|_\Omega = f,\ (u \in S');\ (b)\ u|_\Omega = \varphi. \tag{1}$$

The above formulation actually gives the shortest illustration to the approach used here concerning the problem of Dirichlet (for a given bounded domain Ω, i), for the fractional Poisson equation. We are close now to the key question of existence of a solution S', for the Equation (1), for a sufficiently large class of boundary data. As a first accessory step to that goal, consider the volume-type potential $U_{\beta,f} := \int_\Omega \dfrac{f(y)dy}{|x-y|^\beta}$, $x \in \mathbb{R}^3$, $0 < \beta$, $\beta = \beta(\alpha)$, prompted from the analogy with the conventional case of (1), where $\alpha = 2$, $\beta = 1$. Following [1]. We call such potentials Riesz (volume) potentials, and, enlarging the terminology, the functions of the type $V_{\beta,g} := \int_\Gamma \dfrac{g(y)ds_y}{|x-y|^\beta}$, $(x \in \mathbb{R}^3)$ are called surface (or single layer) Riesz potentials (here $g \in L_2(\Gamma)$, by assumption, and ds_y is the known surface differential element). The right value of β, namely $\beta = 3 - \alpha$, is, however, directly shown from the well-known (e.g., [10,11], and also [1]) Fourier transform relation $\mathbb{F}$: $\frac{c_{3,\alpha}}{|x|^{3-\alpha}} \to \frac{1}{|\xi|^{\alpha}}$, with $c_{3,\alpha} = \frac{\Gamma(\frac{3-\alpha}{2})}{2^{\alpha}\pi^{3/2}\Gamma(\frac{\alpha}{2})}$.Taking the precise expression, instead of $U_{\beta,f}(x)$, we obtain the potential $u_{\alpha,f} := \int_{\Omega} \frac{c_{3,\alpha} f(y)dy}{|x-y|^{3-\alpha}}$, $(x \in \mathbb{R}^3)$, which satisfies the Equation (1). Certainly, $u_{\alpha,f} := \int_{\mathbb{R}^3} \frac{c_{3,\alpha} f^0[f](y)dy}{|x-y|^{3-\alpha}} = \frac{c_{3,\alpha}}{|.|^{3-\alpha}} * f^0[f]$, where $f^0[f](y) = f(y)$ for $y \in \Omega$, and $f^0[f](y) = 0$, when $y \in C\Omega$, $U * V$ is the convolution (see [9] for details) of $U, V \in S'$. Then, $(-\Delta)^{\alpha/2} u_{\alpha,f} = \left((-\Delta)^{\alpha/2} \frac{c_{3,\alpha}}{|.|^{3-\alpha}}\right) * f^0[f] = \delta * f^0 = f^0$, here $\delta = \delta(x)$, $(x \in \mathbb{R}^3)$ is supported by the point $x = 0$ Dirac delta function; i.e., $\left((-\Delta)^{\alpha/2} u_{\alpha,f}\right)(x) = f^0[f](x)$, $x \in \mathbb{R}^3$ (therefore, $(-\Delta)^{\alpha/2} u_{\alpha,f}|_{\Omega} = f$). These calculations evidently also remain valid for the generalized expression of $u_{\alpha,f}$,

$$u_{\alpha,f} := \int_{\mathcal{D}} \frac{c_{3,\alpha} f(y)dy}{|x-y|^{3-\alpha}},\ x \in \mathbb{R}^3,\ \mathcal{D} = \Omega \ or\ \mathcal{D} = C\Omega. \tag{2}$$

From now on we assume the following requirements for the function $f(x)$ are fulfilled when defined for $x \in \mathcal{D}$:

$$(a)\ f \in L^{\infty}(\mathcal{D}),\ \mathcal{D} = \Omega;\ (b)\ f \in L^{\infty}(\mathcal{D}) \cap L_1(\mathcal{D}),\ \mathcal{D} = C\Omega. \tag{3}$$

It is not difficult to establish that the given (2) potential $u_{\alpha,f}(x)$ is a continuous function in $\mathcal{D} \cup \Gamma$, and, therefore, the trace $\varphi_{\alpha,f}(x)$, $x \in \Gamma$, $\varphi_{\alpha,f} := u_{\alpha,f}|_{\Gamma}$ is continuous on Γ. Via the problem of Dirichlet, we find that $u_{\alpha,f}$ is a solution of the equation $(-\Delta)^{\alpha/2} u|_{\mathcal{D}} = f$, with $u|_{\Gamma} = \varphi_{\alpha,f}$. The next step is to seek solutions in S' with arbitrarily prescribed data, as assumed in $L_2(\Gamma)$. In this direction, suppose $u \in S'$ is a solution of the above equation. Then $(-\Delta)^{\alpha/2}[u - u_{\alpha,f}] = 0$ in $\Omega \cup C\Omega$ and, therefore $L_{\alpha,f}(x) := (-\Delta)^{\alpha/2}[u - u_{\alpha,f}](x)$, is a distribution supported on the surface Γ. We deal here with the case, $L_{\alpha,f}(x) \equiv \delta_{\Gamma}[g](x)$, $x \in \mathbb{R}^3$; i.e., we are interested in solutions $u \in S'$ of the equation $(-\Delta)^{\alpha/2} u = f^0[f]$ on $\Omega \cup C\Omega$, satisfying the condition:

$$(-\Delta)^{\alpha/2}[u - u_{\alpha,f}] = \delta_{\Gamma}[g] \ in\ S'. \tag{4}$$

Above $\delta_{\Gamma}[g]$ is the supported Γ delta function of Dirac, with a density function $g = g(x) \in L_2(\Gamma)$ (As is known, e.g., [9], the action ($\delta_{\Gamma}[g]$, ϕ) of $\delta_{\Gamma}[g]$ on an arbitrary $\phi \in S$ is defined by the next surface integral, $(\delta_{\Gamma}[g], \phi) := \int_{\Gamma} g(y)\phi(y)ds_y$). The important partial case $f = 0$ ($u_{\alpha,f} = 0$) concerns these distributions $w \in S'$, solving the equation below (for g varying in $L_2(\Gamma)$):

$$(-\Delta)^{\alpha/2} w = \delta_{\Gamma}[g] \ in\ S'. \tag{5}$$

Solutions such as w are called BF harmonic (basic fractional harmonic) functions in $\mathbb{R}^3$, and the family $S'_{\alpha,f}$ of all solutions $u \in S'$ to (4) (with g varying in $L_2(\Gamma)$) can be called the GS (global solutions) family.

Remark 1. *(1) Clearly for each two distributions u_1, $u_2 \in S'_{\alpha,f}$ the difference $u_2 - u_1$ is a BF harmonic function.*
(2) A possibly larger class of solutions $u \in S'$ to the equation $(-\Delta)^{\alpha/2} u = f^0[f]$ could be expected in the case $L_{\alpha,f} = \delta_{\Gamma}[g_0] + \partial_n \delta_{\Gamma}[g_1]$, where g_0, $g_1 \in L_2(\Gamma)$ and $\partial_n \delta_{\Gamma}[g](x)$ is the normal to Γ derivative of $\delta_{\Gamma}[g]$ at the point $x \in \Gamma$.

It is not difficult to obtain a structural description of the family $S'_{\alpha,f}$. After using the Fourier transform in the relation (4), one can directly resolve (4) regarding $u - u_{\alpha,f}$ and find, in this manner, the next general formula:

$$u = \delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} + u_{\alpha,f} \ in\ S'. \tag{6}$$

The above convolution $\delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}}$ evidently introduces the single layer Riesz potential $v_{\alpha,g}(x) := \int_\Gamma \frac{c_{3,\alpha} g(y) dy}{|x-y|^{3-\alpha}}$, $(x \in \mathbb{R}^3)$ which possesses well-defined direct values $\psi_{\alpha,g}(x)$, $x \in \Gamma$, $\psi_{\alpha,g} := v_{\alpha,g}|_\Gamma$, with $\psi_{\alpha,g} \in L_2(\Gamma)$. This holds because the integral operator $B_{\alpha,\Gamma}[g] := \psi_{\alpha,\Gamma}$, $g \in L_2(\Gamma)$, has a weak singularity $(3 - \alpha < 2)$ and, according to the known classical theory (e.g., [10,12,13]), the map $B_{\alpha,\Gamma} : L_2(\Gamma) \to L_2(\Gamma)$ is a bounded linear operator. As seen from (6), each solution $u \in S'_{\alpha,f}$ has an L_2 trace on Γ (for f satisfying (3)). Let us also check that the term $\delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}}$ is a BF harmonic function:

$$\begin{aligned} \left((-\Delta)^{\alpha/2} \delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} \right) &= \delta_\Gamma[g] * (-\Delta)^{\alpha/2} \frac{c_{3,\alpha}}{|.|^{3-\alpha}} \\ &= \delta_\Gamma[g] * \delta = \delta_\Gamma[g]. \end{aligned}$$

Concluding the above results, we have already found that $(-\Delta)^{\alpha/2} u_{\alpha,f} = f^0[f]$ and

$$\left((-\Delta)^{\alpha/2} \delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} \right) = \delta_\Gamma[g]$$

(both in S'), i. e. $(-\Delta)^{\alpha/2} u = \delta_\Gamma[g] + f^0[f]$ in S' (for each $g \in L_2(\Gamma)$), u given by (6), and $u|_\Gamma = \psi_{\alpha,g} + \varphi_{\alpha,f}$ (f satisfying (3)).

Now the final question is whether a possibly unique $g \in L_2(\Gamma)$ can be determined, corresponding to φ, for arbitrary φ in a suitable sub-space of $L_2(\Gamma)$. Then, by means of the Formula (6) we could obtain a solution of the basic problem:

$$(a)\ (-\Delta)^{\alpha/2} u|_{\mathcal{D}} = f\ ,\ b)\ u|_\Gamma = \varphi. \tag{7}$$

This solution is expected to be unique in the family $S'_{\alpha,f}$. We provide a positive answer to this question by introducing, in Section 3 (below), the sub-space $H^1_\alpha \subset L_2(\Gamma)$ (coincident with the map image of $B_{\alpha,\Gamma}[L_2(\Gamma)]$), and then find a unique $g \in L_2(\Gamma)$, such that $B_{\alpha,\Gamma}[g] = \varphi - \varphi_{\alpha,f}$, for $\varphi \in L_2(\Gamma) : \varphi - \varphi_{\alpha,f} \in H^1_\alpha(\Gamma)$. The key instrument for obtaining the answer is contained in the properties of the boundary operator $B_{\alpha,\Gamma}$, analyzed primarily in the next Section 2.

In the present paper we propose a new approach for solving the problem of Dirichlet for the fractional Poisson equation with local nonzero boundary data, valid both for bounded ($3D$) domains and their unbounded exteriors. It is illustrated by several key results essentially different from the known ones:

(1) The problem whether zero is an eigenvalue of the boundary integral operator

$$B_{\alpha,\Gamma} : L_2(\Gamma) \to L_2(\Gamma)$$

is solved.

(2) By obtaining explicit formulae, consisting in two Riesz-type potentials, a single layer and a volume one, well-posedness is established regarding solutions from the families $S'_{\alpha,f}$.

(3) The basic properties of the found solutions for regularity and asymptotic behavior (far from the boundary), as well as the inherent a-posterior estimates, are proved, including, in particular, the infinite interior regularity of the fractional harmonic functions.

The article is organized as follows. In Section 2 we study the question of the zero kernel of $B_{\alpha,\Gamma}$ and prove the crucial fact that zero is not an eigenvalue of $B_{\alpha,\Gamma}$. Section 3 includes the main well-posedness result (based on explicitly expressed solutions) and the theorems concerning the solution's asymptotic behavior and their regularity in $\mathbb{R}^3 \setminus \Gamma$–classical and in H^s_{loc} sense, the proper estimates as well. In the Appendix A we consider certain inherent cases of singular boundary data for the $1D$ fractional Poisson equation, at $0 < \alpha < 1$.

2. The Zero Kernel of the Boundary Integral Operator

It turns out that the kernel of the operator $B_{\alpha,\Gamma}$ (acting from $L_2(\Gamma)$ into $L_2(\Gamma)$) consists only of the zero element $g = 0$, i.e., the unique solution of the equation $B_{\alpha,\Gamma}[g] = 0$ is $g = 0$. The key to this very important property lies in a simple, but essential, relation in the form: $I_\alpha(\infty) = const.J_{\Gamma,\alpha}$, where $I_\alpha(\infty) = \lim_{r\to\infty} I_\alpha(r)$ and the terms $I_\alpha(r)$, $J_{\Gamma,\alpha}$ present, respectively, the integrals:

$$I_\alpha(r) = \int_{|\xi|\le r} |\hat{\delta}_\Gamma[g]|^2 |\xi|^{-\alpha} d\xi;$$
$$J_{\Gamma,\alpha} = \int_\Gamma g(x)(\overline{\delta}_\Gamma * |.|^{\alpha-3})(x) ds_x.$$

Clearly, the above relation (when it holds) means, in particular, that the integral $I_\alpha(\infty) = \int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g]|^2 |\xi|^{-\alpha} d\xi$ converges. (Here $\hat{\delta}_\Gamma[g](\xi)$ is the Fourier image of $\delta_\Gamma[g](x)$.) The mentioned equality shopuld be found as a specific consequence of the well known Parseval equality (e.g., [9,14]). To this goal we begin by considering a complement to Parseval's equality idea.

Proposition 1. *(The boundary Parseval formula.) The following relation is valid, for each function $\psi \in C^\infty(\mathbb{R}^3)$, with $\hat{\psi} \in L_1(\mathbb{R}^3)$:*

$$(2\pi)^3(\delta_\Gamma[g], \overline{\psi}) = (\hat{\delta}_\Gamma[g], \hat{\overline{\psi}}). \tag{8}$$

Proof. Note, firstly, that $\overline{\psi}$ is the complex conjugated quantity to ψ and recall that the notation $(\delta_\Gamma[g], \overline{\psi})$ expresses the action of $\delta_\Gamma[g]$, as a distribution in S', on the function $\overline{\psi}$ – as an arbitrary element of S. Thus, $(\delta_\Gamma[g], \overline{\psi}) = \int_\Gamma g(x)\overline{\psi}(x) ds_x$, and, by analogy, the notation $(\hat{\delta}_\Gamma[g], \hat{\overline{\psi}})$, i.e.,

$$\begin{aligned} (\hat{\delta}_\Gamma[g], \hat{\overline{\psi}}) &= \int_{\mathbb{R}^3} \hat{\delta}[g](\xi)\hat{\overline{\psi}}(\xi) d\xi \\ &= \int_{\mathbb{R}^3} \hat{\overline{\psi}}(\xi) \int_\Gamma g(x) \exp(-i\langle x, \xi\rangle) ds_x d\xi. \end{aligned} \tag{9}$$

The proof uses the approximation approach to (8) the following two-step scheme: first, obtain (8) with $w \in C_0^\infty(\mathbb{R}^3)$ instead of $\delta_\Gamma[g]$, and $C_0^\infty(\mathbb{R}^3)$ is the space of the compactly supported infinitely smooth functions. Then, apply an approximation procedure with w_n ($w_n \in C_0^\infty(\mathbb{R}^3)$, $n = 1, 2, \ldots$) tending to $\delta_\Gamma[g]$, at $n \to \infty$. The first step is done in the given lemma. □

Lemma 1. *The next Parseval equality is valid for each pair $w \in C_0^\infty(\mathbb{R}^3)$ and $\psi \in C^\infty(\mathbb{R}^3)$, with $\hat{\psi} \in L_1(\mathbb{R}^3)$:*

$$(2\pi)^3(w, \overline{\psi}) = (\hat{w}, \hat{\overline{\psi}}). \tag{10}$$

At the beginning of the proof of (10), note, as above, that the notation $(w, \overline{\psi})$ is used in the known distribution (S') sense, with $(w, \overline{\psi}) = \int_{\mathbb{R}^3} w(x)\overline{\psi}(x) dx$, and, by analogy, the notation $(\hat{w}, \hat{\overline{\psi}})$. Now, let us introduce the function $\phi_0(x) \in C_0^\infty$: $\phi_0(x) = \phi_0(|x|)$, $1 \ge \phi_0(x) \ge 0$, $\forall x$, $\phi_0(x) \equiv 1$ for $|x| \le r^0/2$, $\phi_0(x) \equiv 0$ for $|x| \ge r^0$, with a fixed $r^0 > 0$, such that $\int_{\mathbb{R}^3} \phi_0(x) dx = 1$. With the real parameter $s \in (0, 1]$ we deal with $\phi_0(sx)$ and its Fourier map $\mathbb{F}[\phi_0(s.)](\xi) = \frac{1}{(2\pi s)^3}\hat{\phi}_0(s^{-1}\xi)$. Then, the conventional Parseval formula yields the identity $Q(s) = (2\pi)^{-3}\tilde{Q}(s)$, for $s \in (0, 1]$, where $Q(s) := (w, \overline{\psi}\phi_0(s.))$ and $\tilde{Q}(s) := (\hat{w}, \hat{\overline{\psi}} * (2\pi s)^{-3}\hat{\phi}_0(./s))$. For our goal, we had to compare the limit values of $Q(s)$ and $\tilde{Q}(s)$ at $s \to 0$. Function $Q(s)$ was actually defined and continuous in $[0, 1]$, i.e., its limit value ($s \to 0$) was $Q(0)$, while, concerning the $\lim\limits_{s\to 0} \tilde{Q}(s)$, we needed some reworking of the integral for $\tilde{Q}(s)$. Starting from the initial expression of $\tilde{Q}(s)$, and applying the linear transform $\theta = s^{-1}(\xi - \eta)$ in the repeated integral (below), we consecutively found the following relations:

$$\begin{aligned} \tilde{Q}(s) &= \int_{\mathbb{R}^3} \hat{w}(\xi)\hat{\overline{\psi}} * (2\pi s)^{-3}\hat{\phi}_0(s^{-1}.)(\xi)d\xi \\ &= \int_{\mathbb{R}^3} \hat{w}(\xi) \int_{\mathbb{R}^3} \hat{\overline{\psi}}(\eta)(2\pi s)^{-3}\hat{\phi}_0(s^{-1}(\xi-\eta))d\eta d\xi \\ &= \int_{\mathbb{R}^3} \hat{\overline{\psi}}(\eta) \int_{\mathbb{R}^3} \hat{w}(\eta + s\theta)\frac{\hat{\phi}_0(\theta)}{(2\pi)^3}d\theta d\eta. \end{aligned}$$

The above integral $\int_{\mathbb{R}^3} \hat{w}(\eta + s\theta)\hat{\phi}_0(\theta)d\theta$ is uniformly convergent for, respectively, the parameters $(\eta, s) \in K^0 \times [0, 1]$, for each compact $K^0 \subset \mathbb{R}^3$ (This clearly holds, because $\hat{w}(\xi)$ is a bounded function). Therefore, $\tilde{F}^0_w(\eta, s) := \int_{\mathbb{R}^3} \hat{w}(\eta + s\theta)\hat{\phi}_0(\theta)d\theta$ is a continuous, bounded function in $\mathbb{R}^3 \times [0, 1]$, and, repeating the same argument (now that $\int_{\mathbb{R}^3} \hat{\overline{\psi}}(\eta)\tilde{F}^0_w(\eta, s)d\eta$ is also a uniformly convergent integral), we obtain $\tilde{f}^0_w(s) := \int_{\mathbb{R}^3} \hat{\overline{\psi}}(\eta)\tilde{F}^0_w(\eta, s)d\eta$ is a continuous function in $[0, 1]$. However, $\tilde{Q}(s)$ is identical with $(2\pi)^{-3}\tilde{f}^0_w(s)$ for $0 < s \le 1$, and $\lim\limits_{s\to 0} \tilde{Q}(s) = \tilde{f}^0_w(0)/(2\pi)^3 = (\hat{w}, \hat{\overline{\psi}})$, i.e., $\lim\limits_{s\to 0} \tilde{Q}(s) = (\hat{w}, \hat{\overline{\psi}})$. (We used $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \hat{\phi}_0(\theta)d\theta = \phi_0(0) = 1$). Thus, letting $s \to 0$ in the equality $Q(s) = (2\pi)^{-3}\tilde{Q}(s)$, we obtain the necessary Formula (10).

The approximation step is now performed. Suppose $\{w_n(x)\}$, $n = 1, 2, \ldots$, is an infinite family of functions $w_n \in C^\infty_0(\mathbb{R}^3)$, such that the family of the Fourier maps $\{w_n\}$ is uniformly bounded and $\lim_{n\to\infty} w_n = \delta_\Gamma[g]$ (in S'). An easy direct construction of such a family is given by the convolution $w_n := \delta_\Gamma[g] * n^3\phi_0(nx)$. In this case, it is well-known (and can be easily verified) that $\lim\limits_{n\to\infty} w_n = \delta_\Gamma[g]$ in S' , and the assumption for an uniformly bounded $\{\hat{w}_n\}$ is directly seen from $\hat{w}_n = \hat{\delta}_\Gamma[g]\hat{\phi}_0(./n)$ (clearly $\hat{\delta}_\Gamma[g]$ and $\hat{\phi}_0$ are bounded functions). Letting $n \to \infty$ in the equality $(2\pi)^3(w_n, \overline{\psi}) = (\hat{w}_n, \hat{\overline{\psi}})$ (see (10)), we, respectively, obtain: $\lim\limits_{n\to\infty}(w_n, \overline{\psi}) = (\delta_\Gamma[g], \overline{\psi})$, and $\lim\limits_{n\to\infty}(\hat{w}_n, \hat{\overline{\psi}}) = (\hat{\delta}_\Gamma[g], \hat{\overline{\psi}})$, for $\hat{w}_n = \hat{\delta}_\Gamma[g]\hat{\phi}_0(./n)$. Here we take into account the equality $(\hat{w}_n, \hat{\overline{\psi}}) = \int_{\mathbb{R}^3} \hat{\delta}_\Gamma[g](\xi)\hat{\phi}_0(\xi/n)\hat{\overline{\psi}}(\xi)d\xi$, combined with the estimate $|\hat{\delta}_\Gamma[g](\xi)\hat{\phi}_0(\xi/n)\hat{\overline{\psi}}(\xi)| \le (mes(\Gamma))^{1/2}||g||_{L_2(\Gamma)}|\hat{\overline{\psi}}(\xi)|$, $\xi \in \mathbb{R}^3$, and then apply the well-known Lebesgue dominated convergence theorem (e.g., [15]), to find:

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \hat{\delta}_\Gamma[g](\xi)\hat{\phi}_0(\xi/n)\hat{\overline{\psi}}(\xi)d\xi = \int_{\mathbb{R}^3} \hat{\delta}_\Gamma[g](\xi)\hat{\overline{\psi}}(\xi)d\xi = (\hat{\delta}_\Gamma[g], \hat{\overline{\psi}}).$$

(Above $mes(\Gamma)$ is the measure of Γ and $||g||_{L_2(\Gamma)}$ is the L_2 norm of density g), This proves the boundary Parseval Formula (8).

Below, we add a consequence of (8), useful for the basic result in this section.

Corollary 1. *For each $\phi \in C^\infty_0(\mathbb{R}^3)$ the next Parseval-type relation holds, with $\phi_\mathbb{F} = \mathbb{F}^{-1}[\overline{\phi}]$:*

$$\int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g](\xi)|^2.\frac{\phi(\xi)}{|\xi|^\alpha}d\xi = (2\pi)^3\int_\Gamma g(x)\left(\overline{\delta}_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} * \overline{\phi}_\mathbb{F}\right)(x)ds_x. \tag{11}$$

Proof. Let us set $\psi_g(x) = \left(\delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} * \phi_\mathbb{F}\right)(x)$. Then, $\hat{\overline{\psi}}_g(x) = \hat{\overline{\delta}}_\Gamma[g](\xi)\frac{\phi(\xi)}{|\xi|^\alpha}$ and $(\hat{\delta}_\Gamma[g], \hat{\overline{\psi}}_g) = \int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g](\xi)|^2.\frac{\phi(\xi)}{|\xi|^\alpha}d\xi$. In addition $(\delta_\Gamma[g], \overline{\psi}_g)$ evidently equals the right hand integral above. Clearly, it is not difficult to check the two assumptions regarding ψ_g, First, it is directly seen that $\hat{\psi}_g \in L_1(\mathbb{R}^3)$, and second, from $\psi_g = \psi_{g,\alpha} * \phi_F$, where $\psi_{g,\alpha} := \delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}}$ is a distribution in $L_1^{loc}(\mathbb{R}^3)$, the validation as to whether $\psi_g \in C^\infty(\mathbb{R}^3)$ is obvious. Thus, the proof of (11) follows directly from (8).

Now, the basic result in Section 2 can be presented. □

Theorem 1. *(The kernel of $B_{\alpha,\Gamma}$.) The zero is not an eigen value of the boundary integral operator*

$$B_{\alpha,\Gamma} \,:\, L_2(\Gamma) \to L_2(\Gamma),$$

i.e., the only solution of the equation $B_{\alpha,\Gamma}[g] = 0$ is $g = 0$.

Proof. Using (11) with $\phi(\xi) \equiv \phi_0(\sigma\xi)$, $\xi \in \mathbb{R}^3$, where $\sigma \in (0, 1]$ is a real parameter, we get the formula:

$$\int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g](\xi)|^2 . \frac{\phi_0(\sigma\xi)}{|\xi|^\alpha} d\xi = \int_\Gamma g(x) \left(\overline{\delta}_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} * \sigma^{-3}\hat{\phi}_0(./\sigma) \right)(x) ds_x. \tag{12}$$

(Note here hat $\hat{\phi}_0$ is a real valued function), For the auxiliary assumption for g as a continuous function on Γ we first analyze the limit values of the integrals above, for $\sigma \to 0$. Clearly, the limit expression of (12) is expected in the form:

$$\int_{\mathbb{R}^3} \frac{|\hat{\delta}_\Gamma[g](\xi)|^2}{|\xi|^\alpha} d\xi = (2\pi)^3 \int_\Gamma g(x) \left(\overline{\delta}_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} \right)(x) ds_x. \tag{13}$$

We start with the integral $J^0_{\Gamma,\alpha}(\sigma) := \int_\Gamma g(x) \left(\overline{\delta}_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} * \sigma^{-3}\hat{\phi}_0(./\sigma) \right)(x) ds_x$ (The left integral $I^0_{\Gamma,\alpha}(\sigma)$ in (12), with $I^0_{\Gamma,\alpha}(\sigma) := \int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g](\xi)|^2 . \frac{\phi_0(\sigma\xi)}{|\xi|^\alpha} d\xi$, is commented on later). From the simplified expression $J^0_{\Gamma,\alpha}(\sigma) = \int_\Gamma g(x) J^0_{\Gamma,\phi}(x;\sigma) ds_x$, where

$$\begin{aligned} J^0_{\Gamma,\phi}(x;\sigma) &:= \left(\overline{\delta}_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} * \sigma^{-3}\hat{\phi}_0(./\sigma) \right)(x) \\ &= \int_\Gamma \overline{g}(y) \int_{\mathbb{R}^3} \frac{c_{3,\alpha}\sigma^{-3}\hat{\phi}_0(t\sigma^{-1})}{|x-y-t|^{3-\alpha}} dt ds_y, \end{aligned}$$

it is directly seen that $J^0_{\Gamma,\alpha}(\sigma)$ can be presented as follows (applying the substitution $t\sigma^{-1} = \tau$):

$$\begin{aligned} J^0_{\Gamma,\alpha}(\sigma) &= \int_\Gamma g(x) \int_\Gamma \overline{g}(y) \int_{\mathbb{R}^3} \frac{c_{3,\alpha}\sigma^{-3}\hat{\phi}_0(t\sigma^{-1})}{|x-y-t|^{3-\alpha}} dt ds_y ds_x \\ &= \int_{\mathbb{R}^3} \hat{\phi}_0(\tau) \int_\Gamma g(x) \int_\Gamma \frac{c_{3,\alpha}\overline{g}(y) ds_y}{|x-\sigma\tau-y|^{3-\alpha}} ds_x d\tau. \end{aligned} \tag{14}$$

Thus, $J^0_{\Gamma,\alpha}(\sigma) = \int_{\mathbb{R}^3} \hat{\phi}_0(\tau) \int_\Gamma g(x) F_g(x-\sigma\tau) ds_x d\tau$, with $F_g(\theta) := \int_\Gamma \frac{c_{3,\alpha}\overline{g}(y) ds_y}{|\theta-y|^{3-\alpha}}$. Note that $F_g(\theta)$ is a bounded and continuous function for $\theta \in \mathbb{R}^3$, because the given single layer Riesz potential (defining F_g) is uniformly convergent regarding θ, for $\theta \in K \subset \mathbb{R}^3$, K an arbitrarily fixed compact set, containing the closed surface Γ, under the assumption of continuous surface density g and the second order regularity of Γ. This holds by the same arguments that are well known from classical potential theory (e.g., [12]) of the single layer potential (the case of $\alpha = 2$). Next, the found properties of $F_g(\theta)$ yield the automatic conclusion that the function $G(\theta) := \int_\Gamma g(x) F_g(x-\theta) ds_x$ is also bounded and continuous, $\theta \in \mathbb{R}^3$. Then, again by the mentioned Lebesgue theorem, we see that the integral $\int_{\mathbb{R}^3} \hat{\phi}_0(\tau) G(\sigma\tau) d\tau$ is uniformly convergent regarding $\sigma \in [0, 1]$, i.e.,

$J^0_{\Gamma,\alpha}(\sigma) = \int_{\mathbb{R}^3} \hat{\phi}_0(\tau) G(\sigma\tau) d\tau$ is a continuous function in $[0, 1]$. We get this way: $\exists \lim_{\sigma \to 0} J^0_{\Gamma,\alpha}(\sigma) = J^0_{\Gamma,\alpha}(0)$. As is clear from (14), $J^0_{\Gamma,\alpha}(0) = \int_{\mathbb{R}^3} \hat{\phi}_0(\tau) d\tau \int_\Gamma g(x) \int_\Gamma \frac{c_{3,\alpha} \overline{g}(y) ds_y}{|x-y|^{3-\alpha}} ds_x$, i.e., (because of equality $\int_{\mathbb{R}^3} \hat{\phi}_0(\tau) d\tau = (2\pi)^3$) $J^0_{\Gamma,\alpha}(0) = (2\pi)^3 \int_\Gamma g(x) (\overline{\delta}_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}})(x) ds_x)$ (see the right hand side of (13)). In addition (12) also yields: $\exists \lim_{\sigma \to 0} I^0_{\Gamma,\alpha}(\sigma) = J^0_{\Gamma,\alpha}(0)$. On the other hand, from the estimates $I^0_{\Gamma,\alpha}(r^0 r^{-1}) \le I_\alpha(r) \le I^0_{\Gamma,\alpha}(r^0 r^{-1}/2)$ we establish that there exists the limit value $I_\alpha(\infty) := \lim_{r \to \infty} I_\alpha(r)$, i.e., the integral $\int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g]|^2 |\xi|^{-\alpha} d\xi$ converges and its value $I_\alpha(\infty)$ equals to $J^0_{\Gamma,\alpha}(0)$. Thus, (13) is proven.

Let us look at whether Formula (13) remains valid in the general case $g \in L_2(\Gamma)$. Actually it is enough to establish the next inequality:

$$I_\alpha(r)[g] \le (2\pi)^3 \int_\Gamma g(x) B_{\alpha,\Gamma}[\overline{g}](x) ds_x,\ g \in L_2(\Gamma). \tag{15}$$

Here $r > 0$ is an arbitrary fixed, $I_\alpha(r)[g]$ is the previously given integral $I_\alpha(r)$, and for $x \in \Gamma$: $B_{\alpha,\Gamma}[\overline{g}](x) \equiv (\overline{\delta}_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}})(x)$. Note firstly that the integrals $I_\alpha(r)[g]$ and $\int_\Gamma g(x) B_{\alpha,\Gamma}[\overline{g}](x) ds_x$ are correctly defined $\forall\, g \in L_2(\Gamma)$. Choosing now an arbitrary approximating sequence $\{g_n\} : g_n \to g,\ n \to \infty$ in $L_2(\Gamma)$, g_n – continuous ($\forall\, n = 1, 2, \ldots$), we evidently have from (13) the estimate:

$$I_\alpha(r)[g_n] \le (2\pi)^3 \int_\Gamma g_n(x) B_{\alpha,\Gamma}[\overline{g}_n](x) ds_x. \tag{16}$$

Then, let $n \to \infty$ in (16), to provided preliminary verification that $I_\alpha(r)[g_n]$ and $\int_\Gamma g_n(x) B_{\alpha,\Gamma}[\overline{g}_n](x) ds_x$, respectively, tend to $I_\alpha(r)[g]$ and $\int_\Gamma g(x) B_{\alpha,\Gamma}[\overline{g}](x) ds_x$. Certainly, first of all, the below relations evidently hold,

$$\begin{aligned} |\hat{\delta}_\Gamma[g](\xi) - \hat{\delta}_\Gamma[g_n](\xi)| &= |\int_\Gamma [g(x) - g_n(x)] \exp(-i\langle x, \xi \rangle) ds_x| \\ &\le (mes(\Gamma))^{1/2} ||g - g_n||_{L_2(\Gamma)}, \end{aligned}$$

consequently, $|\hat{\delta}_\Gamma[g_n](\xi)|$ uniformly tends (at $n \to \infty$) to $|\hat{\delta}_\Gamma[g](\xi)|$, for $|\xi| \le r$, and the same is valid concerning $|\hat{\delta}_\Gamma[g_n](\xi)|^2$ and $|\hat{\delta}_\Gamma[g](\xi)|^2$. Therefore, $\lim_{n \to \infty} I_\alpha(r)[g_n] = I_\alpha(r)[g]$. On the other hand, it is not difficult to find:

$$\begin{aligned} |\int_\Gamma (g(x) B_{\alpha,\Gamma}[\overline{g}](x) - g_n(x) B_{\alpha,\Gamma}[\overline{g}_n](x)) ds_x| &\le ||g - g_n||_{L_2(\Gamma)} \cdot ||B_{\alpha,\Gamma}[\overline{g}]||_{L_2(\Gamma)} \\ &+ ||g_n||_{L_2(\Gamma)} \cdot ||B_{\alpha,\Gamma}|| \cdot ||g - g_n||_{L_2(\Gamma)}, \end{aligned}$$

($||B_{\alpha,\Gamma}||$ is the norm of the operator $B_{\alpha,\Gamma}$); i.e., the integral $\int_\Gamma g_n(x) B_{\alpha,\Gamma}[\overline{g}_n](x) ds_x$ tends to $\int_\Gamma g(x) B_{\alpha,\Gamma}[\overline{g}](x) ds_x$ ($n \to \infty$). Thus, the estimate (15) is proved, and, observing that the function of r $I_\alpha(r)[g]$is monotone, increasing and bounded (because of (15)) we conclude that the integral $\int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g]|^2 |\xi|^{-\alpha} d\xi = I_\alpha(\infty)[g] := \lim_{r \to \infty} I_\alpha(r)[g]$ converges and the following inequality is fulfilled:

$$\int_{\mathbb{R}^3} |\hat{\delta}_\Gamma[g]|^2 |\xi|^{-\alpha} d\xi \le (2\pi)^3 \int_\Gamma g(x) B_{\alpha,\Gamma}[\overline{g}](x) ds_x,\ g \in L_2(\Gamma). \tag{17}$$

Finally, when $B_{\alpha,\Gamma}[g] = 0$ evidently $B_{\alpha,\Gamma}[\overline{g}] = 0$ as well, and (17) shows that $\hat{\delta}_\Gamma[g] = 0$, consequently $\delta_\Gamma[g] = 0$ which automatic yields $g = 0$. This proves the theorem. □

3. Main Results

The found property of the operator $B_{\alpha,\Gamma}$ was certainly of essential importance in our approach for solving the problem of Dirichlet. It is in a direct relation with the well-known

Hilbert–Schmidt theorem (e.g., [14,15]) and, as a first step below, we recall a selected formulation of this theorem.

Theorem 2. *(Hilbert–Schmidt)*
Let $B : H \to H$ be a bounded, compact and symmetrical linear operator in the Hilbert space H, with $h = 0$ as the unique solution of the equation $Bh = 0$, h varying in H. Then there exists a complete orthogonal system $\{h_j\} \subset H$, $||h_j|| = 1$, $j = 1, 2, \ldots$, of eigenvalue elements to B, with a corresponding set of (real) eigenvalues $\{\lambda_j\}$, such that the following expression holds, $\forall\, h \in H$:

(H-S), $h = \sum_{j=1}^{\infty} \sigma_j h_j$, $\sigma_j = \langle h, h_j \rangle$. Here, $\langle . \,, . \rangle$ is the scalar product in H (and $||h|| = \langle h, h\rangle$ is the norm of h).

Preparing to apply Theorem 2 concerning the operator $B_{\alpha,\Gamma}$, we start with the next two initial properties, the first one follows from the classical theory of the weakly singular integral equations, and the second from Theorem 2.

(i*) The integral operator $B_{\alpha,\Gamma} : L_2(\Gamma) \to L_2(\Gamma)$, with

$$B_{\alpha,\Gamma}[g](x) := \int_\Gamma c_{3,\alpha} g(y) |x - y|^{\alpha - 3} ds_y,$$

$x \in \Gamma$, $g \in L_2(\Gamma)$, is bounded, compact and symmetrical.

(ii*) Each function $\mu \in L_2(\Gamma)$ can be uniquely expressed by the decomposition formula below:

$$\mu = \sum_{k=1}^{\infty} \gamma_k \zeta_{k,\alpha}, \text{ in } L_2(\Gamma), \tag{18}$$

where $\{\zeta_{k,\alpha}\}$ is the complete orthogonal system of eigen functions for $B_{\alpha,\Gamma}$ and γ_k are the Fourier coefficients of μ, $\gamma_k := \int_\Gamma \mu(x)\zeta_{k,\alpha}(x) ds_x$. In our basic result we use the already mentioned sub-space $H^1_\alpha(\Gamma) \subset L_2(\Gamma)$.

Definition 1. *Let us set*
$H^1_\alpha(\Gamma) := \{\varphi \in L_2(\Gamma), \varphi = \sum_{k=1}^{\infty} \tau_k \zeta_{k,\alpha} : \sum_{k=1}^{\infty} \tau_k^2 \lambda_{k,\alpha}^{-2} < +\infty\}$, where $\lambda_{k,\alpha}$ are the eigenvalues of $B_{\alpha,\Gamma}$. The scalar product $\langle \varphi, \psi \rangle_{1,\alpha}$ in $H^1_\alpha(\Gamma)$ is defined by the sum $\sum_{k=1}^{\infty} \tau_k \overline{\theta}_k (1 + \lambda_{k,\alpha}^{-2})$, for φ, $\psi \in H^1_\alpha(\Gamma)$: $\varphi = \sum_{k=1}^{\infty} \tau_k \zeta_{k,\alpha}$, $\psi = \sum_{k=1}^{\infty} \theta_k \zeta_{k,\alpha}$.

Note that the inverse operator $B^{-1}_{\alpha,\Gamma}$ of $B_{\alpha,\Gamma}$ is correctly defined on $H^1_\alpha(\Gamma)$, by the evident rule $B^{-1}_{\alpha,\Gamma}[\varphi] := \sum_{k=1}^{\infty} \tau_k \lambda^{-1}_{k,\alpha} \zeta_{k,\alpha}$, for $\varphi = \sum_{k=1}^{\infty} \tau_k \zeta_{k,\alpha}$, $\varphi \in H^1_\alpha(\Gamma)$.Thus, $B^{-1}_{\alpha,\Gamma} : H^1_\alpha(\Gamma) \to L_2(\Gamma)$ is a bounded linear operator.

In the first theorem below, excepting results on existence, uniqueness and continuous data dependent on solutions, additional ones are also included concerning the asymptotic (at $|x| \to \infty$) and $L_1^{loc}(\mathbb{R}^3)$ approximation of solutions (by globally defined continuous functions). As a specific moment, the approximation process is uniquely generated by the corresponding boundary one in $L_2(\Gamma)$. Consider now the central result of our study.

Theorem 3. *Let $f(x)$ be a function, defined on $\mathcal{D}$ and satisfying the assumptions (3). Then, for each data $\varphi(x) \in L_2(\Gamma) : (\varphi - \varphi_{\alpha,f})(x) \in H^1_\alpha(\Gamma)$, the problem of Dirichlet (7) is solvable in $L_1^{loc}(\mathbb{R}^3)$ by the formula:*

$$u(x) = \int_\Gamma \frac{c_{3,\alpha} B^{-1}_{\alpha,\Gamma}[\varphi - \varphi_{\alpha,f}](y) ds_y}{|x-y|^{3-\alpha}} + \int_\mathcal{D} \frac{c_{3,\alpha} f(y) dy}{|x-y|^{3-\alpha}}, \; x \in \mathbb{R}^3. \tag{19}$$

The above function u is the unique solution of (7) in the family $S'_{\alpha,f}$, contained in the class $L_1^{loc}(\mathbb{R}^3)$ and continuous in the two domain components of $\mathbb{R}^3 \setminus \Gamma$. The solution (19) is additionally characterized by the following conventional, but essential, properties.

(P1) In case of $f(x)$ with a compact support in $\mathcal{D}$, when $\mathcal{D} = C\Omega$, the asymptotic relation below holds for $u(x)$:

$$|u(x)| \leq \frac{c_0}{|x|^{3-\alpha}}, \; |x| \to \infty, \tag{20}$$

(i.e., $u(x) = O(1/|x|^{3-\alpha})$ for $|x| \to \infty$), with constant c_0;

(P2) A property for continuous data dependence is valid in the $L_1^{loc}(\mathbb{R}^3)$ sense, expressed by the following assertion: given two systems of data, $\{f_1, f_2\}$ – satisfying (3) and $\{\varphi_1, \varphi_2\} \subset L_2(\Gamma)$, where $(\Delta_f \varphi)_i := \varphi_i - \varphi_{\alpha,f_i} \in H^1_\alpha$, $i = 1, 2$, there exist constants C^0_K, C^*_K so that the difference $u_2 - u_1$ of the solutions, corresponding to the above data, satisfies an estimate in the form:

$$||u_2 - u_1||_{L_1(K)} \leq C^0_K ||(\Delta_f \varphi)_2 - (\Delta_f \varphi)_1||_{H^1_\alpha(\Gamma)} + C^*_K ||f_2 - f_1||_{L_1(\mathcal{D})}, \tag{21}$$

for an arbitrarily chosen compact $K \subset \mathbb{R}^3$.

(P3) Each approximating system $\{\psi_n\} \subset H^1_\alpha(\Gamma)$, $\lim_{n\to\infty} \psi_n = g_f[\varphi]$ in $L_2(\Gamma)$ (where $g_f[\varphi] := B^{-1}_{\alpha,\Gamma}[\varphi - \varphi_{\alpha,f}]$), with continuous functions ψ_n, generates an infinite sequence of continuous approximations u_n to u : $\lim_{n\to\infty} u_n = u$ in $L_1^{loc}(\mathbb{R}^3)$. Moreover, u_n solve the problem (7) at the boundary condition $u|_\Gamma = \varphi_n$, $\lim_{n\to\infty} \varphi_n = \varphi$ in $L_2(\Gamma)$, with $\varphi_n := B_{\alpha,\Gamma}[\psi_n] + \varphi_{\alpha,f}$, and the estimate (22) (below) is valid for each fixed compact $K \subset \mathbb{R}^3$:

$$||u - u_n||_{L_1(K)} \leq C^0_K ||\varphi - \varphi_n||_{H^1_\alpha(\Gamma)}. \tag{22}$$

Proof. Recall that the verification as to whether function $u(x)$ from (19) satisfies the equation $(-\Delta)^{\alpha/2} u|_\mathcal{D} = f$ was done in Section 1: by the notations $v_{\alpha,g}(x)$, $u_{\alpha,f}(x)$, respectively, for the already introduced single layer and volume Riesz potential, with $g = g_f[\varphi]$, Formula (19) is rewritten as $u = v_{\alpha,g} + u_{\alpha,f}$, where $v_{\alpha,g}$ is a BF harmonic function, while $(-\Delta)^{\alpha/2} u_{\alpha,f} = f^0[f]$ (in S'). And for $(-\Delta)^{\alpha/2} u$ we get $(-\Delta)^{\alpha/2} u = \delta_\Gamma[g] + f^0[f]$ (in S'), which evidently means that $(-\Delta)^{\alpha/2} u|_\mathcal{D} = f$. Next, for $x \in \Gamma$ we have: $u|_\Gamma = v_{\alpha,g}|_\Gamma + \varphi_{\alpha,f} = B_{\alpha,\Gamma}[B^{-1}_{\alpha,\Gamma}[\varphi - \varphi_{\alpha,f}]] + \varphi_{\alpha,f} = \varphi$. Thus, the existence assertion is proved (i.e., u is a solution of the problem (7) in $S'_{\alpha,f}$). For the uniqueness of solution (19) in $S'_{\alpha,f}$, assuming existence of two ones, u_1, $u_2 \in S'_{\alpha,f}$ which satisfy (7) (with identical data φ, f), it is directly seen that the difference $U = u_2 - u_1$ is a BF harmonic function, i.e., $(-\Delta)^{\alpha/2} U = \delta_\Gamma[g]$ in S', with a density $g \in L_2(\Gamma)$. To resolve this equation regarding U (recall the analogous comments about (4)) we have evidently to act by the operation $\frac{c_{3,\alpha}}{|.|^{3-\alpha}} *$, finding, thus, the expression $U(x) = (\delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}})(x)$, $x \in \mathbb{R}^3$. Restricted to Γ it yields: $U|_\Gamma = B_{\alpha,\Gamma}[g]$, i.e., $B_{\alpha,\Gamma}[g] = 0$, and, therefore, $g = 0$ (Theorem 1), and (from $U = (\delta_\Gamma[g] * \frac{c_{3,\alpha}}{|.|^{3-\alpha}})$ in S') $U(x) = 0$, $x \in \mathbb{R}^3$. Next, looking at Formula (19) (i.e., $u = v_{\alpha,g} + u_{\alpha,f}$), it is directly seen that $v_{\alpha,g}$, $u_{\alpha,f} \in L_1^{loc}(\mathbb{R}^3)$, and the same for $u(x)$. Moreover, as in the proof of (21) (below), it follows the estimate

$$||u||_{L_1(K)} \leq C^0_K ||\varphi - \varphi_{\alpha,f}||_{H^1_\alpha(\Gamma)} + C^*_K ||f||_{L_1(\mathcal{D})}. \tag{23}$$

(Here $K \subset \mathbb{R}^3$ is an arbitrarily fixed compact, and a choice of constants C_K^0, C_K^* is given concerning (21)). The property $u(x) \in C^0(\mathbb{R}^3 \setminus \Gamma)$ (C^0, the space of the continuous functions), in both the cases $\mathcal{D} = \Omega$ and $\mathcal{D} = C\Omega$ is also an automatic consequence from the clear relations $v_{\alpha,g} \in C^0(K)$, $u_{\alpha,f} \in C^0(K)$, valid for each compact $K \subset \mathbb{R}^3 \setminus \Gamma$.

Consider now the properties (P1)–(P3).The asymptotic relation (20) is actually evident (as a slight consequence of the standard inequality $|x-y| \geq |x| - |y| > 0$, valid at $|x| \to \infty$ and y varying in a compact). For the proof of (21) let us first rewrite (19) with $u_2 - u_1$, $\varphi_2 - \varphi_{\alpha,f_2} - (\varphi_1 - \varphi_{\alpha,f_1})$, $f_2 - f_1$, respectively, instead of u, $\varphi - \varphi_{\alpha,f}$, f. For the sake of convenience we use the notations $\Delta_f\varphi = \varphi - \varphi_{\alpha,f}$, $((\Delta_f\varphi)_i = \varphi_i - \varphi_{\alpha,f_i}$, $i = 1,2)$. After integration of $|u_2 - u_1|$ on a compact $K \subset \mathbb{R}^3$ it easily follows:

$$\begin{aligned} |u_2 - u_1|_{L_1(K)} &\leq c_{3,\alpha}\int_\Gamma W_K(y)|B_{\alpha,\Gamma}^{-1}[(\Delta_f\varphi)_2 - (\Delta_f\varphi)_1](y)|ds_y \\ &+ c_{3,\alpha}\int_{\mathcal{D}} W_K(y)|f_2(y) - f_1(y)|dy, \\ W_K(y) &:= \int_K |x-y|^{\alpha-3}dx,\ y \in \mathbb{R}^3. \end{aligned}$$

In order to rework the above inequality suitably we take into account the estimate

$$||c_{3,\alpha}B_{\alpha,\Gamma}^{-1}[(\Delta_f\varphi)_2 - (\Delta_f\varphi)_1]||_{L_2(\Gamma)} \leq b_{\alpha,\Gamma}^*||(\Delta_f\varphi)_2 - (\Delta_f\varphi)_1||_{H_\alpha^1(\Gamma)},$$

where $b_{\alpha,\Gamma}^*$ is the norm of the operator $c_{3,\alpha}B_{\alpha,\Gamma}^{-1}$. In this way, we come to the next relation for the difference $u_2 - u_1$:

$$\begin{aligned} ||u_2 - u_1||_{L_1(K)} &\leq ||W_K||_{L^\infty(\mathbb{R}^3)}\left(b_{\alpha,\Gamma}^*\sqrt{mes\Gamma}||(\Delta_f\varphi)_2 - (\Delta_f\varphi)_1||_{H_\alpha^1(\Gamma)}\right) \\ &+ ||W_K||_{L^\infty(\mathbb{R}^3)}\left(c_{3,\alpha}||f_2 - f_1||_{L_1(\mathcal{D})}\right). \end{aligned} \tag{24}$$

Afterwards, it remains to set: $C_K^0 = b_{\alpha,\Gamma}^*\sqrt{mes\Gamma}||W_K||_{L^\infty(\mathbb{R}^3)}$, $C_K^* = |c_{3,\alpha}||W_K||_{L^\infty(\mathbb{R}^3)}$. Thus (24) takes the form of (21). □

Remark 2. *The partial case $f_1 = f_2$ could be practically more valuable (then the accent is paid on the boundary data dependence). Now the estimates (24), (21) take, respectively, the forms:*

$$||u_2 - u_1||_{L_1(K)} \leq ||W_K||_{L^\infty(\mathbb{R}^3)}b_{\alpha,\Gamma}^*\sqrt{mes\Gamma}||\varphi_2 - \varphi_1||_{H_\alpha^1(\Gamma)}; \tag{25}$$

$$||u_2 - u_1||_{L_1(K)} \leq C_K^0||\varphi_2 - \varphi_1||_{H_\alpha^1(\Gamma)}. \tag{26}$$

Consider, finally, the proof of (P3). Via the remark above, when the boundary problem (7) is used in a model, the contour L_2 data φ can be preferably changed by suitable continuous approximations $\{\varphi_n\}$ in order to simplify a numerical procedure. In our approach, the boundary operator pair $\{B_{\alpha,\Gamma}, B_{\alpha,\Gamma}^{-1}\}$ suggests seeking $\{\varphi_n\}$ by the map $B_{\alpha,\Gamma}[\psi_n]$, given an arbitrary sequence $\{\psi_n\} \subset H_\alpha^1(\Gamma)$ of continuous L_2 approximations to $g_f[\varphi]$. In the framework of the problem (7) (considered now at boundary data φ_n, regarding an unknown solution u_n) the basic Formula (19) serves as the answer both, for φ_n and u_n, namely $\varphi_n = B_{\alpha,\Gamma}[\psi_n] + \varphi_{\alpha,f}$ and u_n as follows:

$$u_n(x) = \int_\Gamma \frac{c_{3,\alpha}\psi_n(y)ds_y}{|x-y|^{3-\alpha}} + u_{\alpha,f}(x),\ x \in \mathbb{R}^3. \tag{27}$$

The property $u_n \in C^0(\mathbb{R}^3)$ follows (by the integral above) from the continuous assumption for ψ_n, and the same for φ_n (We have again taken into account that the single layer Riesz potentials possess, at $3 - \alpha < 2$, the same continuous properties as in the classical case of $3 - \alpha = 1$). The announced estimate (22) is actually proved by the already shown (26). In conclusion, let us comment on how to construct approximating systems $\{\psi_n\} \subset H^1_\alpha(\Gamma)$ of continuous functions by introducing an arbitrary system of (real) numbers $\{\tau_k\}$, $k = 1,2,\ldots$: $\sum_{k=1}^{\infty} \tau_k^2 \lambda_{k,\alpha}^{-2} < \infty$, we get an element g of the space $H^1_\alpha(\Gamma)$, $g(x) := \sum_{k=1}^{\infty} \tau_k \zeta_{k,\alpha}(x)$ and for an obviously convenient approximating (to g) sequence we have to take $\psi_n(x) := \sum_{k=1}^{n} \tau_k \zeta_{k,\alpha}(x)$, $n = 1,2,\ldots$. Recall that the eigen functions $\zeta_{k,\alpha}$ of $B_{\alpha,\Gamma}$ are continuous (i.e., $\{\zeta_{k,\alpha}\} \subset C^0(\Gamma)$), according to the known classical theorem for the continuous L_2 solutions of weakly singular integral equations (e.g., [11–13]).

Our next result concerns regularity properties of the solution (19), in the interior of $\mathbb{R}^3 \setminus \Gamma$, as a consequence of which $f(x)$. We consider below two cases for the regularity of $f(x)$, assumed with a compact support: $f \in C_0^m(\mathcal{D})$, $m = 1,2,\ldots$, and $f \in L^\infty(\mathcal{D}) \cap \mathfrak{L}'(\mathcal{D})$, at $f^0[f] \in H^s(\mathbb{R}^3)$, $s > 0$. Here, as usual, $C_0^m(\mathcal{D})$ the space of the functions smooths up to order m in $\mathcal{D}$ with compact supports, $H^s(\mathbb{R}^3)$ are the known Sobolev classes, related to $\mathbb{R}^3$ (e.g., [9]), and $\mathfrak{L}'(\mathcal{D})$ is the space of the Lauren Schwartz distributions, defined on $\mathcal{D}$ ([9,14,15]), possessing compact supports therein. Clearly the elements of $\mathfrak{L}'(\mathcal{D})$ are automatically extended (on the whole $\mathbb{R}^3$) as zeros out of their supports, presenting, thus, distributions from $S'(\mathbb{R}^3)$.

Concerning the conventional regularity of the (19) solution $u(x)$ (in the case $f \in C_0^m(\mathcal{D})$) we again apply $L_1(K)$ estimates, now related to the partial derivatives $\partial_x^\beta u(x)$. Recall here that β is a 3D multi index, i.e., $\beta = (\beta_1, \beta_2, \beta_3)$, with β_i (i = 1,2,3) – (non-negative) integers; $\partial_x^\beta u(x)$ is of order k (k = 0, 1, 2, . . .) when $|\beta| = k$, $|\beta| := \beta_1 + \beta_2 + \beta_3$, and a function $F(x)$, defined in a domain $\tilde{\Omega} \subset \mathbb{R}^3$, belongs to the class $C^m(\tilde{\Omega})$ when F possesses continuous in $\tilde{\Omega}$ derivatives of each order k, $k \leq m$. In the case of certain Sobolev regularity for the solution $u(x)$ (assuming $f^0[f] \in H^s(\mathbb{R}^3)$), it is clearly expected to hold that $u \in H^s_{loc}(\mathbb{R}^3 \setminus \Gamma)$. As known, this inclusion is characterized by the property $\theta u \in H^s(\mathbb{R}^3)$, valid for each function $\theta(x) \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$ (at θu automatically extended as zero out of the support of $\theta(x)$). We seek a relevant H^s estimation of θu by the boundary data $\varphi - \varphi_{\alpha,f}$ and f.

For analyzing the H^s properties of $u(x)$ we use the following accessory assertion (e.g., [16]). (The given proof of the lemma is due to university lectures [17].)

Lemma 2. *The map* $\mathbf{M}_\Phi : \mathbf{v} \to \Phi\mathbf{v}$, $\mathbf{v} \in H^s$, *with* $\Phi(x) \in S$, *an arbitrary fixed function, is a continuous operator, acting from* H^s *into itself, for each (fixed) real* s. *(Here* $S = S(\mathbb{R}^3)$ *and the same for* H^s*).*

Proof. As a necessary initial step, recall the very useful representation for the Fourier image $(\hat{\Phi \mathbf{v}})(\xi)$, $\xi \in \mathbb{R}^3$, of the (generalized) function $(\Phi\mathbf{v})(x)$:

$$(\hat{\Phi \mathbf{v}})(\xi) = (2\pi)^{-3}(\hat{\mathbf{v}} * \hat{\Phi})(\xi) = (2\pi)^{-3} \int_{\mathbb{R}^3} \hat{\Phi}(\xi - \eta)\hat{\mathbf{v}}(\eta)d\eta. \quad (28)$$

From (28), taking into account the known Peetre inequality (e.g., [16]), we get:

$$|(1 + |\xi)|^2)^{s/2}(\hat{\Phi \mathbf{v}})(\xi)| \leq \frac{2^{|s/2|}}{(2\pi)^3} \int_{\mathbb{R}^3} (1 + |\xi - \eta|^2)^{|s/2|} |\hat{\Phi}(\xi - \eta)(1 + |\eta)|^2)^{s/2}\hat{\mathbf{v}}(\eta)|d\eta.$$

Applying the Young inequality ([9–11]) in the integral term above, we find the sought estimate:

$$||\mathbf{\Phi v}||_s \leq \mathbf{C_M}||\mathbf{v}||_s,\ \mathbf{v} \in H^s. \quad (29)$$

Here $||.||_s$ is the norm in the space H^s, $||\mathbf{\Phi v}||_s := ||(1 + |\xi)|^2)^{s/2}(\hat{\Phi \mathbf{v}})(\xi)||_{L_2(\mathbb{R}^3)}$, and $\mathbf{C_M} := \frac{2^{|s/2|}}{(2\pi)^3}||(1 + |\xi)|^2)^{|s/2|}(\hat{\Phi})(\xi)||_{L_1(\mathbb{R}^3)}$. We can now formulate and prove our result concerning the regularity of the solution in (19). □

Theorem 4. *Suppose the free term $f(x)$ in the Equation (7) belongs to some of the spaces $C_0^m(\mathcal{D})$, $L^\infty(\mathcal{D}) \cap \mathfrak{L}'(\mathcal{D})$ with $f^0[f] \in H^s$. Then, the (19) solution $u(x)$ has the relevant regularity properties in $\mathbb{R}^3 \setminus \Gamma$, satisfying the attached estimates, as follows:*

(I) When $f \in C_0^m(\mathcal{D})$, it holds that $u \in C^m(\mathbb{R}^3 \setminus \Gamma)$ and the estimate below is valid, for each compact $K \subset \mathbb{R}^3 \setminus \Gamma$, $\forall \beta : |\beta| \le m$:

$$||\partial_x^\beta u||_{L_1(K)} \le \mathbf{C}^0_{K,\beta} ||\varphi - \varphi_{\alpha,f}||_{H^1_\alpha(\Gamma)} + \mathbf{C}^*_K ||\partial_x^\beta f||_{L_1(\mathcal{D})}; \tag{30}$$
$$\mathbf{C}^0_{K,\beta} = b^*_{\alpha,\Gamma} \sqrt{mes\Gamma} ||W_{K,\beta}||_{C^0(\Gamma)}, W_{K,\beta}(y) := \int_K \partial_x^\beta |x-y|^{\alpha-3} dx.$$

(II) When $f \in L^\infty(\mathcal{D}) \cap \mathfrak{L}'(\mathcal{D})$: $f^0[f] \in H^s$, for $1 < \alpha < 3/2$, the inclusion $u \in H^s_{loc}(\mathbb{R}^3 \setminus \Gamma)$ is valid and there exist two constants $\mathbf{c}_{\theta,1}$, $\mathbf{c}_{\theta,2}$ (depending on θ), such that the next estimate is fulfilled, $\forall \theta \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$:

$$||\theta u||_s \le \mathbf{c}_{\theta,1} ||\varphi - \varphi_{\alpha,f}||_{H^1_\alpha(\Gamma)} + \mathbf{c}_{\theta,2}(||f||_{L^\infty} + ||f||_s). \tag{31}$$

(Above $L^\infty = L^\infty(\mathbb{R}^3)$.)

Proof. In both cases (I) and (II) we can clearly and conveniently deal with the short version of (19), i.e., $u = v_{\alpha,g} + u_{\alpha,f}$ (with $g = B^{-1}_{\alpha,\Gamma}[\Delta_f \varphi]$). Acting by the operation ∂_x^β on the components $v_{\alpha,g}$ and $u_{\alpha,f}$, we, respectively, find that:

$$\partial_x^\beta v_{\alpha,g}(x) = \int_\Gamma c_{3,\alpha} B^{-1}_{\alpha,\Gamma}[\Delta_f \varphi](y) \partial_x^\beta |x-y|^{\alpha-3} ds_y, \; and \; \partial_x^\beta u_{\alpha,f}(x) = \left(\partial_x^\beta f * \frac{c_{3,\alpha}}{|.|^{3-\alpha}} \right)(x),$$

for $x \in K$ (K is a compact in $\mathbb{R}^3 \setminus \Gamma$). The property $u \in C^m(\mathbb{R}^3 \setminus \Gamma)$ now becomes clear. In addition the above expression for $\partial_x^\beta v_{\alpha,g}$ suggests introducing the function

$$W_{K,\beta}(y) := \int_K \partial_x^\beta |x-y|^{\alpha-3} dx.$$

Afterwards, to prove the estimate (30) we only have to follow the steps already used from the proof of (21): the constant $\mathbf{C}^0_{K,\beta}$ in (30) is evidently analogous to $\mathbf{C}^0_K$ and $\mathbf{C}^*_K$ is from the estimate (21).

Going to the proof of (II), let us multiply the relation $u = v_{\alpha,g} + u_{\alpha,f}$ by an arbitrary $\theta(x) \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$ and consider next the H^s properties of the terms $\theta v_{\alpha,g}$, $\theta u_{\alpha,f}$. According to the Lemma, for the second product we could conclude that $\theta u_{\alpha,f} \in H^s$ if $u_{\alpha,f} \in H^s$. However, the former certainly holds for $1 < \alpha < 3/2$ (under the assumption $f^0[f] \in H^s$):

$$||u_{\alpha,f}||_s^2 = ||(1+|\xi)|^2)^s |\hat{f}(\xi)|^2 |\xi|^{-2\alpha}||_{L_1(\mathbb{R}^3)} \le ||\hat{f}||^2_{C^0(|\xi|\le 1)} \int_{|\xi|\le 1} \frac{2^s}{|\xi|^{2\alpha}} d\xi + ||f||_s^2.$$

On the other hand, $\theta v_{\alpha,g} \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$, therefore $u \in H^s_{loc}(\mathbb{R}^3 \setminus \Gamma)$. Preparing the final estimate (31), we first specify the above estimate for $||u_{\alpha,f}||^2$, concerning the term with $||\hat{f}||^2_{C^0(|\xi|\le 1)}$, it actually holds that:

$$||\hat{f}||^2_{C^0(|\xi|\le 1)} \int_{|\xi|\le 1} \frac{2^s}{|\xi|^{2\alpha}} d\xi \le \frac{2^{s+2}}{3-2\alpha} \pi mes^2 K_f^0 ||f||^2_{L^\infty}.$$

Here $K_f^0 = supp[f]$ (the supporter of f) and $L^\infty = L^\infty(\mathbb{R}^3)$. Consequently, $||u_{\alpha,f}||_s$ satisfies the inequality

$$||u_{\alpha,f}||_s \le \left(1 + 2^{1+s/2} \frac{\sqrt{\pi}}{\sqrt{3-2\alpha}} mes K_f^0 \right) (||f||_{L^\infty} + ||f||_s). \tag{32}$$

Now, by the Lemma 2 we can easily estimate the product $\theta u_{\alpha,f}$:

$$||\theta u_{\alpha,f}||_s \leq \mathbf{C}_{\mathbf{M},\theta}\mathbf{C}_f^0(||f||_{L^\infty} + ||f||_s). \tag{33}$$

(For $\mathbf{C}_{\mathbf{M},\theta}$, $\mathbf{C}_f^0$ we respectively have: $\mathbf{C}_{\mathbf{M},\theta} := 2^{|s|/2}(2\pi)^{-3}||(1+|\xi)|^2)^{|s|/2}\hat{\theta}(\xi)||_{L_1}$, with $L_1 = L_1(\mathbb{R}^3)$, and $\mathbf{C}_f^0 = (1 + 2^{1+s/2}\frac{\sqrt{\pi}}{\sqrt{3-2\alpha}} mesK_f^0.)$

It remains, then, to estimate the product $\theta v_{\alpha,g}$ ($g = B^{-1}_{\alpha,\Gamma}[\Delta_f, \varphi]$). In order to conveniently express the impact of the boundary data we deal with the norm $||\theta v_{\alpha,g}||_{[s]+1}$ (using $||\theta v_{\alpha,g}||_{[s]} \leq ||\theta v_{\alpha,g}||_{[s]+1}$, where $[s]$ is the integer part of s). As known, $||\theta v_{\alpha,g}||^2_{[s]+1}$ can be expressed taking the sum of addends like $||\partial_x^\beta(\theta v_{\alpha,g})||^2_{L_2}$, where $\partial_x^\beta(\theta v_{\alpha,g})(x) = \int_\Gamma c_{3,\alpha} B^{-1}_{\alpha,\Gamma}[\Delta_f \varphi](y) \partial_x^\beta((\theta v_{\alpha,g})(x)|x-y|^{\alpha-3}) ds_y$. The Cauchy–Schwartz inequality now yields:

$$|\partial_x^\beta(\theta v_{\alpha,g})|^2 \leq ||c_{3,\alpha} B^{-1}_{\alpha,\Gamma}[\Delta_f \varphi]||^2_{L_2(\Gamma)} \int_\Gamma |\partial_x^\beta(\theta(x)|x-y|^{\alpha-3})|^2 ds_y. \tag{34}$$

Summarizing the above on all $\beta : |\beta| = k$, for $k = 0, 1, \dots [s]+1$, and taking an integration $\int_{\mathbb{R}^3} |...|^2 dx$ on the relevant terms, we obtain:

$$||\theta v_{\alpha,g}||^2_{[s]+1} \leq ||c_{3,\alpha} B^{-1}_{\alpha,\Gamma}[\Delta_f \varphi]||^2_{L_2(\Gamma)} \int_\Gamma ||\theta|. - y|^{\alpha-3}||^2_{[s]+1} ds_y. \tag{35}$$

By the notation $W_{\alpha,[s]+1}[\theta](y) := ||\theta|. - y|^{\alpha-3}||_{[s]+1}$ (35) can be evidently rearranged in the next form:

$$||\theta v_{\alpha,g}||_{[s]+1} \leq b^*_{\alpha,\Gamma} ||W_{\alpha,[s]+1}[\theta]||_{L_2(\Gamma)} ||\varphi - \varphi_{\alpha,f}||_{H^1_\alpha(\Gamma)}. \tag{36}$$

Finally, from the initial inequality $||\theta u||_s \leq ||\theta v_{\alpha,g}||_{[s]+1} + ||\theta u_{\alpha,f}||_s$, and the sum of (33), (36) we get the expected estimate (31), with $\mathbf{c}_{\theta,1} = b^*_{\alpha,\Gamma}||W_{\alpha,[s]+1}[\theta]||_{L_2(\Gamma)}$, $\mathbf{c}_{\theta,2} = \mathbf{C}_{\mathbf{M},\theta}\mathbf{C}_f^0$. Thus, the theorem is proved. □

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Appendix A. The Dirichlet Problem for 1*D* Equations

We will comment here the problem of Dirichlet for the $1D$ equations $(-\Delta)^{\alpha/2}u|_{(l_0,+\infty)} = f(x)$, with assumed continuous in $(l_0, +\infty)$ and (for the sake of simplicity) $f(x)$ vanishing out of a compact subinterval of $(l_0, +\infty)$, and $(\Delta)^{\alpha/2}u|_{(-l,l)} = f(x)$, with $f(x)$ continuous in $[-l, l]$. Looking for globally existing solutions, we shall need the respective Riesz potentials:

$$u^0_{\alpha,f}(x) = \int_{l_0}^\infty \frac{c_{1,\alpha} f(y) dy}{|x-y|^{1-\alpha}},\ u_{\alpha,f}(x) = \int_{-l}^l \frac{c_{1,\alpha} f(y) dy}{|x-y|^{1-\alpha}},\ \left(c_{1,\alpha} = \frac{\Gamma(\frac{1-\alpha}{2})}{2^\alpha\sqrt{\pi}\Gamma(\frac{\alpha}{2})}\right).$$

Clearly, for the existence of these potentials it is required that $0 < \alpha < 1$. Suggested from the possible singularities of the types $\frac{1}{|x-l_0|^{1-\alpha}}$ or $\frac{1}{|x\pm l|^{1-\alpha}}$, concerning respectively the first or the second equation above, we shall interesting in solutions $u \in S' = S'(\mathbb{R}^1)$ satisfying the relevant condition:

$$(a)\ (x - l_0)^{1-\alpha}u(x) \in L^\infty(l_0 - 1, l_0 + 1);\ (b)\ (x \pm l)^{1-\alpha}u(x) \in L^\infty(-l - 1, l + 1). \quad \text{(A1)}$$

Below we shall use the notation $|x - l^*|^{1-\alpha}u|_{x=l^*}$ for the limit (assumed existing) $\lim\limits_{x\to l^*} |x - l^*|^{1-\alpha}u(x)$, $l^* \in \mathbb{R}^1$. Consider now the following boundary value problems of Dirichlet type:

$$(-\Delta)^{\alpha/2}u|_{(l_0,+\infty)} = f(x);\ |x - l_0|^{1-\alpha}u|_{x=l_0} = c_0\ (c_0 = const \in \mathbb{R}^1). \quad \text{(A2)}$$

$$(-\Delta)^{\alpha/2}u|_{(-l,l)} = f(x);\ |x \pm l|^{1-\alpha}u|_{x=\mp l} = c^0_\mp\ (c^0_-,\ c^0_+ = const \in \mathbb{R}^1). \quad \text{(A3)}$$

Next, the question for resolving the problems is discussed separately but in a common framework. The relevant two assertions give the essence of the needed answer.

Proposition A1. *For $f(x)$, continuous in $[l_0, +\infty)$ and vanishing out of $[x_1, x_2] \subset [l_0, +\infty)$, and an arbitrary constant c_0 problem (A2) has a unique solution $u \in S'$ satisfying condition (A1), expressed by the formula:*

$$u(x) = \frac{c_0}{|x - l_0|^{1-\alpha}} + \int_{l_0}^{+\infty} \frac{c_{1,\alpha} f(y)dy}{|x - y|^{1-\alpha}},\ (x \in \mathbb{R}^1). \quad \text{(A4)}$$

Proposition A2. *For $f(x)$– continuous function in $[-l, l]$ and c^0_-, c^0_+ – arbitrary constants problem (A3) has a unique solution $u \in S'$ satisfying conditions (A1), which is present by the formula:*

$$u(x) = \frac{c^0_-}{|x + l|^{1-\alpha}} + \frac{c^0_+}{|x - l|^{1-\alpha}} + \int_{-l}^{+l} \frac{c_{1,\alpha} f(y)dy}{|x - y|^{1-\alpha}},\ (x \in \mathbb{R}^1). \quad \text{(A5)}$$

Sketch of proofs: Suppose $u \in S'$ is a solution of the equation from (A2), satisfying condition (A1), i.e., $(-\Delta)^{\alpha/2}u = f^0[f]$ in S', and $(-\Delta)^{\alpha/2}[u - u^0_{\alpha,f}] = 0$ on $\mathbb{R}^1 \setminus \{l_0\}$, therefore $(-\Delta)^{\alpha/2}[u - u^0_{\alpha,f}] = C_0\delta(x - l_0)$, with a constant C_0, because of condition (A1). More accurately, according to the known properties of the compactly supported distributions $u \in S'$ ([9]), instead of $C_0\delta(x - l_0)$ it should be taken a sum of the type $C_0\delta(x - l_0) + \sum_{m=1}^{N} C_m\delta^{(m)}(x - l_0)$. However condition (A1) yields $C_m = 0$ ($m = 1, 2, \ldots, N$). Next, as in the Introduction, by applying the Fourier transform to equation $(-\Delta)^{\alpha/2}[u - u^0_{\alpha,f}] = C_0\delta(x - l_0)$ we resolve it regarding u, finding the relation $u = \dfrac{C_0 c_{1,\alpha}}{|x - l_0|^{1-\alpha}} + u^0_{\alpha,f}$. Rewriting in details the potential $u^0_{\alpha,f}$, we get the following general solution formula (with C_0 as a free constant):

$$u(x) = C_0\frac{c_{1,\alpha}}{|x - l_0|^{1-\alpha}} + \int_{l_0}^{+\infty} \frac{c_{1,\alpha} f(y)dy}{|x - y|^{1-\alpha}},\ (x \in \mathbb{R}^1). \quad \text{(A6)}$$

In the case related to the final interval $(-l, l)$ we only have to use once more the above arguments and to solve now the equation $(-\Delta)^{\alpha/2}[u - u^0_{\alpha,f}] = C_{-l}\delta(x + l) + C_l\delta(x - l)$. The resulting expression for u gives the next general formula:

$$u(x) = C_{-l}\frac{c_{1,\alpha}}{|x + l|^{1-\alpha}} + C_l\frac{c_{1,\alpha}}{|x - l|^{1-\alpha}} + \int_{-l}^{+l} \frac{c_{1,\alpha} f(y)dy}{|x - y|^{1-\alpha}},\ (x \in \mathbb{R}^1). \quad \text{(A7)}$$

Substituting afterwards from (A6) and (A7) respectively in the boundary conditions $|x - l_0|^{1-\alpha}u|_{x=l_0} = c_0$, $|x \pm l|^{1-\alpha}u|_{x=\mp l} = c^0_\mp$, we easily obtain the announced

Formulas (A4) and (A5). Thus we actually get the uniqueness part (of Propositions A1 and A2) and the existence one consists in the verification already known from the Introduction.

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