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Stability of $p(\cdot)$-Integrable Solutions for Fractional Boundary Value Problem via Piecewise Constant Functions

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Abstract: The goal of this work is to study a multi-term boundary value problem (BVP) for fractional differential equations in the variable exponent Lebesgue space ($L^{p(\cdot)}$). Both the existence, uniqueness, and the stability in the sense of Ulam–Hyers are established. Our results are obtained using two fixed-point theorems, then illustrating the results with a comprehensive example.

Keywords: boundary value problem; Lebesgue spaces; variable exponent Lebesgue spaces; fixed point theorem; Ulam–Hyers stability

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1. Introduction

The variable exponent Lebesgue spaces $L^{p(\cdot)}$ have been systemically studied with the foundational publication by KovLa.cik and Rakosn L.k [1] in 1991. Following this work, numerous researchers developed an interest in these spaces and their properties (see [2,3]).

Although this concept seems to be simple, it has remarkable outcomes that accurately describe some physical, energy, electric and velocity phenomena. In recent years, we see many works dealing with this topic (see, for example, [4–9]).

In recent years, many authors have studied the existence and uniqueness of solutions to problems for fractional differential equations in the $C(J,\mathbb{R})$ space of continuous functions; on the other hand, the number of works discussing the existence of solutions in the $L^p(J,\mathbb{R})$ space of integrable functions is quite limited. Burton et al. [10] studied the existence of solutions in the $L^1(J,\mathbb{R})$ space to Caputo derivative problems.

Arshad established L^p-solutions of fractional integral equations involving the Riemann–Liouville integral operator in Banach spaces.

Due to the difficulty of studying the existence of solutions to fractional differential equations in $L^{p(\cdot)}$, there are very few studies in the literature dealing with this topic. In 2018, Dong et al. [11] established that there exists a unique solution to a Cauchy type issue for fractional differential equations in $L^{p(\cdot)}$.

Recently, Refice et al. [12] investigated the following boundary value problem of fractional differential equations in $L^{p(\cdot)}$:

$$\begin{cases} D^{\alpha}_{0^+}y(t) = \Phi(t, y(t)), \ t \in [0,\zeta], \ 0 < \alpha < 1 \\ \gamma y(0) + \mu y(T) = c, \end{cases} \tag{1}$$

where ζ, c, γ, μ, with $\mu \neq 0$ are real constants, and $D^{\alpha}_{0^+}$ is the left Riemann–Liouville fractional derivative of order α for function $y(t)$.

In [13], Refice et al. established the following terminal value problem of fractional differential equations in $L^{p(\cdot)}$.

$$\begin{cases} D^{\alpha}_{0^+}y(t) = \Phi(t, y(t), I^{\alpha}_{0^+}y(t)), \ 0 \le t \le \zeta \\ y(T) = c, \ c \in \mathbb{R} \end{cases} \tag{2}$$

where $D^{\alpha}_{0^+}$, $I^{\alpha}_{0^+}$ are the left Riemann–Liouville fractional derivative and integral of order α for the function y, respectively.

Several authors have studied the Ulam stability properties and their existence for fractional differential equations (see [14–20] and the references therein).

Benchohra et al. [21] investigated the existing solutions to the following BVP in $C([0, \zeta], \mathbb{R})$:

$$\begin{cases} {}^cD^{\alpha}_{a}y(t) = \Phi(t, y(t)), \ 0 \le t \le \zeta \\ \gamma y(0) + \mu y(\zeta) = \varpi, \end{cases}$$

where $0 < \alpha < 1$, and ${}^cD^{\alpha}_{a}$ is the Caputo fractional derivative of order α for function $y(t)$.

Abdulahad et al. [22] proved the existing solutions to the BVP in $L^p([\zeta_1, \zeta_2], \mathbb{R})$ as follows:

$$\begin{cases} {}^cD^{\alpha}_{a}y(t) = \Phi(t, y(t)), \ \zeta_1 \le t \le \zeta_2, \ 0 < \alpha < 1, \\ \gamma y(\zeta_1) + \mu y(\zeta_2) = \varpi, \end{cases}$$

where ${}^cD^{\alpha}_{a}$ is the Caputo fractional derivative of order α for function $y(t)$.

Inspired by [12,13,21,22], we consider the existing solutions to the following BVP in $L^{p(\cdot)}(J, \mathbb{R})$:

$$\begin{cases} D^{\alpha}_{0^+}y(t) = \Phi(t, y(t), I^{\alpha}_{0^+}y(t)), \ t \in J := [0, \zeta] \\ \gamma y(0) + \mu y(\zeta) = c, \end{cases} \tag{3}$$

where $0 < \alpha < 1$, $\Phi(., y(\cdot), I^{\alpha}_{0^+}y(\cdot)) \in L^{p(\cdot)}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $y \in L^{p(\cdot)}(J, \mathbb{R})$, ζ, c, γ, μ are real constants with $\mu \neq 0$ and $D^{\alpha}_{0^+}$, $I^{\alpha}_{0^+}$ are the the left Riemann–Liouville fractional derivative and integral of order α for function $y(t)$, respectively.

This paper can be considered as a generalization and extension of several articles. For example, if we choose the function Φ as a two-variable function t, $y(t)$ only and independently of the integration $I^{\alpha}_{0^+}y(t)$, our problem reduces to the problem studied in [12], and, if we choose the constants $\gamma = 0$ and $\mu = 1$, our problem reduces to the terminal value problem studied in [13].

2. Preliminaries

In this section, we list some definitions and lemmas that are used in the following sections.

Definition 1 ([11]). *Letting $0 < \alpha < 1$, the left Riemann–Liouville fractional derivative of order α for function $y(t)$ in $L^{p(\cdot)}$ is defined by:*

$$(D^{\alpha}_{0^+}y)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s) ds, \tag{4}$$

and the left Riemann–Liouville fractional integral for function $y(t)$ of order α in $L^{p(\cdot)}$ is defined by:

$$I^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds. \tag{5}$$

Definition 2 ([23,24]). *Letting* $1 \leq p \leq \infty$, *we denote by* $L^p([\zeta_1, \zeta_2], \mathbb{R})$ *the Lebesgue space* $L^p([\zeta_1, \zeta_2], \mathbb{R})$={ $\Phi : [\zeta_1, \zeta_2] \to \mathbb{R}$ *a measurable functions such that* $\|\Phi\|_p < \infty\}$, *with the norm*

$$\|\Phi\|_p = \left(\int_{\zeta_1}^{\zeta_2} |\Phi(t)|^p dt\right)^{\frac{1}{p}} \quad if \quad 1 \leq p < \infty$$

and

$$\|\Phi\|_\infty = esssup_{\zeta_1 \leq t \leq \zeta_2} |\Phi(t)| \quad if \quad p = \infty.$$

Next, we will present some important properties of $D^\alpha_{\zeta_1^+}$, $I^\alpha_{\zeta_1^+}$.

Lemma 1 ([23,24]). *Let* $1 \leq p < \infty$, $\Phi_1, \Phi_2 \in L^p([\zeta_1, \zeta_2], \mathbb{R})$ *and* $0 < \alpha, \beta < 1$
Then,
(1) $I^\alpha_{\zeta_1^+} I^\beta_{\zeta_1^+} \Phi_1(t) = I^{\alpha+\beta}_{\zeta_1^+} \Phi_1(t)$
(2) $I^\alpha_{\zeta_1^+} [\Phi_1(t) + \Phi_2(t)] = I^\alpha_{\zeta_1^+} \Phi_1(t) + I^\alpha_{\zeta_1^+} \Phi_2(t)$
(3) $D^\alpha_{\zeta_1^+} I^\alpha_{\zeta_1^+} \Phi_1(t) = \Phi_1(t)$
(4) $\|I^\alpha_{\zeta_1^+} \Phi_1\|_p \leq \frac{(\zeta_2 - \zeta_1)^\alpha}{\Gamma(\alpha+1)} \|\Phi_1\|_p$.

Lemma 2 ([25]). *Let* $1 \leq p < \infty$, $0 < \alpha < 1$, *then* $I^\alpha_{a^+} \Phi \in L^p([\zeta_1, \zeta_2], \mathbb{R})$ *for any* $\Phi \in L^p([\zeta_1, \zeta_2], \mathbb{R})$.

Lemma 3 ([23]). *Let* $0 < \alpha < 1$, *then the differential equation*

$$D^\alpha_{\zeta_1^+} h = 0$$

has a unique solution

$$h(t) = \omega_1 (t - \zeta_1)^{\alpha - 1}$$

$\omega_1 \in \mathbb{R}$.

Lemma 4 ([23]). *Let* $0 < \alpha < 1$, $\zeta_1 > 0$, $h \in L^1(J, \mathbb{R})$, $D^\alpha_{\zeta_1^+} h \in L^1(J, \mathbb{R})$, *then*

$$I^\alpha_{\zeta_1^+} D^\alpha_{\zeta_1^+} h(t) = h(t) + \omega_1 (t - \zeta_1)^{\alpha - 1}$$

$\omega_1 \in \mathbb{R}$.

Lemma 5 ([26] (Hölder's inequality)). *Let* p *and* q *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$ *such that,* $1 < p < \infty$, $1 < q < \infty$. *If* $\Phi_1 \in L^p(J, \mathbb{R})$ *and* $\Phi_2 \in L^q(J, \mathbb{R})$, *then* $\Phi_1 \Phi_2$ *belongs to* $L^1(J, \mathbb{R})$ *and satisfies*

$$\int_J |\Phi_1 \Phi_2| dx \leq \left[\int_J |\Phi_1|^p dx\right]^{\frac{1}{p}} \left[\int_J |\Phi_2|^q dx\right]^{\frac{1}{q}}.$$

Definition 3 ([27]). *For $\Omega \subseteq \mathbb{R}^n$ to be an open set in $\mathbb{R}^n$, we denote by $L^{p(\cdot)}(\Omega,\mathbb{R})$ the set $L^{p(\cdot)}(\Omega,\mathbb{R}) = \{\Phi : \Omega \to \mathbb{R}$, which is a measurable function such that $I_{p(\cdot)}(\Phi) = \int_\Omega |\Phi(t)|^{p(t)} dt < \infty$, where $p(t) : \Omega \to [1,\infty)$ is a measurable function $\}$.*

Note that the set $L^{p(\cdot)}(\Omega,\mathbb{R})$ is a Banach space with a norm defined as

$$\|\Phi\|_{p(\cdot)} = inf\{\eta > 0 : I_{p(\cdot)}(\Phi/\eta) \leq 1\}.$$

We use the following notation:

$$p_- = inf_{t\in\Omega} p(t), \;\; p_+ = sup_{t\in\Omega} p(t),$$

and $q(\cdot)$ is denoted by the conjugate exponent of $p(\cdot)$ with

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

Consider the sets of functions

$$\mathcal{P}(\Omega) = \{p : \Omega \to [1,\infty), p \; is \; a \; bounded \; measurable \; function\},$$

and

$$\mathcal{P}^{log}(\Omega) = \{p \in \mathcal{P}(\Omega) : |p(t) - p(s)| \leq \frac{A_p}{-log|t-s|}, |t-s| \leq \frac{1}{2}, \;\; t,s \in \Omega,$$
$$such \; that A_p > 0 \; is \; independent \; of \; t \; and \; s\}$$

Lemma 6 ([28] (Hölder's inequality in $L^{p(\cdot)}(\Omega,\mathbb{R})$)). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\Phi_1 \in L^{p(\cdot)}(J,\mathbb{R})$, $\Phi_2 \in L^{q(\cdot)}(J,\mathbb{R})$ with $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. Then, we have*

$$\int_\Omega |\Phi_1(t)\Phi_2(t)| dt \leq r\|\Phi_1\|_{p(\cdot)} \|\Phi_2\|_{q(\cdot)},$$

where $r = sup_{t\in\Omega} \frac{1}{p(t)} + sup_{t\in\Omega} \frac{1}{q(t)}$.

Theorem 1 ([11]). *$I^\alpha_{0^+}$ is bounded in $L^{p(\cdot)}([0,\zeta],\mathbb{R})$ for any $p(\cdot) \in \mathcal{P}[0,\zeta]$ and $0 < \frac{1}{p_-} < \alpha < 1$.*

Definition 4 ([29]). *Let $J \subset \mathbb{R}$, $n \in \mathbb{N}$ and the finite set $\mathcal{P}$ of generalized intervals $E_i, i = 1,2,...,n$. We say that the $\mathcal{P}$ is a partition of J if E_i are pairwise disjointed and J is the union of E_i exactly. Additionally, the function $p : J \to \mathbb{R}$ is piecewise constant with respect to $\mathcal{P}$ if p adopts a value on E_i, for any E_i of $\mathcal{P}$.*

Theorem 2 ((Riesz compactaness Criteria) [30]). *$M \subset L^p([t_0,t_1],\mathbb{R})$ is relatively compact if and only if the following hold:*
(1) M is bounded in L^p,
(2) $\int_{t_0}^{t_1} |x(t+h) - x(t)|^p dt \to 0$ as $h \to 0$.

The following fixed point theorem will be used in the proof of our main results.

Theorem 3 ([23])**.** *Let* Υ *be a convex subset of a Banach space* Θ *and* $\Lambda : \Upsilon \longrightarrow \Upsilon$ *is a continuous, compact map. Thus,* Λ *has a fixed point in* Υ*.*

Theorem 4 ([23])**.** *Let* (Υ, d) *be a complete metric space,* $\Lambda : \Upsilon \longrightarrow \Upsilon$ *a contraction. Then,* Λ *has a unique fixed point in* Υ*.*

Finally, we will provide a definition of the stability of the BVP (3) as follows:

Definition 5 ([14])**.** *The BVP* (3) *is Ulam–Hyers stable **(UH)** if there exists* $c_\Phi > 0$*, so that, for every* $\epsilon > 0$*, for any solution* $z \in L^p(J, \mathbb{R})$ *of the following inequality*

$$|D^{\alpha}_{0^+} z(t) - \Phi(t, z(t))| \leq \epsilon, \ t \in J \tag{6}$$

there exists a solution $y \in L^p(J, \mathbb{R})$ *of BVP* (3) *with*

$$|z(t) - y(t)| \leq c_\Phi \epsilon, \ t \in J.$$

3. Existence of Solutions

Throughout this paper, we assume that:

(H1) Let $n \in \mathbb{N}$, $\mathcal{P} = \{J_\kappa := (\zeta_{\kappa-1}, \ \zeta_\kappa], \kappa = 1, 2, ..., n\}$ be a partition of the interval J, such that, $\zeta_0 = 0, \zeta_n = \zeta$ and let $u(t) : J \to [1, \infty)$ be a piecewise constant function with respect to $\mathcal{P}$, i.e., $p(t) = \sum_{\kappa=1}^{n} p_\kappa I_\kappa(t)$, where

$$I_\kappa(t) = \begin{cases} 1, & for \ t \in J_\kappa \\ 0, & for \ elsewhere, \end{cases} \quad \kappa = 1, 2, ..., n$$

and $1 \leq p_\kappa < \infty$ are constants.

For $\kappa = 1, 2, ..., n$, $E_\kappa = L^{p_\kappa}(J_\kappa \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ represents the Banach space of measurable functions from J_κ into $\mathbb{R}$ with the norm:

$$\|y\|_{E_\kappa} = \left(\int_{J_\kappa} |y|^{p_\kappa} dx \right)^{\frac{1}{p_\kappa}} < \infty.$$

(H2) Let $\Phi \in C(J_\kappa \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist constants, $M_1, \ M_2 > 0$, such that

$$|\Phi(t, y_1, z_1) - \Phi(t, y_2, z_2)| \leq M_1 |y_1 - y_2| + M_2 |z_1 - z_2|,$$

for every $y_i, \ z_i \in L^{p_\kappa}(J_\kappa \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ i=1, 2 and $t \in J_\kappa$.

(H3) Let $\Phi : J_\kappa \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that Φ is strongly measurable in t and continuous in y and z, and there exist constants, $b, \ d > 0$, such that

$$\|\Phi(t, y(), z())\|_{E_\kappa} \leq a(t) + b\|y\|_{E_\kappa} + d\|z\|_{E_\kappa},$$

for any $y, \ z \in L^{p_\kappa}(J_\kappa, \mathbb{R})$ and $t \in J_\kappa$, and $a \in L^{p_\kappa}(J_\kappa, \mathbb{R})$.

Thus, for $t \in J_\kappa$, $1 \leq \kappa \leq n$, the left Riemann–Liouville fractional derivative of order α for function $y(t)$ defined by (4), is provided by

$$(D^\alpha_{0^+} y)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s) ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \Big(\sum_{\iota=1}^{\kappa-1} \frac{d}{dt} \int_{\zeta_{\iota-1}}^{\zeta_\iota} (t-s)^{-\alpha} y(s) ds + \frac{d}{dt} \int_{\zeta_{\kappa-1}}^{t} (t-s)^{-\alpha} y(s) ds \Big). \tag{7}$$

Thus, we can write the BVP (3) in the form:

$$\frac{1}{\Gamma(1-\alpha)} \Big(\sum_{\iota=1}^{\kappa-1} \frac{d}{dt} \int_{\zeta_{\iota-1}}^{\zeta_\iota} (t-s)^{-\alpha} y(s) ds + \frac{d}{dt} \int_{\zeta_{\kappa-1}}^{t} (t-s)^{-\alpha} y(s) ds \Big) = \Phi(t, y(t), I^\alpha_{0^+} y(t)). \tag{8}$$

For $t \in [0, \zeta_{\kappa-1}]$, we use $y \equiv 0$, then (8) is written as

$$(D^\alpha_{\zeta_{\kappa-1}} y)(t) = \Phi(t, y(t), I^\alpha_{\zeta_{\kappa-1}} y(t)), \quad t \in J_\kappa.$$

For any $1 \leq \kappa \leq n$, we deal with the following BVP:

$$\begin{cases} D^\alpha_{\zeta_{\kappa-1}} y(t) = \Phi(t, y(t), I^\alpha_{\zeta_{\kappa-1}} y(t)), \quad t \in J_\kappa \\ \gamma y(\zeta_{\kappa-1}) + \mu y(\zeta_\kappa) = c. \end{cases} \tag{9}$$

Lemma 7. *Let $1 \leq \kappa \leq n$, $0 < \alpha < 1$, $\Phi \in L^{p_\kappa}(J_\kappa \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*
Then, the solution y of (9) is written as the following integral equation:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^t (t-s)^{\alpha-1} \Phi(s, y(s), I^\alpha_{\zeta_{\kappa-1}} y(s)) ds - \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha} (t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)}$$

$$\int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{\alpha-1} \Phi(s, y(s), I^\alpha_{\zeta_{\kappa-1}} y(s)) ds + \frac{c(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha} (t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}. \tag{10}$$

Proof. Let y be a solution of the BVP (9), we find (see Lemma 4)

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^t (t-s)^{\alpha-1} \Phi(s, y(s), I^\alpha_{\zeta_{\kappa-1}} y(s)) ds - \omega_1 (t - \zeta_{\kappa-1})^{\alpha-1}.$$

By the value boundary of (9), we obtain:

$$\omega_1 = \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha}}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{\alpha-1} \Phi(s, y(s), I^\alpha_{\zeta_{\kappa-1}} y(s)) ds - \frac{c(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha}}{\mu}$$

Hence, we obtain Equation (10).
Inversely, it is clear that y satisfies the BVP (9). □

Our first existence result is based on Theorem 2.

Theorem 5. *Assume that (H2), (H3), and inequality*

$$L(\alpha, p_\kappa, \zeta_{\kappa-1}, \zeta_\kappa) < 1, \tag{11}$$

holds, where:

$$L(\alpha, p_\kappa, \zeta_{\kappa-1}, \zeta_\kappa)$$

$$= 2^{p_\kappa - 1} \Bigg[\frac{2^{2p_\kappa - 2}}{\Gamma^{p_\kappa}(\alpha)} \Bigg(\frac{(\zeta_{\kappa-1} - \zeta_\kappa)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{\big(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}\big)\Big(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\Big)} + \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{\big(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}\big)\Big(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\Big)} \Bigg)$$

$$+ \Big(\frac{2^{3p_\kappa - 3} (\zeta_\kappa - \zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)} \Big) \Bigg(\frac{(\zeta_\kappa - \zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}} \Bigg) \Big(\frac{(\zeta_\kappa - \zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1} \Big) \Big(b + \frac{d(\zeta_\kappa - \zeta_{\kappa-1})^\alpha}{\Gamma(\alpha+1)} \Big) \Bigg].$$

Then, the BVP (9) has at least one solution on E_κ.

Proof. For any function $y \in E_\kappa$ we define the operator

$$\begin{aligned} Sy(t) &= \frac{1}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^{t} (t-s)^{\alpha-1} \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s)) ds \\ &- \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha} (t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{\alpha-1} \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s)) ds \\ &+ \frac{c(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha} (t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}. \end{aligned}$$

By the properties of fractional integrals, the operator $S : E_\kappa \to E_\kappa$ is well defined.

Let

$$R_\kappa \geq \frac{A}{B}$$

where:

$$\begin{aligned} A &= 2^{p_\kappa - 1} \Bigg[\frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)} \left(\frac{(-1)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}} \Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1) I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)} a^{p_\kappa}(\zeta_\kappa)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}} \right) \\ &+ \frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)} \left(\frac{(-1)^{\frac{-((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}} \Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1) I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)} a^{p_\kappa}(\zeta_\kappa-1)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}} \right) \\ &+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)} \right) \left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}} \right) \\ &\left((-1)^{-(\alpha-1)p_\kappa} \Gamma((\alpha-1)p_\kappa) I_{\zeta_{\kappa-1}}^{(\alpha-1)p_\kappa} a^{p_\kappa}(\zeta_\kappa) \right) \\ &+ \left(\frac{2^{2p_\kappa-2} c^{p_\kappa} (\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa}}{\mu^{p_\kappa} \alpha^{p_\kappa}} \right) \left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1} \right) \Bigg]^{\frac{1}{p_\kappa}} \end{aligned}$$

and

$$\begin{aligned} B &= \Bigg[1 - 2^{p_\kappa-1} \Bigg[\frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)} \Bigg(\frac{(\zeta_{\kappa-1}-\zeta_\kappa)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}) \left(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1 \right)} \\ &+ \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}) \left(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1 \right)} \Bigg) \\ &+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)} \right) \left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}} \right) \\ &\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1} \right) \left(b + \frac{d(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha}}{\Gamma(\alpha+1)} \right) \Bigg] \Bigg]^{\frac{1}{p_\kappa}} \end{aligned}$$

We consider the set

$$B_{R_\kappa} = \{ y \in E_\kappa, \ \|y\|_{E_\kappa} \leq R_\kappa \}.$$

Obviously, B_{R_κ} is nonempty, bounded, convex, and closed.
Now, it must be shown that S fulfills the premise of Theorem 2.

STEP 1: $S(B_{R_\kappa}) \subseteq (B_{R_\kappa})$

For $y \in B_{R_\kappa}$, using Holder's inequality and by (H3), we obtain

$$
\begin{aligned}
\left\|Sy(t)\right\|_{E_\kappa}^{p_\kappa} &= \Big\| \tfrac{1}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^{t} (t-s)^{\alpha-1}\Phi(s,y(s),I_{\zeta_{\kappa-1}}^{\alpha}y(s))ds \\
&- \tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa-s)^{\alpha-1}\Phi(s,y(s),I_{\zeta_{\kappa-1}}^{\alpha}y(s))ds \\
&+ \tfrac{c(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}\Big\|_{E_\kappa}^{p_\kappa} \\
&\le \tfrac{2^{p_\kappa-1}}{\Gamma^{p_\kappa}(\alpha)}\Big\|\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-s)^{\alpha-1}\Phi(s,y(s),I_{\zeta_{\kappa-1}}^{\alpha}y(s))ds\Big\|_{E_\kappa}^{p_\kappa} \\
&+ \tfrac{2^{2p_\kappa-2}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\Big\|(t-\zeta_{\kappa-1})^{\alpha-1}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Phi(s,y(s),I_{\zeta_{\kappa-1}}^{\alpha}y(s))ds\Big\|_{E_\kappa}^{p_\kappa} \\
&+ \tfrac{2^{2p_\kappa-2}c^{p_\kappa}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\mu^{p_\kappa}}\|(t-\zeta_{\kappa-1})^{\alpha-1}\|_{E_\kappa}^{p_\kappa} \\
&\le \tfrac{2^{p_\kappa-1}}{\Gamma^{p_\kappa}(\alpha)}\Big\|\Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-s)^{(\alpha-1)q_\kappa}ds\Big)^{\frac{1}{q_\kappa}}\|\Phi(s,y,I_{\zeta_{\kappa-1}}^{\alpha}y)\|_{E_\kappa}\Big\|_{E_\kappa}^{p_\kappa} \\
&+ \tfrac{2^{2p_\kappa-2}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\Big\|(t-\zeta_{\kappa-1})^{\alpha-1}\Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{(\alpha-1)q_\kappa}ds\Big)^{\frac{1}{q_\kappa}}\|\Phi(s,y,I_{\zeta_{\kappa-1}}^{\alpha}y)\|_{E_\kappa}\Big\|_{E_\kappa}^{p_\kappa} \\
&+ \tfrac{2^{2p_\kappa-2}c^{p_\kappa}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\mu^{p_\kappa}}\|(t-\zeta_{\kappa-1})^{\alpha-1}\|_{E_\kappa}^{p_\kappa} \\
&\le \tfrac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(-1)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)}a^{p_\kappa}(\zeta_\kappa)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}}\right) \\
&+ \tfrac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(-1)^{\frac{-((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)}a^{p_\kappa}(\zeta_\kappa-1)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}}\right) \\
&+ \tfrac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(\zeta_{\kappa-1}-\zeta_\kappa)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa})\Big(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\Big)} + \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa})\Big(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\Big)}\right) \\
&\quad \Big(b\|y\|_{E_\kappa}+d\|I_{\zeta_{\kappa-1}}^{\alpha}y\|_{E_\kappa}\Big)^{p_\kappa} \\
&+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\right) \\
&\quad \Big((-1)^{-(\alpha-1)p_\kappa}\Gamma((\alpha-1)p_\kappa)I_{\zeta_{\kappa-1}}^{(\alpha-1)p_\kappa}a^{p_\kappa}(\zeta_\kappa)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\right) \\
&\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1}\right)\left(b\|y\|_{E_\kappa}+d\|I^{\alpha}_{\zeta_{\kappa-1}}y\|_{E_\kappa}\right)^{p_\kappa} \\
&+ \left(\frac{2^{2p_\kappa-2}c^{p_\kappa}(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa}}{\mu^{p_\kappa}\alpha^{p_\kappa}}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1}\right) \\
&\leq \frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(-1)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)}a^{p_\kappa}(\zeta_\kappa)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}}\right) \\
&+ \frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(-1)^{\frac{-((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)}a^{p_\kappa}(\zeta_\kappa-1)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}}\right) \\
&+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\right) \\
&\left((-1)^{-(\alpha-1)p_\kappa}\Gamma((\alpha-1)p_\kappa)I^{(\alpha-1)p_\kappa}_{\zeta_{\kappa-1}}a^{p_\kappa}(\zeta_\kappa)\right) \\
&+ \left(\frac{2^{2p_\kappa-2}c^{p_\kappa}(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa}}{\mu^{p_\kappa}\alpha^{p_\kappa}}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1}\right) \\
&+ \Bigg[\frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(\zeta_{\kappa-1}-\zeta_\kappa)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa})\left(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\right)}+\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa})\left(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\right)}\right) \\
&+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1}\right) \\
&\left(b+\frac{d(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha}}{\Gamma(\alpha+1)}\right)\Bigg]\|y\|^{p_\kappa}_{E_\kappa} \\
&\leq \frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(-1)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)}a^{p_\kappa}(\zeta_\kappa)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}}\right) \\
&+ \frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(-1)^{\frac{-((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\Gamma(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)I_{\zeta_{\kappa-1}}^{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1)}a^{p_\kappa}(\zeta_\kappa-1)}{((\alpha-1)q_\kappa+1)^{\frac{p_\kappa}{q_\kappa}}}\right) \\
&+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\right) \\
&\left((-1)^{-(\alpha-1)p_\kappa}\Gamma((\alpha-1)p_\kappa)I^{(\alpha-1)p_\kappa}_{\zeta_{\kappa-1}}a^{p_\kappa}(\zeta_\kappa)\right) \\
&+ \left(\frac{2^{2p_\kappa-2}c^{p_\kappa}(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa}}{\mu^{p_\kappa}\alpha^{p_\kappa}}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1}\right) \\
&+ \Bigg[\frac{2^{2p_\kappa-2}}{\Gamma^{p_\kappa}(\alpha)}\left(\frac{(\zeta_{\kappa-1}-\zeta_\kappa)^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa})\left(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\right)}+\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1}}{(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa})\left(\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}+1\right)}\right) \\
&+ \left(\frac{2^{3p_\kappa-3}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\right)\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}}{\frac{((\alpha-1)q_\kappa+1)p_\kappa}{q_\kappa}}\right) \\
&\left(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa+1}}{(\alpha-1)p_\kappa+1}\right)\left(b+\frac{d(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha}}{\Gamma(\alpha+1)}\right)\Bigg]R^{p_\kappa}_\kappa
\end{aligned}
$$

which means that $S(B_{R_\kappa}) \subseteq B_{R_\kappa}$.

STEP 2: S is continuous.

Assume the sequence (y_n) is convergent to y in E_κ. Then, for $t \in J_\kappa$, by (H2) and Holder's inequality we have

$$
\begin{aligned}
& \|S(y_n(t)) - S(y(t))\|_{E_\kappa}^{p_\kappa} \\
= \; & \| \frac{1}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^{t} (t-s)^{\alpha-1} \big(\Phi(s, y_n(s), I_{\zeta_{\kappa-1}}^{\alpha} y_n(s)) - \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s))\big) ds \\
- \; & \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha} (t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \\
& \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{\alpha-1} \Big(\Phi(s, y_n(s), I_{\zeta_{\kappa-1}}^{\alpha} y_n(s)) - \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s))\Big) ds \|_{E_\kappa}^{p_\kappa} \\
= \; & \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big| \frac{1}{\Gamma(\alpha)} \int_{\zeta_{\kappa-1}}^{t} (t-s)^{\alpha-1} \Big(\Phi(s, y_n(s), I_{\zeta_{\kappa-1}}^{\alpha} y_n(s)) - \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s))\Big) ds \\
- \; & \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{1-\alpha} (t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \\
& \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{\alpha-1} \Big(\Phi(s, y_n(s), I_{\zeta_{\kappa-1}}^{\alpha} y_n(s)) - \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s)) ds\Big) \Big|^{p_\kappa} dt \\
\leq \; & \frac{2^{p_\kappa - 1}}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big| \int_{\zeta_{\kappa-1}}^{t} (t-s)^{\alpha-1} \Big(\Phi(s, y_n(s), I_{\zeta_{\kappa-1}}^{\alpha} y_n(s)) - \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s))\Big) ds \Big|^{p_\kappa} dt \\
+ \; & \frac{2^{p_\kappa - 1} (\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \\
& \Big(\Phi(s, y_n(s), I_{\zeta_{\kappa-1}}^{\alpha} y_n(s)) - \Phi(s, y(s), I_{\zeta_{\kappa-1}}^{\alpha} y(s))\Big) ds \Big|^{p_\kappa} dt \\
\leq \; & \frac{(2^{p_\kappa - 1} M_1^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big(\int_{\zeta_{\kappa-1}}^{t} (t-s)^{\alpha-1} \Big| y_n(s) - y(s) \Big| ds \Big)^{p_\kappa} dt \\
+ \; & \frac{(2^{1-\frac{1}{p_\kappa}} M_2)^{p_\kappa}}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big(\int_{\zeta_{\kappa-1}}^{t} (t-s)^{\alpha-1} \Big| I_{\zeta_{\kappa-1}}^{\alpha} y_n(s) - I_{\zeta_{\kappa-1}}^{\alpha} y(s) \Big| ds \Big)^{p_\kappa} dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_1^{p_\kappa})(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big((t-\zeta_{\kappa-1})^{\alpha-1} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{\alpha-1} \Big| y_n(s) - y(s) \Big| ds \Big)^{p_\kappa} dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_2^{p_\kappa})(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \\
& \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big((t-\zeta_{\kappa-1})^{\alpha-1} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{\alpha-1} \Big| I_{\zeta_{\kappa-1}}^{\alpha} y_n(s) - I_{\zeta_{\kappa-1}}^{\alpha} y(s) \Big| ds \Big)^{p_\kappa} dt \\
\leq \; & \frac{(2^{p_\kappa - 1} M_1^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big[\Big(\int_{\zeta_{\kappa-1}}^{t} (t-s)^{q_\kappa(\alpha-1)} ds \Big)^{\frac{1}{q_\kappa}} \Big(\int_{\zeta_{\kappa-1}}^{t} \Big| y_n(s) - y(s) \Big|^{p_\kappa} ds \Big)^{\frac{1}{p_\kappa}} \Big]^{p_\kappa} dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_2^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big[\Big(\int_{\zeta_{\kappa-1}}^{t} (t-s)^{q_\kappa(\alpha-1)} ds \Big)^{\frac{1}{q_\kappa}} \\
& \Big(\int_{\zeta_{\kappa-1}}^{t} \Big| I_{\zeta_{\kappa-1}}^{\alpha} y_n(s) - I_{\zeta_{\kappa-1}}^{\alpha} y(s) \Big|^{p_\kappa} ds \Big)^{\frac{1}{p_\kappa}} \Big]^{p_\kappa} dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_1^{p_\kappa})(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)} \Big[\Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{q_\kappa(\alpha-1)} ds \Big)^{\frac{1}{q_\kappa}} \\
& \Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big| y_n(s) - y(s) \Big|^{p_\kappa} ds \Big)^{\frac{1}{p_\kappa}} \Big]^{p_\kappa} dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_2^{p_\kappa})(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)} \Big[\Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (\zeta_\kappa - s)^{q_\kappa(\alpha-1)} ds \Big)^{\frac{1}{q_\kappa}} \\
& \Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \Big| I_{\zeta_{\kappa-1}}^{\alpha} y_n(s) - I_{\zeta_{\kappa-1}}^{\alpha} y(s) \Big|^{p_\kappa} ds \Big)^{\frac{1}{p_\kappa}} \Big]^{p_\kappa} dt \\
\leq \; & \frac{(2^{p_\kappa - 1} M_1^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \frac{(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{p_\kappa}{q_\kappa}}} \Big(\int_{\zeta_{\kappa-1}}^{t} \Big| y_n(s) - y(s) \Big|^{p_\kappa} ds \Big) dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_2^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} \frac{(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{p_\kappa}{q_\kappa}}} \Big(\int_{\zeta_{\kappa-1}}^{t} \Big| I_{\zeta_{\kappa-1}}^{\alpha} y_n(s) - I_{\zeta_{\kappa-1}}^{\alpha} y(s) \Big|^{p_\kappa} ds \Big) dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_1^{p_\kappa})(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)} \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}} \|y_n - y\|_{E_\kappa}^{p_\kappa} dt \\
+ \; & \frac{(2^{p_\kappa - 1} M_2^{p_\kappa})(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \\
& \int_{\zeta_{\kappa-1}}^{\zeta_\kappa} (t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)} \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}} \|I_{\zeta_{\kappa-1}}^{\alpha} y_n - I_{\zeta_{\kappa-1}}^{\alpha} y\|_{E_\kappa}^{p_\kappa} dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}\left(\int_{\zeta_{\kappa-1}}^{t}\left|y_n(s)-y(s)\right|^{p_\kappa}ds\right)dt\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_2}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}\\
&\quad\left(\int_{\zeta_{\kappa-1}}^{t}\left|I^{\alpha}_{\zeta_{\kappa-1}}y_n(s)-I^{\alpha}_{\zeta_{\kappa-1}}y(s)\right|^{p_\kappa}ds\right)dt\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\|y_n-y\|^{p_\kappa}_{E_\kappa}dt.\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\|I^{\alpha}_{\zeta_{\kappa-1}}y_n-I^{\alpha}_{\zeta_{\kappa-1}}y\|^{p_\kappa}_{E_\kappa}dt.
\end{aligned}
$$

We put:

$$
\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}\left(\int_{\zeta_{\kappa-1}}^{t}\left|y_n(s)-y(s)\right|^{p_\kappa}ds\right)dt=\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{\theta_\kappa}r_\kappa(t)dt=\Delta
$$

where

$$
\theta_\kappa=\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa},\ r_\kappa(t)=\int_{\zeta_{\kappa-1}}^{t}\left|y_n(s)-y(s)\right|^{p_\kappa}ds.
$$

Integrating by parts, we have:

$$
\begin{aligned}
\Delta &= \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}r_\kappa(\zeta_\kappa)-\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\frac{(t-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}r'_\kappa(t)dt\\
&= \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}\|y_n-y\|^{p_\kappa}_{E_\kappa}-\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\frac{(t-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}r'_\kappa(t)dt.
\end{aligned}
$$

Since the integral is

$$
\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\frac{(t-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}r'_\kappa(t)dt\geq 0,
$$

then,

$$
\begin{aligned}
\|S(y_n(t))-S(y(t))\|^{p_\kappa}_{E_\kappa} &\leq \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}\|y_n-y\|^{p_\kappa}_{E_\kappa}\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_2}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}\|I^{\alpha}_{\zeta_{\kappa-1}}y_n-I^{\alpha}_{\zeta_{\kappa-1}}y\|^{p_\kappa}_{E_\kappa}\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\|y_n-y\|^{p_\kappa}_{E_\kappa}\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_2(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\|I^{\alpha}_{\zeta_{\kappa-1}}y_n-I^{\alpha}_{\zeta_{\kappa-1}}y\|^{p_\kappa}_{E_\kappa}\\
&\leq \left(\left[\frac{2^{1-\frac{1}{p_\kappa}}M_1}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}\right.\\
&+ \left.\left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\right)\|y_n-y\|^{p_\kappa}_{E_\kappa}\\
&+ \left(\left[\frac{2^{1-\frac{1}{p_\kappa}}M_2}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}\right.\\
&+ \left.\left[\frac{2^{1-\frac{1}{p_\kappa}}M_2(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\right)\|I^{\alpha}_{\zeta_{\kappa-1}}y_n-I^{\alpha}_{\zeta_{\kappa-1}}y\|^{p_\kappa}_{E_\kappa}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|S(y_n(t)) - S(y(t))\|_{E_\kappa} \\
\leq\ & \Bigg(\Big[\frac{2^{1-\frac{1}{p_\kappa}} M_1}{\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1} \\
+\ & \Big[\frac{2^{1-\frac{1}{p_\kappa}} M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1} \\
+\ & \Big[\frac{2^{1-\frac{1}{p_\kappa}} M_2}{\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1} \\
+\ & \Big[\frac{2^{1-\frac{1}{p_\kappa}} M_2(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\Big(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa\alpha}}{\Gamma^{p_\kappa}(\alpha+1)}\Big)\Bigg)^{\frac{1}{p_\kappa}} \\
&\|y_n - y\|_{E_\kappa}
\end{aligned}
$$

$$\|(Sy_n) - (Sy)\|_{E_\kappa} \to 0 \ \ as \ \ n \to \infty.$$

Consequently, S is continuous.

STEP 3: S is compact.
We demonstrate that S is completely continuous. By Step 1, we have $S(B_{R_\kappa}) = \{S(y) : y \in B_{R_\kappa}\} \subset B_{R_\kappa}$, which means that $\|S(y)\|_{E_\kappa} \leq R_\kappa$ for any $y \in B_{R_\kappa}$; thus, $S(B_{R_\kappa})$ is uniformly bounded. We need to prove that $S(B_{R_\kappa})$ is relatively compact. We will use the Riesz compactness criteria Theorem.

For $t_1, t_2 \in J_\kappa,\ t_1 < t_2$, and $y \in B_{R_\kappa}$, we have

$$
\begin{aligned}
&\int_{t_1}^{t_2}\Big(y(t+h) - y(t)\Big)^{p_\kappa} dt = \int_{t_1}^{t_2}\Big(\frac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t+h}(t+h-s)^{\alpha-1}\Phi(s, y(s), I^{\alpha}_{\zeta_{\kappa-1}} y(s))ds \\
-\ & \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t+h-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa - s)^{\alpha-1}\Phi(s, y(s), I^{\alpha}_{\zeta_{\kappa-1}} y(s))ds \\
+\ & \frac{c(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t+h-\zeta_{\kappa-1})^{\alpha-1}}{\mu} \\
-\ & \frac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Phi(s, y(s), I^{\alpha}_{\zeta_{\kappa-1}} y(s))ds \\
+\ & \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa - s)^{\alpha-1}\Phi(s, y(s), I^{\alpha}_{\zeta_{\kappa-1}} y(s))ds \\
-\ & \frac{c(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}\Big)^{p_\kappa} dt.
\end{aligned}
$$

Now, using the fact that translations of L^{p_κ} functions are continuous in norm, we see that

$$\int_{t_1}^{t_2}\Big(y(t+h) - y(t)\Big)^{p_\kappa} dt \to 0 \ \ as \ \ t_2 \to t_1$$

uniformly in $y \in B_{R_\kappa}$, and (2) of Theorem 2 is proved.

Therefore, using Theorem 2, we have that $S(B_{R_\kappa})$ is relatively compact. According to Theorem 3, the BVP (8) has at least solution $\widetilde{y_\kappa}$ in B_{R_κ}. □

The following result is depended on the Banach contraction principle.

Theorem 6. *Assume that (H1), (H2), and inequality*

$$W(\alpha, M_1, M_2, p_\kappa, \zeta_{\kappa-1}, \zeta_\kappa) < 1, \tag{12}$$

holds, where:

$$W(\alpha, M_1, M_2, p_\kappa, \zeta_{\kappa-1}, \zeta_\kappa) = \Big[\frac{2^{1-\frac{1}{p_\kappa}} M_1}{\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}$$
$$+ \Big[\frac{2^{1-\frac{1}{p_\kappa}} M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}$$
$$+ \Big[\frac{2^{1-\frac{1}{p_\kappa}} M_2}{\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}$$
$$+ \Big[\frac{2^{1-\frac{1}{p_\kappa}} M_2(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\Big]^{p_\kappa} \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\Big(\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa\alpha}}{\Gamma^{p_\kappa}(\alpha+1)}\Big).$$

Then, the BVP (9) possesses a unique solution on $E_\kappa, \kappa = 1, 2, ..., n$.

Proof. Let $y_1, y_2 \in L^{p_\kappa}(J_\kappa)$; using Holder's inequality and (H2) we have

$$\begin{aligned}
& \|S(y_1(t)) - S(y_2(t))\|^{p_\kappa}_{E_\kappa} \\
= & \|\frac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}(\Phi(s, y_1(s), I^{\alpha}_{\zeta_{\kappa-1}}y_1(s)) - \Phi(s, y_2(s), I^{\alpha}_{\zeta_{\kappa-1}}y_2(s)))ds \\
- & \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \\
& \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Big(\Phi(s, y_1(s), I^{\alpha}_{\zeta_{\kappa-1}}y_1(s)) - \Phi(s, y_2(s), I^{\alpha}_{\zeta_{\kappa-1}}y_2(s))\Big)ds\|^{p_\kappa}_{E_\kappa} \\
= & \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big|\frac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Big(\Phi(s, y_1(s), I^{\alpha}_{\zeta_{\kappa-1}}y_1(s)) - \Phi(s, y_2(s), I^{\alpha}_{\zeta_{\kappa-1}}y_2(s))\Big)ds \\
- & \frac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \\
& \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Big(\Phi(s, y_1(s), I^{\alpha}_{\zeta_{\kappa-1}}y_1(s)) - \Phi(s, y_2(s), I^{\alpha}_{\zeta_{\kappa-1}}y_2(s))ds\Big|^{p_\kappa}dt \\
\leq & \frac{2^{p_\kappa-1}}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big|\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1} \\
& \Big(\Phi(s, y_1(s), I^{\alpha}_{\zeta_{\kappa-1}}y_1(s)) - \Phi(s, y_2(s), I^{\alpha}_{\zeta_{\kappa-1}}y_2(s))\Big)ds\Big|^{p_\kappa}dt \\
+ & \frac{2^{p_\kappa-1}(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big|(t-\zeta_{\kappa-1})^{\alpha-1}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1} \\
& \Big(\Phi(s, y_1(s), I^{\alpha}_{\zeta_{\kappa-1}}y_1(s)) - \Phi(s, y_2(s), I^{\alpha}_{\zeta_{\kappa-1}}y_2(s))\Big)ds\Big|^{p_\kappa}dt \\
\leq & \frac{(2^{p_\kappa-1}M_1^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big(\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Big|y_1(s)-y_2(s)\Big|ds\Big)^{p_\kappa}dt \\
+ & \frac{(2^{p_\kappa-1}M_2^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big(\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Big|I^{\alpha}_{\zeta_{\kappa-1}}y_1(s) - I^{\alpha}_{\zeta_{\kappa-1}}y_2(s)\Big|ds\Big)^{p_\kappa}dt \\
+ & \frac{(2^{p_\kappa-1}M_1^{p_\kappa})(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big((t-\zeta_{\kappa-1})^{\alpha-1}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Big|y_1(s)-y_2(s)\Big|ds\Big)^{p_\kappa}dt \\
+ & \frac{(2^{p_\kappa-1}M_2^{p_\kappa})(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}} \\
& \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big((t-\zeta_{\kappa-1})^{\alpha-1}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Big|I^{\alpha}_{\zeta_{\kappa-1}}y_1(s) - I^{\alpha}_{\zeta_{\kappa-1}}y_2(s)\Big|ds\Big)^{p_\kappa}dt \\
\leq & \frac{(2^{p_\kappa-1}M_1^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big[\Big(\int_{\zeta_{\kappa-1}}^{t}(t-s)^{q_\kappa(\alpha-1)}ds\Big)^{\frac{1}{q_\kappa}}\Big(\int_{\zeta_{\kappa-1}}^{t}\Big|y_1(s)-y_2(s)\Big|^{p_\kappa}ds\Big)^{\frac{1}{p_\kappa}}\Big]^{p_\kappa}dt \\
+ & \frac{(2^{p_\kappa-1}M_2^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big[\Big(\int_{\zeta_{\kappa-1}}^{t}(t-s)^{q_\kappa(\alpha-1)}ds\Big)^{\frac{1}{q_\kappa}} \\
& \Big(\int_{\zeta_{\kappa-1}}^{t}\Big|I^{\alpha}_{\zeta_{\kappa-1}}y_1(s) - I^{\alpha}_{\zeta_{\kappa-1}}y_2(s)\Big|^{p_\kappa}ds\Big)^{\frac{1}{p_\kappa}}\Big]^{p_\kappa}dt \\
+ & \frac{(2^{p_\kappa-1}M_1^{p_\kappa})(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\Big[\Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{q_\kappa(\alpha-1)}ds\Big)^{\frac{1}{q_\kappa}} \\
& \Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big|y_1(s)-y_2(s)\Big|^{p_\kappa}ds\Big)^{\frac{1}{p_\kappa}}\Big]^{p_\kappa}dt \\
+ & \frac{(2^{p_\kappa-1}M_2^{p_\kappa})(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\Big[\Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{q_\kappa(\alpha-1)}ds\Big)^{\frac{1}{q_\kappa}} \\
& \Big(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big|I^{\alpha}_{\zeta_{\kappa-1}}y_1(s) - I^{\alpha}_{\zeta_{\kappa-1}}y_2(s)\Big|^{p_\kappa}ds\Big)^{\frac{1}{p_\kappa}}\Big]^{p_\kappa}dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(2^{p_\kappa-1}M_1^{p_\kappa})}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\frac{(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{p_\kappa}{q_\kappa}}}\left(\int_{\zeta_{\kappa-1}}^{t}\left|y_1(s)-y_2(s)\right|^{p_\kappa}ds\right)dt\\
&+ \frac{(2^{1-\frac{1}{p_\kappa}}M_2)^{p_\kappa}}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\frac{(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{p_\kappa}{q_\kappa}}}\left(\int_{\zeta_{\kappa-1}}^{t}\left|I_{\zeta_{\kappa-1}}^{\alpha}y_1(s)-I_{\zeta_{\kappa-1}}^{\alpha}y_2(s)\right|^{p_\kappa}ds\right)dt\\
&+ \frac{(2^{p_\kappa-1}M_1^{p_\kappa})(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}}\|y_1-y_2\|_{E_\kappa}^{p_\kappa}dt\\
&+ \frac{(2^{1-\frac{1}{p_\kappa}}M_2)^{p_\kappa}(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)}}{(\Gamma(\alpha))^{p_\kappa}}\\
&\quad\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}}\|I_{\zeta_{\kappa-1}}^{\alpha}y_1-I_{\zeta_{\kappa-1}}^{\alpha}y_2\|_{E_\kappa}^{p_\kappa}dt\\
&\leq \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}\left(\int_{\zeta_{\kappa-1}}^{t}\left|y_1(s)-y_2(s)\right|^{p_\kappa}ds\right)dt\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_2}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\\
&\quad\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}\left(\int_{\zeta_{\kappa-1}}^{t}\left|I_{\zeta_{\kappa-1}}^{\alpha}y_1(s)-I_{\zeta_{\kappa-1}}^{\alpha}y_2(s)\right|^{p_\kappa}ds\right)dt\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\|y_1-y_2\|_{E_\kappa}^{p_\kappa}dt.\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\|I_{\zeta_{\kappa-1}}^{\alpha}y_1-I_{\zeta_{\kappa-1}}^{\alpha}y_2\|_{E_\kappa}^{p_\kappa}dt.
\end{aligned}
$$

In a similar way to Step 2, we find:

$$
\begin{aligned}
&\|S(y_1(t))-S(y_2(t))\|_{E_\kappa}^{p_\kappa} \leq \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}\|y_1-y_2\|_{E_\kappa}^{p_\kappa}\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_2}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\theta_\kappa+1}}{\theta_\kappa+1}\|I_{\zeta_{\kappa-1}}^{\alpha}y_1-I_{\zeta_{\kappa-1}}^{\alpha}y_2\|_{E_\kappa}^{p_\kappa}\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\|y_1-y_2\|_{E_\kappa}^{p_\kappa}\\
&+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_2(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\|I_{\zeta_{\kappa-1}}^{\alpha}y_1-I_{\zeta_{\kappa-1}}^{\alpha}y_2\|_{E_\kappa}^{p_\kappa}\\
&\leq \left(\left[\frac{2^{1-\frac{1}{p_\kappa}}M_1}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}\right.\\
&\left.+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_1(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\right)\|y_1-y_2\|_{E_\kappa}^{p_\kappa}\\
&+ \left(\left[\frac{2^{1-\frac{1}{p_\kappa}}M_2}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}\right.\\
&\left.+ \left[\frac{2^{1-\frac{1}{p_\kappa}}M_2(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\right)\|I_{\zeta_{\kappa-1}}^{\alpha}y_1-I_{\zeta_{\kappa-1}}^{\alpha}y_2\|_{E_\kappa}^{p_\kappa},
\end{aligned}
$$

Then,

$$
\|S(y_1(t))-S(y_2(t))\|_{E_\kappa} \leq \left(\left[\frac{2^{1-\frac{1}{p_\kappa}}M_1}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}}\Gamma(\alpha)}\right]^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}\right.
$$

$$+ \left[\frac{2^{1-\frac{1}{p_\kappa}} M_1 (\zeta_\kappa - \zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}} \Gamma(\alpha)}\right]^{p_\kappa} \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}$$
$$+ \left[\frac{2^{1-\frac{1}{p_\kappa}} M_2}{\left(q_\kappa(\alpha-1)+1\right)^{\frac{1}{q_\kappa}} \Gamma(\alpha)}\right]^{p_\kappa} \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}$$
$$+ \left[\frac{2^{1-\frac{1}{p_\kappa}} M_2 (\zeta_\kappa - \zeta_{\kappa-1})^{(1-\alpha)+\frac{q_\kappa(\alpha-1)+1}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{1}{q_\kappa}} \Gamma(\alpha)}\right]^{p_\kappa} \frac{(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1} \left(\frac{(\zeta_\kappa - \zeta_{\kappa-1})^{p_\kappa \alpha}}{\Gamma^{p_\kappa}(\alpha+1)}\right)\Bigg)^{\frac{1}{p_\kappa}} \|y_1 - y_2\|_{E_\kappa}$$

Hence, by (12) and according the Banach's contraction principle, S has a unique fixed point $\widetilde{y_\kappa} \in L^{p_\kappa}(J_\kappa, \mathbb{R})$, which is a unique solution of the BVP (9). □

Now, we will prove the existence result for BVP (3).

Theorem 7. *Assume that (H1), (H2), and* (11) *hold for any* $1 \le \kappa \le n$. *Thus, the BVP* (3) *has a unique solution on* $L^{p(\cdot)}(J, \mathbb{R})$.

Proof. We have proved in Theorem 6 that the BVP for the Riemann–Liouville fractional differential Equation (9) possesses unique solution $\widetilde{y}_\kappa \in E_\kappa$ for any $1 \le \kappa \le n$.

For $1 \le \kappa \le n$, we define the function:

$$y_\kappa = \begin{cases} 0, & if\ t \in [0\ \ \zeta_{\kappa-1}] \\ \widetilde{y}_\kappa, & if\ t \in J_\kappa. \end{cases}$$

Thus, the function $y_\kappa \in L^p([0,\ \ \zeta_{\kappa-1}], \mathbb{R})$ is a solution to the integral Equation (8) for $t \in J_\kappa$, with

$\gamma y_\kappa(0) + \mu y_\kappa(\zeta_\kappa) = \gamma y_\kappa(0) + \mu \widetilde{y}_\kappa(\zeta_\kappa) = c.$

Then, the function:

$$y(t) = \begin{cases} y_1(t) \in L^{p_1}(J_1, \mathbb{R}), \\ y_2(t) \in L^{p_2}(J_2, \mathbb{R}), \\ . \\ . \\ . \\ y_n(t) \in L^{p_n}(J_n, \mathbb{R}) \end{cases}$$

forms a unique solution of the BVP (3) in $L^{p(\cdot)}(J)$. □

4. Stability of Solutions

Theorem 8. *Let (H2), (H3), and* (11) *be satisfied for every* $1 \le \kappa \le n$. *Thus, BVP* (3) *is* ***(UH)*** *stable.*

Proof. Let $\epsilon > 0$ and the function $z \in L^{p(\cdot)}(J, \mathbb{R})$ be a solution of the inequality (6).

We define the functions $z_1(t) \equiv z(t), t \in [0, \zeta_1]$ and for $\kappa = 2, 3, ..., n$

$$z_\kappa(t) = \begin{cases} 0, & t \in [0, \zeta_{\kappa-1}] \\ z(t), & t \in J_\kappa. \end{cases} \tag{13}$$

For $t \in J_\kappa, \kappa \in \{1, 2, ..., n\}$ and according to (7), we obtain

$$(D^\alpha_{0^+} z_\kappa)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{\zeta_{\kappa-1}}^t (t-s)^{-\alpha} z(s) ds.$$

Using $I^{\alpha}_{\zeta^+_{\kappa-1}}$ of both parts of (6), we obtain

$$\begin{aligned} \Big|z_\kappa(t) \quad - & \quad \tfrac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s))ds + \tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \\ & \quad \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s))ds - \tfrac{c(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}\Big| \\ \leq & \quad \epsilon\tfrac{(t-\zeta_{\kappa-1})^{\alpha}}{\Gamma(\alpha+1)} \\ \leq & \quad \epsilon\tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

By Theorem 7, BVP (3) possesses unique solution $y \in L^{p(\cdot)}(J)$ defined by $y(t) = y_\kappa(t)$ for $t \in J_\kappa,\ \kappa = 1, 2, ..., n$, where

$$y_\kappa = \begin{cases} 0, & t \in [0, \zeta_{\kappa-1}] \\ \widetilde{y}_\kappa, & t \in J_\kappa \end{cases} \tag{14}$$

and $\widetilde{y}_\kappa \in E_\kappa$ is a unique solution of BVP (9) defined by

$$\widetilde{y}_\kappa(t) = \tfrac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))ds - \tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)}$$

$$\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))ds + \frac{c(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}. \tag{15}$$

For $t \in J_\kappa,\ \kappa = 1, 2, ..., n$, by (13), (14) we obtain,

$$|z(t) - y(t)| = |z(t) - y_\kappa(t)| = |z_\kappa(t) - \widetilde{y}_\kappa(t)|.$$

Hence, by (15) we have

$$\begin{aligned} & \|z-y\|^{p_\kappa}_{E_\kappa} = \|z-y_\kappa\|^{p_\kappa}_{E_\kappa} = \|z_\kappa-\widetilde{y}_\kappa\|^{p_\kappa}_{E_\kappa} \\ = \ & \|z_\kappa(t) - \tfrac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))ds \\ + \ & \tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))ds \\ - \ & \tfrac{c(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}\|^{p_\kappa}_{p_\kappa} \\ \leq \ & 2^{p_\kappa-1}\|z_\kappa(t) - \tfrac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s))ds \\ + \ & \tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s))ds \\ - \ & \tfrac{c(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\mu}\|^{p_\kappa}_{E_\kappa} \\ + \ & 2^{p_\kappa-1}\|\tfrac{1}{\Gamma(\alpha)}\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}(\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s)) - \Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s)))ds\|^{p_\kappa}_{E_\kappa} \\ + \ & 2^{p_\kappa-1}\|\tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{1-\alpha}(t-\zeta_{\kappa-1})^{\alpha-1}}{\Gamma(\alpha)} \\ & \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}(\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s)) - \Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s)))ds\|^{p_\kappa}_{E_\kappa} \\ \leq \ & 2^{p_\kappa-1}\epsilon^{p_\kappa}\tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa+1}}{\Gamma^{p_\kappa}(\alpha+1)} + \tfrac{2^{p_\kappa-1}}{\Gamma^{p_\kappa}(\alpha)} \\ & \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\int_{\zeta_{\kappa-1}}^{t}(t-s)^{\alpha-1}|\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s)) - \Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))|ds)^{p_\kappa}dt \\ + \ & \tfrac{2^{p_\kappa-1}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{(\alpha-1)p_\kappa} \\ & (\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{\alpha-1}|\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s)) - \Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))|ds)^{p_\kappa}dt \\ \leq \ & 2^{p_\kappa-1}\epsilon^{p_\kappa}\tfrac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa+1}}{\Gamma^{p_\kappa}(\alpha+1)} + \tfrac{2^{p_\kappa-1}}{\Gamma^{p_\kappa}(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big[\Big(\int_{\zeta_{\kappa-1}}^{t}(t-s)^{q_\kappa(\alpha-1)}ds\Big)^{\frac{1}{q_\kappa}} \\ & \Big(\int_{\zeta_{\kappa-1}}^{t}\Big|\Phi(s,z_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}z_\kappa(s)) - \Phi(s,\widetilde{y}_\kappa(s),I^{\alpha}_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))\Big|^{p_\kappa}ds\Big)^{\frac{1}{p_\kappa}}\Big]^{p_\kappa}dt \end{aligned}$$

$$
\begin{aligned}
&+ \frac{2^{p_\kappa-1}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)} \int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)}\Bigg[\bigg(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(\zeta_\kappa-s)^{q_\kappa(\alpha-1)}ds\bigg)^{\frac{1}{q_\kappa}} \\
&\bigg(\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\Big|\Phi(s,z_\kappa(s),I^\alpha_{\zeta_{\kappa-1}}z_\kappa(s))-\Phi(s,\widetilde{y}_\kappa(s),I^\alpha_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))\Big|^{p_\kappa}ds\bigg)^{\frac{1}{p_\kappa}}\Bigg]^{p_\kappa}dt \\
&\le 2^{p_\kappa-1}\epsilon^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa+1}}{\Gamma^{p_\kappa}(\alpha+1)} + \frac{2^{p_\kappa-1}}{\Gamma^{p_\kappa}(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}\frac{(t-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{p_\kappa}{q_\kappa}}} \\
&\bigg(\int_{\zeta_{\kappa-1}}^{t}\Big|\Phi(s,z_\kappa(s),I^\alpha_{\zeta_{\kappa-1}}z_\kappa(s))-\Phi(s,\widetilde{y}_\kappa(s),I^\alpha_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))\Big|^{p_\kappa}ds\bigg)dt \\
&+ \frac{2^{p_\kappa-1}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\int_{\zeta_{\kappa-1}}^{\zeta_\kappa}(t-\zeta_{\kappa-1})^{p_\kappa(\alpha-1)} \\
&\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}}\big|\Phi(s,z_\kappa(s),I^\alpha_{\zeta_{\kappa-1}}z_\kappa(s))-\Phi(s,\widetilde{y}_\kappa(s),I^\alpha_{\zeta_{\kappa-1}}\widetilde{y}_\kappa(s))\big|^{p_\kappa}_{p_\kappa}dt \\
&\le 2^{p_\kappa-1}\epsilon^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa+1}}{\Gamma^{p_\kappa}(\alpha+1)} + \frac{2^{p_\kappa-1}}{\Gamma^{p_\kappa}(\alpha)\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{p_\kappa}{q_\kappa}}}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1} \\
&(2^{p_\kappa-1}M_1^{p_\kappa}\|z_\kappa-\widetilde{y}_\kappa\|^{p_\kappa}_{E_\kappa}+2^{p_\kappa-1}M_2^{p_\kappa}\|I^\alpha_{\zeta_{\kappa-1}}z_\kappa-I^\alpha_{\zeta_{\kappa-1}}\widetilde{y}_\kappa\|^{p_\kappa}_{E_\kappa}) \\
&+ \frac{2^{p_\kappa-1}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1} \\
&(2^{p_\kappa-1}M_1^{p_\kappa}\|z_\kappa-\widetilde{y}_\kappa\|^{p_\kappa}_{E_\kappa}+2^{p_\kappa-1}M_2^{p_\kappa}\|I^\alpha_{\zeta_{\kappa-1}}z_\kappa-I^\alpha_{\zeta_{\kappa-1}}\widetilde{y}_\kappa\|^{p_\kappa}_{E_\kappa}) \\
&\le 2^{p_\kappa-1}\epsilon^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa+1}}{\Gamma^{p_\kappa}(\alpha+1)} + \Bigg[\frac{2^{p_\kappa-1}\Big(2^{p_\kappa-1}M_1^{p_\kappa}+2^{p_\kappa-1}M_2^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa}}{\Gamma^{p_\kappa}(\alpha+1)}\Big)}{\Gamma^{p_\kappa}(\alpha)\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{p_\kappa}{q_\kappa}}}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1} \\
&+ \frac{2^{p_\kappa-1}\Big(2^{p_\kappa-1}M_1^{p_\kappa}+2^{p_\kappa-1}M_2^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa}}{\Gamma^{p_\kappa}(\alpha+1)}\Big)(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa}}{\Gamma^{p_\kappa}(\alpha)} \\
&\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}}}{(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{p_\kappa(1-\alpha)+1}}{p_\kappa(1-\alpha)+1}\Bigg]\|z_\kappa-\widetilde{y}_\kappa\|^{p_\kappa}_{E_\kappa} \\
&\le 2^{p_\kappa-1}\epsilon^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa+1}}{\Gamma^{p_\kappa}(\alpha+1)} + \vartheta\|z-y\|^{p_\kappa}_{E_\kappa},
\end{aligned}
$$

where

$$
\begin{aligned}
\vartheta = \max_{\kappa=1,2,\ldots,n}&\Bigg[\frac{\Big(2^{p_\kappa-1}M_1^{p_\kappa}+2^{p_\kappa-1}M_2^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa}}{\Gamma^{p_\kappa}(\alpha+1)}\Big)2^{p_\kappa-1}}{\Gamma^{p_\kappa}(\alpha)\Big(q_\kappa(\alpha-1)+1\Big)^{\frac{p_\kappa}{q_\kappa}}}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1}}{\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+1} \\
&+ \frac{\Big(2^{p_\kappa-1}M_1^{p_\kappa}+2^{p_\kappa-1}M_2^{p_\kappa}\frac{(\zeta_\kappa-\zeta_{\kappa-1})^{\alpha p_\kappa}}{\Gamma^{p_\kappa}(\alpha+1)}\Big)2^{p_\kappa-1}(\zeta_\kappa-\zeta_{\kappa-1})^{(1-\alpha)p_\kappa+\frac{p_\kappa(q_\kappa(\alpha-1)+1)}{q_\kappa}+p_\kappa(1-\alpha)+1}}{\Gamma^{p_\kappa}(\alpha)(q_\kappa(\alpha-1)+1)^{\frac{p_\kappa}{q_\kappa}}p_\kappa(1-\alpha)+1}\Bigg].
\end{aligned}
$$

Then,

$$
\|z-y\|_{E_\kappa} \le \frac{2(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{\alpha p_\kappa+1}{p_\kappa}}}{(1-\vartheta)^{\frac{1}{p_\kappa}}\Gamma(\alpha+1)}\epsilon.
$$

Therefore,

$$
\|z-y\|_p \le \frac{1}{\Gamma(\alpha+1)}\bigg(\sum_{\kappa=1}^{\kappa=n}\frac{2(\zeta_\kappa-\zeta_{\kappa-1})^{\frac{\alpha p_\kappa+1}{p_\kappa}}}{(1-\vartheta)^{\frac{1}{p_\kappa}}}\bigg)\epsilon := c_\Phi\epsilon.
$$

As a consequence, the BVP (3) is **(UH)** stable. □

5. Example

Let the following BVP:

$$
\begin{cases} D^{0.5}y(t) = \frac{1}{(2+e^t)(1+y(t)+I_0^{0.5}y(t))}, \ t \in J := [0,2] \\ y(0)+y(2) = 0. \end{cases} \tag{16}
$$

For $(t,y,z) \in ([0,2] \times [1,+\infty) \times [1,+\infty))$, we define the function g as follows

$$\Phi(t,y,z) = \frac{1}{(2+e^t)(1+y(t)+z)}.$$

Thus, we obtain

$$\begin{array}{rcl} |\Phi(t,y_1,z_1) - \Phi(t,y_2,z_2)| & = & \frac{1}{(2+e^t)}\left|\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right| \\ & \leq & \frac{1}{2}|y_2 - y_1| + \frac{1}{2}|z_2 - z_1|. \end{array}$$

Hence, the condition (H2) holds with $M_1 = M_2 = \frac{1}{2}$.

Let

$$p(t) = \begin{cases} p_1 = 2.7, & if\ t \in [0,1], \\ p_2 = 2.8, & if\ t \in]1,2]. \end{cases} \tag{17}$$

We consider two auxiliary BVP:

$$\begin{cases} D^{0.5}y(t) = \frac{1}{(2+e^t)(1+y(t))}, & t \in J_1 := [0,1], \\ y(0) + y(1) = 0, \end{cases} \tag{18}$$

and

$$\begin{cases} D^{0.5}y(t) = \frac{1}{(2+e^t)(1+y(t))}, & ,t \in J_2 :=]1,2], \\ y(1) + y(2) = 0. \end{cases} \tag{19}$$

For $\kappa = 1, p_1 = 2.7$, we have

$$W_{\alpha,M,p_1,\zeta_0,\zeta_1} = 0,015523456 < 1.$$

Ergo, the condition (12) is fulfilled.
Using Theorem (6), the BVP (18) possesses a unique solution $\widetilde{y}_1 \in L^{2.7}(J_1,\mathbb{R})$.
For $\kappa = 2, p_2 = 2.8$, we have

$$W_{\alpha,M,p_2,\zeta_1,\zeta_2} = 0,499862644 < 1.$$

This means that the condition (12) is fulfilled.
According to Theorem 6, the BVP (19) possesses a unique solution $\widetilde{y}_2 \in L^{2.8}(J_2,\mathbb{R})$.
As a result, using Theorem (7), the BVP (19) has a unique solution.

$$y(t) = \begin{cases} \widetilde{y}_1(t) \in L^{2.7}(J_1,\mathbb{R}), \\ y_2(t) \in L^{2.8}(J_2,\mathbb{R}), \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{y}_2(t), & t \in J_2. \end{cases}$$

As a result, according to Theorem 8, BVP (16) is **(UH)** stable.

6. Conclusions

The existence of a unique solution to the multiterm BVP (3) for fractional differential equations in Lebesgue spaces with variable exponent ($L^{p}(\cdot)$) has been discussed in this paper. Based on the main difference between the classical Lebesgue spaces and the variable exponent Lebesgue spaces, we analyzed the BVP (3) and showed its existence and uniqueness. Further, the stability in the sense of Ulam–Hyers was examined. Lastly, an illustrative example was presented to demonstrate the correctness and applicability of the findings.

The obtained results in this paper generalize and extend the existing results on fractional differential equations in Lebesgue spaces with variable exponents ($L^{p}(\cdot)$). In addition, they are important and have numerous applications in the qualitative study of such systems.

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