fractal and fractional

Article

On a System of Sequential Caputo Fractional Differential Equations with Nonlocal Boundary Conditions

Alexandru Tudorache [1] and Rodica Luca [2,*]

[1] Department of Computer Science and Engineering, Gh. Asachi Technical University, 700050 Iasi, Romania
[2] Department of Mathematics, Gh. Asachi Technical University, 700506 Iasi, Romania
* Correspondence: rluca@math.tuiasi.ro

Abstract: We obtain existence and uniqueness results for the solutions of a system of Caputo fractional differential equations which contain sequential derivatives, integral terms, and two positive parameters, supplemented with general coupled Riemann–Stieltjes integral boundary conditions. The proofs of our results are based on the Banach fixed point theorem and the Leray–Schauder alternative.

Keywords: Caputo fractional differential equations; integral boundary conditions; existence of solutions; uniqueness of solutions

MSC: 34A08; 34B15; 45G15

Citation: Tudorache, A.; Luca, R. On a System of Sequential Caputo Fractional Differential Equations with Nonlocal Boundary Conditions. *Fractal Fract.* **2023**, *7*, 181. https://doi.org/10.3390/fractalfract7020181

Academic Editor: Ivanka Stamova

Received: 7 January 2023
Revised: 21 January 2023
Accepted: 10 February 2023
Published: 12 February 2023

1. Introduction

We consider in this paper the system of nonlinear Caputo fractional differential equations containing sequential derivatives:

$$\begin{cases} ({}^cD^{\alpha} + \lambda\, {}^cD^{\alpha-1})x(t) = \mathfrak{f}(t, x(t), y(t), I_{0+}^{p_1}x(t), I_{0+}^{p_2}y(t)), & t \in (0,1), \\ ({}^cD^{\beta} + \mu\, {}^cD^{\beta-1})y(t) = \mathfrak{g}(t, x(t), y(t), I_{0+}^{q_1}x(t), I_{0+}^{q_2}y(t)), & t \in (0,1), \end{cases} \tag{1}$$

supplemented with the general coupled integral boundary conditions

$$\begin{cases} x(0) = x'(0) = 0, \;\; x'(1) = 0, \;\; x(1) = \int_0^1 x(s)\, d\mathfrak{H}_1(s) + \int_0^1 y(s)\, d\mathfrak{H}_2(s), \\ y(0) = y'(0) = 0, \;\; y'(1) = 0, \;\; y(1) = \int_0^1 x(s)\, d\mathfrak{K}_1(s) + \int_0^1 y(s)\, d\mathfrak{K}_2(s). \end{cases} \tag{2}$$

Here, $\alpha,\ \beta \in (3,4]$, $\lambda,\ \mu > 0$, $p_1,\ q_1,\ p_2,\ q_2 > 0$, ${}^cD^{\kappa}$ represents the Caputo fractional derivative of order θ (for $\theta = \alpha,\ \beta,\ \alpha-1,\ \beta-1$), $\mathfrak{f},\ \mathfrak{g} : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions, I_{0+}^{j} denotes the fractional Riemann–Liouville integral of order ν (for $\nu = p_1,\ q_1,\ p_2,\ q_2$), and in the last conditions of (2), we have the Riemann–Stieltjes integrals with bounded variation functions $\mathfrak{H}_1$, $\mathfrak{K}_1$, $\mathfrak{H}_2$, $\mathfrak{K}_2$.

If the functions $\mathfrak{f}$ and $\mathfrak{g}$ satisfy some assumptions, we will prove that problem (1) and (2) has at least one solution. We will present next some papers connected with our problem. The notion of sequential fractional derivative $\mathcal{D}_{a+}^{k\alpha}$, $k = 1, 2, \ldots$, is presented in the monograph of Miller and Ross [1] (page 209). The relation between the sequential fractional derivatives and the non sequential Riemann-Liouville derivatives is described in the papers [2,3]. In [2], the authors investigated the existence of a minimal solution and a maximal solution, and the uniqueness of solution to an initial value fractional problem with Riemann-Liouville sequential fractional derivative $\mathcal{D}_{0+}^{2\alpha}$, $\alpha \in (0,1]$. They used the upper and lower solutions method and the associated monotone iterative method. In [3], the authors studied the existence of solution for a periodic boundary value fractional problem with a sequential Riemann-Liouville fractional derivative $\mathcal{D}_{0+}^{2\alpha}$, $\alpha \in (0,1]$, by applying the upper and lower solutions method and the Schauder fixed point theorem. In [4] the author showed the existence of solutions for a nonlinear impulsive fractional differential equation with Riemann-Liouville sequential fractional derivative subject to periodic boundary conditions by using the monotone iterative method. The nonexistence of solutions in $L^p((1,\infty),\mathbb{R})$ for an initial value problem with linear sequential fractional differential equations involving a Riemann-Liouville derivative and a classical first order derivative is investigated in [5]. By using the contraction principle, Klimek proved in [6] the existence and uniqueness of the solution of a class of nonlinear Hadamard sequential fractional differential equations supplemented with a set of initial conditions involving fractional derivatives. In our paper the sense of the word *sequential* is that the operator ${}^cD^\alpha + \lambda {}^cD^{\alpha-1}$ can be written as a composition of the operators ${}^cD^{\alpha-1}(D+\lambda)$, where D is the ordinary derivative. This kind of operator was introduced by Ahmad and Nieto in [7], where they studied the existence of solutions and the uniqueness of solutions for the Caputo sequential fractional differential equation

$$\begin{cases} {}^cD^\alpha(D+\lambda)x(t) = \mathfrak{f}(t,x(t)), \;\; 0<t<1, \;\; \alpha\in(1,2], \\ x(0)=0, \;\; x'(0)=0, \;\; x(1)=\beta x(\eta), \;\; \eta\in(0,1). \end{cases} \tag{3}$$

The authors applied in [7] the contraction Banach mapping principle and the fixed point theorem of Krasnosel'skii for sums of two operators. For $\alpha=2$, problem (3) arises in the study of Cauchy problems for nano boundary layer fluid flows, hyperbolic conservation laws, and physical phenomena in fluctuating environments (see [7] and its references). In [8], by using some tools from the fixed point theory, the authors proved the existence of solutions for the sequential integrodifferential equation

$$({}^cD^\alpha + k{}^cD^{\alpha-1})u(t) = p\mathfrak{f}(t,u(t)) + qI^\beta\mathfrak{g}(t,u(t)), \;\; t\in(0,1),$$

where $\alpha\in(1,2],\;\beta\in(0,1)$, subject to the conditions $u(0)=0,\;u(1)=0$, or $u'(0)+ku(0)=\mathfrak{a},\;u(1)=\mathfrak{b},\;\mathfrak{a},\mathfrak{b}\in\mathbb{R}$, or $u(0)=\mathfrak{a},\;u'(0)=u'(1),\;\mathfrak{a}\in\mathbb{R}$, and I^β is the fractional integral of Riemann-Liouville type with order β. In [9] the authors investigated the existence of solutions of the sequential Caputo fractional differential equation with boundary conditions which contain a fractional integral of Riemann-Liouville type

$$\begin{cases} ({}^cD^\alpha + k{}^cD^{\alpha-1})x(t) = \mathfrak{f}(t,x(t)), \;\; t\in(0,1), \;\; \alpha\in(2,3], \\ x(0)=0, \;\; x'(0)=0, \;\; x(\zeta)=\mathfrak{a}I^\beta x(\eta), \end{cases}$$

where $\beta>0$, $0<\eta<\zeta<1$, and k, $\mathfrak{a}>0$. In [10], the authors proved the existence of solutions of the sequential fractional differential inclusion of Caputo type, with boundary conditions containing a fractional integral of Riemann-Liouville type

$$\begin{cases} ({}^cD^\alpha + k{}^cD^{\alpha-1})x(t) \in \mathfrak{F}(t,x(t)), \;\; t\in(0,1), \\ x(0)=0, \;\; x'(0)=0, \;\; x(\varsigma)=\mathfrak{a}I^\beta x(\eta), \end{cases}$$

where $\alpha\in(2,3]$, $0<\eta<\varsigma<1$, $\mathfrak{F}:[0,1]\times\mathbb{R}\to P(\mathbb{R})$ is a multivalued map, $P(\mathbb{R})$ denoted the set of all nonempty subsets of $\mathbb{R}$, and k, $\mathfrak{a}$, β are positive numbers. By using some theorems from the fixed point theory, they studied the convex case and the non-convex case of multivalued maps. In [11] the authors obtained several existence and uniqueness results for the Caputo type sequential fractional differential equation

$$({}^cD^\gamma + k{}^cD^{\gamma-1})u(t) = \mathfrak{f}(t,u(t)), \;\; t\in(0,T),$$

where $\gamma\in(1,2],\;k>0$, with the anti-periodic boundary conditions

$$\alpha_1u(0)+\rho_1u(T)=\beta_1, \;\; \alpha_2u'(0)+\rho_2u'(T)=\beta_2,$$

or to the anti-periodic boundary conditions with integral terms

$$\alpha_1u(0)+\rho_1u(T)=\lambda_1\int_0^\eta u(s)\,ds+\lambda_2, \;\; \alpha_2u'(0)+\rho_2u'(T)=\mu_1\int_\xi^T u(s)\,ds+\mu_2,$$

where $0 < \eta < \xi < T$, $\alpha_1, \beta_1, \alpha_2, \beta_2, \rho_1, \lambda_1, \mu_1, \rho_2, \lambda_2, \mu_2 \in \mathbb{R}$, $\alpha_1 + \rho_1 \neq 0$, and $\alpha_2 + \rho_2 e^{-kT} \neq 0$. In [12] the authors proved the existence of solutions for the fractional sequential differential equation with nonlocal boundary conditions

$$\begin{cases} ({}^cD^q + k{}^cD^{q-1})\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t})), \ \ \mathfrak{t} \in (0, T), \ \ q \in (2, 3], \\ \alpha_1 \mathfrak{u}(0) + \sum_{i=1}^m a_i \mathfrak{u}(\eta_i) + \gamma_1 \mathfrak{u}(T) = \beta_1, \\ \alpha_2 \mathfrak{u}'(0) + \sum_{i=1}^m b_i \mathfrak{u}'(\eta_i) + \gamma_2 \mathfrak{u}'(T) = \beta_2, \\ \alpha_3 \mathfrak{u}''(0) + \sum_{i=1}^m c_i \mathfrak{u}''(\eta_i) + \gamma_3 \mathfrak{u}''(T) = \beta_3, \end{cases}$$

where $q, \alpha_k, \beta_k, \gamma_k \in \mathbb{R}$, $k = 1, \ldots, 3$, $a_j, b_j, c_j \in \mathbb{R}$, $j = 1, \ldots, m$, and $k > 0$. In [13] the authors obtained existence results for the solutions of Caputo fractional sequential integro-differential equation and inclusion

$$({}^cD^\alpha + \mu{}^cD^{\alpha-1})\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), {}^cD^p\mathfrak{u}(\mathfrak{t}), I^q\mathfrak{u}(\mathfrak{t})), \ \ \mathfrak{t} \in (0, 1), \tag{4}$$

$$({}^cD^\alpha + \mu{}^cD^{\alpha-1})\mathfrak{u}(\mathfrak{t}) \in \mathfrak{F}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), I^q\mathfrak{u}(\mathfrak{t})), \ \ \mathfrak{t} \in (0, 1), \tag{5}$$

with the boundary conditions

$$\mathfrak{u}(0) = h(\mathfrak{u}), \ \ \mathfrak{u}'(0) = \mathfrak{u}''(0) = 0, \ \ \mathfrak{a}I^\beta\mathfrak{u}(\xi) = \int_0^1 \mathfrak{u}(s)\, dH(s), \tag{6}$$

where $\alpha \in (3, 4]$, $\xi \in (0, 1]$, $q > 0$, $p \in (0, 1)$, $\mu > 0$, $\mathfrak{a} \in \mathbb{R}$, $\beta > 0$. In the proof of the main results for (4) and (6), they applied the fixed point theorem of Krasnosel'skii for the sum of two operators and the contraction mapping principle, and in the proof of the main theorems obtained for (5) and (6) they used the Covitz-Nadler fixed point theorem and the nonlinear alternative of Leray-Schauder type for Kakutani maps. Relying on some fixed point theorems, in [14], the authors proved the existence of solutions for sequential Caputo fractional differential equation and inclusion

$$\begin{aligned} &({}^cD^q + \mu{}^cD^{q-1})\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), {}^cD^\delta\mathfrak{u}(\mathfrak{t}), I^\gamma\mathfrak{u}(\mathfrak{t})), \ \ \mathfrak{t} \in [0, 1], \\ &({}^cD^q + \mu{}^cD^{q-1})\mathfrak{u}(\mathfrak{t}) \in \mathfrak{F}(\mathfrak{t}, \mathfrak{u}(t), {}^cD^\delta\mathfrak{u}(\mathfrak{t}), I^\gamma\mathfrak{u}(\mathfrak{t})), \ \ \mathfrak{t} \in [0, 1], \end{aligned}$$

with the semi-periodic and integral-multipoint boundary conditions

$$\mathfrak{u}(0) = \mathfrak{u}(1), \ \ \mathfrak{u}'(0) = 0, \ \ \sum_{i=1}^m a_i \mathfrak{u}(\zeta_i) = \lambda I^\beta \mathfrak{u}(\eta),$$

where $0 < \eta < \zeta_1 < \cdots < \zeta_m < 1$, $q \in (2, 3]$, $\mu > 0$, $\delta, \gamma \in (0, 1)$, and $\beta > 0$.

The coupled systems of fractional differential equations appear in many problems of applied nature, mainly in biosciences (see [15] and its references). We will mention below some of these fractional systems related to our problem (1) and (2). By using the Banach contraction mapping principle and the Leray-Schauder alternative, in [15], the authors obtained existence and uniqueness results for the solutions of the nonlinear system of sequential Caputo fractional differential equations

$$\begin{cases} ({}^cD^q + \mu{}^cD^{q-1})\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t})), \ \ \mathfrak{t} \in [0, 1], \\ ({}^cD^p + \mu{}^cD^{p-1})\mathfrak{v}(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t})), \ \ \mathfrak{t} \in [0, 1], \end{cases}$$

supplemented with the integral boundary conditions

$$\mathfrak{u}(0) = \mathfrak{u}'(0) = 0, \ \ \mathfrak{u}(\zeta) = aI^\beta\mathfrak{u}(\eta), \ \ \mathfrak{v}(0) = \mathfrak{v}'(0) = 0, \ \ \mathfrak{v}(z) = bI^\gamma\mathfrak{v}(\theta),$$

where $p,\ q \in (2,3],\ \mu > 0,\ \beta,\gamma > 0,\ 0 < \theta < z < 1,\ 0 < \eta < \zeta < 1$. In [16], the authors proved the existence and uniqueness results for the solutions of the system of Hadamard type fractional sequential differential equations subject to nonlocal coupled strip conditions

$$\begin{cases} ({}^HD^q + \mu {}^HD^{q-1})\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), {}^HD^{\alpha}\mathfrak{v}(\mathfrak{t})),\ \ \mathfrak{t} \in (1,e), \\ ({}^HD^p + \mu {}^HD^{p-1})\mathfrak{v}(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), {}^HD^{\delta}\mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t})),\ \ \mathfrak{t} \in (1,e), \\ \mathfrak{u}(1) = 0,\ \ \mathfrak{u}(e) = {}^H I^{\gamma}\mathfrak{v}(\eta),\ \ \mathfrak{v}(1) = 0,\ \ \mathfrak{v}(e) = {}^H I^{\beta}\mathfrak{u}(\zeta), \end{cases}$$

where $\mu > 0,\ q \in (1,2],\ \alpha \in (0,1),\ p \in (1,2],\ \delta \in (0,1),\ \gamma > 0,\ \eta \in (1,e),\ \beta > 0,\ \zeta \in (1,e)$, ${}^HD^{(\cdot)}$ is the Hadamard fractional derivative and ${}^HI^{(\cdot)}$ is Hadamard fractional integral. In [17], the authors investigated the system of sequential Caputo fractional integro-differential equations

$$\begin{cases} ({}^cD^q + \mu {}^cD^{q-1})\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), {}^cD^{\alpha}\mathfrak{v}(\mathfrak{t}), I^{\alpha_1}\mathfrak{v}(\mathfrak{t})),\ \mathfrak{t} \in [0,1], \\ ({}^cD^p + \mu {}^cD^{p-1})\mathfrak{v}(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), {}^cD^{\delta}\mathfrak{u}(\mathfrak{t}), I^{\delta_1}\mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t})),\ \mathfrak{t} \in [0,1], \end{cases}$$

with the coupled Riemann-Liouville integral boundary conditions

$$\begin{cases} \mathfrak{u}(0) = 0,\ \mathfrak{u}'(0) = 0,\ a_1\mathfrak{u}(1) + a_2\mathfrak{u}(\zeta) = aI^{\beta}\mathfrak{v}(\eta), \\ \mathfrak{v}(0) = 0,\ \mathfrak{v}'(0) = 0,\ b_1\mathfrak{v}(1) + b_2\mathfrak{v}(z) = bI^{\gamma}\mathfrak{u}(\theta), \end{cases}$$

where $\mu > 0,\ p, q \in (2,3],\ \alpha, \alpha_1 \in (0,1),\ \delta, \delta_1 \in (0,1),\ \beta, \gamma > 0,\ z,\theta,\zeta,\eta \in (0,1)$, and $a,\ a_i,\ b,\ b_i \in \mathbb{R},\ i = 1,2$. Our problem (1) and (2) generalizes the problem investigated in [18]. In the paper [18] the authors obtained existence results for the solutions of the system of Caputo fractional differential equations

$$\begin{cases} ({}^cD^{\mathfrak{q}+1} + {}^cD^{\mathfrak{q}})x(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t})),\ \ \mathfrak{t} \in (0,1), \\ ({}^cD^{\mathfrak{p}+1} + {}^cD^{\mathfrak{p}})y(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t})),\ \ \mathfrak{t} \in (0,1), \end{cases} \tag{7}$$

with the coupled integral boundary conditions

$$\begin{cases} x(0) = x'(0) = 0,\ \ x'(1) = 0, \\ x(1) = k\int_0^{\rho} y(s)\,dA(s) + \sum_{i=1}^{n-2}\alpha_i y(\sigma_i) + k_1\int_{\nu}^1 y(s)\,dA(s), \\ y(0) = y'(0) = 0,\ \ y'(1) = 0, \\ y(1) = h\int_0^{\rho} x(s)\,dA(s) + \sum_{i=1}^{n-2}\beta_i x(\sigma_i) + h_1\int_{\nu}^1 x(s)\,dA(s), \end{cases} \tag{8}$$

where $\mathfrak{p},\ \mathfrak{q} \in (2,3],\ 0 < \rho < \sigma_i < \nu < 1,\ k, k_1, h, h_1, \alpha_i, \beta_i \in \mathbb{R},\ i = 1,2,\ldots,n-2,\ \mathfrak{f},\ \mathfrak{g} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and the function A has bounded variation. The condition from the second line of (8) can be written as $x(1) = \int_0^1 y(s)\,d\widetilde{A}(s)$, where $\widetilde{A}$ is given by

$$\widetilde{A}(s) = \begin{cases} kA(s),\ \ s \in [0,\rho], \\ 0,\ \ s \in (\rho,\sigma_1], \\ \alpha_1,\ \ s \in (\sigma_1,\sigma_2], \\ \alpha_1 + \alpha_2,\ \ s \in (\sigma_2,\sigma_3], \\ \vdots \\ \alpha_1 + \cdots + \alpha_{n-3},\ \ s \in (\sigma_{n-3},\sigma_{n-2}], \\ \alpha_1 + \cdots + \alpha_{n-2},\ \ s \in (\sigma_{n-2},\nu), \\ k_1A(s),\ \ s \in [\nu,1]. \end{cases}$$

In a similar way, the last condition from (8) can be written as $y(1) = \int_0^1 x(s)\, d\widetilde{B}(s)$, where $\widetilde{B}$ is given by

$$\widetilde{B}(s) = \begin{cases} hA(s), & s \in [0, \rho], \\ 0, & s \in (\rho, \sigma_1], \\ \beta_1, & s \in (\sigma_1, \sigma_2], \\ \beta_1 + \beta_2, & s \in (\sigma_2, \sigma_3], \\ \vdots \\ \beta_1 + \cdots + \beta_{n-3}, & s \in (\sigma_{n-3}, \sigma_{n-2}], \\ \beta_1 + \cdots + \beta_{n-2}, & s \in (\sigma_{n-2}, \nu), \\ h_1 A(s), & s \in [\nu, 1]. \end{cases}$$

So we see that in the boundary conditions (8), x in the point 1 is dependent only of function y, and y in the point 1 is dependent only of function x. In our boundary conditions (2), $x(1)$ and $y(1)$ are dependent on both the functions x and y. In addition, in our system (1) the nonlinearities $\mathfrak{f}$ and $\mathfrak{g}$ are dependent on some integral terms, and in (7) there is no such dependence. We also have in (1) two parameters λ and μ, and in (7) there are no parameters. Another paper connected with our problem (1) and (2) is [19]. In [19] the authors investigated the system of nonlinear Caputo fractional integro-differential equations

$$\begin{cases} ({}^c\mathcal{D}^{\alpha} + \lambda\, {}^c\mathcal{D}^{\alpha-1})x(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), {}^cD^{p_1}y(\mathfrak{t}), I_{0+}^{q_1}y(\mathfrak{t})), & \mathfrak{t} \in (0,1), \\ ({}^c\mathcal{D}^{\beta} + \mu\, {}^c\mathcal{D}^{\beta-1})y(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}, x(\mathfrak{t}), {}^cD^{p_2}x(\mathfrak{t}), I_{0+}^{q_2}x(\mathfrak{t}), y(\mathfrak{t})), & \mathfrak{t} \in (0,1), \end{cases} \tag{9}$$

with the coupled boundary conditions

$$\begin{cases} x(0) = x'(0) = x''(0) = 0, \ \ x(1) = \int_0^1 x(s)\, d\mathfrak{H}_1(s) + \int_0^1 y(s)\, d\mathfrak{H}_2(s), \\ y(0) = y'(0) = y''(0) = 0, \ \ y(1) = \int_0^1 x(s)\, d\mathfrak{K}_1(s) + \int_0^1 y(s)\, d\mathfrak{K}_2(s), \end{cases} \tag{10}$$

where $p_1,\, p_2 \in (0,1)$, $q_1,\, q_2 > 0$, $\alpha,\, \beta \in (3,4]$, $\lambda,\, \mu > 0$. We see that there are differences between our problem (1) and (2) and the above problem (9) and (10) related to the dependence of the nonlinearities $\mathfrak{f}$ and $\mathfrak{g}$ on various fractional derivatives and integrals. Another important difference between these two problems is given by the conditions $x'(1) = y'(1) = 0$ in (2), and $x''(0) = y''(0) = 0$ in (10), which conducts us to differences between the associated integral operators. So the novelty of our problem (1) and (2) is given by the existence of the integral terms and different positive parameters in the system (1), and by the general coupled integral boundary conditions (2). For new results obtained in recent years and for the applications of fractional calculus and fractional differential equations in varied fields, we recall the books [20–30] and their references.

The paper is arranged as follows. In Section 2 we investigate a linear fractional boundary value problem which is associated to our problem (1) and (2). Section 3 is concerned with the main existence results for (1) and (2), and in Section 4 we present two examples illustrating our results. Finally, in Section 5 we give the conclusions of our paper.

2. Auxiliary Results

In this section we study the system of linear fractional differential equations

$$\begin{cases} ({}^c\mathcal{D}^{\alpha} + \lambda\, {}^c\mathcal{D}^{\alpha-1})x(\mathfrak{t}) = h(\mathfrak{t}), & \mathfrak{t} \in (0,1), \\ ({}^c\mathcal{D}^{\beta} + \mu\, {}^c\mathcal{D}^{\beta-1})y(\mathfrak{t}) = k(\mathfrak{t}), & \mathfrak{t} \in (0,1), \end{cases} \tag{11}$$

supplemented with the boundary conditions (2), with h, $k \in C[0,1]$. We denote by

$$\begin{aligned}
A_1 &= \frac{1}{\lambda}(1-e^{-\lambda}), \;\; A_2 = \frac{1}{\lambda^2}(2\lambda - 2 + 2e^{-\lambda}),\\
A_3 &= \frac{1}{\mu}(1-e^{-\mu}), \;\; A_4 = \frac{1}{\mu^2}(2\mu - 2 + 2e^{-\mu}),\\
A_5 &= \frac{1}{\lambda^2}(\lambda - 1 + e^{-\lambda}) - \frac{1}{\lambda^2}\int_0^1 (\lambda s - 1 + e^{-\lambda s})\, d\mathfrak{H}_1(s),\\
A_6 &= \frac{1}{\lambda^3}(\lambda^2 - 2\lambda + 2 - 2e^{-\lambda}) - \frac{1}{\lambda^3}\int_0^1 (\lambda^2 s^2 - 2\lambda s + 2 - 2e^{-\lambda s})\, d\mathfrak{H}_1(s),\\
A_7 &= \frac{1}{\mu^2}\int_0^1 (\mu s - 1 + e^{-\mu s})\, d\mathfrak{H}_2(s),\\
A_8 &= \frac{1}{\mu^3}\int_0^1 (\mu^2 s^2 - 2\mu s + 2 - 2e^{-\mu s})\, d\mathfrak{H}_2(s),\\
A_9 &= \frac{1}{\lambda^2}\int_0^1 (\lambda s - 1 - e^{-\lambda s})\, d\mathfrak{K}_1(s),\\
A_{10} &= \frac{1}{\lambda^3}\int_0^1 (\lambda^2 s^2 - 2\lambda s + 2 - 2e^{-\lambda s})\, d\mathfrak{K}_1(s),\\
A_{11} &= \frac{1}{\mu^2}(\mu - 1 + e^{-\mu}) - \frac{1}{\mu^2}\int_0^1 (\mu s - 1 + e^{-\mu s})\, d\mathfrak{K}_2(s),\\
A_{12} &= \frac{1}{\mu^3}(\mu^2 - 2\mu + 2 - 2e^{-\mu}) - \frac{1}{\mu^3}\int_0^1 (\mu^2 s^2 - 2\mu s + 2 - 2e^{-\mu s})\, d\mathfrak{K}_2(s),
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
E_1 &= \lambda \int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} h(s)\, ds - I_{0+}^{\alpha-1} h(1),\\
E_2 &= \mu \int_0^1 e^{-\mu(1-s)} I_{0+}^{\beta-1} k(s)\, ds - I_{0+}^{\beta-1} k(1),\\
E_3 &= -\int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} h(s)\, ds + \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta\right) d\mathfrak{H}_1(s)\\
&\quad + \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta\right) d\mathfrak{H}_2(s),\\
E_4 &= -\int_0^1 e^{-\mu(1-s)} I_{0+}^{\beta-1} k(s)\, ds + \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta\right) d\mathfrak{K}_1(s)\\
&\quad + \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta\right) d\mathfrak{K}_2(s),\\
\Delta_1 &= (A_2A_5 - A_1A_6)(A_4A_{11} - A_3A_{12}) - (A_2A_9 - A_1A_{10})(A_4A_7 - A_3A_8),\\
\Delta &= A_2A_4\Delta_1.
\end{aligned} \tag{13}$$

Lemma 1. *If $\Delta_1 \neq 0$, then the solution $(x,y) \in (C^4[0,1])^2$ of the boundary value problem (11) and (2) is given by*

$$\begin{cases} x(t) = \displaystyle\int_0^t e^{-\lambda(t-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta + \sum_{i=1}^4 S_i(t)E_i, \;\; t \in [0,1],\\ y(t) = \displaystyle\int_0^t e^{-\mu(t-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta + \sum_{i=1}^4 T_i(t)E_i, \;\; t \in [0,1], \end{cases} \tag{14}$$

where

$$\begin{aligned}
S_i(t) &= \Lambda_i \frac{1}{\lambda^2}(\lambda t - 1 + e^{-\lambda t}) + \Theta_i \frac{1}{\lambda^3}(\lambda^2 t^2 - 2\lambda t + 2 - 2e^{-\lambda t}), \;\; i = 1,\ldots,4,\\
T_i(t) &= \Xi_i \frac{1}{\mu^2}(\mu t - 1 + e^{-\mu t}) + \Upsilon_i \frac{1}{\mu^3}(\mu^2 t^2 - 2\mu t + 2 - 2e^{-\mu t}), \;\; i = 1,\ldots,4,
\end{aligned} \tag{15}$$

and

$$
\begin{aligned}
&\Lambda_1 = \frac{1}{\Delta_1}[-A_6(A_4A_{11} - A_3A_{12}) + A_{10}(A_4A_7 - A_3A_8)], \;\; \Lambda_2 = \frac{A_2}{\Delta_1}(A_8A_{11} - A_7A_{12}),\\
&\Lambda_3 = \frac{A_2}{\Delta_1}(A_4A_{11} - A_3A_{12}), \;\; \Lambda_4 = \frac{A_2}{\Delta_1}(A_4A_7 - A_3A_8),\\
&\Theta_1 = \frac{1}{\Delta_1}[A_5(A_4A_{11} - A_3A_{12}) - A_9(A_4A_7 - A_3A_8)], \;\; \Theta_2 = \frac{A_1}{\Delta_1}(-A_8A_{11} + A_7A_{12}),\\
&\Theta_3 = \frac{A_1}{\Delta_1}(A_3A_{12} - A_4A_{11}), \;\; \Theta_4 = \frac{A_1}{\Delta_1}(A_3A_8 - A_4A_7),\\
&\Xi_1 = \frac{A_4}{\Delta_1}(A_5A_{10} - A_6A_9), \;\; \Xi_2 = \frac{1}{\Delta_1}[-A_{12}(A_2A_5 - A_1A_6) + A_8(A_2A_9 - A_1A_{10})],\\
&\Xi_3 = \frac{A_4}{\Delta_1}(A_2A_9 - A_1A_{10}), \;\; \Xi_4 = \frac{A_4}{\Delta_1}(A_2A_5 - A_1A_6),\\
&\Upsilon_1 = \frac{A_3}{\Delta_1}(A_6A_9 - A_5A_{10}), \;\; \Upsilon_2 = \frac{1}{\Delta_1}[A_{11}(A_2A_5 - A_1A_6) - A_7(A_2A_9 - A_1A_{10})],\\
&\Upsilon_3 = \frac{A_3}{\Delta_1}(A_1A_{10} - A_2A_9), \;\; \Upsilon_4 = \frac{A_3}{\Delta_1}(A_1A_6 - A_2A_5).
\end{aligned} \tag{16}
$$

Proof. The system (11) can be equivalently written as the following system

$$
\begin{cases} {}^cD^{\alpha}(x(t) + \lambda\, {}^cD^{-1}x(t)) = h(t), \;\; t \in (0,1),\\ {}^cD^{\beta}(y(t) + \mu\, {}^cD^{-1})y(t) = k(t), \;\; t \in (0,1), \end{cases} \tag{17}
$$

where ${}^c D^{-1}$ is the integral operator I. The general solution of system (17) is

$$
\begin{cases} x(t) + \lambda^c D^{-1}x(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\zeta)^{\alpha-1}h(\zeta)\, d\zeta + a_0 + a_1 t + a_2 t^2 + a_3 t^3, \;\; t \in [0,1],\\ y(t) + \mu^c D^{-1}y(t) = \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-\zeta)^{\beta-1}k(\zeta)\, d\zeta + b_0 + b_1 t + b_2 t^2 + b_3 t^3, \;\; t \in [0,1], \end{cases}
$$

or equivalently

$$
\begin{cases} x(t) = -\lambda \displaystyle\int_0^t x(\zeta)\, d\zeta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\zeta)^{\alpha-1}h(\zeta)\, d\zeta + a_0 + a_1 t + a_2 t^2 + a_3 t^3, \;\; t \in [0,1],\\ y(t) = -\mu \displaystyle\int_0^t y(\zeta)\, d\zeta + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-\zeta)^{\beta-1}k(\zeta)\, d\zeta + b_0 + b_1 t + b_2 t^2 + b_3 t^3 \;\; t \in [0,1], \end{cases}
$$

with $a_i,\, b_i \in \mathbb{R}, \;\; i = 0, \ldots, 3$. We differentiate the above relations and we find

$$
\begin{cases} x'(t) = -\lambda x(t) + \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-\zeta)^{\alpha-2}h(\zeta)\, d\zeta + a_1 + 2a_2 t + 3a_3 t^2,\\ y'(t) = -\mu y(t) + \dfrac{1}{\Gamma(\beta-1)} \displaystyle\int_0^t (t-\zeta)^{\beta-2}k(\zeta)\, d\zeta + b_1 + 2b_2 t + 3b_3 t^2, \end{cases}
$$

and then

$$
\begin{cases} (e^{\lambda t}x(t))' = \dfrac{e^{\lambda t}}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-s)^{\alpha-2}h(s)\, ds + a_1 e^{\lambda t} + 2a_2 t e^{\lambda t} + 3a_3 t^2 e^{\lambda t},\\ (e^{\mu t}y(t))' = \dfrac{e^{\mu t}}{\Gamma(\beta-1)} \displaystyle\int_0^t (t-s)^{\beta-2}k(s)\, ds + b_1 e^{\mu t} + 2b_2 t e^{\mu t} + 3b_3 t^2 e^{\mu t}. \end{cases}
$$

We integrate now the above relations and we obtain

$$
\begin{aligned}
x(t) &= c_0 e^{-\lambda t} + \frac{c_1}{\lambda}(-e^{-\lambda t} + 1) + \frac{c_2}{\lambda^2}(-1 + \lambda t + e^{-\lambda t}) \\
&\quad + \frac{c_3}{\lambda^3}(\lambda^2 t^2 - 2\lambda t + 2 - 2e^{-\lambda t}) + e^{-\lambda t}\int_0^t e^{\lambda s} I_{0+}^{\alpha-1} h(s)\,ds, \\
y(t) &= d_0 e^{-\mu t} + \frac{d_1}{\mu}(-e^{-\mu t} + 1) + \frac{d_2}{\mu^2}(-1 + \mu t + e^{-\mu t}) \\
&\quad + \frac{d_3}{\mu^3}(\mu^2 t^2 - 2\mu t + 2 - 2e^{-\mu t}) + e^{-\mu t}\int_0^t e^{\mu s} I_{0+}^{\beta-1} k(s)\,ds,
\end{aligned}
$$

with $c_0 = x(0)$, $c_1 = a_1$, $c_2 = 2a_2$, $c_3 = 3a_3$, $d_0 = y(0)$, $d_1 = b_1$, $d_2 = 2b_2$, $d_3 = 3b_3$.

Using the boundary conditions $x(0) = x'(0) = 0$, $y(0) = y'(0) = 0$ from (2), we deduce that $c_0 = c_1 = 0$ and $d_0 = d_1 = 0$. So we conclude

$$
\begin{cases}
x(t) = \dfrac{c_2}{\lambda^2}(\lambda t - 1 + e^{-\lambda t}) + \dfrac{c_3}{\lambda^3}(\lambda^2 t^2 - 2\lambda t + 2 - 2e^{-\lambda t}) + \displaystyle\int_0^t e^{-\lambda(t-s)} I_{0+}^{\alpha-1} h(s)\,ds, \\
y(t) = \dfrac{d_2}{\mu^2}(\mu t - 1 + e^{-\mu t}) + \dfrac{d_3}{\mu^3}(\mu^2 t^2 - 2\mu t + 2 - 2e^{-\mu t}) + \displaystyle\int_0^t e^{-\mu(t-s)} I_{0+}^{\beta-1} k(s)\,ds.
\end{cases}
\tag{18}
$$

We differentiate the system (18) and we find

$$
\begin{cases}
x'(t) = \dfrac{c_2}{\lambda^2}(\lambda - \lambda e^{-\lambda t}) + \dfrac{c_3}{\lambda^3}(2\lambda^2 t - 2\lambda + 2\lambda e^{-\lambda t}) \\
\qquad -\lambda \displaystyle\int_0^t e^{-\lambda(t-s)} I_{0+}^{\alpha-1} h(s)\,ds + I_{0+}^{\alpha-1} h(t), \\
y'(t) = \dfrac{d_2}{\mu^2}(\mu - \mu e^{-\mu t}) + \dfrac{d_3}{\mu^3}(2\mu^2 t - 2\mu + 2\mu e^{-\mu t}) \\
\qquad -\mu \displaystyle\int_0^t e^{-\mu(t-s)} I_{0+}^{\beta-1} k(s)\,ds + I_{0+}^{\beta-1} k(t).
\end{cases}
$$

By imposing the conditions $x'(1) = y'(1) = 0$ (from (2)), we obtain

$$
\begin{cases}
\dfrac{c_2}{\lambda}(1 - e^{-\lambda}) + \dfrac{c_3}{\lambda^2}(2\lambda - 2 + 2e^{-\lambda}) = \lambda \displaystyle\int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} h(s)\,ds - I_{0+}^{\alpha-1} h(1), \\
\dfrac{d_2}{\mu}(1 - e^{-\mu}) + \dfrac{d_3}{\mu^2}(2\mu - 2 + 2e^{-\mu}) = \mu \displaystyle\int_0^1 e^{-\mu(1-s)} I_{0+}^{\beta-1} k(s)\,ds - I_{0+}^{\beta-1} k(1).
\end{cases}
\tag{19}
$$

Now using the last boundary conditions from (2), namely $x(1) = \int_0^1 x(s)\,d\mathfrak{H}_1(s) + \int_0^1 y(s)\,d\mathfrak{H}_2(s)$ and $y(1) = \int_0^1 x(s)\,d\mathfrak{K}_1(s) + \int_0^1 y(s)\,d\mathfrak{K}_2(s)$, by (18) we deduce

$$
\begin{aligned}
& c_2\left[\frac{1}{\lambda^2}(\lambda - 1 + e^{-\lambda}) - \frac{1}{\lambda^2}\int_0^1 (\lambda s - 1 + e^{-\lambda s})\,d\mathfrak{H}_1(s)\right] \\
& + c_3\left[\frac{1}{\lambda^3}(\lambda^2 - 2\lambda + 2 - 2e^{-\lambda}) - \frac{1}{\lambda^3}\int_0^1 (\lambda^2 s^2 - 2\lambda s + 2 - 2e^{-\lambda s})\,d\mathfrak{H}_1(s)\right] \\
& - d_2 \frac{1}{\mu^2}\int_0^1 (\mu s - 1 + e^{-\mu s})\,d\mathfrak{H}_2(s) \\
& - d_3 \frac{1}{\mu^3}\int_0^1 (\mu^2 s^2 - 2\mu s + 2 - 2e^{-\mu s})\,d\mathfrak{H}_2(s) \\
& = -\int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} h(s)\,ds + \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\,d\zeta\right) d\mathfrak{H}_1(s) \\
& + \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} k(\zeta)\,d\zeta\right) d\mathfrak{H}_2(s),
\end{aligned}
$$

$$
\begin{aligned}
&-c_2\frac{1}{\lambda^2}\int_0^1(\lambda s-1-e^{-\lambda s})\,d\mathfrak{K}_1(s)\\
&-c_3\frac{1}{\lambda^3}\int_0^1(\lambda^2s^2-2\lambda s+2-2e^{-\lambda s})\,d\mathfrak{K}_1(s)\\
&+d_2\left[\frac{1}{\mu^2}(\mu-1+e^{-\mu})-\frac{1}{\mu^2}\int_0^1(\mu s-1+e^{-\mu s})\,d\mathfrak{K}_2(s)\right]\\
&+d_3\left[\frac{1}{\mu^3}(\mu^2-2\mu+2-2e^{-\mu})-\frac{1}{\mu^3}\int_0^1(\mu^2s^2-2\mu s-2e^{-\mu s}+2)\,d\mathfrak{K}_2(s)\right]\\
&=-\int_0^1 e^{-\mu(1-s)}I_{0+}^{\beta-1}k(s)\,ds+\int_0^1\left(\int_0^s e^{-\lambda(s-\zeta)}I_{0+}^{\alpha-1}h(\zeta)\,d\zeta\right)d\mathfrak{K}_1(s)\\
&+\int_0^1\left(\int_0^s e^{-\mu(s-\zeta)}I_{0+}^{\beta-1}k(\zeta)\,d\zeta\right)d\mathfrak{K}_2(s).
\end{aligned}
\tag{20}
$$

Therefore by (19), (20), (12) and (13) we find the system in the unknowns c_2, c_3, d_2 and d_3:

$$
\begin{cases}
A_1c_2+A_2c_3=E_1,\\
A_3d_2+A_4d_3=E_2,\\
A_5c_2+A_6c_3-A_7d_2-A_8d_3=E_3,\\
-A_9c_2-A_{10}c_3+A_{11}d_2+A_{12}d_3=E_4.
\end{cases}
\tag{21}
$$

By the first two equations of (21) we find $c_3=\frac{E_1-A_1c_2}{A_2}$ and $d_3=\frac{E_2-A_3d_2}{A_4}$, $(A_2,\ A_4>0)$. Introducing these values of c_3 and d_3 in the last two equations of (21) we deduce the system in the unknowns c_2 and d_2:

$$
\begin{cases}
c_2A_4(A_2A_5-A_1A_6)-d_2A_2(A_4A_7-A_3A_8)=A_2A_4E_3-A_4A_6E_1+A_2A_8E_2,\\
-c_2A_4(A_2A_9-A_1A_{10})+d_2A_2(A_4A_{11}-A_3A_{12})=A_2A_4E_4+A_4A_{10}E_1-A_2A_{12}E_2.
\end{cases}
\tag{22}
$$

The determinant of the system (22) is $\Delta=A_2A_4\Delta_1$, where Δ_1 is given by (13). By assumption of this lemma, $\Delta_1\neq 0$, and then $\Delta\neq 0$. Therefore the solution of system (22) is

$$
\begin{aligned}
c_2=&\frac{A_2}{\Delta}\{E_1[-A_4A_6(A_4A_{11}-A_3A_{12})+A_4A_{10}(A_4A_7-A_3A_8)]\\
&+E_2[A_2A_8(A_4A_{11}-A_3A_{12})-A_2A_{12}(A_4A_7-A_3A_8)]\\
&+E_3A_2A_4(A_4A_{11}-A_3A_{12})+E_4A_2A_4(A_4A_7-A_3A_8)\}\\
=&\Lambda_1E_1+\Lambda_2E_2+\Lambda_3E_3+\Lambda_4E_4,\\
d_2=&\frac{A_4}{\Delta}\{E_1[A_4A_{10}(A_2A_5-A_1A_6)-A_4A_6(A_2A_9-A_1A_{10})]\\
&+E_2[-A_2A_{12}(A_2A_5-A_1A_6)+A_2A_8(A_2A_9-A_1A_{10})]\\
&+E_3A_2A_4(A_2A_9-A_1A_{10})+E_4A_2A_4(A_2A_5-A_1A_6)\}\\
=&\Xi_1E_1+\Xi_2E_2+\Xi_3E_3+\Xi_4E_4,
\end{aligned}
$$

where $\Lambda_i,\ \Xi_i,\ i=1,\ldots,4$ are given by (16). Therefore for the constants c_3 and d_3 we obtain

$$
\begin{aligned}
c_3=&\frac{1}{A_2\Delta}\{E_1A_2A_4[(A_2A_5-A_1A_6)(A_4A_{11}-A_3A_{12})\\
&-(A_2A_9-A_1A_{10})(A_4A_7-A_3A_8)]\\
&-E_1A_1A_2[-A_4A_6(A_4A_{11}-A_3A_{12})+A_4A_{10}(A_4A_7-A_3A_8)]\\
&-A_1A_2E_2[A_2A_8(A_4A_{11}-A_3A_{12})-A_2A_{12}(A_4A_7-A_3A_8)]\\
&-E_3A_1A_2^2A_4(A_4A_{11}-A_3A_{12})-E_4A_1A_2^2A_4(A_4A_7-A_3A_8)\}\\
=&\Theta_1E_1+\Theta_2E_2+\Theta_3E_3+\Theta_4E_4,\\
d_3=&\frac{1}{A_4\Delta}\{-E_1A_3A_4[A_4A_{10}(A_2A_5-A_1A_6)-A_4A_6(A_2A_9-A_1A_{10})]\\
&+E_2A_2A_4[(A_2A_5-A_1A_6)(A_4A_{11}-A_3A_{12})-(A_2A_9-A_1A_{10})(A_4A_7-A_3A_8)]\\
&-E_2A_3A_4[-A_2A_{12}(A_2A_5-A_1A_6)+A_2A_8(A_2A_9-A_1A_{10})]\\
&-E_3A_2A_3A_4^2(A_2A_9-A_1A_{10})-E_4A_2A_3A_4^2(A_2A_5-A_1A_6)\}\\
=&\Upsilon_1E_1+\Upsilon_2E_2+\Upsilon_3E_3+\Upsilon_4E_4,
\end{aligned}
$$

where Θ_i, Υ_i, $i = 1, \ldots, 4$ are given by (16).

By replacing the above constants c_2, d_2, c_3 and d_3 in the system (18), we find the solution of problem (11) and (2), namely

$$\begin{aligned} x(\mathfrak{t}) &= \int_0^{\mathfrak{t}} e^{-\lambda(\mathfrak{t}-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta + \sum_{i=1}^{4} \bigg[\Lambda_i \frac{1}{\lambda^2}(-1+\lambda\mathfrak{t}+e^{-\lambda\mathfrak{t}}) \\ &\quad + \Theta_i \frac{1}{\lambda^3}(\lambda^2\mathfrak{t}^2 - 2\lambda\mathfrak{t} - 2e^{-\lambda\mathfrak{t}} + 2) \bigg] E_i \\ &= \int_0^{\mathfrak{t}} e^{-\lambda(\mathfrak{t}-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta + \sum_{i=1}^{4} S_i(\mathfrak{t}) E_i, \\ y(\mathfrak{t}) &= \int_0^{\mathfrak{t}} e^{-\mu(\mathfrak{t}-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta + \sum_{i=1}^{4} \bigg[\Xi_i \frac{1}{\mu^2}(-1+\mu\mathfrak{t}+e^{-\mu\mathfrak{t}}) \\ &\quad + \Upsilon_i \frac{1}{\mu^3}(\mu^2\mathfrak{t}^2 - 2\mu\mathfrak{t} - 2e^{-\mu\mathfrak{t}} + 2) \bigg] E_i \\ &= \int_0^{\mathfrak{t}} e^{-\mu(\mathfrak{t}-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta + \sum_{i=1}^{4} T_i(\mathfrak{t}) E_i, \end{aligned}$$

where S_i, T_i, $i = 1, \ldots, 4$ are given by (15). The converse of this result is obtained by direct computations. □

Lemma 2. *([19]) For $h, k \in C[0,1]$ with $\|h\| = \sup_{\zeta\in[0,1]} |h(\zeta)|$ and $\|k\| = \sup_{\zeta\in[0,1]} |k(\zeta)|$, the following inequalities hold:*

(a) $|I_{0+}^{\alpha-1} h(\mathfrak{t})| \le \frac{1}{\Gamma(\alpha)} \mathfrak{t}^{\alpha-1} \|h\| \le \frac{\|h\|}{\Gamma(\alpha)}, \quad \forall\, \mathfrak{t} \in [0,1];$

(b) $|I_{0+}^{\beta-1} k(\mathfrak{t})| \le \frac{1}{\Gamma(\beta)} \mathfrak{t}^{\beta-1} \|k\| \le \frac{\|k\|}{\Gamma(\beta)}, \quad \forall\, \mathfrak{t} \in [0,1];$

(c) $\left| \int_0^1 e^{-\lambda(1-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta \right| \le \frac{1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda})\|h\|;$

(d) $\left| \int_0^1 e^{-\mu(1-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta \right| \le \frac{1}{\mu\Gamma(\beta)}(1-e^{-\mu})\|k\|;$

(e) $\left| \int_0^{\mathfrak{t}} e^{-\lambda(\mathfrak{t}-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta \right| \le \frac{1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda})\|h\|, \ \forall\, \mathfrak{t} \in [0,1];$

(f) $\left| \int_0^{\mathfrak{t}} e^{-\mu(\mathfrak{t}-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta \right| \le \frac{1}{\mu\Gamma(\beta)}(1-e^{-\mu})\|k\|, \ \forall\, \mathfrak{t} \in [0,1];$

(g) $\left| \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta \right) d\mathfrak{H}_1(s) \right| \le \frac{\|h\|}{\lambda\Gamma(\alpha)} \left| \int_0^1 s^{\alpha-1}(1-e^{-\lambda s}) d\mathfrak{H}_1(s) \right|;$

(h) $\left| \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} h(\zeta)\, d\zeta \right) d\mathfrak{K}_1(s) \right| \le \frac{\|h\|}{\lambda\Gamma(\alpha)} \left| \int_0^1 s^{\alpha-1}(1-e^{-\lambda s}) d\mathfrak{K}_1(s) \right|;$

(i) $\left| \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta \right) d\mathfrak{H}_2(s) \right| \le \frac{\|k\|}{\mu\Gamma(\beta)} \left| \int_0^1 s^{\beta-1}(1-e^{-\mu s}) d\mathfrak{H}_2(s) \right|;$

(j) $\left| \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} k(\zeta)\, d\zeta \right) d\mathfrak{K}_2(s) \right| \le \frac{\|k\|}{\mu\Gamma(\beta)} \left| \int_0^1 s^{\beta-1}(1-e^{-\mu s}) d\mathfrak{K}_2(s) \right|.$

3. Main Results

We introduce the Banach space $\mathcal{X} = C[0,1]$ equipped with the supremum norm $\|\mathfrak{u}\| = \sup_{\zeta\in[0,1]} |\mathfrak{u}(\zeta)|$, for $\mathfrak{u} \in \mathcal{X}$, and $\mathcal{Y} = \mathcal{X} \times \mathcal{X}$ with the norm $\|(\mathfrak{u}, \mathfrak{v})\|_{\mathcal{Y}} = \|\mathfrak{u}\| + \|\mathfrak{v}\|$ for $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{Y}$.

By using Lemma 1, we define the operator $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$ by $\mathcal{A}(x,y) = (\mathcal{A}_1(x,y), \mathcal{A}_2(x,y))$, where the operators $\mathcal{A}_1 : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{A}_2 : \mathcal{Y} \to \mathcal{X}$ are given by

$$\begin{aligned}
\mathcal{A}_1(x,y)(\mathfrak{t}) &= \int_0^{\mathfrak{t}} e^{-\lambda(\mathfrak{t}-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta \\
&+ S_1(\mathfrak{t}) \left[\lambda \int_0^1 e^{-\lambda(1-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta - I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(1) \right] \\
&+ S_2(\mathfrak{t}) \left[\mu \int_0^1 e^{-\mu(1-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta - I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(1) \right] \\
&+ S_3(\mathfrak{t}) \left[- \int_0^1 e^{-\lambda(1-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta \right. \\
&+ \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta \right) d\mathfrak{H}_1(s) \\
&\left. + \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta \right) d\mathfrak{H}_2(s) \right] \\
&+ S_4(\mathfrak{t}) \left[- \int_0^1 e^{-\mu(1-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta \right. \\
&+ \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta \right) d\mathfrak{K}_1(s) \\
&\left. + \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta \right) d\mathfrak{K}_2(s) \right], \quad \mathfrak{t} \in [0,1], \\
\mathcal{A}_2(x,y)(\mathfrak{t}) &= \int_0^{\mathfrak{t}} e^{-\mu(\mathfrak{t}-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta \\
&+ T_1(\mathfrak{t}) \left[\lambda \int_0^1 e^{-\lambda(1-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta - I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(1) \right] \\
&+ T_2(\mathfrak{t}) \left[\mu \int_0^1 e^{-\mu(1-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta - I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(1) \right] \\
&+ T_3(\mathfrak{t}) \left[- \int_0^1 e^{-\lambda(1-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta \right. \\
&+ \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta \right) d\mathfrak{H}_1(s) \\
&\left. + \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta \right) d\mathfrak{H}_2(s) \right] \\
&+ T_4(\mathfrak{t}) \left[- \int_0^1 e^{-\mu(1-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta \right. \\
&+ \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\zeta)\, d\zeta \right) d\mathfrak{K}_1(s) \\
&\left. + \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\, d\zeta \right) d\mathfrak{K}_2(s) \right], \quad \mathfrak{t} \in [0,1],
\end{aligned}$$

for $(x,y) \in \mathcal{Y}$, with $\mathfrak{f}_{x,y}^{p_1,p_2}(s) = \mathfrak{f}(s, x(s), y(s), I_{0+}^{p_1} x(s), I_{0+}^{p_2} y(s))$ and $\mathfrak{g}_{x,y}^{q_1,q_2}(s) = \mathfrak{g}(s, x(s), y(s), I_{0+}^{q_1} x(s), I_{0+}^{q_2} y(s))$ for $s \in [0,1]$, and $S_i,\ T_i,\ i = 1, \ldots, 4$ are given in (15).

We give now the assumptions we will use in this section.

$(H1)$ $\alpha,\ \beta \in (3,4]$, $\lambda,\ \mu > 0$, $p_1,\ q_1,\ p_2,\ q_2 > 0$, $\Delta_1 \neq 0$ (given by (13)), and the functions $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{K}_1, \mathfrak{K}_2 : [0,1] \to \mathbb{R}$ have bounded variations.

$(H2)$ $\mathfrak{f},\ \mathfrak{g} : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, and there exist the constants $\mathfrak{a}_0 > 0$, $\mathfrak{b}_0 > 0$ and $\mathfrak{a}_i,\ \mathfrak{b}_i \geq 0$, $i = 1, ..., 4$ such that

$$|\mathfrak{f}(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4)| \leq \mathfrak{a}_0 + \sum_{i=1}^{4} \mathfrak{a}_i |\mathfrak{u}_i|, \quad |\mathfrak{g}(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4)| \leq \mathfrak{b}_0 + \sum_{i=1}^{4} \mathfrak{b}_i |\mathfrak{u}_i|,$$

for all $\mathfrak{t} \in [0,1]$ and $\mathfrak{u}_j \in \mathbb{R},\ j = 1, \ldots, 4.$

$(H3)$ $\mathfrak{f}$, $\mathfrak{g} : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, and there exist the constants $\mathfrak{c}_0 > 0$ and $\mathfrak{d}_0 > 0$ such that

$$|\mathfrak{f}(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) - \mathfrak{f}(\mathfrak{t}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4)| \le \mathfrak{c}_0 \sum_{i=1}^{4} |\mathfrak{u}_i - \mathfrak{v}_i|,$$
$$|\mathfrak{g}(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) - \mathfrak{g}(\mathfrak{t}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4)| \le \mathfrak{d}_0 \sum_{i=1}^{4} |\mathfrak{u}_i - \mathfrak{v}_i|,$$

for all $\mathfrak{t} \in [0,1]$ and $\mathfrak{u}_j$, $\mathfrak{v}_j \in \mathbb{R}$, $j = 1, \dots, 4$.

We denote by $\widetilde{S}_i = \sup_{\zeta \in [0,1]} |S_i(\zeta)|$, $\widetilde{T}_i = \sup_{\zeta \in [0,1]} |T_i(\zeta)|$, for $i = 1, \dots, 4$,

$$\begin{aligned}
U_1 = {} & \frac{1}{\lambda\Gamma(\alpha)}(1 - e^{-\lambda}) + \widetilde{S}_1 \frac{1}{\Gamma(\alpha)}(2 - e^{-\lambda}) + \widetilde{S}_3 \left[\frac{1}{\lambda\Gamma(\alpha)}(1 - e^{-\lambda}) \right. \\
& \left. + \frac{1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1}(1 - e^{-\lambda\zeta})\, d\mathfrak{H}_1(\zeta) \right| \right] \\
& + \widetilde{S}_4 \frac{1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1}(1 - e^{-\lambda\zeta})\, d\mathfrak{K}_1(\zeta) \right|, \\
U_2 = {} & \widetilde{S}_2 \frac{1}{\Gamma(\beta)}(2 - e^{-\mu}) + \widetilde{S}_3 \frac{1}{\mu\Gamma(\beta)} \left| \int_0^1 \zeta^{\beta-1}(1 - e^{-\mu\zeta})\, d\mathfrak{H}_2(\zeta) \right| \\
& + \widetilde{S}_4 \left[\frac{1}{\mu\Gamma(\beta)}(1 - e^{-\mu}) + \frac{1}{\mu\Gamma(\beta)} \left| \int_0^1 \zeta^{\beta-1}(1 - e^{-\mu\zeta})\, d\mathfrak{K}_2(\zeta) \right| \right], \\
V_1 = {} & \widetilde{T}_1 \frac{1}{\Gamma(\alpha)}(2 - e^{-\lambda}) + \widetilde{T}_3 \frac{1}{\lambda\Gamma(\alpha)} \left[(1 - e^{-\lambda}) + \left| \int_0^1 \zeta^{\alpha-1}(1 - e^{-\lambda\zeta})\, d\mathfrak{H}_1(\zeta) \right| \right] \\
& + \widetilde{T}_4 \frac{1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1}(1 - e^{-\lambda\zeta})\, d\mathfrak{K}_1(\zeta) \right|, \\
V_2 = {} & \frac{1}{\mu\Gamma(\beta)}(1 - e^{-\mu}) + \widetilde{T}_2 \frac{1}{\Gamma(\beta)}(2 - e^{-\mu}) \\
& + \widetilde{T}_3 \frac{1}{\mu\Gamma(\beta)} \left| \int_0^1 \zeta^{\beta-1}(1 - e^{-\mu\zeta})\, d\mathfrak{H}_2(\zeta) \right| \\
& + \widetilde{T}_4 \frac{1}{\mu\Gamma(\beta)} \left[(1 - e^{-\mu}) + \left| \int_0^1 \zeta^{\beta-1}(1 - e^{-\mu\zeta})\, d\mathfrak{K}_2(\zeta) \right| \right],
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\Psi_0 &= (U_1 + V_1)\mathfrak{a}_0 + (U_2 + V_2)\mathfrak{b}_0, \\
\Psi_1 &= (U_1 + V_1)\left(\mathfrak{a}_1 + \frac{\mathfrak{a}_3}{\Gamma(p_1 + 1)} \right) + (U_2 + V_2)\left(\mathfrak{b}_1 + \frac{\mathfrak{b}_3}{\Gamma(q_1 + 1)} \right), \\
\Psi_2 &= (U_1 + V_1)\left(\mathfrak{a}_2 + \frac{\mathfrak{a}_4}{\Gamma(p_2 + 1)} \right) + (U_2 + V_2)\left(\mathfrak{b}_2 + \frac{\mathfrak{b}_4}{\Gamma(q_2 + 1)} \right).
\end{aligned} \tag{24}$$

Our first main result is the theorem below which gives us the existence of solutions for our problem (1) and (2), based on the Leray-Schauder alternative theorem.

Theorem 1. *We suppose that* $(H1)$ *and* $(H2)$ *hold. If*

$$\max\{\Psi_1, \Psi_2\} < 1, \tag{25}$$

then the boundary value problem (1) and (2) has at least one solution $(x(\mathfrak{t}), y(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$.

Proof. We show firstly that $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$ is a completely continuous operator. Because $\mathfrak{f}$ and $\mathfrak{g}$ are continuous functions, we conclude that $\mathcal{A}_1$ and $\mathcal{A}_2$ are continuous operators, and so the operator $\mathcal{A}$ is also continuous.

We prove now that $\mathcal{A}$ is uniformly bounded. Let $\mathcal{D} \subset \mathcal{Y}$ be a bounded set. Then there exists a positive constant r_0 such that $\|(x,y)\|_{\mathcal{Y}} \leq r_0$ for all $(x,y) \in \mathcal{D}$. So $\|x\| \leq r_0$, $\|y\| \leq r_0$, and by Lemma 2 we find

$$|I_{0+}^{p_1} x(\mathfrak{t})| \leq \frac{r_0}{\Gamma(p_1+1)}, \quad |I_{0+}^{p_2} y(\mathfrak{t})| \leq \frac{r_0}{\Gamma(p_2+1)}, \quad \forall\, \mathfrak{t} \in [0,1],$$
$$|I_{0+}^{q_1} x(\mathfrak{t})| \leq \frac{r_0}{\Gamma(q_1+1)}, \quad |I_{0+}^{q_2} y(\mathfrak{t})| \leq \frac{r_0}{\Gamma(q_2+1)}, \quad \forall\, \mathfrak{t} \in [0,1].$$

Hence there exist positive constants C_1 and C_2 such that

$$|\mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{p_1} x(\mathfrak{t}), I_{0+}^{p_2} y(\mathfrak{t}))| \leq C_1, \quad |\mathfrak{g}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{q_1} x(\mathfrak{t}), I_{0+}^{q_2} y(\mathfrak{t})| \leq C_2, \tag{26}$$

for all $\mathfrak{t} \in [0,1]$ and $(x,y) \in \mathcal{D}$.

Therefore by using (26) and Lemma 2, for any $(x,y) \in \mathcal{D}$ and $\mathfrak{t} \in [0,1]$ we obtain

$$\begin{aligned}
|\mathcal{A}_1(x,y)(\mathfrak{t})| &\leq \int_0^{\mathfrak{t}} e^{-\lambda(\mathfrak{t}-s)} I_{0+}^{\alpha-1} |\mathfrak{f}_{x,y}^{p_1,p_2}|(s)\, ds \\
&\quad + |S_1(\mathfrak{t})| \left[\lambda \int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} |\mathfrak{f}_{x,y}^{p_1,p_2}|(s)\, ds + I_{0+}^{\alpha-1} |\mathfrak{f}_{x,y}^{p_1,p_2}|(1) \right] \\
&\quad + |S_2(\mathfrak{t})| \left[\mu \int_0^1 e^{-\mu(1-s)} I_{0+}^{\beta-1} |\mathfrak{g}_{x,y}^{q_1,q_2}|(s)\, ds + I_{0+}^{\beta-1} |\mathfrak{g}_{x,y}^{q_1,q_2}|(1) \right] \\
&\quad + |S_3(\mathfrak{t})| \left[\int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} |\mathfrak{f}_{x,y}^{p_1,p_2}|(s)\, ds \right. \\
&\quad + \left| \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} |\mathfrak{f}_{x,y}^{p_1,p_2}|(\zeta)\, d\zeta \right) d\mathfrak{H}_1(s) \right| \\
&\quad \left. + \left| \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} |\mathfrak{g}_{x,y}^{q_1,q_2}|(\zeta)\, d\zeta \right) d\mathfrak{H}_2(s) \right| \right] \\
&\quad + |S_4(\mathfrak{t})| \left[\int_0^1 e^{-\mu(1-\zeta)} I_{0+}^{\beta-1} |\mathfrak{g}_{x,y}^{q_1,q_2}|(\zeta)\, d\zeta \right. \\
&\quad + \left| \int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)} I_{0+}^{\alpha-1} |\mathfrak{f}_{x,y}^{p_1,p_2}|(\zeta)\, d\zeta \right) d\mathfrak{K}_1(s) \right| \\
&\quad \left. + \left| \int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)} I_{0+}^{\beta-1} |\mathfrak{g}_{x,y}^{q_1,q_2}|(\zeta)\, d\zeta \right) d\mathfrak{K}_2(s) \right| \right] \\
&\leq \frac{C_1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda}) + \widetilde{S}_1 \frac{C_1}{\Gamma(\alpha)}(2-e^{-\lambda}) + \widetilde{S}_2 \frac{C_2}{\Gamma(\beta)}(2-e^{-\mu}) \\
&\quad + \widetilde{S}_3 \left[C_1 \frac{1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda}) + C_1 \frac{1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\, d\mathfrak{H}_1(\zeta) \right| \right. \\
&\quad \left. + C_2 \frac{1}{\mu\Gamma(\beta)} \left| \int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\, d\mathfrak{H}_2(\zeta) \right| \right] \\
&\quad + \widetilde{S}_4 \left[C_2 \frac{1}{\mu\Gamma(\beta)}(1-e^{-\mu}) + C_1 \frac{1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\, d\mathfrak{K}_1(\zeta) \right| \right. \\
&\quad \left. + C_2 \frac{1}{\mu\Gamma(\beta)} \left| \int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\, d\mathfrak{K}_2(\zeta) \right| \right] \\
&= C_1 \left\{ \frac{1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda}) + \widetilde{S}_1 \frac{1}{\Gamma(\alpha)}(2-e^{-\lambda}) + \widetilde{S}_3 \left[\frac{1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda}) \right. \right. \\
&\quad \left. + \frac{1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\, d\mathfrak{H}_1(\zeta) \right| \right] \\
&\quad \left. + \widetilde{S}_4 \frac{1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\, d\mathfrak{K}_1(\zeta) \right| \right\} \\
&\quad + C_2 \left\{ \widetilde{S}_2 \frac{1}{\Gamma(\beta)}(2-e^{-\mu}) + \widetilde{S}_3 \frac{1}{\mu\Gamma(\beta)} \left| \int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\, d\mathfrak{H}_2(\zeta) \right| \right. \\
&\quad \left. + \widetilde{S}_4 \left[\frac{1}{\mu\Gamma(\beta)}(1-e^{-\mu}) + \frac{1}{\mu\Gamma(\beta)} \left| \int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\, d\mathfrak{K}_2(\zeta) \right| \right] \right\} \\
&= C_1 U_1 + C_2 U_2,
\end{aligned}$$

where U_1 and U_2 are defined by (23).

Then we conclude

$$\|\mathcal{A}_1(x,y)\| \le C_1 U_1 + C_2 U_2, \;\; \forall\,(x,y) \in \mathcal{D}. \tag{27}$$

In a similar manner, for any $(x,y) \in \mathcal{D}$ and $\mathfrak{t} \in [0,1]$, we find

$$\begin{aligned}
|\mathcal{A}_2(x,y)(\mathfrak{t})| &\le \int_0^{\mathfrak{t}} e^{-\mu(\mathfrak{t}-s)}\, I_{0+}^{\beta-1}|\mathfrak{g}_{x,y}^{q_1,q_2}|(s)\,ds \\
&\quad + |T_1(\mathfrak{t})|\left[\lambda \int_0^1 e^{-\lambda(1-s)}\, I_{0+}^{\alpha-1}|\mathfrak{f}_{x,y}^{p_1,p_2}|(s)\,ds + I_{0+}^{\alpha-1}|\mathfrak{f}_{x,y}^{p_1,p_2}|(1)\right] \\
&\quad + |T_2(\mathfrak{t})|\left[\mu \int_0^1 e^{-\mu(1-s)}\, I_{0+}^{\beta-1}|\mathfrak{g}_{x,y}^{q_1,q_2}|(s)\,ds + I_{0+}^{\beta-1}|\mathfrak{g}_{x,y}^{q_1,q_2}|(1)\right] \\
&\quad + |T_3(\mathfrak{t})|\left[\int_0^1 e^{-\lambda(1-s)}\, I_{0+}^{\alpha-1}|\mathfrak{f}_{x,y}^{p_1,p_2}|(s)\,ds \right. \\
&\quad + \left|\int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)}\, I_{0+}^{\alpha-1}|\mathfrak{f}_{x,y}^{p_1,p_2}|(\zeta)\,d\zeta\right) d\mathfrak{H}_1(s)\right| \\
&\quad \left. + \left|\int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)}\, I_{0+}^{\beta-1}|\mathfrak{g}_{x,y}^{q_1,q_2}|(\zeta)\,d\zeta\right) d\mathfrak{H}_2(s)\right|\right] \\
&\quad + |T_4(\mathfrak{t})|\left[\int_0^1 e^{-\mu(1-\zeta)}\, I_{0+}^{\beta-1}|\mathfrak{g}_{x,y}^{q_1,q_2}|(\zeta)\,d\zeta \right. \\
&\quad + \left|\int_0^1 \left(\int_0^s e^{-\lambda(s-\zeta)}\, I_{0+}^{\alpha-1}|\mathfrak{f}_{x,y}^{p_1,p_2}|(\zeta)\,d\zeta\right) d\mathfrak{K}_1(s)\right| \\
&\quad \left. + \left|\int_0^1 \left(\int_0^s e^{-\mu(s-\zeta)}\, I_{0+}^{\beta-1}|\mathfrak{g}_{x,y}^{q_1,q_2}|(\zeta)\,d\zeta\right) d\mathfrak{K}_2(s)\right|\right] \\
&\le C_2 \frac{1}{\mu\Gamma(\beta)}(1-e^{-\mu}) + \widetilde{T}_1 \frac{C_1}{\Gamma(\alpha)}(2-e^{-\lambda}) + \widetilde{T}_2 \frac{C_2}{\Gamma(\beta)}(2-e^{-\mu}) \\
&\quad + \widetilde{T}_3\left[C_1 \frac{1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda}) + C_1 \frac{1}{\lambda\Gamma(\alpha)}\left|\int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\,d\mathfrak{H}_1(\zeta)\right| \right. \\
&\quad \left. + C_2 \frac{1}{\mu\Gamma(\beta)}\left|\int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\,d\mathfrak{H}_2(\zeta)\right|\right] \\
&\quad + \widetilde{T}_4\left[C_2 \frac{1}{\mu\Gamma(\beta)}(1-e^{-\mu}) + C_1 \frac{1}{\lambda\Gamma(\alpha)}\left|\int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\,d\mathfrak{K}_1(\zeta)\right| \right. \\
&\quad \left. + C_2 \frac{1}{\mu\Gamma(\beta)}\left|\int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\,d\mathfrak{K}_2(\zeta)\right|\right] \\
&= C_1\left\{\widetilde{T}_1 \frac{1}{\Gamma(\alpha)}(2-e^{-\lambda}) + \widetilde{T}_3 \frac{1}{\Gamma(\alpha)}\left[\frac{1}{\lambda}(1-e^{-\lambda}) + \frac{1}{\lambda}\left|\int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\,d\mathfrak{H}_1(\zeta)\right|\right] \right. \\
&\quad \left. + \widetilde{T}_4 \frac{1}{\lambda\Gamma(\alpha)}\left|\int_0^1 \zeta^{\alpha-1}(1-e^{-\lambda\zeta})\,d\mathfrak{K}_1(\zeta)\right|\right\} \\
&\quad + C_2\left\{\frac{1}{\mu\Gamma(\beta)}(1-e^{-\mu}) + \widetilde{T}_2 \frac{1}{\Gamma(\beta)}(2-e^{-\mu}) \right. \\
&\quad + \widetilde{T}_3 \frac{1}{\mu\Gamma(\beta)}\left|\int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\,d\mathfrak{H}_2(\zeta)\right| \\
&\quad \left. + \widetilde{T}_4 \frac{1}{\mu\Gamma(\beta)}\left[(1-e^{-\mu}) + \left|\int_0^1 \zeta^{\beta-1}(1-e^{-\mu\zeta})\,d\mathfrak{K}_2(\zeta)\right|\right]\right\} \\
&= C_1 V_1 + C_2 V_2,
\end{aligned}$$

where V_1 and V_2 are defined by (23).

Hence we obtain

$$\|\mathcal{A}_2(x,y)\| \le C_1 V_1 + C_2 V_2, \;\; \forall\,(x,y) \in \mathcal{D}. \tag{28}$$

From the inequalities (27) and (28) we deduce that $\mathcal{A}_1$ and $\mathcal{A}_2$ are uniformly bounded, and then

$$\|\mathcal{A}(x,y)\|_{\mathcal{Y}} = \|\mathcal{A}_1(x,y)\| + \|\mathcal{A}_2(x,y)\| \le C_1(U_1+V_1) + C_2(U_2+V_2), \;\; \forall\,(x,y) \in \mathcal{D},$$

that is the operator $\mathcal{A}$ is uniformly bounded.

Next we will establish that $\mathcal{A}$ is an equicontinuous operator. Let $(x,y) \in \mathcal{D}$ and $\mathfrak{t}_1,\ \mathfrak{t}_2 \in [0,1]$, $\mathfrak{t}_1 < \mathfrak{t}_2$. Therefore we find

$$
\begin{aligned}
& |\mathcal{A}_1(x,y)(\mathfrak{t}_2) - \mathcal{A}_1(x,y)(\mathfrak{t}_1)| \\
& \leq \left| \int_0^{\mathfrak{t}_1} \left[e^{-\lambda(\mathfrak{t}_2 - s)} - e^{-\lambda(\mathfrak{t}_1 - s)} \right] I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(s)\, ds \right| \\
& \quad + \left| \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} e^{-\lambda(\mathfrak{t}_2 - s)} \mathfrak{f}_{x,y}^{p_1,p_2}(s)\, ds \right| \\
& \quad + |S_1(\mathfrak{t}_2) - S_1(\mathfrak{t}_1)| \left[\left| \lambda \int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(s)\, ds \right| + \left| I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(1) \right| \right] \\
& \quad + |S_2(\mathfrak{t}_2) - S_2(\mathfrak{t}_1)| \left[\left| \mu \int_0^1 e^{-\mu(1-s)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(s)\, ds \right| + \left| I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(1) \right| \right] \\
& \quad + |S_3(\mathfrak{t}_2) - S_3(\mathfrak{t}_1)| \left[\left| \int_0^1 e^{-\lambda(1-s)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(s)\, ds \right| \right. \\
& \quad + \left| \int_0^1 \left(\int_0^s e^{-\lambda(s-\tau)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\tau)\, d\tau \right) d\mathfrak{H}_1(s) \right| \\
& \quad \left. + \left| \int_0^1 \left(\int_0^s e^{-\mu(s-\tau)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\tau)\, d\tau \right) d\mathfrak{H}_2(s) \right| \right] \\
& \quad + |S_4(\mathfrak{t}_2) - S_4(\mathfrak{t}_1)| \left[\left| \int_0^1 e^{-\mu(1-s)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(s)\, ds \right| \right. \\
& \quad + \left| \int_0^1 \left(\int_0^s e^{-\lambda(s-\tau)} I_{0+}^{\alpha-1} \mathfrak{f}_{x,y}^{p_1,p_2}(\tau)\, d\tau \right) d\mathfrak{K}_1(s) \right| \\
& \quad \left. + \left| \int_0^1 \left(\int_0^s e^{-\mu(s-\tau)} I_{0+}^{\beta-1} \mathfrak{g}_{x,y}^{q_1,q_2}(\tau)\, d\tau \right) d\mathfrak{K}_2(s) \right| \right] \\
& \leq \frac{C_1}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda \mathfrak{t}_1}) (1 - e^{-\lambda(\mathfrak{t}_2 - \mathfrak{t}_1)} \right) + \frac{C_1}{\lambda} \left(1 - e^{-\lambda(\mathfrak{t}_2 - \mathfrak{t}_1)} \right) \\
& \quad + |S_1(\mathfrak{t}_2) - S_1(\mathfrak{t}_1)| \frac{C_1}{\Gamma(\alpha)} (2 - e^{-\lambda}) + |S_2(\mathfrak{t}_2) - S_2(\mathfrak{t}_1)| \frac{C_2}{\Gamma(\beta)} (2 - e^{-\mu}) \\
& \quad + |S_3(\mathfrak{t}_2) - S_3(\mathfrak{t}_1)| \left[\frac{C_1}{\lambda\Gamma(\alpha)} (1 - e^{-\lambda}) + \frac{C_1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1} (1 - e^{-\lambda\zeta})\, d\mathfrak{H}_1(\zeta) \right| \right. \\
& \quad \left. + \frac{C_2}{\mu\Gamma(\beta)} \left| \int_0^1 s^{\beta-1} (1 - e^{-\mu s})\, d\mathfrak{H}_2(s) \right| \right] \\
& \quad + |S_4(\mathfrak{t}_2) - S_4(\mathfrak{t}_1)| \left[\frac{C_2}{\mu\Gamma(\beta)} (1 - e^{-\mu}) + \frac{C_1}{\lambda\Gamma(\alpha)} \left| \int_0^1 \zeta^{\alpha-1} (1 - e^{-\lambda\zeta})\, d\mathfrak{K}_1(\zeta) \right| \right. \\
& \quad \left. + \frac{C_2}{\mu\Gamma(\beta)} \left| \int_0^1 s^{\beta-1} (1 - e^{-\mu s})\, d\mathfrak{K}_2(s) \right| \right].
\end{aligned}
$$

Because

$$
\begin{aligned}
& |S_i(\mathfrak{t}_2) - S_i(\mathfrak{t}_1)| = \left| \frac{1}{\lambda^2} (\lambda \mathfrak{t}_2 - 1 + e^{-\lambda \mathfrak{t}_2}) \Lambda_i + \frac{1}{\lambda^3} (\lambda^2 \mathfrak{t}_2^2 - 2\lambda \mathfrak{t}_2 + 2 - 2e^{-\lambda \mathfrak{t}_2}) \Theta_i \right. \\
& \quad \left. - \frac{1}{\lambda^2} (\lambda \mathfrak{t}_1 - 1 + e^{-\lambda \mathfrak{t}_1}) \Lambda_i - \frac{1}{\lambda^3} (\lambda^2 \mathfrak{t}_1^2 - 2\lambda \mathfrak{t}_1 + 2 - 2e^{-\lambda \mathfrak{t}_1}) \Theta_i \right| \\
& \leq \frac{1}{\lambda^2} |\Lambda_i| \left| \lambda(\mathfrak{t}_2 - \mathfrak{t}_1) - e^{-\lambda \mathfrak{t}_1} + e^{-\lambda \mathfrak{t}_2} \right| \\
& \quad + \frac{1}{\lambda^3} |\Theta_i| \left| \lambda^2 (\mathfrak{t}_2^2 - \mathfrak{t}_1^2) - 2(e^{-\lambda \mathfrak{t}_2} - e^{-\lambda \mathfrak{t}_1}) - 2\lambda(\mathfrak{t}_2 - \mathfrak{t}_1) \right| \to 0, \text{ as } \mathfrak{t}_2 \to \mathfrak{t}_1,\ i = 1, \ldots, 4,
\end{aligned}
$$

we deduce that $|\mathcal{A}_1(x,y)(\mathfrak{t}_2) - \mathcal{A}_1(x,y)(\mathfrak{t}_1)| \to 0$ as $\mathfrak{t}_2 \to \mathfrak{t}_1$.

In a similar manner we obtain

$$
\begin{aligned}
&|\mathcal{A}_2(x,y)(\mathrm{t}_2)-\mathcal{A}_2(x,y)(\mathrm{t}_1)| \\
&\le \left|\int_0^{\mathrm{t}_1}\left(e^{-\mu(\mathrm{t}_2-s)}-e^{-\mu(\mathrm{t}_1-s)}\right)I_{0+}^{\beta-1}\mathfrak{g}_{x,y}^{q_1,q_2}(s)\,ds\right| \\
&\quad+\left|\int_{\mathrm{t}_1}^{\mathrm{t}_2}e^{-\mu(\mathrm{t}_2-s)}I_{0+}^{\beta-1}\mathfrak{g}_{x,y}^{q_1,q_2}(s)\,ds\right| \\
&\quad+|T_1(\mathrm{t}_2)-T_1(\mathrm{t}_1)|\left[\left|\lambda\int_0^1 e^{-\lambda(1-s)}I_{0+}^{\alpha-1}\mathfrak{f}_{x,y}^{p_1,p_2}(s)\,ds\right|+\left|I_{0+}^{\alpha-1}\mathfrak{f}_{x,y}^{p_1,p_2}(1)\right|\right] \\
&\quad+|T_2(\mathrm{t}_2)-T_2(\mathrm{t}_1)|\left[\left|\mu\int_0^1 e^{-\mu(1-s)}I_{0+}^{\beta-1}\mathfrak{g}_{x,y}^{q_1,q_2}(s)\,ds\right|+\left|I_{0+}^{\beta-1}\mathfrak{g}_{x,y}^{q_1,q_2}(1)\right|\right] \\
&\quad+|T_3(\mathrm{t}_2)-T_3(\mathrm{t}_1)|\left[\left|\int_0^1 e^{-\lambda(1-s)}I_{0+}^{\alpha-1}\mathfrak{f}_{x,y}^{p_1,p_2}(s)\,ds\right|\right. \\
&\quad+\left|\int_0^1\left(\int_0^s e^{-\lambda(s-\tau)}I_{0+}^{\alpha-1}\mathfrak{f}_{x,y}^{p_1,p_2}(\tau)\,d\tau\right)d\mathfrak{H}_1(s)\right| \\
&\quad\left.+\left|\int_0^1\left(\int_0^s e^{-\mu(s-\tau)}I_{0+}^{\beta-1}\mathfrak{g}_{x,y}^{q_1,q_2}(\tau)\,d\tau\right)d\mathfrak{H}_2(s)\right|\right] \\
&\quad+|T_4(\mathrm{t}_2)-T_4(\mathrm{t}_1)|\left[\left|\int_0^1 e^{-\mu(1-\zeta)}I_{0+}^{\beta-1}\mathfrak{g}_{x,y}^{q_1,q_2}(\zeta)\,d\zeta\right|\right. \\
&\quad+\left|\int_0^1\left(\int_0^s e^{-\lambda(s-\tau)}I_{0+}^{\alpha-1}\mathfrak{f}_{x,y}^{p_1,p_2}(\tau)\,d\tau\right)d\mathfrak{K}_1(s)\right| \\
&\quad\left.+\left|\int_0^1\left(\int_0^s e^{-\mu(s-\tau)}I_{0+}^{\beta-1}\mathfrak{g}_{x,y}^{q_1,q_2}(\tau)\,d\tau\right)d\mathfrak{K}_2(s)\right|\right] \\
&\le \frac{C_2}{\mu\Gamma(\beta)}(1-e^{-\mu\mathrm{t}_1})\left(1-e^{-\mu(\mathrm{t}_2-\mathrm{t}_1)}\right)+\frac{C_2}{\mu\Gamma(\beta)}\left(1-e^{-\mu(\mathrm{t}_2-\mathrm{t}_1)}\right) \\
&\quad+|T_1(\mathrm{t}_2)-T_1(\mathrm{t}_1)|\frac{C_1}{\Gamma(\alpha)}(2-e^{-\lambda})+|T_2(\mathrm{t}_2)-T_2(\mathrm{t}_1)|\frac{C_2}{\Gamma(\beta)}(2-e^{-\mu}) \\
&\quad+|T_3(\mathrm{t}_2)-T_3(\mathrm{t}_1)|\left[\frac{C_1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda})+\frac{C_1}{\lambda\Gamma(\alpha)}\left|\int_0^1\zeta^{\alpha-1}(1-e^{-\lambda\zeta})\,d\mathfrak{H}_1(\zeta)\right|\right. \\
&\quad\left.+\frac{C_2}{\mu\Gamma(\beta)}\left|\int_0^1 s^{\beta-1}(1-e^{-\mu s})\,d\mathfrak{H}_2(s)\right|\right] \\
&\quad+|T_4(\mathrm{t}_2)-T_4(\mathrm{t}_1)|\left[\frac{C_2}{\mu\Gamma(\beta)}(1-e^{-\mu})+\frac{C_1}{\lambda\Gamma(\alpha)}\left|\int_0^1\zeta^{\alpha-1}(1-e^{-\lambda\zeta})\,d\mathfrak{K}_1(\zeta)\right|\right. \\
&\quad\left.+\frac{C_2}{\mu\Gamma(\beta)}\left|\int_0^1 s^{\beta-1}(1-e^{-\mu s})\,d\mathfrak{K}_2(s)\right|\right].
\end{aligned}
$$

Because

$$
\begin{aligned}
|T_i(\mathrm{t}_2)-T_i(\mathrm{t}_1)| &= \left|\frac{1}{\mu^2}(\mu\mathrm{t}_2-1+e^{-\mu\mathrm{t}_2})\Xi_i+\frac{1}{\mu^3}(\mu^2\mathrm{t}_2^2-2\mu\mathrm{t}_2+2-2e^{-\mu\mathrm{t}_2})\Upsilon_i\right. \\
&\quad\left.-\frac{1}{\mu^2}(\mu\mathrm{t}_1-1+e^{-\mu\mathrm{t}_1})\Xi_i-\frac{1}{\mu^3}(\mu^2\mathrm{t}_1^2-2\mu\mathrm{t}_1+2-2e^{-\mu\mathrm{t}_1})\Upsilon_i\right| \\
&\le \frac{1}{\mu^2}|\Xi_i|\left|\mu(\mathrm{t}_2-\mathrm{t}_1)+e^{-\mu\mathrm{t}_2}-e^{-\mu\mathrm{t}_1}\right| \\
&\quad+\frac{1}{\mu^3}|\Upsilon_i|\left|\mu^2(\mathrm{t}_2^2-\mathrm{t}_1^2)-2(e^{-\mu\mathrm{t}_2}-e^{-\mu\mathrm{t}_1})-2\mu(\mathrm{t}_2-\mathrm{t}_1)\right|\to 0,\ \text{as}\ \mathrm{t}_2\to\mathrm{t}_1,\ i=1,\ldots,4,
\end{aligned}
$$

we conclude that $|\mathcal{A}_2(x,y)(\mathrm{t}_2)-\mathcal{A}_2(x,y)(\mathrm{t}_1)|\to 0$ as $\mathrm{t}_2\to\mathrm{t}_1$.

We deduce that $\mathcal{A}_1$ and $\mathcal{A}_2$ are equicontinuous operators, and so the operator $\mathcal{A}$ is also equicontinuous. Therefore, by Arzela-Ascoli theorem, we deduce that $\mathcal{A}$ is compact, and hence $\mathcal{A}$ is completely continuous.

Next we will prove that the set $\mathcal{F}=\{(x,y)\in\mathcal{Y},\ (x,y)=\zeta\mathcal{A}(x,y),\ 0<\zeta<1\}$ is bounded. For this, let $(x,y)\in\mathcal{F}$, then $(x,y)=\zeta\mathcal{A}(x,y)$ for some $\zeta\in(0,1)$. Hence for any $\mathrm{t}\in[0,1]$, we have $x(\mathrm{t})=\zeta\mathcal{A}_1(x,y)(\mathrm{t})$, $y(\mathrm{t})=\zeta\mathcal{A}_2(x,y)(\mathrm{t})$, and so $|x(\mathrm{t})|\le|\mathcal{A}_1(x,y)(\mathrm{t})|$,

$|y(t)| \le |\mathcal{A}_2(x,y)(t)|$ for all $t \in [0,1]$. Using some inequalities proved at the beginning of the proof, we find

$$|x(t)| \le |\mathcal{A}_1(x,y)(t)| \le U_1\left(\mathfrak{a}_0 + \mathfrak{a}_1\|x\| + \mathfrak{a}_2\|y\| + \frac{\mathfrak{a}_3}{\Gamma(p_1+1)}\|x\| + \frac{\mathfrak{a}_4}{\Gamma(p_2+1)}\|y\|\right)$$
$$+U_2\left(\mathfrak{b}_0 + \mathfrak{b}_1\|x\| + \mathfrak{b}_2\|y\| + \frac{\mathfrak{b}_3}{\Gamma(q_1+1)}\|x\| + \frac{\mathfrak{b}_4}{\Gamma(q_2+1)}\|y\|\right), \ \forall t \in [0,1],$$

and so

$$\begin{aligned}\|x\| \le\ & U_1\left[\mathfrak{a}_0 + \left(\mathfrak{a}_1 + \frac{\mathfrak{a}_3}{\Gamma(p_1+1)}\right)\|x\| + \left(\mathfrak{a}_2 + \frac{\mathfrak{a}_4}{\Gamma(p_2+1)}\right)\|y\|\right] \\ &+U_2\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \frac{\mathfrak{b}_3}{\Gamma(q_1+1)}\right)\|x\| + \left(\mathfrak{b}_2 + \frac{\mathfrak{b}_4}{\Gamma(q_2+1)}\right)\|y\|\right].\end{aligned} \tag{29}$$

In a similar manner we deduce

$$|y(t)| \le |\mathcal{A}_2(x,y)(t)| \le V_1\left(\mathfrak{a}_0 + \mathfrak{a}_1\|x\| + \mathfrak{a}_2\|y\| + \frac{\mathfrak{a}_3}{\Gamma(p_1+1)}\|x\| + \frac{\mathfrak{a}_4}{\Gamma(p_2+1)}\|y\|\right)$$
$$+V_2\left(\mathfrak{b}_0 + \mathfrak{b}_1\|x\| + \mathfrak{b}_2\|y\| + \frac{\mathfrak{b}_3}{\Gamma(q_1+1)}\|x\| + \frac{\mathfrak{b}_4}{\Gamma(q_2+1)}\|y\|\right), \ \forall t \in [0,1],$$

and then

$$\begin{aligned}\|y\| \le\ & V_1\left[\mathfrak{a}_0 + \left(\mathfrak{a}_1 + \frac{\mathfrak{a}_3}{\Gamma(p_1+1)}\right)\|x\| + \left(\mathfrak{a}_2 + \frac{\mathfrak{a}_4}{\Gamma(p_2+1)}\right)\|y\|\right] \\ &+V_2\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \frac{\mathfrak{b}_3}{\Gamma(q_1+1)}\right)\|x\| + \left(\mathfrak{b}_2 + \frac{\mathfrak{b}_4}{\Gamma(q_2+1)}\right)\|y\|\right].\end{aligned} \tag{30}$$

Therefore by (29) and (30) we conclude that

$$\begin{aligned}\|(x,y)\|_{\mathcal{Y}} &= \|x\| + \|y\| \le (U_1+V_1)\mathfrak{a}_0 + (U_2+V_2)\mathfrak{b}_0 \\ &+\|x\|\left[(U_1+V_1)\left(\mathfrak{a}_1 + \frac{\mathfrak{a}_3}{\Gamma(p_1+1)}\right) + (U_2+V_2)\left(\mathfrak{b}_1 + \frac{\mathfrak{b}_3}{\Gamma(q_1+1)}\right)\right] \\ &+\|y\|\left[(U_1+V_1)\left(\mathfrak{a}_2 + \frac{\mathfrak{a}_4}{\Gamma(p_2+1)}\right) + (U_2+V_2)\left(\mathfrak{b}_2 + \frac{\mathfrak{b}_4}{\Gamma(q_2+1)}\right)\right] \\ &= \Psi_0 + \Psi_1\|x\| + \Psi_2\|y\| \le \Psi_0 + \max\{\Psi_1,\Psi_2\}\|(x,y)\|_{\mathcal{Y}},\end{aligned}$$

where $\Psi_i, \ i = 0,1,2$ are given by (24). By (25) we deduce that

$$\|(x,y)\|_{\mathcal{Y}} \le \frac{\Psi_0}{1-\max\{\Psi_1,\Psi_2\}},$$

which means that the set $\mathcal{F}$ is bounded. By applying the Leray-Schauder alternative, we deduce that operator $\mathcal{A}$ has at least one fixed point (x,y). So there exists at least one solution $(x(t),y(t)), \ t \in [0,1]$ of problem (1) and (2). □

Now we introduce the constants

$$\begin{aligned}&\mathfrak{f}_0 = \sup_{t\in[0,1]}|\mathfrak{f}(t,0,0,0,0)|, \ \ \mathfrak{g}_0 = \sup_{t\in[0,1]}|\mathfrak{g}(t,0,0,0,0)|, \\ &\rho_1 = \max\left\{1 + \frac{1}{\Gamma(p_1+1)}, 1 + \frac{1}{\Gamma(p_2+1)}\right\}, \\ &\rho_2 = \max\left\{1 + \frac{1}{\Gamma(q_1+1)}, 1 + \frac{1}{\Gamma(q_2+1)}\right\}, \\ &D_1 = \mathfrak{c}_0\rho_1U_1 + \mathfrak{d}_0\rho_2U_2, \ \ D_2 = \mathfrak{c}_0\rho_1V_1 + \mathfrak{d}_0\rho_2V_2, \\ &G_1 = \mathfrak{f}_0U_1 + \mathfrak{g}_0U_2, \ \ G_2 = \mathfrak{f}_0V_1 + \mathfrak{g}_0V_2,\end{aligned} \tag{31}$$

where U_1, U_2, V_1, V_2 are given by (23).

In the second result will prove give the existence and uniqueness of solution of problem (1) and (2) and it relies on the Banach contraction mapping principle.

Theorem 2. *We suppose that* $(H1)$ *and* $(H3)$ *are satisfied, and*

$$D_1 + D_2 < 1. \tag{32}$$

Then problem (1) and (2) has a unique solution $(x(\mathfrak{t}), y(\mathfrak{t})),\ \mathfrak{t} \in [0,1]$.

Proof. By using condition (32), we define the positive number

$$r = \frac{G_1 + G_2}{1 - D_1 - D_2},$$

where G_1, G_2, D_1, D_2 are given by (31). We will prove that $\mathcal{A}(\overline{B}_r) \subset \overline{B}_r$, where $\overline{B}_r = \{(x,y) \in \mathcal{Y},\ \|(x,y)\|_{\mathcal{Y}} \leq r\}$. For $(x,y) \in \overline{B}_r$ and $\mathfrak{t} \in [0,1]$ we obtain

$$\begin{aligned}
&|\mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{p_1} x(\mathfrak{t}), I_{0+}^{p_2} y(\mathfrak{t}))| \\
&\leq |\mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{p_1} x(\mathfrak{t}), I_{0+}^{p_2} y(\mathfrak{t})) - \mathfrak{f}(\mathfrak{t},0,0,0,0)| + |\mathfrak{f}(\mathfrak{t},0,0,0,0)| \\
&\leq \mathfrak{c}_0(|x(\mathfrak{t})| + |y(\mathfrak{t})| + |I_{0+}^{p_1} x(\mathfrak{t})| + |I_{0+}^{p_2} y(\mathfrak{t})|) + \mathfrak{f}_0 \\
&\leq \mathfrak{c}_0\left(\|x\| + \|y\| + \frac{1}{\Gamma(p_1+1)}\|x\| + \frac{1}{\Gamma(p_2+1)}\|y\| \right) + \mathfrak{f}_0 \\
&\leq \mathfrak{c}_0 \max\left\{ 1 + \frac{1}{\Gamma(p_1+1)}, 1 + \frac{1}{\Gamma(p_2+1)} \right\} \|(x,y)\|_{\mathcal{Y}} + \mathfrak{f}_0 \\
&\leq \mathfrak{c}_0 \max\left\{ 1 + \frac{1}{\Gamma(p_1+1)}, 1 + \frac{1}{\Gamma(p_2+1)} \right\} r + \mathfrak{f}_0 \\
&= \mathfrak{c}_0 \rho_1 r + \mathfrak{f}_0,
\end{aligned}$$

and

$$\begin{aligned}
&|\mathfrak{g}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{q_1} x(\mathfrak{t}), I_{0+}^{q_2} y(\mathfrak{t}))| \\
&\leq |\mathfrak{g}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{q_1} x(\mathfrak{t}), I_{0+}^{q_2} y(\mathfrak{t})) - \mathfrak{g}(\mathfrak{t},0,0,0,0)| + |\mathfrak{g}(\mathfrak{t},0,0,0,0)| \\
&\leq \mathfrak{d}_0(|x(\mathfrak{t})| + |y(\mathfrak{t})| + |I_{0+}^{q_1} x(\mathfrak{t})| + |I_{0+}^{q_2} y(\mathfrak{t})|) + \mathfrak{g}_0 \\
&\leq \mathfrak{d}_0\left(\|x\| + \|y\| + \frac{1}{\Gamma(q_1+1)}\|x\| + \frac{1}{\Gamma(q_2+1)}\|y\| \right) + \mathfrak{g}_0 \\
&\leq \mathfrak{d}_0 \max\left\{ 1 + \frac{1}{\Gamma(q_1+1)}, 1 + \frac{1}{\Gamma(q_2+1)} \right\} \|(x,y)\|_{\mathcal{Y}} + \mathfrak{g}_0 \\
&\leq \mathfrak{d}_0 \max\left\{ 1 + \frac{1}{\Gamma(q_1+1)}, 1 + \frac{1}{\Gamma(q_2+1)} \right\} r + \mathfrak{g}_0 \\
&= \mathfrak{d}_0 \rho_2 r + \mathfrak{g}_0.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
|\mathcal{A}_1(x,y)(\mathfrak{t})| &\leq (\mathfrak{c}_0 \rho_1 r + \mathfrak{f}_0) U_1 + (\mathfrak{d}_0 \rho_2 r + \mathfrak{g}_0) U_2 \\
&= (\mathfrak{c}_0 \rho_1 U_1 + \mathfrak{d}_0 \rho_2 U_2) r + \mathfrak{f}_0 U_1 + \mathfrak{g}_0 U_2 = D_1 r + G_1,
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
|\mathcal{A}_2(x,y)(\mathfrak{t})| &\leq (\mathfrak{c}_0 \rho_1 r + \mathfrak{f}_0) V_1 + (\mathfrak{d}_0 \rho_2 r + \mathfrak{g}_0) V_2 \\
&= (\mathfrak{c}_0 \rho_1 V_1 + \mathfrak{d}_0 \rho_2 V_2) r + \mathfrak{f}_0 V_1 + \mathfrak{g}_0 V_2 = D_2 r + G_2.
\end{aligned} \tag{34}$$

Therefore, by (33) and (34) and the definition of r, we conclude that

$$\|\mathcal{A}(x,y)\|_{\mathcal{Y}} = \|\mathcal{A}_1(x,y)\| + \|\mathcal{A}_2(x,y)\| \leq (D_1 + D_2) r + G_1 + G_2 = r, \ \ \forall\, (x,y) \in \overline{B}_r,$$

which gives us $\mathcal{A}(\overline{B}_r) \subset \overline{B}_r$.

We will prove next that $\mathcal{A}$ is a contraction operator. By using $(H3)$, for $(x_i, y_i) \in \overline{B}_r,\ i = 1,2$, and for any $\mathfrak{t} \in [0,1]$ we find

$$\begin{aligned}
&|\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(\mathfrak{t}) - \mathfrak{f}^{p_1,p_2}_{x_2,y_2}(\mathfrak{t})| = |\mathfrak{f}(\mathfrak{t}, x_1(\mathfrak{t}), y_1(\mathfrak{t}), I^{p_1}_{0+}x_1(\mathfrak{t}), I^{p_2}_{0+}y_1(\mathfrak{t})) \\
&\quad - \mathfrak{f}(\mathfrak{t}, x_2(\mathfrak{t}), y_2(\mathfrak{t}), I^{p_1}_{0+}x_2(\mathfrak{t}), I^{p_2}_{0+}y_2(\mathfrak{t}))| \\
&\leq \mathfrak{c}_0(|x_1(\mathfrak{t}) - x_2(\mathfrak{t})| + |y_1(\mathfrak{t}) - y_2(\mathfrak{t})| \\
&\quad + |I^{p_1}_{0+}x_1(\mathfrak{t}) - I^{p_1}_{0+}x_2(\mathfrak{t})| + |I^{p_2}_{0+}y_1(\mathfrak{t}) - I^{p_2}_{0+}y_2(\mathfrak{t})|\Big) \\
&\leq \mathfrak{c}_0\bigg(\|x_1 - x_2\| + \|y_1 - y_2\| + \frac{1}{\Gamma(p_1+1)}\|x_1 - x_2\| + \frac{1}{\Gamma(p_2+1)}\|y_1 - y_2\|\bigg) \\
&= \mathfrak{c}_0\bigg(\bigg(1 + \frac{1}{\Gamma(p_1+1)}\bigg)\|x_1 - x_2\| + \bigg(1 + \frac{1}{\Gamma(p_2+1)}\bigg)\|y_1 - y_2\|\bigg) \\
&\leq \mathfrak{c}_0\rho_1(\|x_1 - x_2\| + \|y_1 - y_2\|),
\end{aligned} \tag{35}$$

and in a similar manner

$$|\mathfrak{g}^{q_1,q_2}_{x_1,y_1}(\mathfrak{t}) - \mathfrak{g}^{q_1,q_2}_{x_2,y_2}(\mathfrak{t})| \leq \mathfrak{d}_0\rho_2(\|x_1 - x_2\| + \|y_1 - y_2\|). \tag{36}$$

Because

$$\begin{aligned}
\Big|I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(\mathfrak{t}) - I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_2,y_2}(\mathfrak{t})\Big| &\leq \frac{1}{\Gamma(\alpha)}\|\mathfrak{f}^{p_1,p_2}_{x_1,y_1} - \mathfrak{f}^{p_1,p_2}_{x_2,y_2}\|, \\
\Big|I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_1,y_1}(\mathfrak{t}) - I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_2,y_2}(\mathfrak{t})\Big| &\leq \frac{1}{\Gamma(\beta)}\|\mathfrak{g}^{q_1,q_2}_{x_1,y_1} - \mathfrak{g}^{q_1,q_2}_{x_2,y_2}\|,
\end{aligned}$$

we deduce by using (35) and (36), that

$$\begin{aligned}
&|\mathcal{A}_1(x_1,y_1)(\mathfrak{t}) - \mathcal{A}_1(x_2,y_2)(\mathfrak{t})| \\
&\leq \int_0^{\mathfrak{t}} e^{-\lambda(\mathfrak{t}-s)}\Big|I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(s) - I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_2,y_2}(s)\Big|ds \\
&\quad + |S_1(\mathfrak{t})|\bigg[\lambda\int_0^1 e^{-\lambda(1-s)}\Big|I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(s) - I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_2,y_2}(s)\Big|ds \\
&\quad + \Big|I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(1) - I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_2,y_2}(1)\Big|\bigg] \\
&\quad + |S_2(\mathfrak{t})|\bigg[\mu\int_0^1 e^{-\mu(1-s)}\Big|I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_1,y_1}(s) - I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_2,y_2}(s)\Big| \\
&\quad + \Big|I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_1,y_1}(1) - I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_2,y_2}(1)\Big|\bigg] \\
&\quad + |S_3(\mathfrak{t})|\bigg[\int_0^1 e^{-\lambda(1-s)}\Big|I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(s) - I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_2,y_2}(s)\Big|ds \\
&\quad + \bigg|\int_0^1\bigg(\int_0^s e^{-\lambda(s-\tau)}\Big|I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(\tau) - I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_2,y_2}(\tau)\Big|d\tau\bigg)d\mathfrak{H}_1(s)\bigg| \\
&\quad + \bigg|\int_0^1\bigg(\int_0^s e^{-\mu(s-\tau)}\Big|I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_1,y_1}(\tau) - I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_2,y_2}(\tau)\Big|d\tau\bigg)d\mathfrak{H}_2(s)\bigg|\bigg] \\
&\quad + |S_4(\mathfrak{t})|\bigg[\int_0^1 e^{-\mu(1-s)}\Big|I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_1,y_1}(s) - I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_2,y_2}(s)\Big|ds \\
&\quad + \bigg|\int_0^1\bigg(\int_0^s e^{-\lambda(s-\tau)}\Big|I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_1,y_1}(\tau) - I^{\alpha-1}_{0+}\mathfrak{f}^{p_1,p_2}_{x_2,y_2}(\tau)\Big|d\tau\bigg)d\mathfrak{K}_1(s)\bigg| \\
&\quad + \bigg|\int_0^1\bigg(\int_0^s e^{-\mu(s-\tau)}\Big|I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_1,y_1}(\tau) - I^{\beta-1}_{0+}\mathfrak{g}^{q_1,q_2}_{x_2,y_2}(\tau)\Big|d\tau\bigg)d\mathfrak{K}_2(s)\bigg|\bigg] \\
&\leq \mathfrak{c}_0\rho_1 U_1(\|x_1 - x_2\| + \|y_1 - y_2\|) + \mathfrak{d}_0\rho_2 U_2(\|x_1 - x_2\| + \|y_1 - y_2\|) \\
&= (\mathfrak{c}_0\rho_1 U_1 + \mathfrak{d}_0\rho_2 U_2)\|(\|x_1 - x_2\| + \|y_1 - y_2\|) = D_1(\|x_1 - x_2\| + \|y_1 - y_2\|),
\end{aligned}$$

and in a similar manner as above

$$\begin{aligned}
&|\mathcal{A}_2(x_1,y_1)(\mathfrak{t}) - \mathcal{A}_2(x_2,y_2)(\mathfrak{t})| \\
&\leq (\mathfrak{c}_0\rho_1 V_1 + \mathfrak{d}_0\rho_2 V_2)\|(\|x_1 - x_2\| + \|y_1 - y_2\|) = D_2(\|x_1 - x_2\| + \|y_1 - y_2\|).
\end{aligned}$$

So we have

$$\|\mathcal{A}(x_1,y_1) - \mathcal{A}(x_2,y_2)\|_{\mathcal{Y}} \leq (D_1 + D_2)\|(x_1,y_1) - (x_2,y_2)\|_{\mathcal{Y}}.$$

Because $D_1 + D_2 < 1$, (by (32)), we deduce that $\mathcal{A}$ is a contraction operator. Then by the Banach fixed point theorem, we deduce that problem (1) and (2) has a unique solution $(x(\mathfrak{t}), y(\mathfrak{t})),\ \mathfrak{t} \in [0,1]$. □

4. Examples

Let $\alpha = \frac{10}{3}$, $\beta = \frac{7}{2}$, $p_1 = \frac{5}{4}$, $p_2 = \frac{25}{6}$, $q_1 = \frac{11}{5}$, $q_2 = \frac{22}{7}$, $\lambda = 2$, $\mu = 3$, $\mathfrak{H}_1(s) = \frac{1}{2}s^3$, $s \in [0,1]$, $\mathfrak{H}_2(s) = \{1,\ s \in [0,1/2);\ 13/5,\ s \in [1/2,1]$, $\mathfrak{K}_1(s) = \{0,\ s \in [0,1/5);\ 9/4$, $s \in [1/5,1]$, $\mathfrak{K}_2(s) = \frac{1}{3}s^2,\ s \in [0,1]$.

We consider the system of fractional differential equations

$$\begin{cases} ({}^c\mathcal{D}^{10/3} + 2\,{}^c\mathcal{D}^{7/3})x(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{5/4}x(\mathfrak{t}), I_{0+}^{25/6}y(\mathfrak{t})),\ \ \mathfrak{t} \in (0,1), \\ ({}^c\mathcal{D}^{7/2} + 3\,{}^c\mathcal{D}^{5/2})y(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t}, x(\mathfrak{t}), y(\mathfrak{t}), I_{0+}^{11/5}x(\mathfrak{t}), I_{0+}^{22/7}y(\mathfrak{t})),\ \ \mathfrak{t} \in (0,1), \end{cases} \tag{37}$$

with the boundary conditions

$$\begin{cases} x(0) = x'(0) = 0,\ \ x'(1) = 0,\ \ x(1) = \dfrac{3}{2}\displaystyle\int_0^1 s^2 x(s)\,ds + \dfrac{8}{5}y\left(\dfrac{1}{2}\right), \\ y(0) = y'(0) = 0,\ \ y'(1) = 0,\ \ y(1) = \dfrac{9}{4}x\left(\dfrac{1}{5}\right) + \dfrac{2}{3}\displaystyle\int_0^1 sy(s)\,ds. \end{cases} \tag{38}$$

We have here

$$\begin{aligned} A_1 &= \frac{1}{2}\left(1 - e^{-2}\right) \approx 0.43233236,\ \ A_2 = \frac{1}{2}\left(1 + e^{-2}\right) \approx 0.56766764, \\ A_3 &= \frac{1}{3}\left(1 - e^{-3}\right) \approx 0.31673764,\ \ A_4 = \frac{2}{9}\left(2 + e^{-3}\right) \approx 0.45550824, \\ A_5 &= \frac{1}{4}\left(1 + e^{-2}\right) - \frac{3}{8}\int_0^1 s^2\left(2s - 1 + e^{-2s}\right) ds \approx 0.19102223, \\ A_6 &= \frac{1}{4}\left(1 - e^{-2}\right) - \frac{3}{8}\int_0^1 s^2\left(2s^2 - 2s + 1 - e^{-2s}\right) ds \approx 0.15897777, \\ A_7 &= \frac{8}{45}\left(\frac{1}{2} + e^{-3/2}\right) \approx 0.12855647,\ \ A_8 = \frac{8}{135}\left(\frac{5}{4} - 2e^{-3/2}\right) \approx 0.04762902, \\ A_9 &= \frac{9}{16}\left(-\frac{3}{5} + 2e^{-2/5}\right) \approx 0.41661005,\ \ A_{10} = \frac{9}{32}\left(\frac{34}{25} - 2e^{-2/5}\right) \approx 0.00544497, \\ A_{11} &= \frac{1}{9}\left(2 + e^{-3}\right) - \frac{2}{27}\int_0^1 \left(3s^2 - s + se^{-3s}\right) ds \approx 0.18412571, \\ A_{12} &= \frac{1}{27}\left(5 - 2e^{-3}\right) - \frac{2}{81}\int_0^1 \left(9s^3 - 6s^2 + 2s - 2se^{-3s}\right) ds \approx 0.1550273, \end{aligned}$$

$\Delta_1 \approx -0.00879828 \neq 0$, and $\Delta = A_2 A_4 \Delta_1 \approx -0.00227504 \neq 0$. So assumption $(H1)$ is satisfied.

Then we obtain $\Lambda_1 \approx 0.60132182$, $\Lambda_2 \approx 0.72004874$, $\Lambda_3 \approx -2.24322795$, $\Lambda_4 \approx -2.80486639$, $\Xi_1 \approx 3.37512736$, $\Xi_2 \approx -0.56788972$, $\Xi_3 \approx -12.12209555$, $\Xi_4 \approx -2.05567055$, $\Theta_1 \approx 1.30363097$, $\Theta_2 \approx -0.54838491$, $\Theta_3 \approx 1.70842929$, $\Theta_4 \approx 2.13616985$, $\Upsilon_1 \approx -2.34689474$, $\Upsilon_2 \approx 2.59023208$, $\Upsilon_3 \approx 8.4290989$, $\Upsilon_4 \approx 1.42941047$.

In addition, we find

$$\begin{aligned} S_i(\mathfrak{t}) &= \Lambda_i \frac{1}{4}\left(2\mathfrak{t} - 1 + e^{-2\mathfrak{t}}\right) + \Theta_i \frac{1}{4}\left(2\mathfrak{t}^2 - 2\mathfrak{t} + 1 - e^{-2\mathfrak{t}}\right),\ \mathfrak{t} \in [0,1],\ \ i = 1, \ldots, 4, \\ T_i(\mathfrak{t}) &= \Xi_i \frac{1}{9}\left(3\mathfrak{t} - 1 + e^{-3\mathfrak{t}}\right) + \Upsilon_i \frac{1}{27}\left(9\mathfrak{t}^2 - 6\mathfrak{t} + 2 - 2e^{-3\mathfrak{t}}\right),\ \mathfrak{t} \in [0,1],\ \ i = 1, \ldots, 4, \\ \widetilde{S}_1 &= \sup_{\zeta \in [0,1]} |S_1(\zeta)| \approx 0.45247639,\ \ \widetilde{S}_2 = \sup_{\zeta \in [0,1]} |S_2(\zeta)| \approx 0.08583191, \\ \widetilde{S}_3 &= \sup_{\zeta \in [0,1]} |S_3(\zeta)| \approx 0.26739933,\ \ \widetilde{S}_4 = \sup_{\zeta \in [0,1]} |S_4(\zeta)| \approx 0.33434827, \end{aligned}$$

$$\begin{aligned}
&\widetilde{T}_1 = \sup_{\zeta\in[0,1]} |T_1(\zeta)| \approx 0.34274421, \ \widetilde{T}_2 = \sup_{\zeta\in[0,1]} |T_2(\zeta)| \approx 0.34078079, \\
&\widetilde{T}_3 = \sup_{\zeta\in[0,1]} |T_3(\zeta)| \approx 1.23099888, \ \widetilde{T}_4 = \sup_{\zeta\in[0,1]} |T_4(\zeta)| \approx 0.20875336, \\
&U_1 = \frac{1}{2\Gamma(10/3)}\left(1-e^{-2}\right) + \widetilde{S}_1\frac{1}{\Gamma(10/3)}\left(2-e^{-2}\right) + \widetilde{S}_3\left[\frac{1}{2\Gamma(10/3)}\left(1-e^{-2}\right)\right. \\
&\quad \left. + \frac{1}{2\Gamma(10/3)}\frac{3}{2}\int_0^1 s^{7/3}\left(1-e^{-2s}\right)s^2\,ds\right] \\
&\quad + \widetilde{S}_4\frac{1}{2\Gamma(10/3)}\frac{9}{4}\left(\frac{1}{5}\right)^{7/3}\left(1-e^{-2/5}\right) \approx 0.51289045, \\
&U_2 = \widetilde{S}_2\frac{1}{\Gamma(7/2)}\left(2-e^{-3}\right) + \widetilde{S}_3\frac{1}{3\Gamma(7/2)}\frac{8}{5}\left(\frac{1}{2}\right)^{5/2}\left(1-e^{-3/2}\right) \\
&\quad + \widetilde{S}_4\left[\frac{1}{3\Gamma(7/2)}\left(1-e^{-3}\right) + \frac{1}{3\Gamma(7/2)}\frac{2}{3}\int_0^1 s^{5/2}\left(1-e^{-3s}\right)s\,ds\right] \approx 0.09261299, \\
&V_1 = \widetilde{T}_1\frac{1}{\Gamma(10/3)}\left(2-e^{-2}\right) + \widetilde{T}_3\frac{1}{2\Gamma(10/3)}\left[1-e^{-2}+\frac{3}{2}\int_0^1 s^{7/3}\left(1-e^{-2s}\right)s^2\,ds\right] \\
&\quad + \widetilde{T}_4\frac{1}{2\Gamma(10/3)}\frac{9}{4}\left(\frac{1}{5}\right)^{7/3}\left(1-e^{-2/5}\right) \approx 0.47253171, \\
&V_2 = \frac{1}{3\Gamma(7/2)}\left(1-e^{-3}\right) + \widetilde{T}_2\frac{1}{\Gamma(7/2)}\left(2-e^{-3}\right) + \widetilde{T}_3\frac{1}{3\Gamma(7/2)}\frac{8}{5}\left(\frac{1}{2}\right)^{5/2}\left(1-e^{-3/2}\right) \\
&\quad + \widetilde{T}_4\frac{1}{3\Gamma(7/2)}\left[1-e^{-3}+\frac{2}{3}\int_0^1 s^{5/2}\left(1-e^{-3s}\right)s\,ds\right] \approx 0.34511088.
\end{aligned}$$

Example 1. We consider the functions

$$\begin{aligned}
\mathfrak{f}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4) &= \frac{\mathfrak{t}}{\mathfrak{t}^3+1}\left(2\cos\mathfrak{t} + \frac{1}{8}\sin(\mathfrak{u}_1+\mathfrak{u}_2)\right) - \frac{1}{9(\mathfrak{t}+2)^2}\mathfrak{u}_3 + \frac{1}{11}\arctan\mathfrak{u}_4, \\
\mathfrak{g}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4) &= \frac{1}{(\mathfrak{t}+3)^3}\left(7e^{-\mathfrak{t}} + \frac{1}{3}\mathfrak{u}_1 + 4\mathfrak{u}_2\right) - \frac{\mathfrak{t}+5}{8}\sin(\mathfrak{u}_3+\mathfrak{u}_4),
\end{aligned} \tag{39}$$

for all $\mathfrak{t}\in[0,1]$, $\mathfrak{u}_j\in\mathbb{R}$, $j=1,\dots,4$.

We obtain the inequalities

$$\begin{aligned}
|\mathfrak{f}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4)| &\le \frac{2\sqrt[3]{4}}{3} + \frac{\sqrt[3]{4}}{24}|\mathfrak{u}_1| + \frac{\sqrt[3]{4}}{24}|\mathfrak{u}_2| + \frac{1}{36}|\mathfrak{u}_3| + \frac{1}{11}|\mathfrak{u}_4|, \\
|\mathfrak{g}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4)| &\le \frac{7}{27} + \frac{1}{81}|\mathfrak{u}_1| + \frac{4}{27}|\mathfrak{u}_2| + \frac{5}{8}|\mathfrak{u}_3| + \frac{5}{8}|\mathfrak{u}_4|,
\end{aligned}$$

for all $\mathfrak{t}\in[0,1]$ and $\mathfrak{u}_j\in\mathbb{R}$, $j=1,\dots,4$. We also have $\mathfrak{a}_0=\frac{2\sqrt[3]{4}}{3}$, $\mathfrak{a}_1=\frac{\sqrt[3]{4}}{24}$, $\mathfrak{a}_2=\frac{\sqrt[3]{4}}{24}$, $\mathfrak{a}_3=\frac{1}{36}$, $\mathfrak{a}_4=\frac{1}{11}$, $\mathfrak{b}_0=\frac{7}{27}$, $\mathfrak{b}_1=\frac{1}{81}$, $\mathfrak{b}_2=\frac{4}{27}$, $\mathfrak{b}_3=\frac{5}{8}$, $\mathfrak{b}_4=\frac{5}{8}$.

We find here $\Psi_1\approx 0.20760463$ and $\Psi_2\approx 0.17091827$. We deduce that the condition (25), that is $\max\{\Psi_1,\Psi_2\}=\Psi_1<1$ is satisfied. Then by Theorem 1 we conclude that the problem (37) and (38) with the nonlinearities (39) has at least one solution $(x(\mathfrak{t}),y(\mathfrak{t}))$, $\mathfrak{t}\in[0,1]$.

Example 2. We consider the functions

$$\begin{aligned}
\mathfrak{f}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4) &= \frac{\mathfrak{t}+1}{3} + \frac{1}{9(\mathfrak{t}+2)^2}\left(-\mathfrak{u}_1 + \frac{|\mathfrak{u}_2|}{1+|\mathfrak{u}_2|}\right) \\
&\quad - \frac{1}{3\sqrt{1+\mathfrak{t}^2}}\cos\mathfrak{u}_3 + \frac{\mathfrak{t}}{4}\arctan\mathfrak{u}_4, \\
\mathfrak{g}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4) &= \frac{\mathfrak{t}^2+2}{\mathfrak{t}^3+4} - \frac{1}{7}\mathfrak{u}_1 + \frac{1}{8}\sin\mathfrak{u}_2 \\
&\quad + \frac{1}{5\sqrt{\mathfrak{t}+3}}\sin^2\mathfrak{u}_3 - e^{-2\mathfrak{t}}\frac{|\mathfrak{u}_4|}{8(1+|\mathfrak{u}_4|)},
\end{aligned} \tag{40}$$

for all $\mathfrak{t}\in[0,1]$ and $\mathfrak{u}_j\in\mathbb{R}$, $j=1,\dots,4$.

We obtain here the following inequalities

$$
\begin{aligned}
&|\mathfrak{f}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4)-\mathfrak{f}(\mathfrak{t},\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)| \leq \frac{1}{36}|\mathfrak{u}_1-\mathfrak{v}_1| + \frac{1}{36}|\mathfrak{u}_2-\mathfrak{v}_2| \\
&\quad + \frac{1}{3}|\mathfrak{u}_3-\mathfrak{v}_3| + \frac{1}{4}|\mathfrak{u}_4-\mathfrak{v}_4| \leq \frac{1}{3}\sum_{i=1}^{4}|\mathfrak{u}_i-\mathfrak{v}_i|, \\
&|\mathfrak{g}(\mathfrak{t},\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4)-\mathfrak{g}(\mathfrak{t},\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)| \leq \frac{1}{7}|\mathfrak{u}_1-\mathfrak{v}_1| + \frac{1}{8}|\mathfrak{u}_2-\mathfrak{v}_2| \\
&\quad + \frac{2}{5\sqrt{3}}|\mathfrak{u}_3-\mathfrak{v}_3| + \frac{1}{8}|\mathfrak{u}_4-\mathfrak{v}_4| \leq \frac{2}{5\sqrt{3}}\sum_{i=1}^{4}|\mathfrak{u}_i-\mathfrak{v}_i|,
\end{aligned}
$$

for all $\mathfrak{t} \in [0,1]$ and $\mathfrak{u}_j, \mathfrak{v}_j \in \mathbb{R},\ j = 1,\ldots,4$. So we have $\mathfrak{c}_0 = \frac{1}{3}$ and $\mathfrak{d}_0 = \frac{2}{5\sqrt{3}}$. In addition we find $\rho_1 = 1 + \frac{1}{\Gamma(9/4)} \approx 1.88261$, $\rho_2 = 1 + \frac{1}{\Gamma(16/5)} \approx 1.41255$, $D_1 \approx 0.35206922$ and $D_2 \approx 0.40911092$. Then $D_1 + D_2 \approx 0.76118013 < 1$, that is the condition (32) is satisfied. Therefore by Theorem 2 we conclude that problem (37) and (38) with the nonlinearities (40) has a unique solution $(x(\mathfrak{t}), y(\mathfrak{t})),\ \mathfrak{t} \in [0,1]$.

5. Conclusions

In this paper we investigated the existence of solutions, and the existence and uniqueness of solution for the system of fractional equations with sequential Caputo derivatives, two positive parameters and nonlinearities which contain various integral terms (1), supplemented with the general Riemann-Stieltjes integral nonlocal boundary conditions (2). We mention that in these conditions the unknown functions x and y in the point 1 are dependent on both x and y in the whole interval $[0,1]$. In the proof of our main theorems we applied the Leray-Schauder alternative theorem and the Banach contraction mapping principle.

Author Contributions: Conceptualization, R.L.; Formal analysis, A.T. and R.L.; Methodology, A.T. and R.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley and Sons: New York, NY, USA, 1993.
2. Wei, Z.; Li, Q.; Che, J. Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative. *J. Math. Anal. Appl.* **2010**, *367*, 260–272. [CrossRef]
3. Wei, Z.; Dong, W. Periodic boundary value problems for Riemann-Liouville sequential fractional differential equations. *Electr. J. Qual. Theory Differ. Equ.* **2011**, *87*, 1–13. [CrossRef]
4. Bai, C. Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative. *J. Math. Anal. Appl.* **2011**, *384*, 211–231. [CrossRef]
5. Baleanu, D.; Mustafa, O.G.; Agarwal, R.P. On L^p-solutions for a class of sequential fractional differential equations. *Appl. Math. Comput.* **2011**, *218*, 2074–2081. [CrossRef]
6. Klimek, M. Sequential fractional differential equations with Hadamard derivative. *Commun. Nonlinear Sci. Numer. Simulat.* **2011**, *16*, 4689–4697. [CrossRef]
7. Ahmad, B.; Nieto, J.J. Sequential fractional differential equations with three-point boundary conditions. *Comput. Math. Appl.* **2012**, *64*, 3046–3052. [CrossRef]
8. Ahmad, B.; Nieto, J.J. Boundary value problems for a class of sequential integrodifferential equations of fractional order. *J. Function Spaces Appl.* **2013**, *2013*, 149659. [CrossRef]
9. Alsaedi, A.; Ntouyas, S.K.; Agarwal, R.P.; Ahmad, B. On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Adv. Diff. Equ.* **2015**, *33*, 1–12. [CrossRef]

10. Ahmad, B.; Ntouyas, S.K. Existence results for Caputo type sequential fractional differential inclusions with nonlocal integral boundary conditions. *J. Appl. Math. Comput.* **2016**, *50*, 157–174. [CrossRef]
11. Aqlan, M.H.; Alsaedi, A.; Ahmad, B.; Nieto, J.J. Existence theory for sequential fractional differential equations with anti-periodic type boundary conditions. *Open Math.* **2016**, *14*, 723–735. [CrossRef]
12. Ahmad, B.; Alsaedi, A.; Aqlan, M.H. Sequential fractional differential equations and unification of anti-periodic and multi-point boundary conditions. *J. Nonlinear Sci. Appl.* **2017**, *10*, 71–83.
13. Ahmad, B.; Luca, R. Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions. *Appl. Math. Comput.* **2018**, *339*, 516–534. [CrossRef]
14. Ahmad, B.; Ntouyas, S.K.; Alsaedi, A. Sequential fractional differential equations and inclusions with semi-periodic and nonlocal integro-multipoint boundary conditions. *J. King Saud Univ. Sc.* **2019**, *31*, 184–193. [CrossRef]
15. Ahmad, B.; Ntouyas, S.K. Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Appl. Math. Comput.* **2015**, *266*, 615–622. [CrossRef]
16. Aljoudi, S.; Ahmad, B.; Nieto, J.J.; Alsaedi, A. A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions. *Chaos Solitons Fractals* **2016**, *91*, 39–46. [CrossRef]
17. Ahmad, B.; Alsaedi, A.; Aljoudi, S.; Ntouyas, S.K. A six-point nonlocal boundary value problem of nonlinear coupled sequential fractional integro-differential equations and coupled integral boundary conditions. *J. Appl. Math. Comput.* **2018**, *56*, 367–389. [CrossRef]
18. Alruwaily, Y.; Ahmad, B.; Ntouyas, S.K.; Alzaidi, A.S.M. Existence results for coupled nonlinear sequential fractional differential equations with coupled Riemann-Stieltjes integro-multipoint boundary conditions. *Fractal Fract.* **2022**, *6*, 123. [CrossRef]
19. Ahmad, B.; Luca, R. Existence of solutions for a sequential fractional integro-differential system with coupled integral boundary conditions. *Chaos Solitons Fractals* **2017**, *104*, 378–388. [CrossRef]
20. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017.
21. Ahmad, B.; Henderson, J.; Luca, R. *Boundary Value Problems for Fractional Differential Equations and Systems*; Trends in Abstract and Applied Analysis 9; World Scientific: Hackensack, NJ, USA, 2021.
22. Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus Models and Numerical Methods*; Series on Complexity, Nonlinearity and Chaos; World Scientific: Boston, MA, USA, 2012.
23. Das, S. *Functional Fractional Calculus for System Identification and Controls*; Springer: New York, NY, USA, 2008.
24. Henderson, J.; Luca, R. *Boundary Value Problems for Systems of Differential, Difference and Fractional Equations. Positive Solutions*; Elsevier: Amsterdam, The Netherlands, 2016.
25. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
26. Klafter, J.; Lim, S.C.; Metzler, R. (Eds.). *Fractional Dynamics in Physics*; World Scientific: Singapore, 2011.
27. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
28. Sabatier, J.; Agrawal, O.P.; Machado, J.A.T. (Eds.). *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*; Springer: Dordrecht, The Netherlands, 2007.
29. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Theory and Applications; Gordon and Breach: Yverdon, Switzerland, 1993.
30. Zhou, Y.; Wang, J.R.; Zhang, L. *Basic Theory of Fractional Differential Equations*, 2nd ed.; World Scientific: Singapore, 2016.