fractal and fractional

Article

On Caputo Fractional Derivatives and Caputo–Fabrizio Integral Operators via (s, m)-Convex Functions

Ammara Nosheen [1], Maria Tariq [1], Khuram Ali Khan [1], Nehad Ali Shah [2] and Jae Dong Chung [2,*]

[1] Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan
[2] Department of Mechanical Engineering, Sejong University, Seoul 05006, Republic of Korea
* Correspondence: jdchung@sejong.ac.kr

Abstract: This paper contains a variety of new integral inequalities for (s, m)-convex functions using Caputo fractional derivatives and Caputo–Fabrizio integral operators. Various generalizations of Hermite–Hadamard-type inequalities containing Caputo–Fabrizio integral operators are derived for those functions whose derivatives are (s, m)-convex. Inequalities involving the digamma function and special means are deduced as applications.

Keywords: (s, m)-convex function; Hermite–Hadamard inequalities; Caputo fractional derivative; Caputo–Fabrizio integral operator

Citation: Nosheen, A.; Tariq, M.; Khan, K.A.; Shah, N.A.; Chung, J.D. On Caputo Fractional Derivatives and Caputo–Fabrizio Integral Operators via (s, m)-Convex Functions. *Fractal Fract.* **2023**, *7*, 187. https://doi.org/10.3390/fractalfract7020187

Academic Editor: Rodica Luca

Received: 13 January 2023
Revised: 7 February 2023
Accepted: 9 February 2023
Published: 13 February 2023

1. Introduction

The idealogy of convex functions has achieved rapid advancement. Applications of convex functions have been discovered in engineering [1], statistics [2], optimization [3], and many others. In [4,5], Khan et al. build up the foresighted estimations by utilizing the definition of convex functions, various inequalities, and the power mean. They offer applications in information theory. Hudzik et al. considered in [6] the class of s-convex functions in the second sense. In 1993, V. Mihesan initiated the class of (s, m)-convex function. In 2014, N. Eftekhari [7] proposed the class of (s, m)-convex functions in the second sense by combining an excerpt of s-convexity in the second sense with m-convexity.

In the study of different classes of equations, inequalities are considered as essential tools. When direct methods of solving problems seem inconvenient, inequalities can provide indirect routes of reasoning. Inequalities are involved in the problems of applied sciences and engineering. An enormous amount of endeavour has been committed to find new sorts of inequalities [8]. In [9], Bainov et al. illustrated the applications of integral inequalities in partial differential equations, impulse differential equations, etc.

The Hermite–Hadamard inequality, a rudimentary result for convex functions, was first investigated by J. Hadamard in 1893. It has a simple geometrical exposition and immense pertinence [10,11]. The classical Hermite Hadamard inequality [12] delivers an appraisal of mean values of convex function $\sigma : I \to \mathbb{R}$,

$$\sigma(\frac{\lambda+\mu}{2}) \leq \frac{1}{\lambda-\mu}\int_{\lambda}^{\mu} \sigma(y)dy \leq \frac{\sigma(\lambda)+\sigma(\mu)}{2}, \tag{1}$$

where $\lambda, \mu \in I$ and I is a closed interval in $\mathbb{R}$. The Hermite Hadamard inequality for s-convex function [13] is

$$2^{s-1}\sigma(\frac{\lambda+\mu}{2}) \leq \frac{1}{\lambda-\mu}\int_{\lambda}^{\mu} \sigma(y)dy \leq \frac{\sigma(\lambda)+\sigma(\mu)}{s+1}, \tag{2}$$

where $s \in (0, 1]$.

In 2022, Khan et al. [14] illustrate analogous inequalities for the (s, m)-convex function as:

If function $\sigma : [0, u] \longrightarrow \mathbb{R}$, $u > 0$ is (s, m)-convex function, then

$$\begin{aligned} &2^s \sigma\Big(\frac{\lambda + m\mu}{2}\Big) \\ &\leq \Big[\frac{1}{m\mu - \lambda} \int_{\lambda}^{m\mu} \sigma(x)dx + \frac{m^2}{m\mu - \lambda} \int_{\frac{\lambda}{m}}^{\mu} \sigma(y)dy\Big] \\ &\leq \Big(\frac{\sigma(\lambda) + m\sigma(\mu)}{s+1}\Big) + m\Big(\frac{\sigma(\mu) + m\sigma(\frac{\lambda}{m^2})}{s+1}\Big). \end{aligned} \tag{3}$$

hold, where $s, m \in (0, 1]$, $\lambda, \mu \in [0, u]$ and $\eta \in [0, 1]$. If we put $m = 1$ in (3), we obtain (2). If we put $s = 1$ and $m = 1$ in (3), we obtain (1). In [15], generalizations of Hermite–Hadamard inequality to n-time differentiable functions, which are s-convex, are established.

Fractional calculus has a remarkable development in the field of mathematics, besides that it is a landmark in physics, biology, economics, and many other fields [16,17]. The anomalous diffusion has been observed in many phenomena with accurate physical measurements [18–20].

Michele Caputo introduced Caputo fractional derivatives in 1967 [21]. The Caputo operator has a non-singular kernel that can be converted to an integral by using Laplace transformation. Usually, the Caputo version is chosen when physical models are presented because the physical interpretation of the given data is unambiguous. In practice, most circumstances when a fractional derivative concept is required are covered by the Caputo fractional derivative; see [22].

In [23], some inequalities are generated using the Caputo–Fabrizio integral operator. In [24,25], Butt et al. gave inequalities that have Caputo fractional integrals for exponential s-convex functions and the Caputo fractional derivative for exponential (s, m)-convex functions. Kemali et al. [26] established Hermite–Hadamard-type inequalities for s-convex functions in the second sense through Caputo derivatives and Caputo–Fabrizio integral operators. In [27], Abbasi et al. provided these inequalities in a generalized form and its bounds, for s-convex functions using Caputo–Fabrizio integral operator. In [28], Li et al. gave analogous inequalities for strongly convex functions.

With the aid of the Caputo fractional derivative, the spreading version of COVID-19 is investigated [29]. Wang et al. [30] initiated a new local fractional modified Benjamin–Bona–Mahony equation that had the local fractional derivative. Ji-Huan He in [31] discussed fractal calculus and its geometrical explanation. In [32], the authors presented a fractional model of a falling object with the aid of the Caputo derivative. Wanassi et al. investigated the world population growth as an application of fractional derivative [33]. The biological model is presented using the Caputo Fabrizio operator in [34]. Areshi et al. investigated wave solutions of the predator–prey model with fractional derivative [35]. Mahatekar et al. [36] acquired a new numerical method for the solution of fractional differential equations that have Caputo–Fabrizio derivatives. From the above-cited work, the primary purpose of this paper is to accomplish various inequalities for the functions whose derivatives are (s, m)-convex; these inequalities involve Caputo fractional derivatives and Caputo–Fabrizio integrals.

The paper is organised as follows. In the main findings, firstly, the inequalities for the functions whose derivatives are (s, m)-convex functions in second sense are established using the Caputo fractional derivative. For (s, m)-convex functions in the second sense, the Hermite–Hadamard inequality involving Caputo–Fabrizio operators is presented. Furthermore, some inequalities for the product of (s, m)-convex functions are constructed. We establish two vital lemmas, which are helpful to construct new inequalities that contain the Caputo–Fabrizio operator. Additionally, some applications to special means are created.

2. Preliminaries

The following are some definitions that are useful in our paper.

Convex function [12]:

A real valued function σ is said to be convex on close interval I if

$$\sigma\Big(\lambda\eta + (1-\eta)\mu\Big) \leq \eta\sigma(\lambda) + (1-\eta)\sigma(\mu) \tag{4}$$

holds, for all λ, μ in I and $\eta \in [0,1]$.

***s*-convex function** [6]:

A function $\sigma : [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in second sense if

$$\sigma\Big(\lambda\eta + (1-\eta)\mu\Big) \leq \eta^s\sigma(\lambda) + (1-\eta)^s\sigma(\mu) \tag{5}$$

holds, provided that all $\lambda, \mu \in [0,\infty)$, $s \in (0,1]$ and $\eta \in [0,1]$.

(s,m)-convex function [17]:

A function $\sigma : [0,u] \longrightarrow \mathbb{R}$, $u > 0$ is said to be (s,m)-convex function in the second sense with $s, m \in (0,1]$, if

$$\sigma\Big(\lambda\eta + m(1-\eta)\mu\Big) \leq \eta^s\sigma(\lambda) + m(1-\eta)^s\sigma(\mu) \tag{6}$$

holds, provided that all $\lambda, \mu \in [0,u]$ and $\eta \in [0,1]$.

Beta function [37]: The integral form of Beta function for $f > 0$ and $l > 0$ is as follows:

$$\beta(f,l) = \int_0^1 k^{f-1}(1-k)^{l-1}dk. \tag{7}$$

Gamma function [37]: Integral form of Gamma function is

$$\Gamma(x) = \int_0^\infty e^{-k}k^{x-1}dk, \tag{8}$$

where $x > 0$.

Digamma function [38]: The integral form of digamma function is

$$\psi(v) = \int_0^1 \frac{1-q^{v-1}}{1-q}dq - \gamma, \tag{9}$$

where $v > 0$ and γ is the Euler–Mascheroni constant.

Caputo fractional derivative [21,24,26]:

Let $AC^n[\lambda,\mu]$ be the space of functions that have nth derivatives absolutely continuous, $\sigma \in AC^n[\lambda,\mu]$, where $n = [\kappa] + 1$, $\kappa \notin \{1,2,3,....\}$ and $[\cdot]$ denotes floor function. The right side Caputo fractional derivative is

$$({}^C D^\kappa_{\lambda^+}\sigma)(z) = \frac{1}{\Gamma(n-\kappa)}\int_\lambda^z \frac{\sigma^{(n)}(v)}{(z-v)^{\kappa-n+1}}dv, \tag{10}$$

$z > \lambda$, the left side Caputo fractional derivative is

$$({}^C D^\kappa_{\mu^-}\sigma)(z) = \frac{(-1)^n}{\Gamma(n-\kappa)}\int_z^\mu \frac{\sigma^{(n)}(v)}{(v-z)^{\kappa-n+1}}dv, \tag{11}$$

$z < \mu$. If $\kappa = n \in \{1, 2, 3...\}$ and usual derivative $\sigma^{(n)}(z)$ of order n exists, then Caputo fractional $({}^{C}D^{n}_{\lambda^+}\sigma)(z)$ matches with $\sigma^{(n)}(z)$, whereas $({}^{C}D^{n}_{\mu^-}\sigma)(z)$ matches with $\sigma^{(n)}(z)$ with exactness to a constant multiplier $(-1)^n$. If $n = 1, \kappa = 0$, then we have

$$({}^{C}D^{0}_{\lambda^+}\sigma)(z) = ({}^{C}D^{0}_{\mu^-}\sigma)(z) = \sigma(z).$$

Caputo–Fabrizio integral operator [16,26]:

Let $H^1(\lambda, \mu)$ be the Sobolev space of order one defined as

$$H^1(\lambda, \mu) = \{g \in L^2(\lambda, \mu) : g' \in L^2(\lambda, \mu)\},$$

where

$$L^2(\lambda, \mu) = \{g(z) : \Big(\int_{\lambda}^{\mu} g^2(z)dz\Big)^{\frac{1}{2}} < \infty\}.$$

Let $\sigma \in H^1(\lambda, \mu), \lambda < \mu$ and $\kappa \in [0, 1]$; then, the left derivative in the sense of Caputo–Fabrizio is defined as

$$({}^{CFD}_{\lambda}D^{\kappa}\sigma)(z) = \frac{B(\kappa)}{1-\kappa}\int_{\lambda}^{z} \sigma'(l)e^{\frac{-\kappa(z-l)^{\kappa}}{1-\kappa}}dl,$$

$z > \kappa$ and the associated integral operator is

$$({}^{CF}_{\lambda}I^{\kappa}\sigma)(z) = \frac{1-\kappa}{B(\kappa)}\sigma(z) + \frac{\kappa}{B(\kappa)}\int_{\lambda}^{z} \sigma(v)dv, \tag{12}$$

where $B(\kappa) > 0$ is the normalization function satisfying B(0) = B(1) = 1. For $\kappa = 0, \kappa = 1$, the left derivative is defined as follows, respectively

$$({}^{CFD}_{\lambda}D^{0}\sigma)(z) = \sigma'(z),$$

$$({}^{CFD}_{\lambda}I^{1}\sigma)(z) = \sigma(z) - \sigma(\lambda).$$

For the right derivative operator

$$({}^{CFD}_{\mu}D^{\kappa}\sigma)(z) = \frac{-B(\kappa)}{1-\kappa}\int_{z}^{\mu} \sigma'(l)e^{\frac{-\kappa(l-z)^{\kappa}}{1-\kappa}}dl,$$

$z < \mu$ and the associated integral operator is

$$({}^{CF}I^{\kappa}_{\mu}\sigma)(z) = \frac{1-\kappa}{B(\kappa)}\sigma(z) + \frac{\kappa}{B(\kappa)}\int_{z}^{\mu} \sigma(v)dv, \tag{13}$$

where $B(\kappa) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

Means [39,40]:

Let $0 < \lambda < \mu$, for $p \in \mathbb{R} - \{0, -1\}$, the arithmetic mean and Stolarsky mean are defined, respectively, as

$$A(\lambda, \mu) = \frac{\lambda + \mu}{2}; \tag{14}$$

$$L_p(\lambda, \mu) = \Big(\frac{\lambda^{p+1} - \mu^{p+1}}{(p+1)(\lambda - \mu)}\Big)^{\frac{1}{p}}. \tag{15}$$

3. Main Results

The following theorem gives inequality for the (s,m)-convex function that has a Caputo fractional derivative.

Theorem 1. *Let $\sigma : [\lambda,\mu] \subset [0,\infty) \to \mathbb{R}$ be n- times differentiable function, where n is a positive integer. If $\sigma^{(n)}(\cdot)$ is (s,m)-convex function, then for $\kappa,\theta > 1$, $x \in [\lambda,\mu]$ with $n > \max\{\kappa,\theta\}$,* (16) *holds.*

$$\begin{aligned}
&\Gamma(n-\kappa+1)({}^{C}D^{\kappa-1}_{\lambda^+}\sigma)(x) + \Gamma(n-\theta+1)({}^{C}D^{\theta-1}_{\mu^-}\sigma)(x) \\
&\leq \Big(\frac{(x-\lambda)^{n-\kappa+1}\sigma^n(\lambda) + (-1)^n(\mu-x)^{n-\theta+1}\sigma^n(\mu)}{s+1}\Big) \\
&+ m\Big(\frac{(x-\lambda)^{n-\kappa+1} + (-1)^n(\mu-x)^{n-\theta+1}}{s+1}\Big)\sigma^n(\frac{x}{m}).
\end{aligned} \tag{16}$$

Proof. For $z \in [\lambda,x]$ and $n > \kappa$, we have

$$(x-z)^{n-\kappa} \leq (x-\lambda)^{n-\kappa}. \tag{17}$$

Let $z = \left(\frac{x-z}{x-\lambda}\right)\lambda + m\left(\frac{z-\lambda}{x-\lambda}\right)(\frac{x}{m})$. Since $\sigma^{(n)}(\cdot)$ is (s,m)- convex function, the following inequality holds:

$$\sigma^{(n)}(z) \leq \left(\frac{x-z}{x-\lambda}\right)^s\sigma^n(\lambda) + m\left(\frac{z-\lambda}{x-\lambda}\right)^s\sigma^{(n)}(\frac{x}{m}). \tag{18}$$

Multiply (17) and (18); then, integrate with respect to z over $[\lambda,x]$ to obtain

$$\int_\lambda^x (x-z)^{n-\kappa}\sigma^{(n)}(z)dz \leq \frac{(x-\lambda)^{n-\lambda}}{(x-\lambda)^s}\Big(\sigma^n(\lambda)\int_\lambda^x (x-z)^s dz + m\sigma^n(\frac{x}{m})\int_\lambda^x (z-\lambda)^s dz\Big).$$

$$\int_\lambda^x (x-z)^{n-\kappa}\sigma^{(n)}(z)dz \leq \frac{(x-\lambda)^{n-\kappa-s-1}}{s+1}\Big(\sigma^{(n)}(\lambda) + m\sigma^{(n)}(\frac{x}{m})\Big).$$

Using (10), we have

$$\Gamma(n-\kappa+1)({}^{C}D^{\kappa-1}_{\lambda^+}\sigma)(x) \leq \frac{(x-\lambda)^{n-\kappa+1}}{s+1}\Big(\sigma^{(n)}(\lambda) + m\sigma^{(n)}(\frac{x}{m})\Big). \tag{19}$$

Now, consider $z \in [x,\mu]$, $n > \theta$; we have

$$(z-x)^{n-\theta} \leq (\mu-x)^{n-\theta}. \tag{20}$$

Let $z = \left(\frac{z-x}{\mu-x}\right)\mu + m\left(\frac{\mu-z}{\mu-x}\right)(\frac{x}{m})$. Since, $\sigma^{(n)}(\cdot)$ is (s,m)-convex.

$$\sigma^{(n)}(z) \leq \left(\frac{z-x}{\mu-x}\right)^s\sigma^n(\mu) + m\left(\frac{\mu-z}{\mu-x}\right)^s\sigma^{(n)}(\frac{x}{m}). \tag{21}$$

Multiply (20) and (21); then, integrate with respect to z over $[x,\mu]$

$$\int_x^\mu (z-x)^{n-\theta}\sigma^{(n)}(z)dz \leq \frac{(\mu-x)^{n-\theta}}{(\mu-x)^s}\Big(\sigma^n(\mu)\int_x^\mu (z-x)^s dz + m\sigma^n(\frac{x}{m})\int_x^\mu (\mu-z)^s dz\Big).$$

$$\int_x^\mu (z-x)^{n-\theta}\sigma^{(n)}(z)dz \leq \frac{(\mu-x)^{n-\theta+1}}{s+1}\Big(\sigma^{(n)}(\mu) + m\sigma^{(n)}(\frac{x}{m})\Big). \tag{22}$$

Multiplying both sides of (22) with $(-1)^n$ and taking into account the oddness and eveness of n, we obtain

$$({}^{C}D^{\theta-1}_{\mu^-}\sigma)(x)\Gamma(n-\theta+1) \leq \frac{(-1)^n(\mu-x)^{n-\theta+1}}{s+1}\Big(\sigma^{(n)}(\mu)+m\sigma^{(n)}(\frac{x}{m})\Big). \tag{23}$$

Add (19) and (23) to obtain (16). □

Remark 1. *Put $m=1$ in (16), we obtain [Theorem 2.1] [26].*

Corollary 1. *If we take $\kappa=\theta$ in (16) , we obtain*

$$\begin{aligned}
&\Gamma(n-\kappa+1)\Big(({}^{C}D^{\kappa-1}_{\lambda^+}\sigma)(x)+({}^{C}D^{\kappa-1}_{\mu^-}\sigma)(x)\Big)\\
&\leq\Big(\frac{(x-\lambda)^{n-\kappa+1}\sigma^n(\lambda)+(-1)^n(\mu-x)^{n-\kappa+1}\sigma^n(\mu)}{s+1}\Big)\\
&+m\Big(\frac{(x-\lambda)^{n-\kappa+1}+(-1)^n(\mu-x)^{n-\kappa+1}}{s+1}\Big)\sigma^n(\frac{x}{m}).
\end{aligned} \tag{24}$$

Remark 2. *If we put $m=1$ in (24), we obtain [Corollary 2.1] [26].*

Corollary 2. *If we substitute $\kappa=\theta$ and $s=1$ in Theorem 1, then (25) is obtained.*

$$\begin{aligned}
&\Gamma(n-\kappa+1)\Big(({}^{C}D^{\kappa-1}_{\lambda^+}\sigma)(x)+({}^{C}D^{\kappa-1}_{\mu^-}\sigma)(x)\Big)\\
&\leq\Big(\frac{(x-\lambda)^{n-\kappa+1}\sigma^n(\lambda)+(-1)^n(\mu-x)^{n-\kappa+1}\sigma^n(\mu)}{2}\Big)\\
&+m\Big(\frac{(x-\lambda)^{n-\kappa+1}+(-1)^n(\mu-x)^{n-\kappa+1}}{2}\Big)\sigma^n(\frac{x}{m}).
\end{aligned} \tag{25}$$

Remark 3. ***(a)*** *If we put $m=1$ in (25), we have [Corollary 2.2] [26].* ***(b)*** *When n is even and $m=1$ in Corollary 2, [Corollary 2.1] [41] is obtained.*

Theorem 2. *Let $\sigma:[\lambda,\mu]\subset[0,\infty)\to\mathbb{R}$ be an n- times differentiable function, where n is a positive integer. If $\sigma^{(n)}(\cdot)$ is (s,m)-convex function and integrable on $[\lambda,\mu]$, then the following inequalities hold:*

$$\begin{aligned}
\frac{2^s}{n-\kappa}\sigma^{(n)}(\frac{\lambda+m\mu}{2}) &\leq \frac{\Gamma(n-\kappa)}{(m\mu-\lambda)^{n-\kappa}}({}^{C}D^{\kappa}_{\lambda^+}\sigma^{(n)})(m\mu)\\
&+m\frac{\Gamma(n-\kappa)}{(\mu-\frac{\lambda}{m})^{n-\kappa}}(-1)^n({}^{C}D^{\kappa}_{\mu^-}\sigma^{(n)})(\frac{\lambda}{m})\\
&\leq\Big[\frac{\sigma^{(n)}(\lambda)}{n-\kappa+s}+m\sigma^{(n)}(\frac{\lambda}{m})\beta(s+1,n-\kappa)\Big]\\
&+\sigma^{(n)}(\mu)\Big[m\beta(s+1,n-\kappa)+\frac{m^2}{n-\kappa+s}\Big].
\end{aligned} \tag{26}$$

Proof. Since $\sigma^{(n)}(\cdot)$ is (s,m)-convex function, we have

$$\sigma^{(n)}(\frac{x+my}{2}) \leq \frac{\sigma^{(n)}(x)+m\sigma^{(n)}(y)}{2^s} \tag{27}$$

for $x,y\in[\lambda,\mu]$. Let $x=z\lambda+m(1-z)\mu$, $y=(1-z)\frac{\lambda}{m}+z\mu$ for $z\in[0,1]$. Then, (27) gives:

$$2^s\sigma^{(n)}(\frac{\lambda+m\mu}{2}) \leq \sigma^{(n)}(z\lambda+m(1-z)\mu)+m\sigma^{(n)}((1-z)\frac{\lambda}{m}+z\mu). \tag{28}$$

Multiply (28) by $z^{n-\kappa-1}$; then, integrate over $[0,1]$

$$
\begin{aligned}
&2^s\sigma^{(n)}(\tfrac{\lambda+m\mu}{2})\int_0^1 z^{n-\kappa-1}dz \\
&\le \int_0^1 z^{n-\kappa-1}\sigma^{(n)}(z\lambda+m(1-z)\mu)dz \\
&+m\int_0^1 z^{n-\kappa-1}\sigma^{(n)}((1-z)\tfrac{\lambda}{m}+z\mu)dz.
\end{aligned}
$$

From which one has

$$
\begin{aligned}
&2^s\sigma^{(n)}(\frac{\lambda+m\mu}{2})\frac{1}{n-\kappa} \\
&\le \frac{\Gamma(n-\kappa)}{(m\mu-\lambda)^{n-\kappa}}({}^C D^{\kappa}_{\lambda^+}\sigma^{(n)})(m\mu) \\
&+\frac{\Gamma(n-\kappa)}{(\mu-\frac{\lambda}{m})^{n-\kappa}}(-1)^n({}^C D^{\kappa}_{\mu^-}\sigma^{(n)})(\frac{\lambda}{m}).
\end{aligned}
\tag{29}
$$

(s,m)-convexity of $\sigma^{(n)}(\cdot)$ gives

$$
\begin{aligned}
&\sigma^{(n)}(z\lambda+m(1-z)\mu)+m\sigma^{(n)}((1-z)\tfrac{\lambda}{m}+z\mu) \\
&\le [\sigma^{(n)}(\lambda)z^s+m\sigma^{(n)}(\tfrac{\lambda}{m})(1-z)^s]+\sigma^{(n)}(\mu)[m(1-z)^s+m^2z^s].
\end{aligned}
\tag{30}
$$

Multiply both sides of (30) by $z^{n-\kappa-1}$; then, integrate with respect to z over $[0,1]$.

$$
\begin{aligned}
&\int_0^1 z^{n-\kappa-1}\sigma^{(n)}(z\lambda+m(1-z)\mu)dz+m\int_0^1 z^{n-\kappa-1}\sigma^{(n)}((1-z)\frac{\lambda}{m}+\mu z)dz \\
&\le \int_0^1 [z^s\sigma^{(n)}(\lambda)+m\sigma^{(n)}(\frac{\lambda}{m})(1-z)^s]z^{n-\kappa-1}dz \\
&+\sigma^{(n)}(\mu)\int_0^1 [m(1-z)^s+m^2z^s]z^{n-\kappa-1}dz \\
&=\Big[\frac{\sigma^{(n)}(\lambda)}{n-\lambda+s}+m\sigma^{(n)}(\frac{\lambda}{m})\beta(s+1,n-\kappa)\Big] \\
&+\sigma^{(n)}(\mu)\Big[m\beta(s+1,n-\kappa)+\frac{m^2}{n-\kappa+s}\Big],
\end{aligned}
$$

from which one obtains

$$
\begin{aligned}
&\frac{\Gamma(n-\kappa)}{(m\mu-\lambda)^{n-\kappa}}({}^C D^{\kappa}_{\lambda^+}\sigma^{(n)})(m\mu)+\frac{\Gamma(n-\kappa)}{(\mu-\frac{\lambda}{m})^{n-\kappa}}(-1)^n({}^C D^{\kappa}_{\mu^-}\sigma^{(n)})(\frac{\lambda}{m}) \\
&\le \sigma^{(n)}(\lambda)\Big[\frac{1}{n-\kappa+s}+m\beta(s+1,n-\kappa)\Big]+\sigma^{(n)}(m\mu)\Big[m\beta(s+1,n-\kappa)+\frac{m^2}{n-\kappa+s}\Big].
\end{aligned}
\tag{31}
$$

Equations (29) and (31) give (26). □

Remark 4. *If we put $m=1$ in (26), we obtain [Theorem 2.2] [26]*

Corollary 3. *If we put $\kappa = s = 1$ and $n = 2$ in (16), we obtain:*

$$\begin{aligned}
2\sigma^{(n)}(\frac{\lambda+m\mu}{2}) &\leq \frac{1)}{(m\mu-\lambda)}({}^{C}D^{1}_{\lambda^{+}}\sigma'')(m\mu) \\
&+ m\frac{1)}{(\mu-\frac{\lambda}{m})}(-1)^{n}({}^{C}D^{\kappa}_{\mu^{-}}\sigma'')(\frac{\lambda}{m}) \\
&\leq \Big[\frac{\sigma''(\lambda)}{n-\kappa+s} + m\sigma''(\frac{\lambda}{m})\beta(s+1,n-\kappa)\Big] \\
&+ \sigma^{(n)}(\mu)\Big[m\beta(s+1,n-\kappa) + \frac{m^{2}}{n-\kappa+s}\Big].
\end{aligned} \tag{32}$$

Remark 5. *If we substitute $m = 1$ in (32), we obtain [Corollary 2.2] [26].*

Theorem 3. *If $\sigma : [\lambda, \mu] \to \mathbb{R}$ be (s, m)-convex function and integrable on $[\lambda, \mu]$, then*

$$\begin{aligned}
2^{s}\sigma(\frac{\lambda+m\mu}{2}) &\leq \frac{B(\kappa)}{\kappa(m\mu-\lambda)}\Big(\Big[({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big] \\
&+ m^{2}\Big[({}^{CF}_{\frac{\lambda}{m}}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{\mu}\sigma)(x)\Big] - \frac{2(1+m^{2})(1-\kappa)}{B(\kappa)}\sigma(x)\Big) \\
&\leq \Big[\Big(\frac{\sigma(\lambda)+m\sigma(\mu)}{s+1}\Big) + m\Big(\frac{\sigma(\mu)+m\sigma(\frac{\lambda}{m^{2}})}{s+1}\Big)\Big]
\end{aligned} \tag{33}$$

hold for $\kappa \in [0, 1]$.

Proof. Multiply (3) by $\dfrac{\kappa(m\mu-\lambda)}{B(\kappa)}$; then, add $\dfrac{2(1+m^{2})(1-\kappa)}{B(\kappa)}\sigma(x)$ to obtain

$$\begin{aligned}
&\frac{\kappa(m\mu-\lambda)}{B(\kappa)}2^{s}\sigma(\frac{\lambda+m\mu}{2}) + \frac{(2+2m^{2})(1-\kappa)}{B(\kappa)}\sigma(x) \\
&\leq \frac{\kappa}{B(\kappa)}\Big[\int_{\lambda}^{m\mu}\sigma(u)du + m^{2}\int_{\frac{\lambda}{m}}^{\mu}\sigma(u)du\Big] + \frac{(2+2m^{2})(1-\kappa)}{B(\kappa)}\sigma(x) \\
&\leq \frac{\kappa(m\mu-\lambda)}{B(\kappa)}\Big[\Big(\frac{\sigma(\lambda)+m\sigma(\mu)}{s+1}\Big) + m\Big(\frac{\sigma(\mu)+m\sigma(\frac{\lambda}{m^{2}})}{s+1}\Big)\Big] \\
&+ \frac{(2+2m^{2})(1-\kappa)}{B(\kappa)}\sigma(x).
\end{aligned} \tag{34}$$

Consider the left side of (34)

$$\begin{aligned}
&\frac{\kappa(m\mu-\lambda)}{B(\kappa)}2^{s}\sigma(\frac{\lambda+m\mu}{2}) + \frac{(2+2m^{2})(1-\kappa)}{B(\kappa)}\sigma(x) \\
&\leq \Big[\frac{\kappa}{B(\kappa)}\int_{\lambda}^{m\mu}\sigma(v)dv + \frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\Big] + m^{2}\Big[\int_{\frac{\lambda}{m}}^{\mu}\sigma(v)dv + \frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\Big] \\
&= \Big[({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big] + m^{2}\Big[({}^{CF}_{\frac{\lambda}{m}}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{\mu}\sigma)(x)\Big].
\end{aligned} \tag{35}$$

Take the right side of (34)

$$\begin{aligned}
&\frac{\kappa(m\mu-\lambda)}{B(\kappa)}\Big[\Big(\frac{\sigma(\lambda)+m\sigma(\mu)}{s+1}\Big)+m\Big(\frac{\sigma(\mu)+m\sigma(\frac{\lambda}{m^2})}{s+1}\Big)\Big]+\frac{(2+2m^2)(1-\kappa)}{B(\kappa)}\sigma(x)\\
&\geq\Big[\frac{\kappa}{B(\kappa)}\int_{\lambda}^{m\mu}\sigma(v)dv+\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\Big]+m^2\Big[\int_{\frac{\lambda}{m}}^{\mu}\sigma(v)dv+\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\Big] \qquad (36)\\
&=\Big[({}^{CF}_{\lambda}I^{\kappa}\sigma)(x)+({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big]+m^2\Big[({}^{CF}_{\frac{\lambda}{m}}I^{\kappa}\sigma)(x)+({}^{CF}I^{\kappa}_{\mu}\sigma)(x)\Big].
\end{aligned}$$

Combine (35) and (36), and further solving yields the required result. □

Remark 6. ***(a)*** *If we put $m=1$ in* (33)*, we obtain [Theorem 2.3] [26].* ***(b)*** *If we put $m=1$ and $s=1$ in* (33)*, we obtain [Theorem2] [42].*

Theorem 4. *Let $\sigma:[\lambda,\mu]\to\mathbb{R}$ be (s_1,m)-convex function and $\chi:[\lambda,\mu]\to\mathbb{R}$ be (s_2,m)-convex function $s_1,s_2\in(0,1]$. If $\sigma\chi$ integrable on $[\lambda,\mu]$, where $\lambda,\mu\in\mathbb{R}$, then*

$$\begin{aligned}
&\frac{B(\kappa)}{\kappa(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma\chi)(x)+({}^{CF}I^{\kappa}_{m\mu}\sigma\chi)(x)\Big)-\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\chi(x)\Big]\\
&\leq\frac{M(\lambda,\mu)}{s_1+s_2+1}+\beta(s_1+1,s_2+1)mN(\lambda,\mu),
\end{aligned} \qquad (37)$$

where $B(\kappa)>0$ is normalization function $M(\lambda,\mu)=\sigma(\lambda)\chi(\lambda)+m^2\sigma(\mu)\chi(\mu)$ and $N(\lambda,\mu)=\sigma(\lambda)\chi(\mu)+\sigma(\mu)\chi(\lambda)$.

Proof. Since $\sigma:[\lambda,\mu]\to\mathbb{R}$ is the (s_1,m)-convex function and $\chi:[\lambda,\mu]\to\mathbb{R}$ is the (s_2,m)-convex function, we have

$$\sigma(\omega\lambda+m(1-\omega)\mu)\leq\omega^{s_1}\sigma(\lambda)+m(1-\omega)^{s_1}\sigma(\mu) \qquad (38)$$

and

$$\chi(\omega\lambda+m(1-\omega)\mu)\leq\omega^{s_2}\chi(\lambda)+m(1-\omega)^{s_2}\chi(\mu), \qquad (39)$$

where $\lambda,\mu\in I$ and $\omega\in[0,1]$.

Multiply (38) and (39) side to side; then, integrate over $[0,1]$ to obtain

$$\begin{aligned}
&\int_0^1\sigma(\omega\lambda+m(1-\omega)\mu)\chi(\omega\lambda+m(1-\omega)\mu)d\omega\\
&\leq\int_0^1[\omega^{s_1}\sigma(\lambda)+m(1-\omega)^{s_1}\sigma(\mu)][\omega^{s_2}\chi(\lambda)+m(1-\omega)^{s_2}\chi(\mu)]d\omega\\
&=\frac{M(\lambda,\mu)}{s_1+s_2+1}+mN(\lambda,\mu)\beta(s_1+1,s_2+1).
\end{aligned}$$

Substitute $\omega\lambda+m(1-\omega)\mu$ = y; this inequality gives the following inequality:

$$\begin{aligned}
&\frac{1}{m\mu-\lambda}\int_{\lambda}^{m\mu}\sigma(y)\chi(y)dy\\
&\leq\frac{M(\lambda,\mu)}{s_1+s_2+1}+mN(\lambda,\mu)\beta(s_1+1,s_2+1).
\end{aligned} \qquad (40)$$

Multiply (40) with $\frac{\kappa(m\mu-\lambda)}{B(\kappa)}$ and add $\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\chi(x)$ to obtain

$$\begin{aligned}&\frac{\kappa}{B(\kappa)}\Big[\int_{\lambda}^{x}\sigma(y)\chi(y)dy+\int_{x}^{m\mu}\sigma(y)\chi(y)dy\Big]+\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\chi(x)\\&\leq\frac{\kappa(m\mu-\lambda)}{B(\kappa)}\Big[\frac{M(\lambda,\mu)}{s_1+s_2+1}+mN(\lambda,\mu)\beta(s_1+1,s_2+1)\Big]+\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\chi(x).\end{aligned}$$

Use (12) and (13) to obtain the following inequality:

$$\begin{aligned}&\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma\chi)(x)+({}^{CF}I^{\kappa}_{m\mu}\sigma\chi)(x)\Big)\\&\leq\frac{\kappa(m\mu-\lambda)}{B(\kappa)}\Big[\frac{M(\lambda,\mu)}{s_1+s_2+1}+mN(\lambda,\mu)\beta(s_1+1,s_2+1)\Big]+\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\chi(x).\end{aligned}\tag{41}$$

Subtract $\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\chi(x)$ from both sides of (41); then, multiply both sides by $\frac{B(\kappa)}{2(1-\kappa)}\sigma(x)\chi(x)$, so that inequality (37) is obtained. □

Corollary 4. *If we choose $s_1=s_2=1$, then we obtain*

$$\begin{aligned}&\frac{B(\kappa)}{\kappa(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma\chi)(x)+({}^{CF}I^{\kappa}_{m\mu}\sigma\chi)(x)\Big)-\frac{2(1-\kappa)}{B(\kappa)}\sigma(x)\chi(x)\Big]\\&\leq\frac{M(\lambda,\mu)}{3}+\frac{1}{6}mN(\lambda,\mu).\end{aligned}$$

Remark 7. *For $m=1$, we have [Corollary 2.4] [26].*

Corollary 5. *Put κ=1, $B(\kappa)=B(1)$=1 in (37); we obtain*

$$\begin{aligned}&\frac{1}{\kappa(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma\chi)(x)+({}^{CF}I^{\kappa}_{m\mu}\sigma\chi)(x)\Big)\Big]\\&\leq\frac{M(\lambda,\mu)}{s_1+s_2+1}+\beta(s_1+1,s_2+1)mN(\lambda,\mu).\end{aligned}\tag{42}$$

Remark 8. *For $m=1$, we have [Theorem 6] [22].*

Lemma 1. *Let $\sigma:[\lambda,m\mu]\to\mathbb{R}$ be a differentiable function on $(\lambda,m\mu)$. If $\sigma'(\cdot)$ is integrable on $[\lambda,m\mu]$, then*

$$\begin{aligned}&\frac{\sigma(\lambda)+\sigma(m\mu)}{2}-\frac{B(\kappa)}{\kappa(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x)+({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big]+\frac{2(1-\kappa)}{(m\mu-\lambda)\kappa}\sigma(x)\\&=\frac{(m\mu-\lambda)}{2}\int_{0}^{1}(1-2y)\sigma'(\lambda y+m(1-y)\mu)dy\end{aligned}$$

holds for $\kappa\in[0,1]$.

Proof. It is easy to see that

$$\int_0^1 (1-2y)\sigma'(\lambda y + m(1-y)\mu)dy = \frac{\sigma(\lambda)+\sigma(m\mu)}{(m\mu-\lambda)} + 2\int_0^1 \frac{\sigma(\lambda y + (1-y)m\mu)}{\lambda - m\mu}dy.$$

Substitute $\lambda y + (1-y)m\mu = u$ in the integral on right side of equation.

$$\int_0^1 (1-2y)\sigma'(\lambda y + m(1-y)\mu)dy = \frac{\sigma(\lambda)+\sigma(m\mu)}{(m\mu-\lambda)} - \frac{2}{(m\mu-\lambda)^2}\int_\lambda^{m\mu} \sigma(u)du. \tag{43}$$

Multiply both sides of (43) by $\frac{\kappa(m\mu-\lambda)^2}{2B(\kappa)}$ and subtract $\frac{2(1-\kappa)\sigma(x)}{B(\kappa)}$ to obtain:

$$\begin{aligned} &\frac{\kappa(m\mu-\lambda)^2}{2B(\kappa)}\int_0^1 (1-2y)\sigma'(\lambda y + m(1-y)\mu)dy - \frac{2(1-\kappa)\sigma(x)}{B(\kappa)} \\ &= \frac{\kappa(m\mu-\lambda)^2}{2B(\kappa)}\Big[\frac{\sigma(\lambda)+\sigma(m\mu)}{(m\mu-\lambda)} - \frac{2}{(m\mu-\lambda)^2}\Big(\int_\lambda^x \sigma(u)du + \\ &\int_x^{m\mu}\sigma(u)du\Big)\Big] - \frac{2(1-\kappa)\sigma(x)}{B(\kappa)}, \end{aligned} \tag{44}$$

where $x \in [\lambda, \mu]$. Further solving (44) leads towards the proof of Lemma 1. □

Remark 9. *Put $m = 1$, in Lemma 1, we obtain [Lemma 2] [42].*

Lemma 2. *Let $\sigma : [\lambda, m\mu] \to \mathbb{R}$ be a differentiable function on $(\lambda, m\mu)$. If $\sigma''(\cdot)$ is integrable on $[\lambda, m\mu]$ and $\kappa \in [0,1]$, then (45) holds.*

$$\begin{aligned} &\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{B(\kappa)}{\kappa(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big] + \frac{2(1-\kappa)}{\kappa(m\mu-\lambda)}\sigma(x) \\ &= \frac{(m\mu-\lambda)^2}{2}\int_0^1 r(1-r)\sigma''(\lambda r + m(1-r)\mu)dr. \end{aligned} \tag{45}$$

Proof. It is easy to show

$$\int_0^1 r(1-r)\sigma''(\lambda r + m(1-r)\mu)dr = \frac{2}{(m\mu-\lambda)^2}\Big[\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \int_0^1 \sigma(\lambda r + m(1-r)\mu)dr\Big]. \tag{46}$$

Substitute $u = \lambda r + (1-r)m\mu$ on the right side of the Equation (46).

$$\int_0^1 r(1-r)\sigma''(\lambda r + m(1-r)\mu)dr = \frac{2}{(m\mu-\lambda)^2}\Big[\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{1}{m\mu-\lambda}\int_\lambda^{m\mu}\sigma(u)du\Big]. \tag{47}$$

Multiply both sides of of (47) by $\frac{\kappa(m\mu-\lambda)^3}{2B(\kappa)}$ and subtract $\frac{2(1-\kappa)\sigma(x)}{B(\kappa)}$ to obtain

$$\begin{aligned}&\frac{\kappa(m\mu-\lambda)^3}{2B(\kappa)}\int_0^1 r(1-r)\sigma''(\lambda r + m(1-r)\mu)dr - \frac{2(1-\kappa)\sigma(x)}{B(\kappa)}\\ &= \frac{\kappa(m\mu-\lambda)^3}{2B(\kappa)}\Big[\frac{2}{(m\mu-\lambda)^2}\Big[\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{1}{m\mu-\lambda}\Big(\int_\lambda^x\sigma(u)du + \int_x^{m\mu}\sigma(u)du\Big)\Big]\Big]\\ &- \frac{2(1-\kappa)\sigma(x)}{B(\kappa)},\end{aligned} \tag{48}$$

where $x \in [\lambda, \mu]$. Further solving (48) leads to the proof of Lemma 2. □

Remark 10. *Put $m = 1$, in Lemma 2, we obtain [Lemma10] [43].*

Theorem 5. *Let $\sigma : [\lambda, m\mu] \to \mathbb{R}$ be a differentiable function on $(\lambda, m\mu)$. If $\sigma'(\cdot)$ is (s,m)-convex function and integrable on $[\lambda, m\mu]$, then (49) holds:*

$$\begin{aligned}&\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{B(\kappa)}{\kappa(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big] + \frac{2(1-\kappa)}{\kappa(m\mu-\lambda)}\sigma(x)\\ &\le \frac{m\mu-\lambda}{2}[m\sigma'(\mu) - \sigma'(\lambda)]\frac{s}{(s+1)(s+2)}.\end{aligned} \tag{49}$$

Proof. Lemma 1 and (s,m)-convexity of $\sigma'(\cdot)$ give

$$\begin{aligned}&\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{B(\sigma)}{\sigma(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big] + \frac{2(1-\kappa)}{\kappa(m\mu-\lambda)}\sigma(x)\\ &\le \frac{m\mu-\lambda}{2}\int_0^1(1-2y)[y^s\sigma'(\lambda) + m(1-y)^s\sigma'(\mu)]dy\\ &= \frac{m\mu-\lambda}{2}[m\sigma'(\mu) - \sigma'(\lambda)]\frac{s}{(s+1)(s+2)}.\end{aligned}$$

□

Corollary 6. *If we put $s = 1$ in Theorem 5, we obtain*

$$\begin{aligned}&\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{B(\sigma)}{\sigma(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big] + \frac{2(1-\kappa)}{\kappa(m\mu-\lambda)}\sigma(x)\\ &\le \frac{m\mu-\lambda}{12}[m\sigma'(\mu) - \sigma'(\lambda)].\end{aligned} \tag{50}$$

Remark 11. *For $m = 1$ in (50), we have [Corollary 2.5] [26].*

Theorem 6. *Let $\sigma : [\lambda, m\mu] \to \mathbb{R}$ be a twice differentiable function on $(\lambda, m\mu)$. If $\sigma''(\cdot)$ is an (s,m)-convex function and an integrable on $[\lambda, m\mu]$, then*

$$\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{B(\sigma)}{\sigma(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big] + \frac{2(1-\kappa)}{\kappa(m\mu-\lambda)}\sigma(x)$$
$$\leq \frac{(m\mu-\lambda)^2}{2}\frac{[\sigma''(\lambda)+m\sigma''(\mu)]}{(s+3)(s+2)}.$$

Proof. Use Lemma 2 and (s,m)-convexity of $\sigma''(\cdot)$

$$\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{B(\sigma)}{\sigma(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big] + \frac{2(1-\kappa)}{\kappa(m\mu-\lambda)}\sigma(x)$$
$$\leq \frac{(m\mu-\lambda)^2}{2}\int_0^1 (y-y^2)[y^s\sigma''(\lambda)+m(1-y)^s\sigma''(\mu)]dy$$
$$= \frac{(m\mu-\lambda)^2}{2}\frac{[\sigma''(\lambda)+m\sigma''(\mu)]}{(s+3)(s+2)}.$$

□

Corollary 7. *If we substitute $s = 1$ in Theorem 6, the following inequality is obtained:*

$$\frac{\sigma(\lambda)+\sigma(m\mu)}{2} - \frac{B(\sigma)}{\sigma(m\mu-\lambda)}\Big[\Big(({}^{CF}_{\lambda}I^{\kappa}\sigma)(x) + ({}^{CF}I^{\kappa}_{m\mu}\sigma)(x)\Big)\Big] + \frac{2(1-\kappa)}{\kappa(m\mu-\lambda)}\sigma(x)$$
$$\leq \frac{(m\mu-\lambda)^2}{24}[\sigma''(\lambda)+m\sigma''(\mu)].$$

Remark 12. *For $m = 1$, we have in [Corollary 2.6] [26].*

Proposition 1. *Let λ, μ be a positive real number with $\lambda < m\mu$. The inequalities*

$$\begin{aligned} 2^{\frac{s}{s+m-1}}A(\lambda, m\mu) &\leq (1+m^{2-s-m})^{\frac{1}{s+m-1}}L_{s+m-1}(\lambda, m\mu) \\ &\leq \Big(\frac{(1+m^{4-2s-2m})\lambda^{s+m-1}+(2m)\mu^{s+m-1}}{s+1}\Big)^{\frac{1}{s+m-1}}, \end{aligned} \tag{51}$$

hold for $s \in (0,1)$, $m \in (0,1]$ and $0 < s+m < 1$.

Proof. Applying Theorem 3 to the (s,m)-convex function $\sigma : [0,\infty) \to [0,\infty)$, $\sigma(x) = x^{s+m-1}$, $0 < s+m < 1$, $\kappa = 1$ and $B(\kappa) = 1$

$$\begin{aligned} 2^s\Big(\frac{\lambda+m\mu}{2}\Big)^{s+m-1} &\leq \frac{m^{2-s-m}+1}{m\mu-\lambda}\Big[\frac{(m\mu)^{s+m}-\lambda^{s+m}}{s+m}\Big] \\ &\leq \Big(\frac{(1+m^{4-2s-2m})\lambda^{s+m-1}+(2m)\mu^{s+m-1}}{s+1}\Big). \end{aligned} \tag{52}$$

Implies that

$$\begin{aligned} 2^{\frac{s}{s+m-1}}\Big(\frac{\lambda+m\mu}{2}\Big) &\leq \Big(\frac{m^{2-s-m}+1}{m\mu-\lambda}\Big[\frac{(m\mu)^{s+m}-\lambda^{s+m}}{s+m}\Big]\Big)^{\frac{1}{s+m-1}} \\ &\leq \Big(\frac{(1+m^{4-2s-2m})\lambda^{s+m-1}+(2m)\mu^{s+m-1}}{s+1}\Big)^{\frac{1}{s+m-1}}. \end{aligned} \tag{53}$$

Use (14) and (15) in (53) to obtain (51). □

Remark 13. *If we put $m = 1$ in (51), we have [Proposition 3.1] [26].*

Proposition 2. *Let $h \in (1,2)$, $m \in (0,1]$. Then,*

$$2^{h-m} - 2^{1-m} \leq [\psi(h) + \gamma][1 + m^{3-h}] \leq \frac{1+m^{6-2h}+2m^{3-h}(h-1)}{h-m},$$

where $\psi(h)$ is digamma function, i.e.,

$$\psi(h) = \frac{\Gamma'(h)}{\Gamma(h)}$$

for $h > 0$ and γ is Euler–Mascheroni constant.

Proof. Substitute $p = \frac{\lambda}{m\mu}$, in (52), where $s \in (0,1)$, $m \in (0,1]$ and $0 < s+m < 1$.

$$\begin{aligned} 2^s \left(\frac{1+p}{2}\right)^{s+m-1} &\leq \left(\frac{1-p^{s+m}}{1-p}\right)\left(\frac{1+m^{2-s-m}}{s+m}\right) \\ &\leq \left(p^{s+m-1}(1+m^{4-2s-2m}) + \frac{2}{m^{s+m-2}}\right)\frac{1}{s+1}. \end{aligned} \tag{54}$$

Integrate (54) with respect to p, over [0, 1], to obtain

$$\begin{aligned} 2^{s+1} - 2^{1-m} &\leq (1+m^{2-s-m})\int_0^1 \frac{1-p^{s+m}}{1-p}dp \\ &\leq \frac{1+m^{4-2s-2m}+2m^{2-s-m}(s+m)}{s+1}. \end{aligned} \tag{55}$$

In (55), use Equation (9) of digamma function:

$$\begin{aligned} 2^{s+1} - 2^{1-m} &\leq \psi(s+m+1) + \gamma \\ &\leq \frac{1+m^{4-2s-2m}+2m^{2-s-m}(s+m)}{s+1}. \end{aligned} \tag{56}$$

The substitution $h = s+m+1$ in (56) for $h \in (1,2)$ leads towards the proof. □

Remark 14. *If we put $m = 1$ in Proposition 2, we obtain double inequalities in the statement of [Proposition 3.2] [26] with $h \in (1,2)$.*

4. Conclusions

This paper presents several inequalities accomplished for the functions whose nth derivatives are (s,m)-convex functions via Caputo fractional derivatives. The paper also includes the outcomes obtained by Caputo–Fabrizio integrals, which depict a generalization of Hermite–Hadamard-type inequalities for the (s,m)-convex function and the product of (s,m)-convex functions. Lemmas 1 and 2 are established to obtain new inequalities involving Caputo–Fabrizio integrals, which are applied to obtain the special means and an inequality involving the digamma function. These lemmas are also convenient to obtain bounds and error estimates. Our results provide the extension of the inequalities presented in [26,41–43]. Other types of inequalities can be obtained with the analogous classes of other convex functions.

Author Contributions: Conceptualization, A.N. and K.A.K.; Methodology, A.N.; Validation, N.A.S.; Formal analysis, N.A.S. and J.D.C.; Investigation, K.A.K.; Writing—original draft, M.T.; Writing—review & editing, K.A.K.; Supervision, A.N.; Funding acquisition, J.D.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by Korea Institute of Energy Technology Evaluation and Planning (KETEP) grant funded by the Korea government (MOTIE) (No. 20192010107020, Development of hybrid adsorption chiller using unutilized heat source of low temperature).

Data Availability Statement: Not applicable.

Acknowledgments: The authors are very grateful for the support of Korea Institute of Energy Technology Evaluation and Planning (KETEP) grant funded by the Korea government (MOTIE) (No. 20192010107020, Development of hybrid adsorption chiller using unutilized heat source of low temperature).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cloud, M.J.; Drachman, B.C.; Lebedev, L.P. *Inequalities with Applications to Engineering*; Springer: Cham, Switzerland; Heidelberg, Germany; New York, NY, USA; Dordrecht, The Netherlands; London, UK, 2014
2. Ullah, H.; Khan, M.A.; Saeed, T.; Sayed, Z.M.M. Some improvements of Jensen's inequality via 4-convexity and applications. *J. Funct. Spaces* **2022**, *2022*, 2157375. [CrossRef]
3. Borwein, J.; Lewi, A. *Convex Analysis and Nonlinear Optimization, Theory and Examples*; Springer: New York, NY, USA, 2000.
4. Khan, M.A.; Ullah, H.; Saeed, T.; Alsulami, H.H.; Sayed, Z.M.M.M.; Alshehri, A.M. Estimations of the Slater Gap via Convexity and Its Applications in Information Theory. *Math. Probl. Eng.* **2022**, *2022*, 1750331.
5. Khan, M.A.; Faisal, S.; Khan, S. Estimation of Jensen's gap through an integral identity with applications to divergence. *Innov. J. Math.* **2022**, *1*, 99–114.
6. Hudzik, H.; Maligranda, L. Some remarks on s-convex functions. *Aequationes Math.* **1994**, *48*, 100–111. [CrossRef]
7. Eftekhari, N. Some remarks on (s,m)-convexity in the second sense. *J. Math. Inequal.* **2014**, *8*, 489–495. [CrossRef]
8. Beckenbach, E.F.; Bellman, R. *Inequalities*; Springer Science & Business Media: Berlin, Germany, 2012; Volume 30.
9. Bainov, D.D.; Simeonov, P.S. *Integral Inequalities and Applications;* Springer Science & Business Media: Berlin, Germany, 2013; Volume 57.
10. Dragomir, S.S.; Pearce, C. Selected topics on Hermite–Hadamard inequalities and applications. *Sci. Direct Work. Pap.* **2003**, *4*, 1574–0358.
11. Pachpatte, B.G. *Mathemematical Inequalities*; Elsevier: Amsterdam, The Netherlands, 2005.
12. El Farissi, A. Simple proof and refinement of Hermite–Hadamard inequality. *J. Math. Inequalities* **2010**, *4*, 365–369. [CrossRef]
13. Barsam, H.; Ramezani, M.S.; Sayyari, Y. On the new Hermite–Hadamard type inequalities for s-convex functions. *Afr. Mat.* **2021**, *13*, 1355–1367. [CrossRef]
14. Khan, K.A.; Ayaz, S.; İşcan, İ.; Shah, N.A.; Weera, W. Applications of Hölder-Iscan inequality for n-times differentiable (s,m)-convex functions. *AIMS Math.* **2022**, *8*, 1620–1635. [CrossRef]
15. Jiang, W.D.; Niu, D.W.; Hua, Y.; Qi, F. Generalizations of Hermite–Hadamard inequality to n-times differentiable function which s-convex in second sense. *Analysis* **2012**, *32*, 209–220. [CrossRef]
16. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
17. Cortez, M.V. Féjer Type inequalities for (s,m)-convex functions in second sense. *Appl. Math. Inf. Sci.* **2016**, *10*, 1689–1696. [CrossRef]
18. Jiang, S.; Zhang, J.; Zhang, Q.; Zhang, Z. Fast evaluation of Caputo fractional derivatives and its applications to fractional diffusion equations. *Commun. Copmutational Phys.* **2017**, *21*, 650–678. [CrossRef]
19. Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Kodamasingh, B.; Nonlaopon, K.; Abualnaja, K.M. Interval valued Hadamard-Fejer and Pachpatte Type inequalities pertaining to a new fractional integral operator with exponential kernel. *AIMS Math.* **2022**, *7*, 15041–15063. [CrossRef]
20. Sahoo, S.K.; Agarwal, R.P.; Mohammed, P.O.; Kodamasingh, B.; Nonlaopon, K.; Abualnaja, K.M. Hadamard-Mercer, Dragomir-Agarwal-Mercer, and Pachpatte-Mercer Type Fractional Inclusions for Convex Functions with an Exponential Kernel and Their Applications. *Symmetry* **2022**, *14*, 836. [CrossRef]
21. Caputo, M. Linear model of dissipation whose Q is almost frequency independent. *Geophys. J. Int.* **1967**, *13*, 529–539. [CrossRef]
22. Kilbas, A.; Srivastava, H.; Trujillo, J. *Theory and Application of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
23. Nchama, G.A.M.; Mecias, A.L.; Richard, M.R. The Caputo–Fabrizio fractional integral to generate some new inequalities. *Inf. Sci. Lett.* **2019**, *8*, 73–80.
24. Butt, S.I.; Nadeem, M.; Farid, G. On Caputo fractional derivatives via exponential s-convex functions. *Turk. J. Sci.* **2020**, *5*, 140–146.
25. Butt, S.I.; Nadeem, M.; Farid, G. On Caputo fractional derivatives via exponential (s,m)-convex functions. *Eng. Appl. Sci. Lett.* **2020**, *3*, 32–39.
26. Kemali, S.; Tinaztepe, G.; Işik, I.Y.; Evcan, S.S. New integral inequalities for s-convex functions in the second sense via Caputo fractional derivative and Caputo–Fabrizio integral operator. *Rocky Mt. J. Math.* **2022**, *6*, 6377–6389.
27. Abbasi, A.M.K.; Anwar, M. Hermite–Hadamard inequality involving Caputo–Fabrizio fractional integrals and related inequalities via s-convex functions in the second sense. *AIMS Math.* **2022**, *7*, 18565–18575. [CrossRef]
28. Li, Q.; Saleem, M.S.; Yan, P.; Zahoor, M.S.; Imran, M. On Strongly Convex Functions via Caputo–Fabrizio-Type Fractional Integral and Some Applications. *J. Math.* **2021**, *2021*, 6625597. [CrossRef]

29. Tuan, N.H.; Mohammadi, H.; Rezapour, S. A mathematical model for COVID-19 transmission by using the caputo fractional derivative. *Chaos Solitons Fractals* **2020**, *140*, 110107. [CrossRef] [PubMed]
30. Wang, K.J.; Shi, F. A new perspective on the exact solutions of the local fractional modified Benjamin–Bona–Mahony equation on cantor sets. *Fractal Fract.* **2023**, *7*, 72. [CrossRef]
31. He, J.H. Fractal calculus and its geometrical explanation. *Results Phys.* **2018**, *10*, 272–276. [CrossRef]
32. Turkyilmazoglu, M.; Altanji, M. Fractional models of falling object with linear and quadratic frictional forces considering Caputo derivative. *Chaos Solitons Fractals* **2023**, *166*, 112980. [CrossRef]
33. Wanassi, O.K.; Torres, D.F. An integral boundary fractional model to the world population growth. *Chaos Solitons Fractals* **2023**, *168*, 113151. [CrossRef]
34. Sajjad, A.; Farman, M.; Hasan, A.; Nisar, K.S. Transmission dynamics of fractional order yellow virus in red chili plants with the Caputo–Fabrizio operator. *Math. Comput. Simul.* **2023**, *207*, 347–368. [CrossRef]
35. Areshi, M.; Seadawy, A.R.; Ali, A.; Alharbi, A.F.; Aljohani, A.F. Analytical Solutions of the Predator-Prey Model with Fractional Derivative Order via Applications of Three Modified Mathematical Methods. *Fractal Fract.* **2023**, *7*, 128. [CrossRef]
36. Mahatekar, Y.; Scindia, P.S.; Kumar, P. A new numerical method to solve fractional differential equations in terms of Caputo–Fabrizio derivatives. *Phys. Scr.* **2023**, *98*, 024001. [CrossRef]
37. Riddhi, D. Beta Function and Its Applications; The University of Tennesse: Knoxville, TN, USA, 2008.
38. Dragomir, S.S.; Agarwal, R.P.; Barnett, S.N. Inequalities for beta and gamma functions via some classical and new integral inequalities. *RGMIA Res. Rep. Collect.* **1999**, *2*, 103–165. [CrossRef]
39. Chu, Y.M.; Long, B.Y. Best possible inequalities between generalized logarithmic mean and classical means. *Abstr. Appl. Anal.* **2010**, *2010*, 303286. [CrossRef]
40. Xi, B.Y.; Qi, F. Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means. *J. Funct. Spaces Appl.* **2012**, *2012*, 980438. [CrossRef]
41. Farid, G. On Caputo fractional derivatives via convexity. *Kragujev. J. Math.* **2020**, *44*, 393–399. [CrossRef]
42. Gürbüz, M.; Akdemir, A.O.; Rashid, S.; Set, E. Hermite–Hadamard inequality for fractional integrals of Caputo–Fabrizio type and related inequalities. *J. Inequal. Appl.* **2020**, *1*, 172 . [CrossRef]
43. Nwaeze, E.R.; Kermausuor, S. Caputo–Fabrizio fractional Hermite–Hadamard type and associated results for strongly convex functions. *J. Anal.* **2021**, *29*, 1351–1365. [CrossRef]