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On the 1st-Level General Fractional Derivatives of Arbitrary Order

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Abstract: In this paper, the 1st-level general fractional derivatives of arbitrary order are defined and investigated for the first time. We start with a generalization of the Sonin condition for the kernels of the general fractional integrals and derivatives and then specify a set of the kernels that satisfy this condition and possess an integrable singularity of the power law type at the origin. The 1st-level general fractional derivatives of arbitrary order are integro-differential operators of convolution type with the kernels from this set. They contain both the general fractional derivatives of arbitrary order of the Riemann–Liouville type and the regularized general fractional derivatives of arbitrary order considered in the literature so far. For the 1st-level general fractional derivatives of arbitrary order, some important properties, including the 1st and the 2nd fundamental theorems of fractional calculus, are formulated and proved.

Keywords: generalized Sonin condition; general fractional integral; general fractional derivative of arbitrary order; regularized general fractional derivative of arbitrary order; 1st-level general fractional derivative; 1st fundamental theorem of fractional calculus; 2nd fundamental theorem of fractional calculus

MSC: 26A33; 26B30; 33E30; 44A10; 44A35; 44A40; 45D05; 45E10; 45J05

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1. Introduction

Within the last few years, a lot of attention in the fractional calculus (FC) literature has been devoted to the so-called generalized Riemann–Liouville fractional derivative, nowadays often referred to as the Hilfer fractional derivative that is defined as follows:

$$(D_{0+}^{\alpha,\beta} f)(t) := (I_{0+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} (I_{0+}^{(1-\beta)(n-\alpha)} f))(t),\ t > 0,\ n-1 < \alpha \le n,\ n \in \mathbb{N}, \tag{1}$$

where the Riemann–Liouville fractional integral I_{0+}^{α} of the order α, $\alpha > 0$ is given by the formula

$$(I_{0+}^{\alpha} f)(t) := \left(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)} * f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ t > 0. \tag{2}$$

Because the operator family I_{0+}^{α} tends to the identity operator, say, in the sense of the L_p-norm as $\alpha \to 0+$, the Riemann–Liouville fractional integral of the order $\alpha = 0$ is defined as the identity operator:

$$(I_{0+}^{0} f)(t) := f(t),\ t > 0. \tag{3}$$

In Ref. [1] (see also [2]), the operator (1) was treated in the case of the derivative order $\alpha \in (0, 1]$ and type $\beta \in [0, 1]$ and in [3] in the case of an arbitrary non-negative order α ($n-1 < \alpha \le n$, $n \in \mathbb{N}$) and type $\beta \in [0, 1]$.

The Hilfer fractional derivative with the order $\alpha \in (0, 1]$ is a particular case of a more general operator introduced in the paper [4] by Djrbashian and Nersessian published in 1968. However, the operator (1) with the derivative order $\alpha > 1$ seems to be not considered in the literature before publication of the paper [3].

It is worth mentioning that in [5], a different parametrization of the Djrbashian–Nersessian operators was introduced in the form

$$(D^{\alpha,(\gamma)}_{nL}\, f)(t) := \left(\prod_{k=1}^{n} (I^{\gamma_k}\, \frac{d}{dt})\right) (I^{n-\alpha-s_n}\, f)(t), \tag{4}$$

where

$$s_k = \sum_{i=1}^{k} \gamma_i,\ k = 1, 2, \ldots, n$$

and the parameters $\alpha \in (0,\, 1]$ and $\gamma_1,\ \gamma_2, \ldots, \gamma_n \in \mathbb{R}$ satisfy the conditions

$$0 \le \gamma_k \text{ and } \alpha + s_k \le k,\ \ k = 1, 2, \ldots, n.$$

Because the operators (4) are compositions of n first-order derivatives and $n+1$ Riemann–Liouville fractional integrals, in [5], they were called the nth-level fractional derivatives of order α, $0 < \alpha \le 1$ and type $\gamma = (\gamma_1,\ \gamma_2, \ldots, \gamma_n)$. It is easy to see that the Hilfer fractional derivative (1) with the order $\alpha \in (0,\, 1]$ can be interpreted as a 1st level fractional derivative. In this paper, we introduce a generalization of the Hilfer fractional derivative of arbitrary order in the form of the integro-differential operators with some general kernels. By analogy with the 1st-level fractional derivative (the Hilfer fractional derivative), we call these operators the 1st-level general fractional derivatives.

The main advantage of the Hilfer fractional derivative (1) is that this operator contains both the Riemann–Liouville fractional derivative D^{α}_{0+} and the Caputo fractional derivative $_*D^{\alpha}_{0+}$ as its particular cases ($n-1 < \alpha \le n,\ n \in \mathbb{N}$):

$$(D^{\alpha,0}_{0+}\, f)(x) = (I^{0}_{0+}\, \frac{d^n}{dt^n}\, I^{n-\alpha}_{0+}\, f)(t) \ = \ \frac{d^n}{dt^n}\, (I^{n-\alpha}_{0+}\, f)(t) \ =: \ (D^{\alpha}_{0+}\, f)(t), \tag{5}$$

$$(D^{\alpha,1}_{0+}\, f)(t) \ = \ (I^{n-\alpha}_{0+}\, \frac{d^n}{dt^n}\, I^{0}_{0+}\, f)(t) \ = \ (I^{n-\alpha}_{0+}\, f^{(n)})(t) \ =: \ (_*D^{\alpha}_{0+}\, f)(t). \tag{6}$$

Moreover, the Hilfer fractional derivative (1) with any value of the parameter $\beta \in [0,\, 1]$ is a left-inverse operator to the Riemann–Liouville fractional integral (2) of the order $\alpha,\ \alpha > 0$ (see, for example, [5]) and thus, it can be interpreted as a fractional derivative of the order α. Because of the relations (5) and (6), any result derived for the Hilfer fractional derivative covers the analogous results for the Riemann–Liouville and for the Caputo fractional derivatives that are often obtained using the different methods and in the separate publications.

Another hot topic in the current FC literature is the so-called general fractional integrals and derivatives with the Sonin kernels that satisfy the Sonin condition ([6]):

$$(\kappa\, *\, k)(t) = \int_0^t \kappa(t-\tau)\, k(\tau)\, d\tau \ = \ \{1\},\ t > 0, \tag{7}$$

where $*$ denotes the Laplace convolution, and $\{1\}$ stands for the function identically equal to one for $t > 0$. For a certain kernel κ, the function k in the relation (7) is called a kernel associated to the kernel κ. A very important pair of the Sonin kernels is provided in terms of the power law functions

$$\kappa(t) = h_{\alpha}(t),\ \ k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1, \tag{8}$$

where

$$h_{\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ \alpha > 0. \tag{9}$$

The relation (7) for the functions h_{α} and $h_{1-\alpha}$ in the form

$$(h_{\alpha}\, *\, h_{1-\alpha})(t) \ = \ h_1(t) \ = \ \{1\},\ t > 0,\ 0 < \alpha < 1 \tag{10}$$

was known already to Abel. In Ref. [7], Abel used this formula to derive a solution to the integro-differential equation (in slightly different notations)

$$f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \phi'(\tau)\, d\tau,\ t > 0,\ 0 < \alpha < 1 \tag{11}$$

with $\phi(0) = 0$ in the form

$$\phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ t > 0. \tag{12}$$

In Ref. [6], Sonin recognized that the same method can be applied to solve the integral equation

$$f(t) = (\kappa * \phi)(t) = \int_0^t \kappa(t-\tau)\phi(\tau)\, d\tau,\ t > 0 \tag{13}$$

with a kernel κ that satisfies the condition (7). Its solution can be represented in the form

$$\phi(t) = \frac{d}{dt}(k * f)(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau)\, d\tau,\ t > 0, \tag{14}$$

where k is the Sonin kernel associated to the kernel κ.

In Ref. [8] (see also [9–11]), Kochubei investigated the properties of the integro-differential operators of the convolution type with the kernels k that satisfy the conditions (K1)–(K4) below:

(K1) The Laplace transform $\tilde{k}$ of k

$$\tilde{k}(p) = (\mathcal{L}\, k)(p) = \int_0^{+\infty} k(t)\, e^{-pt}\, dt \tag{15}$$

exists for all $p > 0$.

(K2) $\tilde{k}(p)$ is a Stieltjes function (see [12] for details regarding the Stieltjes functions).

(K3) $\tilde{k}(p) \to 0$ and $p\tilde{k}(p) \to +\infty$ as $p \to +\infty$.

(K4) $\tilde{k}(p) \to +\infty$ and $p\tilde{k}(p) \to 0$ as $p \to 0$.

It turns out that for any function k that satisfies the properties (K1)–(K4), there exists a function κ such that the Sonin condition (7) is fulfilled. Thus, the functions κ and k are the Sonin kernels. In what follows, the set of such kernels is denoted by $\mathcal{K}$ and referred to as the Kochubei set of the Sonin kernels.

For $(\kappa, k) \in \mathcal{K}$, Kochubei interpreted the operators defined by the right-hand sides of Equations (13) and (14) as the general fractional integral (GFI) and the general fractional derivative (GFD), respectively:

$$(\mathbb{I}_{(\kappa)}\, f)(t) := (\kappa * f)(t) = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau,\ t > 0, \tag{16}$$

$$(\mathbb{D}_{(k)}\, f)(t) := \frac{d}{dt}(k * f)(t) = \frac{d}{dt}\,(\mathbb{I}_{(k)}\, f)(t),\ t > 0. \tag{17}$$

Kochubei also introduced a regularized GFD in the form

$$({}_*\mathbb{D}_{(k)}\, f)(t) := (\mathbb{D}_{(k)}\, f)(t) - f(0)k(t),\ t > 0. \tag{18}$$

For the functions with the integrable first order derivative, this GFD can be also represented as follows:

$$({}_*\mathbb{D}_{(k)}\, f)(t) = (\mathbb{I}_{(k)}\, f')(t),\ t > 0. \tag{19}$$

The GFD (17) and the regularized GFD (18) with the kernels from the Kochubei set $\mathcal{K}$ are the left inverse operators to the GFI defined by (16) [5,8,13]. Moreover, the solutions to the Cauchy problems for the time-fractional ordinary and partial differential equations with the regularized GFD (18) possess some typical features of solutions to the evolution equations; see [8,13–15] for more details. For a treatment of some inverse problems for the fractional differential equations with the GFDs, we refer to the publications [16–18]. The Adams-type predictor–corrector method for the numerical solution of the fractional differential equations with the GFDs was presented in the recent paper [19].

It is worth mentioning that the properties of the GFI defined as in (16) and of the GFDs defined as in (17) and (18) essentially depend on the additional conditions posed of their kernels. In this sense, there exists not just one, but several theories of the GFIs and the GFDs, each one constructed for the operators with the kernels from its own set.

Recently, in a series of papers [20–26] by Luchko, another important set of the Sonin kernels was introduced. The kernels from this set are continuous on the real positive semi-axes and possess an integrable singularity of the power function type at the point zero. In Ref. [27] (see also [28–32]), this set of Sonin kernels was called the Luchko set, and the GFIs and the GFDs with these kernels were referred to as the Luchko GFIs and the Luchko GFDs, respectively. In the papers [20–26], the GFI (16), the GFD (17), and the regularized GFD (18) with the Sonin kernels from the Luchko set were studied on the space of functions that are continuous on the real positive semi-axis and have an integrable singularity of power function type at point zero and its suitable sub-spaces.

In the meantime, the GFI (16), the GFD (17), and the regularized GFD (18) with the Sonin kernels from the Luchko set were already employed in a number of innovative mathematical models of some important physical theories. In particular, in [28–34], Tarasov used these operators for the formulation of general fractional dynamics, general non-Markovian quantum dynamics, general fractional vector calculus, general non-local continuum mechanics, a non-local probability theory, non-local statistical mechanics, and a non-local gravity theory, respectively. It is also worth mentioning that the theory of the GFIs and the GFDs was recently employed in [35] for the construction of the so-called Scarpi variable-order fractional integrals and derivatives.

In Ref. [26], a construction of a GFD that comprises both the GFD (17) and the regularized GFD (18) was introduced for the first time in the FC literature. Following the notation suggested in [5] for the fractional derivative (1) with the order less than or equal to one, this GFD was called the 1st-level GFD. To define the 1st-level GFD, the Sonin condition (7) was extended to the case of three kernels. Then, a suitable set of kernels that satisfy this extended condition was introduced. For the kernels from this set, the GFI was defined as in (16), whereas the 1st-level GFD was introduced as a composition of a GFI, the first-order derivative, and another GFI. In the case of the power law kernels, this 1st-level GFD is reduced to the Hilfer fractional derivative (1) with the order less than or equal to one.

It is worth mentioning that the construction of the 1st-level GFD proposed in [26] is restricted to the case of the "generalized fractional order" less than or equal to one. In the case of the Hilfer fractional derivative, this corresponds to the value $n = 1$ in Equation (1). However, in [21], the GFD of arbitrary order and the regularized GFD of arbitrary order were defined and investigated. Moreover, the case of the general FC operators of arbitrary order is very important for applications of this theory, say, for the construction of the general non-local models of the anomalous diffusion-wave processes.

The main aim of this paper is defining and investigating the 1st-level GFD of arbitrary order that comprises both the GFD and the regularized GFD of arbitrary order that were introduced in [21].

The rest of the paper is organized as follows: Section 2 is devoted to a presentation of the basic results derived in the literature for the GFD of arbitrary order and the regularized GFD of arbitrary order with the kernels from the Luchko set. In Section 3, the 1st-level GFD of arbitrary order is introduced and investigated. For this fractional derivative and the GFI defined as in (16), the 1st and the 2nd fundamental theorems of FC are formulated and proved. As a consequence of the 2nd fundamental theorem of FC, a formula for the projector operator for the 1st-level GFD of arbitrary order is derived. In particular, this formula determines the form of the natural initial conditions for the fractional differential equations with the 1st-level GFDs of arbitrary order. Section 4 contains some discussions and directions for further research.

2. The GFI, the GFD, and the Regularized GFD of Arbitrary Order

In this section, we provide the definitions and the basic properties of the GFI of arbitrary order, the GFD of arbitrary order, and the regularized GFD of arbitrary order that were introduced in [21] for the first time.

The GFI, the GFD, and the regularized GFD are a far reaching generalization of the Riemann–Liouville fractional integral (2), the Riemann–Liouville fractional derivative (5), and the Caputo fractional derivative (6), respectively. However, the GFI (16) and the GFDs (17) and (18) with the Sonin kernels cover only the case of the Riemann–Liouville fractional integral I_{0+}^{α}, the Riemann–Liouville fractional derivative D_{0+}^{α}, and the Caputo fractional derivative ${}_*D_{0+}^{\alpha}$ with the order α from the interval $(0, 1)$:

$$(\mathbb{I}_{(h_\alpha)}\, f)(t) \,=\, (h_\alpha \,*\, f)(t) \,=\, (I_{0+}^{\alpha}\, f)(t),\ 0 < \alpha < 1,\ t > 0, \tag{20}$$

$$(\mathbb{D}_{(h_{1-\alpha})}\, f)(t) \,=\, \frac{d}{dt}(h_{1-\alpha} \,*\, f)(t) \,=\, \frac{d}{dt}(I_{0+}^{1-\alpha}\, f)(t) \,=\, (D_{0+}^{\alpha}\, f)(t),\ 0 < \alpha < 1,\ t > 0, \tag{21}$$

$$({}_*\mathbb{D}_{(h_{1-\alpha})}\, f)(t) \,=\, (h_{1-\alpha} \,*\, f')(t) \,=\, (I_{0+}^{1-\alpha}\, f')(t) \,=\, ({}_*D_{0+}^{\alpha}\, f)(t),\ 0 < \alpha < 1,\ t > 0. \tag{22}$$

The condition $0 < \alpha < 1$ in Equations (20)–(22) is a consequence of the fact that the power law functions $\kappa(t) = h_\alpha(t)$ and $k(t) = h_{1-\alpha}(t)$ are the Sonin kernels if and only if the parameter α satisfies this condition.

It is worth mentioning that the Sonin condition (10) can be extended to the case $\alpha = 0$ or $\alpha = 1$, respectively, in the sense of the generalized functions (the function h_0 plays the role of the Dirac δ-function):

$$h_0 \,*\, h_1 = h_1. \tag{23}$$

The relation (23) immediately leads to the following interpretations of the GFI (16), the GFD (17), and the regularized GFD (18) (or (19)) with the kernels h_0 and h_1:

$$(\mathbb{I}_{(h_0)}\, f)(t) \,=\, (h_0 \,*\, f)(t) \,=\, (I_{0+}^{0}\, f)(t) \,=\, f(t),\ t > 0, \tag{24}$$

$$(\mathbb{I}_{(h_1)}\, f)(t) \,=\, (h_1 \,*\, f)(t) \,=\, (I_{0+}^{1}\, f)(t) \,=\, \int_0^t f(t)\, dt,\ t > 0, \tag{25}$$

$$(\mathbb{D}_{(h_0)}\, f)(t) \,=\, \frac{d}{dt}(h_0 \,*\, f)(t) \,=\, f'(t),\ t > 0, \tag{26}$$

$$(\mathbb{D}_{(h_1)}\, f)(t) \,=\, \frac{d}{dt}(h_1 \,*\, f)(t) \,=\, \frac{d}{dt}\int_0^t f(t)\, dt \,=\, f(t),\ t > 0, \tag{27}$$

$$({}_*\mathbb{D}_{(h_0)}\, f)(t) \,=\, (\mathbb{I}_{(h_0)}\, f')(t) \,=\, f'(t),\ t > 0, \tag{28}$$

$$({}_*\mathbb{D}_{(h_1)}\, f)(t) \,=\, (\mathbb{I}_{(h_1)}\, f')(t) \,=\, \int_0^t f'(t)\, dt \,=\, f(t) - f(0),\ t > 0. \tag{29}$$

Because the right-hand side of the Sonin condition (7) is the function h_1 that corresponds to the definite integral (the Riemann–Liouville fractional integral of the order one), the "generalized fractional orders" of the GFI (16), the GFD (17), and the regularized GFD (18) with the Sonin kernels κ and k are less than or equal to one.

However, it is well known that both the Riemann–Liouville fractional integral and the Riemann–Liouville and Caputo fractional derivatives are well defined for an arbitrary order $\alpha \geq 0$. To introduce the GFI, the GFD, and the regularized GFD of arbitrary order, in [21], an extension of the Sonin condition (7) for the kernels κ and k has been suggested, and a set of the kernels that satisfy the extended Sonin condition and belong to the suitable spaces of functions has been specified. In this section, we represent those results from [21] that will be needed for the further discussions.

Definition 1 ([21])**.** *Let the functions κ and k defined on the positive real semi-axes satisfy the following conditions:*

(L1)

$$(\kappa * k)(t) = \{1\}^{<n>}(t),\ n \in \mathbb{N},\ t > 0, \quad (30)$$

where

$$\{1\}^{<n>}(t) := (\underbrace{\{1\} * \ldots * \{1\}}_{n\ times})(t) = h_n(t) = \frac{t^{n-1}}{(n-1)!}.$$

(L2)

$$\kappa \in C_{-1}(0, +\infty), \quad (31)$$

where

$$C_{-1}(0, +\infty) := \{f :\ f(t) = t^p f_1(t),\ t > 0,\ p > -1,\ f_1 \in C[0, +\infty)\}. \quad (32)$$

(L3)

$$k \in C_{-1,0}(0, +\infty), \quad (33)$$

where

$$C_{-1,0}(0, +\infty) := \{f :\ f(t) = t^p f_1(t),\ t > 0,\ -1 < p < 0,\ f_1 \in C[0, +\infty)\}. \quad (34)$$

The set of the ordered pairs (κ, k) *of such kernels is denoted by* $\mathcal{L}_n$.

Remark 1. *The condition (30) is a generalization of the Sonin condition (7). The kernels* (κ, k) *that satisfy the condition (30) are not the Sonin kernels unless* $n = 1$*. Moreover, in Definition 1, the spaces of functions for the kernels* κ *and* k *are specified in the inclusions (31) and (33). Both the condition (30) and the inclusions (31) and (33) are essential components for the construction of a self-contained theory of the GFIs and the GFDs with the kernels from the set* $\mathcal{L}_n$.

Remark 2. *Evidently, the inclusion* $C_{-1,0}(0, +\infty) \subset C_{-1}(0, +\infty)$ *holds true. Any function from* $C_{-1,0}(0, +\infty)$ *is continuous on the real positive semi-axes and has an integrable singularity at the point zero. In contrast, the functions from* $C_{-1}(0, +\infty)$ *can have an integrable singularity at the point zero or not. Thus, in general, the kernels* κ *and* k *from Definition 1 cannot be interchanged if* $n \geq 2$*. However, in the case* $n = 1$*, both functions* κ *and* k *are the Sonin kernels that have an integrable singularity at the point zero and thus* $\kappa, k \in C_{-1,0}(0, +\infty)$ *if* $(\kappa, k) \in \mathcal{L}_1$.

Remark 3. *The kernels of all reasonable time-fractional integrals and derivatives introduced so far in the FC literature belong to the set* $\mathcal{L}_n$*. In particular, the kernels* $\kappa(t) = h_\alpha(t)$ *and* $k(t) = h_{n-\alpha}(t)$ *of the Riemann–Liouville fractional integral of arbitrary order* $\alpha > 0$ *defined by Equation (2) and the Riemann–Liouville and Caputo fractional derivatives of arbitrary order* $\alpha > 0$ *defined as in Equations (5) and (6), respectively, are from the set* $\mathcal{L}_n$ *if the order* α *satisfies the conditions* $n - 1 < \alpha < n,\ n \in \mathbb{N}$.

Indeed, the inclusions $h_\alpha \in C_{-1}(0, +\infty)$ *and* $h_{n-\alpha} \in C_{-1,0}(0, +\infty)$ *evidently hold true for any* α *under the conditions* $n - 1 < \alpha < n,\ n \in \mathbb{N}$*. The relation*

$$(h_\alpha * h_{n-\alpha})(t) = h_n(t),\ n - 1 < \alpha < n,\ n \in \mathbb{N} \quad (35)$$

is a particular case of the formula

$$(h_\alpha * h_\beta)(t) = h_{\alpha+\beta}(t),\ \alpha, \beta > 0, \quad (36)$$

that immediately follows from the well-known relation between the Euler beta- and gamma functions:

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

For $n = 1$, the condition (30) is just the Sonin condition (7). Thus, the set $\mathcal{L}_1$ contains all Sonin kernels that belong to the space $C_{-1,0}(0,+\infty)$ (see Remark 2). Because the space $C_{-1,0}(0,+\infty)$ is a very general one, almost all of the Sonin kernels introduced so far belong to this space. In particular, in [6], the following family of the Sonin kernels was derived:

$$\kappa(t) = h_\alpha(t) \cdot \kappa_1(t),\ \kappa_1(t) = \sum_{k=0}^{+\infty} a_k t^k,\ a_0 \neq 0,\ 0 < \alpha < 1, \tag{37}$$

$$k(t) = h_{1-\alpha}(t) \cdot k_1(t),\ k_1(t) = \sum_{k=0}^{+\infty} b_k t^k, \tag{38}$$

where κ_1 and k_1 are analytical functions, and the coefficients $a_k,\ b_k,\ k \in \mathbb{N}_0$ satisfy an infinite triangular system of linear equations:

$$a_0 b_0 = 1,\ \sum_{k=0}^{n} \Gamma(k+1-\alpha)\Gamma(\alpha+n-k) a_{n-k} b_k = 0,\ n \geq 1. \tag{39}$$

Evidently, the kernels κ and k specified by Equations (37) and (38) are from the set $\mathcal{L}_1$. In Ref. [6], Sonin derived an important particular case of the kernels in forms (37) and (38):

$$\kappa(t) = t^{\nu/2} J_\nu(2\sqrt{t}),\ k(t) = t^{-\nu/2-1/2} I_{-\nu-1}(2\sqrt{t}),\ -1 < \nu < 0, \tag{40}$$

where

$$J_\nu(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k (t/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)} \quad \text{and} \quad I_\nu(t) = \sum_{k=0}^{+\infty} \frac{(t/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)} \tag{41}$$

are the Bessel and the modified Bessel functions, respectively. For other examples of the Sonin kernels that belong to the set $\mathcal{L}_1$, we refer to [6,8,20,22,36–38].

In Ref. [21], the set $\mathcal{L}_n$ of the Luchko kernels was defined for the first time. Many examples of the kernels from $\mathcal{L}_n$ as well as several methods for their construction based on the kernels from $\mathcal{L}_1$ were presented in [21,27], and we shortly mention here some of them.

The first method employs the relation (30) in the Laplace domain (provided that the Laplace transforms $\tilde{\kappa},\ \tilde{k}$ of the kernels κ and k do exist) in the form

$$\tilde{\kappa}(p) \cdot \tilde{k}(p) = \frac{1}{p^n},\ \Re(p) > p_{\kappa,k} \in \mathbb{R},\ n \in \mathbb{N}, \tag{42}$$

along with the tables of the direct and inverse Laplace transforms (see, for example, [39,40]). In particular, in [21], this method was used to derive the following kernels from the set $\mathcal{L}_n$:

$$\kappa(t) = t^{\nu/2} J_\nu(2\sqrt{t}),\ \ k(t) = t^{n/2-\nu/2-1} I_{n-\nu-2}(2\sqrt{t}),\ n-2 < \nu < n-1,\ n \in \mathbb{N}, \tag{43}$$

where the Bessel function J_ν and the modified Bessel function I_ν are defined as in (41). Please note that for $n = 1$, the kernels from (43) are reduced to the Sonin kernels (40).

Let now κ and k be the kernels from the set $\mathcal{L}_1$, i.e., the kernels that satisfy the Sonin condition (7). Then the kernels $\hat{\kappa} = \kappa^{<n>}$ and $\hat{k} = k^{<n>}$ satisfy the relation (30)

$$(\hat{\kappa} * \hat{k})(t) = (\kappa^{<n>} * k^{<n>})(t) = (\kappa * k)^{<n>}(t) = \{1\}^{<n>}(t).$$

Because of the inclusion $\kappa \in C_{-1}(0,+\infty)$, the kernel $\hat{\kappa} = \kappa^{<n>}$ belongs to the space $C_{-1}(0,+\infty)$ [41]. However, the inclusion $\hat{k} = k^{<n>} \in C_{-1,0}(0,+\infty)$ does not always hold true. Say, in the case of the Riemann–Liouville fractional derivative with the kernel $k(t) = h_{1-\alpha}$, the kernel $k^{<n>}(t) = h_{n(1-\alpha)}(t)$ belongs to the space $C_{-1,0}(0,+\infty)$ if and only if $1 - \frac{1}{n} < \alpha < 1$. Thus, the pair of the kernels $\hat{\kappa} = \kappa^{<n>}$ and $\hat{k} = k^{<n>}$ is from the set $\mathcal{L}_n$ if and only if $k^{<n>} \in C_{-1,0}(0,+\infty)$. On the other hand, if $k^{<n>} \notin C_{-1,0}(0,+\infty)$ and if one of the derivatives $\frac{d^j k^{<n>}}{dt^j}$, $j = 1,\ldots,n-1$ belongs to the space $C_{-1,0}(0,+\infty)$, we obtain the relation

$$(\hat{\kappa} * \frac{d^j}{dt^j} k^{<n>})(t) = \frac{d^j}{dt^j}(\kappa^{<n>} * k^{<n>})(t) = \frac{d^j}{dt^j}\{1\}^{<n>}(t) = \{1\}^{<n-j>}(t).$$

Thus, the pair of the kernels $\hat{\kappa} = \kappa^{<n>}$ and $\frac{d^j}{dt^j}k^{<n>}$ belongs to the set $\mathcal{L}_{n-j}$ (it is easy to see that there exists at most one derivative of the order $j \in \{1, \dots, n-1\}$ that satisfies the inclusion $\frac{d^j}{dt^j}k^{<n>}C_{-1,0}(0, +\infty)$).

To illustrate the procedure described above, let us consider two examples:

(1) $\kappa(t) = h_{2/3}(t)$, $k(t) = h_{1/3}(t)$. Evidently, κ and k are the kernels from the set $\mathcal{L}_1$. Then $\kappa^{<2>}(t) = (\kappa * \kappa)(t) = h_{4/3}(t)$ and $k^{<2>}(t) = (k * k)(t) = h_{2/3}(t)$ (see Equation (36)). Because of the inclusion $k^{<2>}(t) = h_{2/3}(t) \in C_{-1,0}(0, +\infty)$, the kernels $\kappa^{<2>}(t) = h_{4/3}(t)$ and $k^{<2>}(t) = h_{2/3}(t)$ are from the set $\mathcal{L}_2$.

(2) $\kappa(t) = h_{1/3}(t)$, $k(t) = h_{2/3}(t)$. Once again, κ and k are the kernels from the set $\mathcal{L}_1$. Then $\kappa^{<2>}(t) = (\kappa * \kappa)(t) = h_{2/3}(t)$ and $k^{<2>}(t) = (k * k)(t) = h_{4/3}(t)$. This time, $k^{<2>}(t) = h_{4/3}(t) \notin C_{-1,0}(0, +\infty)$. However, the function $\frac{d}{dt}k^{<2>}(t) = \frac{d}{dt}h_{4/3}(t) = h_{1/3}(t)$ is from the set $C_{-1,0}(0, +\infty)$. Thus, the kernels $\kappa^{<2>}(t) = h_{2/3}(t)$ and $\frac{d}{dt}k^{<2>}(t) = h_{1/3}(t)$ are from the set $\mathcal{L}_1$.

In Ref. [27], a generalization of the method presented above is suggested. Let the pairs of the kernels $(\kappa_1,\ k_1), \dots, (\kappa_n,\ k_n)$ belong to the set $\mathcal{L}_1$. Then the kernels $\kappa(t) = (\kappa_1 * \cdots * \kappa_n)(t)$ and $k(t) = (k_1 * \cdots * k_n)(t)$ satisfy the condition (30). The inclusion $\kappa \in C_{-1}(0, +\infty)$ is also fulfilled. Thus, the pair of the kernels $(\kappa,\ k)$ is from the set $\mathcal{L}_n$ if and only if $k \in C_{-1,0}(0, +\infty)$. If $k \notin C_{-1,0}(0, +\infty)$ and there exists $j \in \{1, \dots, n-1\}$ such that $\frac{d^j}{dt^j}k \in C_{-1,0}(0, +\infty)$, the pair of the kernels $(\kappa,\ \frac{d^j}{dt^j}k)$ is from the set $\mathcal{L}_{n-j}$ (see the discussions above).

Another important method for the construction of kernels from the set $\mathcal{L}_n$, $n > 1$ based on the Sonin kernels $(\kappa,\ k) \in \mathcal{L}_1$ is as follows: ([21]):

$$\hat{\kappa}(t) = (\{1\}^{<n-1>} * \kappa)(t),\ \ \hat{k}(t) = k(t). \tag{44}$$

Evidently, the kernels $\hat{\kappa}, \hat{k}$ defined as in (44) satisfy all three conditions from Definition 1 and thus $(\hat{\kappa},\ \hat{k}) \in \mathcal{L}_n$.

In the rest of this section, for the kernels $(\kappa,\ k)$ from the set $\mathcal{L}_n$, we define the GFI of arbitrary order with the kernel κ, the GFD of arbitrary order with the kernel k, and the regularized GFD of arbitrary order with the kernel k and discuss some of their basic properties.

Definition 2 ([21])**.** *Let $(\kappa,\ k)$ be a pair of the kernels from the set $\mathcal{L}_n$. The GFI of arbitrary order with the kernel κ, the GFD of arbitrary order with the kernel k, and the regularized GFD of arbitrary order with the kernel k are defined by the following formulas, respectively:*

$$(\mathbb{I}_{(\kappa)}\, f)(t) := (\kappa * f)(t) = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau,\ t > 0, \tag{45}$$

$$(\mathbb{D}_{(k)}\, f)(t) := \frac{d^n}{dt^n}(k * f)(t) = \frac{d^n}{dt^n}(\mathbb{I}_{(k)}\, f)(t),\ t > 0, \tag{46}$$

$$({}_*\mathbb{D}_{(k)}\, f)(t) := \left(\mathbb{D}_{(k)}\left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot)\right)\right)(t),\ t > 0. \tag{47}$$

It is worth mentioning that under certain conditions, the regularized GFD (47) can be represented in a different form [21]:

$$({}_*\mathbb{D}_{(k)}\, f)(t) = (\mathbb{D}_{(k)}\, f)(t) - \sum_{j=0}^{n-1} f^{(j)}(0) \frac{d^{n-j-1}}{dt^{n-j-1}} k(t),\ t > 0. \tag{48}$$

In particular, the representation (48) is valid for $f,\ k \in C_{-1}^{n-1}(0,+\infty)$, where the space of functions $C_{-1}^{m}(0,+\infty),\ m = 0,1,2,\ldots$ is defined as follows:

$$C_{-1}^{m}(0,+\infty) := \{f : f^{(m)} \in C_{-1}(0,+\infty)\}. \quad (49)$$

For $m = 0$, the space $C_{-1}^{m}(0,+\infty)$ is interpreted as the space $C_{-1}(0,+\infty)$. This means, in particular, that for $n = 1$, the representation (48) is valid without any additional conditions on the kernel k.

Moreover, for $f \in C_{-1}^{n}(0,+\infty)$, the regularized GFD (47) can be represented as follows [21]:

$$({}_*\mathbb{D}_{(k)}\, f)(t) = (\mathbb{I}_{(k)}\, f^{(n)})(t) \;=\; \int_0^t k(t-\tau)\, f^{(n)}(\tau)\, d\tau,\ t > 0. \quad (50)$$

As mentioned in Remark 3, the pair of the kernels $\kappa(t) = h_\alpha(t),\ \alpha > 0$ and $k(t) \;=\; h_{n-\alpha}(t)$ belongs to the set $\mathcal{L}_n$ provided that $n-1 < \alpha < n,\ n \in \mathbb{N}$. This implicates that the GFI (45) with the kernel $\kappa(t) = h_\alpha(t),\ \alpha > 0$ is the Riemann–Liouville fractional integral of the order $\alpha > 0$, whereas the Riemann=-Liouville and the Caputo fractional derivatives of the order $\alpha,\ n-1 < \alpha < n,\ n \in \mathbb{N}$ are particular cases of the GFD (46) and the regularized GFD (47) (or (48) or (50)) with the kernel $k(t) = h_{n-\alpha}(t)$, respectively.

The constructions of the GFD and the regularized GFD of arbitrary order presented in Definition 2 produce the Riemann–Liouville and the Caputo fractional derivatives of the non-integer order α that satisfy the conditions $n-1 < \alpha < n,\ n \in \mathbb{N}$. To define them for $\alpha = n \in \mathbb{N}_0$, we proceed as in the case $n = 1$ (see the discussions at the beginning of this section). Namely, we extend the relation (35) to the case $\alpha = 0$ or $\alpha = n \in \mathbb{N}$, respectively, in the sense of the generalized functions (the function h_0 plays the role of the Dirac δ-function):

$$h_0 \,*\, h_n = h_n,\ n \in \mathbb{N}. \quad (51)$$

Then the GFI (45), the GFD (46), and the regularized GFD (47) with the kernels h_0 and h_n can be interpreted as follows:

$$(\mathbb{I}_{(h_0)}\, f)(t) \;=\; (h_0 \,*\, f)(t) \;=\; f(t),\ t > 0, \quad (52)$$

$$(\mathbb{I}_{(h_n)}\, f)(t) \;=\; (h_n \,*\, f)(t) \;=\; (I_{0+}^n\, f)(t),\ t > 0, \quad (53)$$

$$(\mathbb{D}_{(h_0)}\, f)(t) \;=\; \frac{d^n}{dt^n}(h_0 \,*\, f)(t) \;=\; f^{(n)}(t),\ t > 0, \quad (54)$$

$$(\mathbb{D}_{(h_n)}\, f)(t) \;=\; \frac{d^n}{dt^n}(h_n \,*\, f)(t) \;=\; \frac{d^n}{dt^n}\,(I_{0+}^n\, f)(t) \;=\; f(t),\ t > 0, \quad (55)$$

$$({}_*\mathbb{D}_{(h_0)}\, f)(t) \;=\; \left(\mathbb{D}_{(h_0)}\left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0)h_{j+1}(\cdot)\right)\right)(t) \;=\; f^{(n)}(t),\ t > 0, \quad (56)$$

$$({}_*\mathbb{D}_{(h_n)}\, f)(t) \;=\; \left(\mathbb{D}_{(h_n)}\left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0)h_{j+1}(\cdot)\right)\right)(t) \;=\; f(t) - \sum_{j=0}^{n-1} f^{(j)}(0)h_{j+1}(t),\ t > 0. \quad (57)$$

As an example of a new and non-trivial particular case of the GFI (45) of arbitrary order, the GFD (46) of arbitrary order, and the regularized GFD (47) of arbitrary order, respectively, we mention the following operators constructed for the kernels $(\kappa,\ k) \in \mathcal{L}_n$ given by the formula (43) that is valid under the conditions $n-2 < \nu < n-1,\ n \in \mathbb{N}$ [21]:

$$(\mathbb{I}_{(\kappa)}\, f)(t) = \int_0^t (t-\tau)^{\nu/2}\, J_\nu(2\sqrt{t-\tau}) f(\tau)\, d\tau,\ t > 0, \quad (58)$$

$$(\mathbb{D}_{(k)}\, f)(t) = \frac{d^n}{dt^n}\int_0^t (t-\tau)^{n/2-\nu/2-1} I_{n-\nu-2}(2\sqrt{t-\tau}) f(\tau)\, d\tau,\ t > 0, \quad (59)$$

$$({}_*\mathbb{D}_{(k)}\, f)(t) := \left(\mathbb{D}_{(k)}\left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0)h_{j+1}(\cdot)\right)\right)(t),\ t > 0. \quad (60)$$

Because the kernels $(\kappa,\ k) \in \mathcal{L}_n$ of the GFI (45) of arbitrary order, the GFD (46) of arbitrary order, and the regularized GFD (47) of arbitrary order are from the space $C_{-1}(0,+\infty)$, the natural domains of definition for these operators are the space $C_{-1}(0,+\infty)$ and its suitable sub-spaces. The basic properties of the GFI (45) easily follow from the properties of the Laplace convolution on the space $C_{-1}(0,+\infty)$ (see [41]):

$$\mathbb{I}_{(\kappa)}\,:\; C_{-1}(0,+\infty)\; \to C_{-1}(0,+\infty),\; \kappa \in \mathcal{L}_n \;\; \text{(mapping property)}, \tag{61}$$

$$\mathbb{I}_{(\kappa_1)}\,\mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_2)}\,\mathbb{I}_{(\kappa_1)},\; \kappa_1,\, \kappa_2 \in \mathcal{L}_n \;\; \text{(commutativity law)}, \tag{62}$$

$$\mathbb{I}_{(\kappa_1)}\,\mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_1 * \kappa_2)},\; \kappa_1,\, \kappa_2 \in \mathcal{L}_n \;\; \text{(semi-group property)}. \tag{63}$$

According to the axioms of FC suggested in [42], the GFDs should be left-inverse operators to the corresponding GFIs (1st fundamental theorem of FC). Moreover, as shown in [5,23,24], the compositions of the GFIs and the corresponding GFDs applied to a function f are the so-called convolution polynomials subtracted from the function f (2nd fundamental theorem of FC).

In the rest of this section, we present the fundamental theorems of FC for the GFI (45) of arbitrary order, the GFD (46) of arbitrary order, and for the regularized GFD (47) of arbitrary order. For the proofs of these theorems, we refer to [21,25].

Theorem 1 ([21])**.** *Let $(\kappa,\ k)$ be a pair of the kernels from the set $\mathcal{L}_n$.*

Then the GFD (46) is a left-inverse operator to the GFI (45) on the space $C_{-1}(0,+\infty)$:

$$(\mathbb{D}_{(k)}\,\mathbb{I}_{(\kappa)}\,f)(t) = f(t),\; f \in C_{-1}(0,+\infty),\; t > 0, \tag{64}$$

and the regularized GFD (47) is a left-inverse operator to the GFI (45) on the space $\mathbb{I}_{(k)}(C_{-1}(0,+\infty))$:

$$({}_*\mathbb{D}_{(k)}\,\mathbb{I}_{(\kappa)}\,f)(t) = f(t),\; f \in \mathbb{I}_{(k)}(C_{-1}(0,+\infty)),\; t > 0, \tag{65}$$

where

$$\mathbb{I}_{(k)}(C_{-1}(0,+\infty)) := \{f\,:\; f(t) = (\mathbb{I}_{(k)}\,\phi)(t),\; \phi \in C_{-1}(0,+\infty)\}. \tag{66}$$

Theorem 2 ([21])**.** *Let $(\kappa,\ k)$ be a pair of the kernels from the set $\mathcal{L}_n$.*

Then the relation

$$(\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}\,f)(t) = f(t) - \sum_{j=0}^{n-1} f^{(j)}(0)\,h_{j+1}(t) \tag{67}$$

holds true on the space $C_{-1}^{n}(0,+\infty)$ defined as in (49), and the formula

$$(\mathbb{I}_{(\kappa)}\,\mathbb{D}_{(k)}\,f)(t) = f(t),\; t > 0 \tag{68}$$

is valid for the functions $f \in \mathbb{I}_{(\kappa)}(C_{-1}(0,+\infty))$.

The Equation (68) means that the GFD of arbitrary order is a right-inverse operator to the corresponding GFI on the space $\mathbb{I}_{(\kappa)}(C_{-1}(0,+\infty))$. However, in general, this formula does not hold true if we consider the GFD (46) of arbitrary order on its natural domain of definition, namely, on the following space of functions:

$$C^{1}_{-1,(k)}(0,+\infty) = \{f \in C_{-1}(0,+\infty)\,:\; \mathbb{D}_{(k)}\,f \in C_{-1}(0,+\infty)\}. \tag{69}$$

For the functions from $C^{1}_{-1,(k)}(0,+\infty)$, a more general result compared to the one given by Equation (68) is valid. This result is provided in the theorem below.

Theorem 3 ([25])**.** *Let $(\kappa,\ k)$ be a pair of the kernels from the set $\mathcal{L}_n$ and $\kappa \in C_{-1}^{n-1}(0,+\infty)$.*

Then for any function $f \in C^1_{-1,(k)}(0,+\infty)$, the formula

$$(\mathbb{I}_{(\kappa)}\,\mathbb{D}_{(k)}\,f)(t) = f(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,\mathbb{I}_{(k)}\,f\right)(0)\,\frac{d^{n-j-1}}{dt^{n-j-1}}\,\kappa(t),\ t>0 \tag{70}$$

holds true.

The difference in Equations (68) and (70) is caused by the inclusion $\mathbb{I}_{(\kappa)}(C_{-1}(0,+\infty)) \subset C^1_{-1,(k)}(0,+\infty)$ that immediately follows from Equation (64) of Theorem 1. Moreover, for any function from the space $\mathbb{I}_{(\kappa)}(C_{-1}(0,+\infty))$, a comparison of Equations (68) and (70) leads to the relations

$$\left(\frac{d^j}{dt^j}\,\mathbb{I}_{(k)}\,f\right)(0) = 0,\ j = 0,1,\ldots,n-1. \tag{71}$$

It is worth mentioning that the left-hand sides of the relations (71) can be interpreted as the natural initial conditions while dealing with the fractional differential equations with the GFDs of arbitrary order (see [25]). In particular, in the case of the Riemann–Liouville fractional integral I^α_{0+} with the kernel $\kappa(t) = h_\alpha(t)$, $n-1<\alpha<n$, $n\in\mathbb{N}$ and the Riemann–Liouville fractional derivative D^α_{0+} with the kernel $k(t) = h_{n-\alpha}(t)$, Equation (70) takes the following well-known form [43]:

$$(I^\alpha_{0+}\,D^\alpha_{0+}\,f)(t) = f(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,I^{n-\alpha}_{0+}\,f\right)(0)\,h_{\alpha-n+j+1}(t),\ t>0. \tag{72}$$

Thus, the natural initial conditions for the fractional differential equations with the Riemann–Liouville fractional derivatives are formulated as follows ([43]):

$$\left(\frac{d^j}{dt^j}\,I^{n-\alpha}_{0+}\,f\right)(0) = a_j,\ j=0,1,\ldots,n-1.$$

3. The 1st-Level GFD of Arbitrary Order

Similar to the case of the Riemann–Liouville fractional derivative and the Caputo fractional derivative, both the definitions, the particular cases, and the formulas for the GFD (46) and for the regularized GFD (47) look very different. Moreover, as a rule, these GFDs as well as the fractional differential equations with these derivatives are treated in the separate publications and by employing the unequal methods. As mentioned in the Introduction, the Hilfer fractional derivative (1) is a unification and a generalization of the Riemann–Liouville fractional derivative and the Caputo fractional derivative. In this section, we introduce the 1st-level GFD of arbitrary order that generalizes the Hilfer fractional derivative (1) to the case of the arbitrary kernels that satisfy an extended Sonin condition and investigate its basic properties. The main particular cases of this derivative are the GFD (46) of arbitrary order and the regularized GFD (47) of arbitrary order. The case of the 1st-level GFD with the generalized order less than or equal to one was treated in [26].

We start with a suitable generalization of the kernels from Definition 1 that we call the kernels of the 1st-level GFDs of arbitrary order.

Definition 3. *Let the functions κ, k_1, k_2 : $(0,+\infty)\to\mathbb{R}$ satisfy the following conditions:*
1st condition:

$$(\kappa * k_1 * k_2)(t) = \{1\}^{<n>}(t) = h_n(t) = \frac{t^{n-1}}{(n-1)!},\ n\in\mathbb{N},\ t>0, \tag{73}$$

2nd condition:

$$\kappa \in C_{-1}(0,+\infty), \tag{74}$$

3rd condition:

$$k_1,\ k_2 \in C_{-1,0}(0,+\infty). \tag{75}$$

The set of such triples $(\kappa,\ k_1,\ k_2)$ *is denoted by* $\mathcal{L}_n^1$.

Remark 4. *As suggested in* [26], *the denotation* $\mathcal{L}_n^m$ *stands for the set of the kernels of the mth-level GFDs with the generalized order from the interval* $(n-1,\ n)$. *The GFD* (46) *of arbitrary order and the regularized GFD* (47) *of arbitrary order with the kernels from* $\mathcal{L}_n$ *treated in Section* 2 *can be interpreted as the 0th-level GFDs, and thus we set* $\mathcal{L}_n^0 := \mathcal{L}_n$. *The case of the kernels from the set* $\mathcal{L}_1^1$ *was considered in* [26].

In what follows, we associate the first kernel from the triple $(\kappa,\ k_1,\ k_2) \in \mathcal{L}_n^1$ with the GFI defined as in (16), whereas the kernels k_1 and k_2 are assigned to the 1st-level GFD of arbitrary order that will be defined below. Due to the inclusion $k_1,\ k_2 \in C_{-1,0}(0,+\infty)$, the kernels k_1 and k_2 can be interchanged. However, for $n \geq 2$, the kernels k_1 and k_2 cannot be interchanged with the kernel κ and thus, the triples $(\kappa,\ k_1,\ k_2)$ are partially ordered. It is worth mentioning that as soon as any two kernels from a triple $(\kappa,\ k_1,\ k_2)$ are fixed, the 3rd kernel is uniquely determined by the relation (73). This property is a consequence from Theorem 3.1 in [41] that states that the ring $\mathcal{R}_{-1} = (C_{-1}(0,+\infty),+,*)$ does not have any divisors of zero. For the kernels k_1 and k_2, the kernel κ from the triple $(\kappa,\ k_1,\ k_2)$ is called the 1st-level GFD kernel associated to the kernel pair $(k_1,\ k_2)$.

As an example of the kernels from the set $\mathcal{L}_n^1$ we mention the following power law functions

$$\kappa(t) = h_\alpha(t),\ k_1(t) = h_\gamma(t),\ k_2(t) = h_{n-\alpha-\gamma}(t),\ t > 0$$

that satisfy the conditions from Definition 3 if

$$n-1 < \alpha < n,\ 0 < \gamma < n-\alpha,\ n \in \mathbb{N}. \tag{76}$$

Indeed, the function $\kappa(t) = h_\alpha(t)$ is from the space $C_{-1}(0,+\infty)$, whereas the functions $k_1(t) = h_\gamma(t)$ and $k_2(t) = h_{n-\alpha-\gamma}(t)$ belong to the space $C_{-1,0}(0,+\infty)$ due to the inequalities (76). Equation (36) easily leads to the relation

$$(h_\alpha * h_\gamma * h_{n-\alpha-\gamma})(t) = h_{\alpha+\gamma+n-\alpha-\gamma}(t) = h_n(t),\ t > 0. \tag{77}$$

Thus, we have the inclusion

$$(h_\alpha(t),\ h_\gamma(t),\ h_{n-\alpha-\gamma}(t)) \in \mathcal{L}_n^1. \tag{78}$$

It is worth mentioning that the relation (77) can be also verified by employing the Laplace transform technique. Provided the Laplace transforms of the kernels κ, k_1, and k_2 do exist, application of the Laplace transform to the condition (73) from Definition 3 leads to the relation

$$\tilde{\kappa}(p) \cdot \tilde{k_1}(p) \cdot \tilde{k_2}(p) = \frac{1}{p^n}. \tag{79}$$

In the case of the power law kernels (78), we immediately get the Laplace transform formulas

$$\tilde{\kappa}(p) = \frac{1}{p^\alpha},\ \tilde{k_1}(p) = \frac{1}{p^\gamma},\ \tilde{k_2}(p) = \frac{1}{p^{n-\alpha-\gamma}}$$

and thus the relation (79) evidently holds true:

$$\frac{1}{p^\alpha} \cdot \frac{1}{p^\gamma} \cdot \frac{1}{p^{n-\alpha-\gamma}} = \frac{1}{p^n}.$$

The relation (79) and the tables of the Laplace transforms and the inverse Laplace transforms can be used to determine other triples of the kernels from $\mathcal{L}_n^1$. For other techniques for construction of the Sonin kernels that can be employed for derivation of the 1st-level GFDs kernels, we refer the readers to [37].

Remark 5. *In Definition 3, the kernels κ, k_1, and k_2 are the functions from the spaces $C_{-1}(0,+\infty)$ and $C_{-1,0}(0,+\infty)$, respectively. However, Equation (23) in the sense of generalized functions implicates the following forms of the condition (73) that involve the generalized function h_0:*

$$\kappa * h_0 * k_2 = \kappa * k_2 = h_n, \tag{80}$$

$$\kappa * k_1 * h_0 = \kappa * k_1 = h_n. \tag{81}$$

Equations (80) and (81) can be interpreted as follows: if one of the kernels k_1 or k_2 of the 1st-level GFD of arbitrary order is set to be the generalized function h_0, then the inclusion $(\kappa, k_2) \in \mathcal{L}_n$ or $(\kappa, k_1) \in \mathcal{L}_n$, respectively, holds true, i.e., the pair of the remainder kernels satisfies the conditions from Definition 1.

Let us now proceed with defining the 1st-level GFDs of arbitrary order.

Definition 4. *Let (κ, k_1, k_2) be a triple of the kernels from the set $\mathcal{L}_n^1$.*

The 1st-level GFD of arbitrary order is defined by the formula

$$({}_{1L}\mathbb{D}_{(k_1,k_2)}\, f)(t) := \left(\mathbb{I}_{(k_1)}\, \frac{d^n}{dt^n}\, \mathbb{I}_{(k_2)}\, f\right)(t), \tag{82}$$

whereas the corresponding GFI with the kernel κ is provided by the relation (16).

An important particular case of the 1st-level GFD of arbitrary order is the Hilfer fractional derivative (1). Indeed, let us put the power law kernels k_1 and k_2 from the triple (78) into Equation (82). Because the GFI with a power law kernel is nothing else than the Riemann–Liouville fractional integral (2), Equation (82) takes the form

$$({}_{1L}\mathbb{D}_{(h_\gamma,h_{n-\alpha-\gamma})}\, f)(t) = \left(I_{0+}^{\gamma}\, \frac{d^n}{dt^n}\, I_{0+}^{n-\alpha-\gamma}\, f\right)(t). \tag{83}$$

It is easy to see that the operator at the right-hand side of (83) is the Hilfer fractional derivative (1) in a slightly different parametrization suggested in [5]. Indeed, the Hilfer derivative (1) coincides with the operator at the right-hand side of (83) if we set $\gamma = \beta(n-\alpha)$. The form (83) of the Hilfer fractional derivative with $n = 1$ was called in [5] the 1st-level fractional derivative. As we see, the operator (82) is a natural generalization of the Hilfer derivative (83) to the case of arbitrary kernels from the set $\mathcal{L}_n^1$.

As in the case of the Hilfer fractional derivative that unifies the Riemann–Liouville and the Caputo fractional derivatives in one formula, the main idea behind the 1st level GFD (82) of arbitrary order is that this derivative contains both the GFD (46) of arbitrary order and the regularized GFD of arbitrary order in form (50) as its particular cases.

Indeed, the interpretation (52) of the GFI with the kernel h_0 and the discussions presented in Remark 5 immediately lead to the following particular cases of the 1st-level GFD:

I. For $k_1 = h_0$, the 1st level GFD (82) of arbitrary order is reduced to the GFD (46) of arbitrary order with the kernel k_2:

$$({}_{1L}\mathbb{D}_{(h_0,k_2)}\, f)(t) = \left(\mathbb{I}_{(h_0)}\, \frac{d^n}{dt^n}\, \mathbb{I}_{(k_2)}\, f\right)(t) = \left(\frac{d^n}{dt^n}\, \mathbb{I}_{(k_2)}\, f\right)(t) = (\mathbb{D}_{(k_2)}\, f)(t). \tag{84}$$

II. For $k_2 = h_0$, the 1st level GFD (82) of arbitrary order is reduced to the regularized GFD (50) of arbitrary order with the kernel k_1:

$$({}_{1L}\mathbb{D}_{(k_1,h_0)}\, f)(t) = \left(\mathbb{I}_{(k_1)}\, \frac{d^n}{dt^n}\, \mathbb{I}_{(h_0)}\, f\right)(t) = \left(\mathbb{I}_{(k_1)}\frac{d^n}{dt^n}\, f\right)(t) = ({}_*\mathbb{D}_{(k_1)}\, f)(t). \quad (85)$$

In the publications devoted to the GFDs and the regularized GFDs, their properties were investigated separately and often by employing unequal methods for the GFDs and the regularized GFDs, respectively. The same statement is valid for the publications devoted to the fractional differential equations with the GFDs and the regularized GFDs. The concept of the 1st-level GFD introduced in Definition 4 opens a gateway for the derivation of formulas that are valid both for the GFDs and for the regularized GFDs. In the rest of this section, we formulate and prove some results for the 1st-level GFDs of arbitrary order, including the 1st and the 2nd fundamental theorems of FC. In particular, these results are valid both for the GFDs and for the regularized GFDs of arbitrary order.

We start with the 1st fundamental theorem of FC for the 1st-level GFDs of arbitrary order.

Theorem 4. *Let a triple of the kernels $(\kappa,\, k_1,\, k_2)$ belong to the set $\mathcal{L}_n^1$.*

Then the 1st-level GFD (82) of arbitrary order is a left inverse operator to the GFI (16) on the space $\mathbb{I}_{(k_1)}(C_{-1}(0,+\infty))$ defined as in (66):

$$({}_{1L}\mathbb{D}_{(k_1,k_2)}\, \mathbb{I}_{(\kappa)}\, f)(t) = f(t),\ f \in \mathbb{I}_{(k_1)}(C_{-1}(0,+\infty)),\ t > 0. \quad (86)$$

Proof. By definition, any function f from the space $\mathbb{I}_{(k_1)}(C_{-1}(0,+\infty))$ can be represented in the form

$$f(t) = (\mathbb{I}_{(k_1)}\, \phi)(t) = (k_1 * \phi)(t),\ \phi \in C_{-1}(0,+\infty).$$

For the kernels $(\kappa,\, k_1,\, k_2) \in \mathcal{L}_1^1$, the condition (73) is fulfilled. Then we obtain the following chain of the equations

$$({}_{1L}\mathbb{D}_{(k_1,k_2)}\, \mathbb{I}_{(\kappa)}\, f)(t) = \left(\mathbb{I}_{(k_1)}\, \frac{d^n}{dt^n}\, \mathbb{I}_{(k_2)}\, \mathbb{I}_{(\kappa)}\, f\right)(t) = \left(\mathbb{I}_{(k_1)}\, \frac{d^n}{dt^n}\, \mathbb{I}_{(k_2)}\, \mathbb{I}_{(\kappa)}\, \mathbb{I}_{(k_1)}\, \phi\right)(t) =$$

$$\left(\mathbb{I}_{(k_1)}\, \frac{d^n}{dt^n}\, (k_2 * \kappa * k_1 * \phi)(t)\right)(t) = \left(\mathbb{I}_{(k_1)}\, \frac{d^n}{dt^n}\, (\{1\}^{<n>} * \phi)(t)\right)(t) = (\mathbb{I}_{(k_1)}\, \phi)(t) = f(t)$$

which proves Equation (86). □

According to Theorem 1, the formula of type (86) is valid both for the GFD (46) of arbitrary order and for the regularized GFD (47) of arbitrary order. Both these formulas are particular cases of Equation (86). Indeed, setting $k_1 = h_0$ in (86) leads to Equation (64) for the GFD (46) of arbitrary order with the kernel $k = k_2$ on the space $C_{-1}(0,+\infty)$, whereas a substitution $k_2 = h_0$ in (86) results in Equation (65) for the regularized GFD (47) of arbitrary order with the kernel $k = k_1$ on the space $\mathbb{I}_{(k)}(C_{-1}(0,+\infty))$. Thus, Theorem 4 covers the results presented in Theorem 1, including the spaces of functions used in its formulation.

The next important result is a formula for a composition of the GFI of arbitrary order with the kernel κ and the 1st-level GFD of arbitrary order with the pair of the associated kernels $(k_1,\, k_2)$. This result is referred to as the 2nd fundamental theorem of FC for the 1st-level GFD of arbitrary order.

Theorem 5. *Let a triple of the kernels $(\kappa,\, k_1,\, k_2)$ belong to the set $\mathcal{L}_n^1$.*

Then the formula

$$(\mathbb{I}_{(\kappa)}\, {}_{1L}\mathbb{D}_{(k_1,k_2)}\, f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{d^j}{dt^j}(\mathbb{I}_{(k_2)}\, f)(0)\, \frac{d^{n-1-j}}{dt^{n-1-j}}(\kappa * k_1)(t),\ t > 0 \quad (87)$$

holds valid for any function $f \in \mathbb{I}^n_{(k_2)}(C_{-1}(0,+\infty))$ *under the condition that*

$$\kappa * k_1 \in C^{n-1}_{-1}(0,+\infty), \tag{88}$$

where the space $C^{n-1}_{-1}(0,+\infty)$ *is defined as in* (49) *and*

$$\mathbb{I}^m_{(k)}(C_{-1}(0,+\infty)) := \{f \in C_{-1}(0,+\infty) : \frac{d^m}{dt^m}(\mathbb{I}_{(k)} f)(t) \in C_{-1}(0,+\infty)\}. \tag{89}$$

Proof. First, we determine the null-space of the 1st-level GFD ${}_{1L}\mathbb{D}_{(k_1,k_2)}$. Because the GFI (16) is an injection [20], we obtain the following chain of relations:

$$({}_{1L}\mathbb{D}_{(k_1,k_2)} f)(t) = 0 \Leftrightarrow \frac{d^n}{dt^n}\mathbb{I}_{(k_2)} f = 0 \Leftrightarrow (\mathbb{I}_{(k_2)} f)(t) = (k_2 * f)(t) = \sum_{j=0}^{n-1} a_j h_{j+1}(t).$$

Now we apply the GFI with the kernel $\kappa * k_1$ to the relation deduced above and obtain

$$(\mathbb{I}_{\kappa * k_1} f)(t) = (\kappa * k_1 * k_2 * f)(t) = (\sum_{j=0}^{n-1} a_j h_{j+1}(\cdot) * \kappa * k_1)(t) \Leftrightarrow$$
$$(\{1\}^{<n>} * f)(t) = (\sum_{j=0}^{n-1} a_j h_{j+1}(\cdot) * \kappa * k_1)(t).$$

n-times differentiation of the last formula that is allowed under the condition (88) leads to the representation

$$f(t) = \sum_{j=0}^{n-1} a_j \frac{d^{n-1-j}}{dt^{n-1-j}}(\kappa * k_1)(t)$$

and we arrive at the following description of the null-space of the 1st-level GFD of arbitrary order:

$$\ker {}_{1L}\mathbb{D}_{(k_1,k_2)} = \{f : f(t) = \sum_{j=0}^{n-1} a_j \frac{d^{n-1-j}}{dt^{n-1-j}}(\kappa * k_1)(t),\ a_j \in \mathbb{R},\ j = 0,1,\ldots,n-1\}. \tag{90}$$

Now we introduce an auxiliary function ϕ as follows:

$$\phi(t) := (\mathbb{I}_{(\kappa)}\, {}_{1L}\mathbb{D}_{(k_1,k_2)} f)(t). \tag{91}$$

Because of the inclusion $f \in \mathbb{I}^n_{(k_2)}(C_{-1}(0,+\infty))$, the 1st-level GFD $\mathbb{D}_{(k_1,k_2)} f$ does exist and is a function from the space $C_{-1}(0,+\infty)$. Thus, we obtain the inclusion $\phi \in \mathbb{I}_\kappa(C_{-1}(0,+\infty))$. The application of Theorem 4 results in the formula

$$({}_{1L}\mathbb{D}_{(k_1,k_2)}\, \phi)(t) = ({}_{1L}\mathbb{D}_{(k_1,k_2)}\, \mathbb{I}_{(\kappa)}\, {}_{1L}\mathbb{D}_{(k_1,k_2)} f)(t) = ({}_{1L}\mathbb{D}_{(k_1,k_2)} f)(t) \tag{92}$$

which can be rewritten as follows:

$$({}_{1L}\mathbb{D}_{(k_1,k_2)}\, (\phi - f))(t) = 0,\ t > 0. \tag{93}$$

Due to Equation (90) for the null-space of the 1st level GFD, we obtain the representation

$$\phi(t) = f(t) + \sum_{j=0}^{n-1} a_j \frac{d^{n-1-j}}{dt^{n-1-j}}(\kappa * k_1)(t),\ t > 0. \tag{94}$$

Applying the GFI $\mathbb{I}_{(k_2)}$ to the left-hand side of (94), i.e., to the function ϕ, leads to the formula

$$(\mathbb{I}_{(k_2)}\, \phi)(t) = (\mathbb{I}_{(k_2)}\, \mathbb{I}_{(\kappa)}\, \mathbb{D}_{(k_1,k_2)} f)(t) = (\mathbb{I}_{(k_2)}\, \mathbb{I}_{(\kappa)}\, \mathbb{I}_{(k_1)} \frac{d^n}{dt^n}\, \mathbb{I}_{(k_2)} f)(t) =$$

$$(k_2 * \kappa * k_1 * (\frac{d^n}{dt^n}\mathbb{I}_{(k_2)}f))(t) = (\{1\}^{<n>} * (\frac{d^n}{d^n}\mathbb{I}_{(k_2)}f))(t) =$$
$$(I^n_{0+}\frac{d^n}{dt^n}\mathbb{I}_{(k_2)}f)(t) = (\mathbb{I}_{(k_2)}\, f)(t) - \sum_{j=0}^{n-1}\frac{d^j}{dt^j}(\mathbb{I}_{(k_2)}\, f)(0)h_{j+1}(t).$$

Thus, we arrive at the representation

$$(\mathbb{I}_{(k_2)}\,\phi)(t) = (\mathbb{I}_{(k_2)}\, f)(t) - \sum_{j=0}^{n-1}\frac{d^j}{dt^j}(\mathbb{I}_{(k_2)}\, f)(0)h_{j+1}(t). \tag{95}$$

Now we apply the GFI $\mathbb{I}_{(k_2)}$ to the right-hand side of Equation (94):

$$(\mathbb{I}_{(k_2)}\,\phi)(t) = (\mathbb{I}_{(k_2)}\, f(\cdot) + \sum_{j=0}^{n-1} a_j \frac{d^{n-1-j}}{dt^{n-1-j}}(\kappa * k_1)(\cdot))(t) =$$
$$(\mathbb{I}_{(k_2)}\, f)(t) + \sum_{j=0}^{n-1} a_j \frac{d^{n-1-j}}{dt^{n-1-j}}(k_2 * \kappa * k_1)(t) =$$
$$(\mathbb{I}_{(k_2)}\, f)(t) + \sum_{j=0}^{n-1} a_j \frac{d^{n-1-j}}{dt^{n-1-j}}\{1\}^{<n>}(t) = (\mathbb{I}_{(k_2)}\, f)(t) + \sum_{j=0}^{n-1} a_j h_{j+1}(t).$$

The last formula along with Equation (95) leads to the following expressions for the coefficients a_j:

$$a_j = -\frac{d^j}{dt^j}(\mathbb{I}_{(k_2)}\, f)(0),\ j = 0, 1, \ldots, n-1. \tag{96}$$

The 2nd fundamental theorem of FC for the 1st-level GFD of arbitrary order (Equation (87) from Theorem 5) immediately follows from Equations (91), (94) and (96), and that completes the proof. □

It is worth mentioning that Theorem 5 contains a very general result that covers many known formulas for particular cases of the 1st-level GFD of arbitrary order.

If we set $k_2 = h_0$ in Equation (87), then the condition $(\kappa * k_1 * k_2)(t) = \{1\}^{<n>}(t)$ is reduced to the condition $(\kappa * k_1)(t) = \{1\}^{<n>}(t)$. Thus, $(\kappa, k_1) \in \mathcal{L}_n$ and the 1st-level GFD is the regularized GFD with the kernel k_1. The space of functions $\mathbb{I}^n_{(k_2)}(C_{-1}(0,+\infty))$ becomes the space $C^n_{-1}(0,+\infty)$, and the condition (88) from Theorem 5 is automatically satisfied:

$$(\kappa * k_1)(t) = \{1\}^{<n>}(t) \in C^{n-1}_{-1}(0,+\infty).$$

Equation (87) then takes the form

$$(\mathbb{I}_{(\kappa)}\, {}_*\mathbb{D}_{(k_1)}\, f)(t) = f(t) - \sum_{j=0}^{n-1} f^{(j)}(0)\, h_{j+1}(t), \tag{97}$$

that is exactly Equation (67) for the regularized GFD with the kernel k_1 from Theorem 2.

In the case of the power law kernels $\kappa(t) = h_\alpha(t)$ and $k_1(t) = h_{n-\alpha}(t)$, $n-1 < \alpha \le n$, the regularized GFD is the Caputo fractional derivative ${}_*D^\alpha_{0+}$ of the order α, and Equation (97) takes the well-known form

$$(I^\alpha_{0+}\, {}_*D^\alpha_{0+}\, f)(t) = f(t) - \sum_{j=0}^{n-1} f^{(j)}(0)\, h_{j+1}(t).$$

For $k_1 = h_0$, the 1st-level GFD is reduced to the GFD with the kernel k_2, where $(\kappa, k_2) \in \mathcal{L}_n$. Taking into account the relation $\kappa * k_1 = \kappa * h_0 = \kappa$, Equation (87) can be rewritten as follows:

$$(\mathbb{I}_{(\kappa)}\,\mathbb{D}_{(k_2)}\,f)(t) = f(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,\mathbb{I}_{(k_2)}\,f\right)(0)\,\frac{d^{n-j-1}}{dt^{n-j-1}}\,\kappa(t),\ t > 0. \tag{98}$$

According to the results presented in Theorem 3, this formula is valid under the condition $\kappa \in C_{-1}^{n-1}(0,+\infty)$. Because of the inclusion $(\kappa, k_2) \in \mathcal{L}_n$, the space $\mathbb{I}^n_{(k_2)}(C_{-1}(0,+\infty))$ mentioned in Theorem 5 becomes the space $C^1_{-1,(k_2)}(0,+\infty)$ from Theorem 3.

In the case of the power law kernels $\kappa(t) = h_\alpha(t)$ and $k_2(t) = h_{n-\alpha}(t)$, $n-1 < \alpha \le n$, the GFD with the kernel k_2 is the Riemann–Liouville fractional derivative D^α_{0+} of the order α. Equation (98) in this case is well known:

$$(I^\alpha_{0+}\,D^\alpha_{0+}\,f)(t) = f(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,I^{n-\alpha}_{0+}\,f\right)(0)\,h_{\alpha-n+j+1}(t),\ t > 0.$$

Another important particular case of Equation (87) concerns the Hilfer fractional derivative (1). To generate the Hilfer fractional derivative, we put the kernels (κ, k_1, k_2) that are the power law functions defined as in (78) into the formula for the 1st-level GFD of arbitrary order. For these kernels, the GFIs from the definition of the 1st-level GFD of arbitrary order are reduced to the Riemann–Liouville fractional integrals:

$$(\mathbb{I}_{(k_1)}\,f)(t) = (I^\gamma_{0+}\,f)(t),\ (\mathbb{I}_{(k_2)}\,f)(t) = (I^{n-\alpha-\gamma}_{0+}\,f)(t).$$

Moreover, for the power law kernels, the relations

$$(h_\alpha * h_\gamma)(t) = h_{\alpha+\gamma}(t),\ \frac{d}{dt}\,h_\alpha(t) = h_{\alpha-1}(t)$$

hold true. Substituting the formulas presented above into Equation (87), we arrive at the 2nd fundamental theorem for the Hilfer fractional derivative (1) with the order α ($n-1 < \alpha \le n$, $n \in \mathbb{N}$) and type β ($0 \le \beta \le 1$):

$$(I^\alpha_{0+}\,D^{\alpha,\beta}_{0+}\,f)(t) = f(t) - \sum_{j=0}^{n-1}\frac{d^j}{dt^j}(I^{(1-\beta)(n-\alpha)}_{0+}f)(0)\,h_{j-(1-\beta)(n-\alpha)}(t),\ t > 0. \tag{99}$$

This formula was derived for the first time in [3].

Finally, we mention that Equation (87) can be rewritten in terms of the projector operator P_{1L} of the 1st-level GFD of arbitrary order:

$$(P_{1L}\,f)(t) := f(t) - (\mathbb{I}_{(\kappa)}\,{}_{1L}\mathbb{D}_{(k_1,k_2)}\,f)(t) = \sum_{j=0}^{n-1}\frac{d^j}{dt^j}(\mathbb{I}_{(k_2)}\,f)(0)\,\frac{d^{n-1-j}}{dt^{n-1-j}}(\kappa * k_1)(t),\ t > 0. \tag{100}$$

The coefficients $\frac{d^j}{dt^j}(\mathbb{I}_{(k_2)}\,f)(0)$ by the functions $\frac{d^{n-1-j}}{dt^{n-1-j}}(\kappa * k_1)(t)$, $j = 0,1,\ldots,n-1$ at the right-hand side of Equation (100) determine the form of the natural initial conditions for the fractional differential equations with the 1st-level GFDs of arbitrary order. In particular, it is well known that the initial conditions for the fractional differential equations with the Caputo fractional derivatives are posed in the form $y^{(j)}(0) = a_j$, $j = 0,1,\ldots,n-1$, whereas the initial conditions for the fractional differential equations with the Riemann–Liouville fractional derivatives are formulated as follows: $\left(\frac{d^j}{dt^j}\,I^{n-\alpha}_{0+}\,f\right)(0) = a_j$, $j = 0,1,\ldots,n-1$.

4. Conclusions and Directions for Further Research

In this paper, the 1st-level GFDs of arbitrary order were defined and investigated for the first time in the FC literature. These derivatives can be interpreted as a generalization of the Hilfer derivative to the case of some general kernels of the Sonin type.

To define the 1st-level GFDs of arbitrary order, a suitable generalization of the Sonin condition for their kernels was introduced in the form

$$(\kappa * k_1 * k_2)(t) = h_n(t) = \frac{t^{n-1}}{(n-1)!}, \ n \in \mathbb{N}.$$

Then we specified a set of the kernels that satisfy this condition and possess an integrable singularity of the power law type at the origin. This set was denoted by $\mathcal{L}_n^1$ (kernels of the 1st level general fractional derivatives of the generalized order from the interval $(n-1,\ n)$).

The 1st-level GFDs of arbitrary order are integro-differential operators of the convolution type with the kernels from the set $\mathcal{L}_n^1$ defined by

$$({}_{1L}\mathbb{D}_{(k_1,k_2)}\, f)(t) = \left(\mathbb{I}_{(k_1)}\, \frac{d^n}{dt^n}\, \mathbb{I}_{(k_2)}\, f\right)(t),$$

where $\mathbb{I}_{(k_1)}$ and $\mathbb{I}_{(k_2)}$ are the integral operators of convolution type with the kernels k_1 and k_2, respectively.

The main advantage of these derivatives is that both the GFDs of arbitrary order of the Riemann–Liouville type and the regularized GFDs of arbitrary order considered in the literature so far are their particular cases. In this paper, some important properties, including the 1st and the 2nd fundamental theorems of fractional calculus for the 1st level GFDs of arbitrary order were formulated and proved.

As for the directions for further research, the notion of the 1st-level GFDs can be extended to the case of the 2nd- and even the mth-level GFDs following the procedure presented in [5] for the case of the power law kernels. For instance, the kernels of the mth-level GFDs of the generalized order from the interval $(0,\ 1)$ should satisfy the generalized Sonin condition in the form

$$(\kappa * k_1 * \ldots * k_{m+1})(t) = h_1(t) = \{1\},\ t > 0.$$

As in the case of the 1st-level GFDs, a set $\mathcal{L}_1^m$ of the kernels that satisfy this condition and possess an integrable singularity of power law type at the origin can be defined (kernels of the mth-level GFDs of the generalized order from the interval $(0,\ 1)$).

A natural definition of the mth-level GFDs with the kernels from $\mathcal{L}_1^m$ is as follows:

$$({}_{mL}\mathbb{D}_{(k_1,\ldots,k_{m+1})}\, f)(t) := \left(\mathbb{I}_{(k_1)}\, \frac{d}{dt}\, \ldots\, \mathbb{I}_{(k_n)}\, \frac{d}{dt}\, \mathbb{I}_{(k_{m+1})}\, f\right)(t).$$

By analogy with the construction presented above, the mth-level GFDs of the generalized order from the interval $(n-1,\ n)$ with the kernels from the set $\mathcal{L}_n^m$ can be also defined and studied.

All these and further related topics will be considered elsewhere.

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