fractal and fractional

Article

Some Local Fractional Hilbert-Type Inequalities

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Abstract: The main purpose of this paper is to prove some new local fractional Hilbert-type inequalities. Our general results are applicable to homogeneous kernels. Furthermore, the best possible constants in terms of local fractional hypergeometric function are obtained. The obtained results prove that the employed method is very simple and effective for treating various kinds of local fractional Hilbert-type inequalities.

Keywords: local fractional calculus; local fractional integral; local fractional hypergeometric function

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1. Introduction

If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then follows (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left[\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right]^{\frac{1}{2}}, \quad (1)$$

where the constant π is the best possible. Inequality (1) is known as Hilbert's integral inequality, important in the field of Mathematical Analysis and its applications.

Hilbert-type inequalities can be divided into three parts: Hilbert's inequalities (1908), Hardy-Hilbert-type inequalities (1934) and Yang-Hilbert-type inequalities (2009). For example, in the book [2], the authors investigated a great deal of integral Discrete Hilbert-type inequalities including Hilbert's inequality are important in mathematical analysis and its applications. Adiyasuren [3,4] derived several Hilbert-type inequalities with a differential operator and certain differential operators for a general homogeneous kernel for the best constants. AlNemer et al. [5] developed some new generalizations of dynamic Hilbert-type inequalities containing some integral and discrete inequalities. Over the past century, Chen and Yang [6] presented the introductory overview of the theory of Hilbert-type inequalities. By employing the weight coefficients and classical analytic methods, Yang et al. [7] considered an extension of a Hardy-Hilbert-type inequality with the best possible constant factor and multiparameter.

On the other hand, Yang [8,9] introduced the definition of the local fractional derivative and integral. Since fractional calculus on fractal sets can better describe natural phenomena, it has been widely used in the fields of science and engineering. Recently, many scholars have applied fractional calculus to fractal sets to investigate some classical inequalities. By using the properties of the generalized convex function, Mo et al. [10] established the generalized local fractional Jensen's inequality and generalized Hermite–Hadamard's inequality on a fractal set. Samet and Mehmet [11] obtained some generalized Pompeiu-type inequalities for local fractional integrals and applications. By making the fractal theory and the methods of weight function, Liu and Sun [12] established a Hilbert-type fractal integral inequality and its equivalent form with the best constant factors and applications. According to the general local fractional integral identity and the generalized harmonically convex function on fractal sets, Sun [13] obtained some generalized Hermite–Hadamard, Ostrowski and Simpson-type inequalities. By using the generalized power mean inequality and generalized Hölder inequality, Sun and Liu [14] investigated the local fractional

Hermite–Hadamard type integral inequalities for generalized harmonically convex functions. For more details, see in [15–20].

First, we introduce basic definitions and results of the local fractional calculus (see [8,9]). For $0 < \alpha \leq 1$ we define the following fractal sets:

- $\mathbb{Z}^\alpha$ is defined by
$$\mathbb{Z}^\alpha := \{0^\alpha\} \cup \{\pm m^\alpha : m \in \mathbb{N}_+\};$$
- $\mathbb{Q}^\alpha$ is defined by
$$\mathbb{Q}^\alpha := \{q^\alpha : q \in \mathbb{Q}\} = \left\{\left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z}, n \in \mathbb{N}_+\right\};$$
- $\mathbb{J}^\alpha$ is defined by
$$\mathbb{J}^\alpha := \{r^\alpha : r \in \mathbb{J}\} = \left\{r^\alpha \neq \left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z}, n \in \mathbb{N}_+\right\};$$
- $\mathbb{R}^\alpha$ is defined by
$$\mathbb{R}^\alpha = \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha.$$

Additionally, we list properties for $\mathbb{R}^\alpha$. If $a^\alpha, b^\alpha, c^\alpha \in \mathbb{R}^\alpha$, then

(a) $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$, $a^\alpha b^\alpha \in \mathbb{R}^\alpha$.
(b) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$.
(c) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$.
(d) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$.
(e) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$.
(f) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$.
(g) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.
(h) For each $a^\alpha \in \mathbb{R}^\alpha$, its inverse element $(-a)^\alpha$ may be written as $-a^\alpha$; for each $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, its inverse element $(1/b)^\alpha$ may be written as $1^\alpha / b^\alpha$ but not as $1/b^\alpha$ (see Proposition 1, [20]).
(i) $a^\alpha < b^\alpha$ if and only if $a < b$ (see [20]).
(j) $a^\alpha = b^\alpha$ if and only if $a = b$.

Further, we define the local fractional derivative and integral.

Definition 1. *A non-differentiable function $f(x)$ is said to be local fractional continuous at $x = x_0$ if for each $\varepsilon > 0$, there exists for $\delta > 0$ such that*
$$|f(x) - f(x_0)| < \varepsilon^\alpha,$$
holds for $0 < |x - x_0| < \delta$. If a function f is local continuous on the interval (a,b), we denote $f \in C_\alpha(a,b)$.

Definition 2. *Let $f(x) \in C_\alpha[a,b]$. Local fractional derivative of the function $f(x)$ at $x = x_0$ is given by*
$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Gamma(1+\alpha)(f(x) - f(x_0))}{(x - x_0)^\alpha}.$$

Definition 3. *Let $f(x) \in C_\alpha[a,b]$ and let $P = \{t_0, t_1, \ldots, t_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a,b]$ such that $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Further, for this partition P, let $\Delta t_j = t_{j+1} - t_j$, $j = 0, \ldots, N-1$, and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_{N-1}\}$. Then the local fractional integral of f on the interval $[a,b]$ of order α (denoted by ${}_aI_b^{(\alpha)} f(x)$) is defined by*
$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha.$$

The above definition implies that ${}_aI_b^{(\alpha)}f(x) = 0$ *if* $a = b$, *and* ${}_aI_b^{(\alpha)}f(x) = -{}_bI_a^{(\alpha)}f(x)$ *if* $a < b$. *If for any* $x \in [a,b]$, *there exists* ${}_aI_x^{(\alpha)}f(x)$, *then we denote by* $f(x) \in I_x^{(\alpha)}[a,b]$.

At the end of this summary, we give some useful formulas:

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}, \quad k > 0;$$
$$\frac{d^\alpha E_\alpha((cx)^\alpha)}{dx^\alpha} = c^\alpha E_\alpha((cx)^\alpha),$$

where $E_\alpha(\cdot)$ denotes the Mittag–Leffler function given by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)};$$
$$\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha);$$
$$\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}(b^{(k+1)\alpha} - a^{(k+1)\alpha}), \quad k > 0;$$
$$B_\alpha(a,b) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{x^{\alpha(b-1)}}{(1^\alpha + x^\alpha)^{a+b}}(dx)^\alpha,$$

where $B_\alpha(a,b)$ denotes local fractional Beta function.

From now on we use the following abbreviated notations:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad (d\mathbf{x})^\alpha = \prod_{i=1}^{n}(dx_i)^\alpha, \quad (\hat{d^i}\mathbf{x})^\alpha = \prod_{j=1, j\neq i}^{n}(dx_j)^\alpha.$$

Remind the reader: the function $K \in C_\alpha(a,b)$ is said to be homogeneous of degree $-\alpha\lambda$, $\lambda > 0$, if $K(t\mathbf{x}) = t^{-\alpha\lambda}K(\mathbf{x})$ for all $t > 0$. Furthermore, for $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{c}) = {}_a I_b^{((n-1)\alpha)} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{\alpha c_j}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\hat{\mathbf{u}}^i = (u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n)$, and provided that the above integral converges.

In the continuation of the research, we will use the following result of Krnić and Vuković from [19]: Let $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \ldots, n$, and let $K \in C_\alpha(0,\infty)^n$ be a non-negative function homogeneous function of degree $-\alpha s$, $s > 0$. If $f_i \in C_\alpha(0,\infty)$ are non-negative functions then hold the inequality

$$\frac{1}{\Gamma^n(1+\alpha)} \int_{(0,\infty)^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)(d\mathbf{x})^\alpha \tag{3}$$
$$\leq L \prod_{i=1}^{n} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x_i^{\alpha(p_i - s - 1)} f_i^{p_i}(x_i)(dx_i)^\alpha \right)^{\frac{1}{p_i}},$$

where $L = k_1(\frac{s-p_2}{p_2}, \frac{s-p_3}{p_3}, \ldots, \frac{s-p_n}{p_n})$ is the best possible.

The paper is divided into two sections as follows: in the introductory part, in Section 1 we give a brief overview of basic definitions and properties of the local fractional calculus. In Section 2, we derive our main result—a fractal Hilbert-type inequality with a general kernel and weight functions. In addition, we obtained the best possible constants in terms of local fractional hypergeometric function.

2. Main Results

Below we introduce some lemmas necessary to prove our main result.

Lemma 1. *Let $s > 0$ and $b > a \geq 0$. Let $\beta_1, \beta_2 > -1$, $\beta_1 + \beta_2 < s - 2$ and*

$$k_1(\beta_1, \beta_2) := \frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty \frac{t_1^{\beta_1 \alpha} t_2^{\beta_2 \alpha}}{(a \min\{1, t_1, t_2\} + b \max\{1, t_1, t_2\})^{\alpha s}} (dt_1)^\alpha (dt_2)^\alpha.$$

Then

$$\begin{aligned} k_1(\beta_1, \beta_2) &= b^{-\alpha s} \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i \alpha)}{\Gamma(1+(\beta_i+1)\alpha)} B_\alpha(\beta_{i+1}+1, 1) \\ &\times {}_2F_1^\alpha\left(s, \beta_{i+1}+1; \beta_{i+1}+2; -\frac{a}{b}\right) \\ &- b^{-\alpha s} B_\alpha(\beta_1+\beta_2+2, 1)\, {}_2F_1^\alpha\left(s, \beta_1+\beta_2+2; \beta_1+\beta_2+3; -\frac{a}{b}\right) \\ &\times \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i \alpha)}{\Gamma(1+(\beta_i+1)\alpha)} \\ &+ b^{-\alpha s} B_\alpha(s-\beta_1-\beta_2-2, 1)\, {}_2F_1^\alpha\left(s, s-\beta_1-\beta_2-2; s-\beta_1-\beta_2-1; -\frac{a}{b}\right) \\ &\times \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i \alpha)}{\Gamma(1+(\beta_i+1)\alpha)} \\ &- b^{-\alpha s} \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i \alpha)}{\Gamma(1+(\beta_i+1)\alpha)} B_\alpha(s-\beta_{i+1}-1, 1) \\ &\times {}_2F_1^\alpha\left(s, s-\beta_{i+1}-1; s-\beta_{i+1}; -\frac{a}{b}\right) \\ &+ b^{-\alpha s} \sum_{i=1}^{2} \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_i^{\beta_i \alpha}\, {}_2F_1^\alpha\left(s, s-\beta_{i+1}-1; s-\beta_{i+1}; -\frac{a}{b} t_i\right) (dt_i)^\alpha, \end{aligned} \tag{4}$$

where ${}_2F_1^\alpha(a, b; c; z)$ denotes the local fractional hypergeometric function defined by

$${}_2F_1^\alpha(a, b; c; z) = \frac{1}{B_\alpha(b, c-b)} \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{(b-1)\alpha} (1-t)^{(c-b-1)\alpha} (1-zt)^{-a\alpha} (dt)^\alpha,$$

$c > b > 0$, $|z| \leq 1$. *The indices are taken modulo 2.*

Proof. Consider the identity $k_1(\beta_1, \beta_2) = I_1 + I_2$, where

$$I_1 = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1 \alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{t_2^{\beta_2 \alpha}}{(a \min\{t_1, t_2\} + b \max\{1, t_2\})^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha$$

and

$$I_2 = \frac{1}{\Gamma(1+\alpha)} \int_1^\infty t_1^{\beta_1 \alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{t_2^{\beta_2 \alpha}}{(a \min\{1, t_2\} + b \max\{t_1, t_2\})^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha.$$

In what follows we shall express the integral I_1 by a local fractional hypergeometric function. Further, set $I_1 = I_{11} + I_{12}$, where

$$I_{11} = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1 \alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{t_2^{\beta_2 \alpha}}{(a \min\{t_1, t_2\} + b)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha$$

and

$$I_{12} = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_1^\infty \frac{t_2^{\beta_2\alpha}}{(at_1+bt_2)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha.$$

In view of the above identities, the integral I_{11} can be written as

$$\begin{aligned} I_{11} = &\frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{t_1} \frac{t_2^{\beta_2\alpha}}{(at_2+b)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha \\ &+ \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^1 \frac{t_2^{\beta_2\alpha}}{(at_1+b)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha. \end{aligned} \tag{5}$$

Applying classical real analysis we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{t_1} \frac{t_2^{\beta_2\alpha}}{(at_2+b)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_2^{\beta_2\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_{t_2}^1 \frac{t_1^{\beta_1\alpha}}{(at_2+b)^{\alpha s}} (dt_1)^\alpha \right) (dt_2)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_2^{\beta_2} (at_2+b)^{-\alpha s} \left(\frac{1}{\Gamma(1+\alpha)} \int_{t_2}^1 t_1^{\beta_1\alpha} (dt_1)^\alpha \right) (dt_2)^\alpha \\ &= \frac{b^{-\alpha s}\Gamma(1+\beta_1\alpha)}{\Gamma(1+(\beta_1+1)\alpha)} \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_2^{\beta_2\alpha} (1^\alpha - t_2^{(\beta_1+1)\alpha}) \left(1+\frac{a}{b}t_2\right)^{-\alpha s} (dt_2)^\alpha \\ &= \frac{b^{-\alpha s}\Gamma(1+\beta_1\alpha)}{\Gamma(1+(\beta_1+1)\alpha)} \left[B_\alpha(\beta_2+1,1)\, {}_2F_1^\alpha\left(s,\beta_2+1;\beta_2+2;-\frac{a}{b}\right) \right. \\ &\quad \left. - B_\alpha(\beta_1+\beta_2+2,1)\, {}_2F_1^\alpha\left(s,\beta_1+\beta_2+2;\beta_1+\beta_2+3;-\frac{a}{b}\right) \right], \end{aligned} \tag{6}$$

and similarly

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^1 \frac{t_2^{\beta_2\alpha}}{(at_1+b)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha \\ &= \frac{b^{-\alpha s}\Gamma(1+\beta_2\alpha)}{\Gamma(1+(\beta_2+1)\alpha)} \left[B_\alpha(\beta_1+1,1)\, {}_2F_1^\alpha\left(s,\beta_2+1;\beta_2+2;-\frac{a}{b}\right) \right. \\ &\quad \left. - B_\alpha(\beta_1+\beta_2+2,1)\, {}_2F_1^\alpha\left(s,\beta_1+\beta_2+2;\beta_1+\beta_2+3;-\frac{a}{b}\right) \right]. \end{aligned} \tag{7}$$

Applying (6), (7) in (5), we obtain

$$\begin{aligned} I_{11} = &b^{-\alpha s} \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i\alpha)}{\Gamma(1+(\beta_i+1)\alpha)} B_\alpha(\beta_{i+1}+1,1) \\ &\times {}_2F_1^\alpha\left(s,\beta_{i+1}+1;\beta_{i+1}+2;-\frac{a}{b}\right) \\ &- b^{-\alpha s} B_\alpha(\beta_1+\beta_2+2,1)\, {}_2F_1^\alpha\left(s,\beta_1+\beta_2+2;\beta_1+\beta_2+3;-\frac{a}{b}\right) \\ &\times \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i\alpha)}{\Gamma(1+(\beta_i+1)\alpha)}. \end{aligned} \tag{8}$$

In addition, letting $u = 1/t_2$, we find

$$I_{12} = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_1^\infty \frac{t_2^{\beta_2\alpha}}{(at_1+bt_2)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha \tag{9}$$
$$= \frac{b^{-\alpha s}}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 u^{\alpha s - \beta_2\alpha - 2\alpha} \left(1 + \frac{a}{b} t_1 u\right)^{-\alpha s} (du)^\alpha \right) (dt_1)^\alpha$$
$$= \frac{b^{-\alpha s}}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \, {}_2F_1^\alpha\left(s, s-\beta_2-1; s-\beta_2; -\frac{a}{b} t_1\right) (dt_1)^\alpha.$$

Furthemore, from (8) and (9) we get

$$\begin{aligned} I_1 &= I_{11} + I_{12} \\ &= b^{-\alpha s} \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i\alpha)}{\Gamma(1+(\beta_i+1)\alpha)} B_\alpha(\beta_{i+1}+1, 1) \\ &\quad \times {}_2F_1^\alpha\left(s, \beta_{i+1}+1; \beta_{i+1}+2; -\frac{a}{b}\right) \\ &\quad - b^{-\alpha s} B_\alpha(\beta_1+\beta_2+2, 1) \, {}_2F_1^\alpha\left(s, \beta_1+\beta_2+2; \beta_1+\beta_2+3; -\frac{a}{b}\right) \\ &\quad \times \sum_{i=1}^{2} \frac{\Gamma(1+\beta_i\alpha)}{\Gamma(1+(\beta_i+1)\alpha)} \\ &\quad + \frac{b^{-\alpha s}}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \, {}_2F_1^\alpha\left(s, s-\beta_2-1; s-\beta_2; -\frac{a}{b} t_1\right) (dt_1)^\alpha. \end{aligned}$$

At last, if we continue with the described procedure for the integrals

$$I_{21} = \frac{1}{\Gamma(1+\alpha)} \int_1^\infty t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{t_2^{\beta_2\alpha}}{(at_2+bt_1)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha$$

and

$$I_{22} = \frac{1}{\Gamma(1+\alpha)} \int_1^\infty t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_1^\infty \frac{t_2^{\beta_2\alpha}}{(a+b\max\{t_1,t_2\})^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha,$$

such that $I_{21} + I_{22} = I_2$, we get the identity (4). □

For the homogeneous function of degree $-\alpha s$, $K(x_1,x_2,x_3) = (a\min\{x_1,x_2,x_3\} + b\max\{x_1,x_2,x_3\})^{-\alpha s}$, $b > a \geq 0$, combining (3) and Lemma 1 we have the following result.

Theorem 1. *If $f_i \in C_\alpha(0,\infty)$, $i = 1,2,3$, are non-negative functions then hold the inequality*

$$\frac{1}{\Gamma^3(1+\alpha)} \int_{(0,\infty)^3} \frac{f_1(x_1)f_2(x_2)f_3(x_3)}{(a\min\{x_1,x_2,x_3\} + b\max\{x_1,x_2,x_3\})^{\alpha s}} (dx_1)^\alpha (dx_2)^\alpha (dx_3)^\alpha$$
$$\leq L_1 \prod_{i=1}^{3} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x_i^{\alpha(p_i-s-1)} f_i^{p_i}(x_i) (dx_i)^\alpha \right)^{\frac{1}{p_i}}, \tag{10}$$

where

$$
\begin{aligned}
L_1 = {} & b^{-\alpha s}\sum_{i=2}^{3}\frac{\Gamma(1+(\frac{s}{p_i}-1)\alpha)}{\Gamma(1+\frac{s}{p_i}\alpha)}B_\alpha\left(\frac{s}{p_{i+1}},1\right) \\
& \times {}_2F_1^\alpha\left(s,\frac{s}{p_{i+1}};\frac{s}{p_{i+1}}+1;-\frac{a}{b}\right) \\
& -b^{-\alpha s}B_\alpha\left(s-\frac{s}{p_1},1\right){}_2F_1^\alpha\left(s,s-\frac{s}{p_1};s-\frac{s}{p_1}+1;-\frac{a}{b}\right) \\
& \times\sum_{i=2}^{3}\frac{\Gamma(1+(\frac{s}{p_i}-1)\alpha)}{\Gamma(1+\frac{s}{p_i}\alpha)} \\
& +b^{-\alpha s}B_\alpha\left(\frac{s}{p_1},1\right){}_2F_1^\alpha\left(s,\frac{s}{p_1};\frac{s}{p_1}+1;-\frac{a}{b}\right) \\
& \times\sum_{i=2}^{3}\frac{\Gamma(1+(\frac{s}{p_i}-1)\alpha)}{\Gamma(1+\frac{s}{p_i}\alpha)} \\
& -b^{-\alpha s}\sum_{i=2}^{3}\frac{\Gamma(1+(\frac{s}{p_i}-1)\alpha)}{\Gamma(1+\frac{s}{p_i}\alpha)}B_\alpha\left(\frac{s}{p_1}+\frac{s}{p_i}+1,1\right) \\
& \times {}_2F_1^\alpha\left(s,\frac{s}{p_1}+\frac{s}{p_i}+1;\frac{s}{p_1}+\frac{s}{p_i};-\frac{a}{b}\right) \\
& +b^{-\alpha s}\sum_{i=2}^{3}\frac{1}{\Gamma(1+\alpha)}\int_0^1 t_i^{\alpha\frac{s-p_i}{p_i}}\,{}_2F_1^\alpha\left(s,\frac{s}{p_1}+\frac{s}{p_i}+1;\frac{s}{p_1}+\frac{s}{p_i};-\frac{a}{b}t_i\right)(dt_i)^\alpha
\end{aligned} \tag{11}
$$

and the indices are taken modulo 2. The constant L_1 is the best possible in inequality (10).

Proof. The proof follows easily from inequality (3). Namely, setting the kernel $K(x_1,x_2,x_3)$ $= (a\min\{x_1,\,x_2,x_3\}+b\max\{x_1,x_2,x_3\})^{-\alpha s}$ in (3), it is enough to calculate the constant $L_1=k_1\left(\frac{s-p_2}{p_2},\frac{s-p_3}{p_3}\right)$. For this purpose, by using Lemma 1 we get (11). □

For the next application of Theorem 1, we need a technical lemma.

Lemma 2. *Let $s>0$, $\beta_1,\beta_2>-1$, $\beta_1+\beta_2<s-2$ and*

$$k_2(\beta_1,\beta_2):=\frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty\frac{t_1^{\beta_1\alpha}t_2^{\beta_2\alpha}}{(1+t_1+t_2-\min\{1,t_1,t_2\})^{\alpha s}}(dt_1)^\alpha(dt_2)^\alpha.$$

Then

$$
\begin{aligned}
k_2(\beta_1,\beta_2) = {} & B_\alpha(\beta_1+\beta_2+2,1)\,{}_2F_1^\alpha(s,\beta_1+\beta_2+2;\beta_1+\beta_2+3;-1) \\
& \times\sum_{i=1}^{2}\frac{\Gamma(1+\beta_i\alpha)}{\Gamma(1+(\beta_i+1)\alpha)} \\
& + B_\alpha(s-\beta_1-\beta_2-2,1)\,{}_2F_1^\alpha(s,s-\beta_1-\beta_2-2;s-\beta_1-\beta_2-1;-1) \\
& \times\sum_{i=1}^{2}\frac{\Gamma(1+\beta_i\alpha)}{\Gamma(1+(\beta_i+1)\alpha)}.
\end{aligned} \tag{12}
$$

Proof. As in the proof of Lemma 1 we start with the identity $k_2(\beta_1,\beta_2)=I_1+I_2$, where

$$I_1=\frac{1}{\Gamma(1+\alpha)}\int_0^1 t_1^{\beta_1\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\frac{t_2^{\beta_2\alpha}}{(1+t_1+t_2-\min\{t_1,t_2\})^{\alpha s}}(dt_2)^\alpha\right)(dt_1)^\alpha$$

and

$$I_2 = \frac{1}{\Gamma(1+\alpha)} \int_1^\infty t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{t_2^{\beta_2\alpha}}{(1+t_1+t_2-\min\{t_1,t_2\})^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha.$$

Further, we have $I_1 = I_{11} + I_{12}$, where

$$I_{11} = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{t_2^{\beta_2\alpha}}{(1+t_1+t_2-\min\{t_1,t_2\})^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha$$

and

$$I_{12} = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_1^\infty \frac{t_2^{\beta_2\alpha}}{(1+t_2)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha.$$

Easiliy we can get

$$\begin{aligned} I_{11} =& \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{t_1} \frac{t_2^{\beta_2\alpha}}{(1+t_1)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha \\ &+ \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^1 \frac{t_2^{\beta_2\alpha}}{(1+t_2)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha. \end{aligned} \tag{13}$$

Applying local fractional calculus we obtain

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{t_1} \frac{t_2^{\beta_2\alpha}}{(1+t_1)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1}(1+t_1)^{-\alpha s} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{t_1} t_2^{\beta_2\alpha} (dt_2)^\alpha \right) (dt_1)^\alpha \\ &= \frac{\Gamma(1+\beta_2\alpha)}{\Gamma(1+(\beta_2+1)\alpha)} \frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{(\beta_1+\beta_2+1)\alpha}(1+t_1)^{-\alpha s}(dt_1)^\alpha \\ &= \frac{\Gamma(1+\beta_2\alpha)}{\Gamma(1+(\beta_2+1)\alpha)} B_\alpha(\beta_1+\beta_2+2,1)\ {}_2F_1^\alpha(s,\beta_1+\beta_2+2;\beta_1+\beta_2+3;-1), \end{aligned} \tag{14}$$

and similarly

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 t_1^{\beta_1\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^1 \frac{t_2^{\beta_2\alpha}}{(1+t_2)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha \\ &= \frac{\Gamma(1+\beta_1\alpha)}{\Gamma(1+(\beta_1+1)\alpha)} B_\alpha(\beta_1+\beta_2+2,1)\ {}_2F_1^\alpha(s,\beta_1+\beta_2+2;\beta_1+\beta_2+3;-1). \end{aligned} \tag{15}$$

From (13)–(15) we get

$$\begin{aligned} I_{11} =& B_\alpha(\beta_1+\beta_2+2,1)\ {}_2F_1^\alpha(s,\beta_1+\beta_2+2;\beta_1+\beta_2+3;-1) \\ &\times \sum_{i=1}^2 \frac{\Gamma(1+\beta_i\alpha)}{\Gamma(1+(\beta_i+1)\alpha)}. \end{aligned}$$

By using (9) and the substitution $u = \frac{1}{t_2}$ (see also proof of Lemma 1) we have

$$I_{12} = \frac{\Gamma(1+\beta_1\alpha)}{\Gamma(1+(\beta_1+1)\alpha)} B_\alpha(s-\beta_2-1,1)\ {}_2F_1^\alpha(s,s-\beta_2-1;s-\beta_2;-1).$$

In the same way we obtain the expresion of integral $I_2 = I_{21} + I_{22}$, where

$$I_{21} = \frac{1}{\Gamma(1+\alpha)} \int_1^\infty t_1^{\beta_1 \alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{t_2^{\beta_2 \alpha}}{(1+t_1)^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha$$

and

$$I_{22} = \frac{1}{\Gamma(1+\alpha)} \int_1^\infty t_1^{\beta_1 \alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_1^\infty \frac{t_2^{\beta_2 \alpha}}{(1+t_1+t_2-\min\{t_1,t_2\})^{\alpha s}} (dt_2)^\alpha \right) (dt_1)^\alpha.$$

□

If we put $K(x_1, x_2, x_3) = (x_1 + x_2 + x_3 - \min\{x_1, x_2, x_3\})^{-\alpha s}$, $s > 0$, in Theorem 1 and applying Lemma 2 we get the following result.

Corollary 1. *If $f_i \in C_\alpha(0,\infty)$, $i = 1,2,3$, are non-negative functions then holds the inequality*

$$\frac{1}{\Gamma^3(1+\alpha)} \int_{(0,\infty)^3} \frac{f_1(x_1) f_2(x_2) f_3(x_3)}{(x_1+x_2+x_3-\min\{x_1,x_2,x_3\})^{\alpha s}} (dx_1)^\alpha (dx_2)^\alpha (dx_3)^\alpha \leq L_2 \prod_{i=1}^{3} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x_i^{\alpha(p_i - s - 1)} f_i^{p_i}(x_i) (dx_i)^\alpha \right)^{\frac{1}{p_i}},$$

where the constant

$$\begin{aligned} L_2 = {} & B_\alpha\left(s - \frac{s}{p_1}, 1\right) {}_2F_1^\alpha\left(s, s - \frac{s}{p_1}; s - \frac{s}{p_1} + 1; -1\right) \\ & \times \sum_{i=2}^{3} \frac{\Gamma(1 + (\frac{s}{p_i} - 1)\alpha)}{\Gamma(1 + \frac{s}{p_i}\alpha)} \\ & + B_\alpha\left(\frac{s}{p_1}, 1\right) {}_2F_1^\alpha\left(s, \frac{s}{p_1}; \frac{s}{p_1} + 1; -1\right) \\ & \times \sum_{i=2}^{3} \frac{\Gamma(1 + (\frac{s}{p_i} - 1)\alpha)}{\Gamma(1 + \frac{s}{p_i}\alpha)} \end{aligned}$$

is the best possible.

3. Conclusions

In this paper, motivated by the mentioned papers, we obtained some new Hilbert-type inequalities via the local fractional integrals. As an application, a Hilbert-type inequality with constant in terms of local fractional hypergeometric function is also presented. In order to obtain the best possible constant in terms of local fractional hypergeometric function, the technique that combines local fractional calculus and new results related to Hilbert-type inequalities in a local fractional sense was applied.

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