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A Fractional Chemotaxis Navier–Stokes System with Matrix-Valued Sensitivities and Attractive–Repulsive Signals

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Abstract: In this paper, we considered a fractional chemotaxis fluid system with matrix-valued sensitivities and attractive–repulsive signals on a two-dimensional periodic torus $\mathbb{T}^2$. This model describes the interaction between a type of cell that proliferates following a logistic law, and the diffusion of cells is fractional Laplace diffusion. The cells and attractive–repulsive signals are transported by a viscous incompressible fluid under the influence of a force due to the aggregation of cells. We proved the existence and uniqueness of the global classical solution on the matrix-valued sensitivities, and the initial data satisfied the regular conditions. Moreover, by using energy functionals, the stabilization of global bounded solutions of the system was proven.

Keywords: chemotaxis Navier–Stokes; matrix-valued sensitivities; fractional diffusion; global classical solution; asymptotic stability

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1. Introduction

We investigated a fractional chemotaxis fluid system with matrix-valued sensitivities and attractive–repulsive signals on a two-dimensional periodic torus $\mathbb{T}^2 = [-\pi, \pi]^2$ in the present paper. This model describes the interaction between a type of cell that can proliferate following a logistic law, and the diffusion of cells is fractional Laplace diffusion. The cells and attractive–repulsive signals are transported by a viscous incompressible fluid under the influence of a force due to the aggregation of cells, where the attractive signal and the repulsive signal are produced by the cells themselves and also degrade at a constant rate. This model is represented by the following system:

$$\begin{cases} c_t + \mathbf{u} \cdot \nabla c = -(-\Delta)^\alpha c - \nabla \cdot (cS_1(x,c,v)\nabla v) + \nabla \cdot (cS_2(x,c,w)\nabla w) & \\ +\mu c - \nu c^2, & \text{in } \mathbb{T}^2 \times (0,\infty), \\ v_t + \mathbf{u} \cdot \nabla v = \Delta v + \alpha_1 c - \beta_1 v, & \text{in } \mathbb{T}^2 \times (0,\infty), \\ w_t + \mathbf{u} \cdot \nabla w = \Delta w + \alpha_2 c - \beta_2 w, & \text{in } \mathbb{T}^2 \times (0,\infty), \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta \mathbf{u} - \nabla P + c\nabla\phi + g, & \text{in } \mathbb{T}^2 \times (0,\infty), \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \mathbb{T}^2 \times (0,\infty), \\ c(x,0) = n_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), \mathbf{u}(x,0) = \mathbf{u}_0(x), & \text{in } \mathbb{T}^2. \end{cases} \tag{1}$$

$c(x,t)$ is the cell density; $v(x,t)$, $w(x,t)$, P, and $\mathbf{u}(x,t)$ represent the chemical concentrations of the attractant and repellent, the hydrostatic pressure, and the velocity field of the fluid, respectively. $S_1(x,c,v) = (s'_{ij}(x,c,v))_{i,j\in\{1,2\}}$ and $S_2(x,c,w) = (s''_{ij}(x,n,w))_{i,j\in\{1,2\}}$ are matrix-valued sensitivities functions, and we imposed the conditions:

$$S_1(x,c,v),\ S_2(x,c,w) \in C^2(\mathbb{T}^2 \times [0,\infty) \times [0,\infty); \mathbb{R}^{2\times 2}) \tag{2}$$

and

$$|S_1(x,c,v)| \le C_{S_1},\ \ |S_2(x,c,w)| \le C_{S_2} \tag{3}$$

for some positive constants C_{S_i} $(i = 1, 2)$. $\phi(x)$ and $g(x,t)$ are the gravitational potential function and the external force, respectively, which satisfy

$$\begin{cases} \phi \in W^{1,\infty}(\mathbb{T}^2), \\ g \in C^1(\mathbb{T}^2 \times [0,\infty)) \cap L^\infty(\mathbb{T}^2 \times (0,\infty)) \cap L^2(\mathbb{T}^2 \times (0,\infty)), \end{cases} \tag{4}$$

as well as the initial data:

$$\begin{cases} c_0 \in L^\infty(\mathbb{T}^2), & n_0 > 0 \text{ in } \mathbb{T}^2, \\ v_0 \in W^{1,\infty}(\mathbb{T}^2), & v_0 \geq 0 \text{ in } \mathbb{T}^2, \\ w_0 \in W^{1,\infty}(\mathbb{T}^2), & w_0 \geq 0 \text{ in } \mathbb{T}^2, \\ \mathbf{u}_0 \in L^\infty(\mathbb{T}^2),\ \nabla \cdot \mathbf{u}_0 = 0, & \text{in } \mathbb{T}^2. \end{cases} \tag{5}$$

Here, we write $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ and define

$$\widehat{\Lambda^\alpha c(\xi)} = |\xi|^\alpha \hat{c}(\xi),$$

where $\hat{\cdot}$ represents the usual Fourier transform, and the differential operator Λ^α has the following kernel representation:

$$\Lambda^\alpha c(x) = C_{\alpha,2}\,\text{P.V.} \int_{\mathbb{T}^2} \frac{c(x) - c(y)}{|x-y|^{2+\alpha}} \mathrm{d}y + C_{\alpha,2} \sum_{n \in \mathbb{Z}^2, n \neq 0} \int_{\mathbb{T}^2} \frac{c(x) - c(y)}{|x - y + 2n\pi|^{2+\alpha}} \mathrm{d}y,$$

where $C_{\alpha,2} = \frac{2^\alpha \Gamma(\frac{2+\alpha}{2})}{\pi \left|\Gamma(-\frac{\alpha}{2})\right|} > 0$ is a normalization constant. The notation P.V. means that the integral is taken in the Cauchy principle value sense (see, for example, [1]). The positive constants α_1, α_2, β_1, and β_2 denote the production of the chemoattractant and chemorepellent, the chemoattractant's decay, and the chemorepellent's decay, respectively. The logistic source $\mu c - \nu c^2$ describes the local dynamics of the mobile species, where $\mu \geq 0$ is the intrinsic growth rate of cells and $\nu > 0$ is the intraspecific competition of cells.

Next, we discuss the inspirations and developments of Problem (1) and, finally, list the main results.

1.1. The Classical Chemotaxis System with Attractive–Repulsive Signals

Chemotaxis refers to the movement of cells towards the concentration gradient of chemicals in a certain environment. The the movement of cells in the direction of the increasing concentration of a signal is called chemotactic attraction, whereas chemotactic repulsion means that cells move along the decreasing concentration of a cue [2–5]. These results have led some authors to consider the following attraction–repulsion chemotaxis model with a logistic source:

$$\begin{cases} c_t = \Delta c - \nabla \cdot (cS_1(x,c,v)\nabla v) + \nabla \cdot (cS_2(x,c,w)\nabla v) + f(c), & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + \alpha_1 c - \beta_1 v, & \text{in } \Omega \times (0,\infty), \\ w_t = \Delta w + \alpha_2 c - \beta_2 w, & \text{in } \Omega \times (0,\infty), \end{cases} \tag{6}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. If $S_1(x,c,v) = const = \chi$ and $S_2(x,c,w) = const = \xi$, Li et al. [6,7] proved the existence of a unique global classical solution for (6) in bounded domains of $\mathbb{R}^N$, $N = 1, 2$. If $N = 3$, $f(c) = c - \mu c^{1+\gamma}$, $c \geq 0$, $\mu > 0$, and $\gamma \geq 1$. Li et al. [8] showed that (6) possesses a unique global bounded classical solution under the conditions $\beta_1, \beta_2 \geq \frac{1}{2}$ and $\mu \geq \max\{(\frac{41}{2}\chi\alpha_1 + 9\xi\alpha_2)^\gamma, (9\chi\alpha_1 + \frac{41}{2}\xi\alpha_2)^\gamma\}$. Moreover, whenever $c_0 \not\equiv 0$ and for any $\gamma \in N$, the solution of the system approaches the steady state $\left((\frac{1}{\mu})^{\frac{1}{\gamma}}, (\frac{1}{\mu})^{\frac{1}{\gamma}} \frac{\alpha_1}{\beta_1}, (\frac{1}{\mu})^{\frac{1}{\gamma}} \frac{\alpha_2}{\beta_2}\right)$ in the norm of $L^\infty(\Omega)$ as $t \to \infty$. Furthermore, for $N \geq 3$, Zheng, Mu, and Hu [9] proved that the system admits a unique global bounded classical solution provided that $f(c) \leq \mu c - \nu c^2$

with $\mu \geq 0$, $\nu > 0$, and $\beta_1 = \beta_2$ and there exists $\theta_0 > 0$ such that $\frac{\chi\alpha_1 + \xi\alpha_2}{\nu} > \theta_0$. More recently, the research results of Shi, Liu, and Jin [10] implied that, when repulsion cancels attraction (i.e., $\chi\alpha_1 = \xi\alpha_2$), the logistic source plays an important role in the solution behavior of (6).

1.2. The Classical Chemotaxis (Navier) Stokes System with Matrix-Valued Sensitivities

More recent observations have shown that chemotactic migration does not necessarily follow the gradient of the direction of a chemical substance, but may involve rotating flux components. This requires the sensitivity function S_i $(i = 1, 2)$ to be a matrix possibly containing nontrivial off-diagonal entries [11]. Adjusting the classical model accordingly, some scientists shall subsequently consider the chemotaxis (Navier) Stokes system involving matrix-valued sensitivities:

$$\begin{cases} c_t + \mathbf{u} \cdot \nabla c = \Delta c - \nabla \cdot (cS(x,c,v)\nabla v) + f(c), & \text{in } \Omega \times (0,\infty), \\ v_t + \mathbf{u} \cdot \nabla v = \Delta v + c - v, & \text{in } \Omega \times (0,\infty), \\ \mathbf{u}_t + \kappa(\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta \mathbf{u} + \nabla P + c\nabla\phi, \ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \times (0,\infty), \\ \nabla v \cdot \nu = (D(c)\nabla c - cS(x,c,v)\nabla v) \cdot \nu = 0, \ \mathbf{u} = 0, & \text{in } \partial\Omega \times (0,\infty). \end{cases} \tag{7}$$

Next, we briefly introduce some previous results for (7) in the literature. For the case $|S(x,c,v)| \leq C_S$: if $f(c) \equiv 0$ and $\mathbf{u} \equiv 0$, Cao [12] asserted that the above problem possesses a global classical solution, which is bounded and converges to the constant steady state $(\frac{\int_\Omega c_0}{|\Omega|}, \frac{\int_\Omega c_0}{|\Omega|})$, for which $\|c_0\|_{L^{\frac{N}{2}}(\Omega)}$ and $\|v_0\|_{L^N(\Omega)}$ $(N \geq 2)$ are sufficiently small. If $+c - v$ of $(7)_2$ is replaced by $-vc$ and $N = 2, 3$, Cao and Lankeit [13] showed that (7) has a global classical solution if the initial data satisfy certain smallness conditions and give decay properties of these solution. By applying the results of [13], Cao [14] proved that, under a mild assumption $|S(x,c,v)| \leq S_0(v_0)$ with some non-decreasing function $S_0 \in C^2((0,\infty))$, the chemotaxis (Navier) Stokes system has a global classical solution under a smallness assumption on $\|v_0\|_{L^\infty(\Omega)}$, and moreover, they obtained the boundedness and large time convergence for the solution. For the case $|S(x,c,v)| \leq C_S(1+c)^{-\gamma}$: if $f(c) \equiv 0$ and $\mathbf{u} \not\equiv 0$, when $N = 2$ and $\kappa = 0$, Wang and Xiang [15] established the existence of a global bounded classical solution for arbitrarily large initial data. Moreover, Wang et al. [16] extended the result of [15] to the chemotaxis Navier–Stokes model with $0 < \gamma < \frac{1}{2}$. When $N = 3$, Wang and Xiang [17] developed a method to establish the existence and boundedness of a global classical solution of (7) under the assumption $\kappa = 0$ and $\gamma > \frac{1}{2}$. Meanwhile, for $\kappa = 1$ and $\gamma > \frac{1}{3}$, Wang [18] defined a weak solution, which requires the solution to satisfy very mild regularity hypotheses only, and they obtained that (7) has a global weak solution. Furthermore, if $\kappa \neq 0$ and $\gamma \geq \frac{3}{7}$, Liu and Wang [19] proved that (7) admits at least one global weak solution. Recently, Ke and Zheng [20] improved the above results, and they optimized the parametric conditions that $\kappa \in \mathbb{R}$ and $\gamma > \frac{1}{3}$. If $f(c) = \mu c - \nu c^2$, $\mathbf{u} \equiv 0$, and Δc are replaced by $\nabla \cdot (c^{m-1}\nabla c)$, Yi et al. [21] considered an attraction–repulsion chemotaxis model with matrix-valued sensitivities, for $S_1(x,c,v) \leq C_{S_1}(1+c)^{-\gamma_1}$ and $S_2(x,c,w) \leq C_{S_2}(1+c)^{-\gamma_2}$, and they proved that, under the conditions $m > 0$ and $\min\{m + 2\gamma_1, m + 2\gamma_2\} > \frac{2N}{N+2}$, the corresponding initial boundary value problem possessed at least one global bounded weak solution. For more results about matrix-valued sensitivities, interested readers can refer to [22–29] for more details.

1.3. The Fractional Chemotaxis System

By recent research, we know that, in nature, the behavior of many organisms can no longer be accurately described by classical chemotactic models. The research results of Garfinkel et al. [30] showed that mesenchymal cells move due to the attraction of certain chemicals, which does not fit the classical chemotaxis model. Therefore, Escudero [31] improved the classical chemotaxis model by replacing the classical Laplace diffusion with the fractional Laplace diffusion. Since then, scientists began to use fractional operators to describe the diffusion of cells. In recent years, fractional chemotaxis models have been studied extensively by scientists. Among them, it is worth noting that Burczak and Granero-Belinchón [32–37] conducted a series of studies on the fractional chemotaxis system on the periodic torus $\mathbb{T}^N = [-\pi, \pi]^N$. The readers can refer to [38–45] and the references therein for more details.

To the best of our knowledge, there are few studies on fractional chemotaxis Navier–Stokes models. In 2019, Zhu et al. [46] dealt with a fractional chemotaxis fluid model in $\mathbb{R}^3$. They obtained the existence, uniqueness, and asymptotic stability of a global solution without a logistic source and with small initial data. Jiang et al. [47] investigated a fractional double-chemotaxis model under the effect of the Navier–Stokes fluid in $\mathbb{R}^N$, $N \geq 3$. They developed a framework for a *unified* treatment of the existence, uniqueness, and decay estimates of the global mild solution to this problem under the assumption that the initial data were small enough. Nie and Zheng [48] obtained the global-in-time existence and uniqueness of a weak solution to the equations for a class of large initial data of two-dimensional incompressible chemotaxis Navier–Stokes equations with the lower fractional diffusion. Recently, Lei et al. [49] investigated the following fractional chemotaxis fluid system with a logistic source:

$$\begin{cases} c_t + \mathbf{u} \cdot \nabla c = -(-\Delta)^\alpha c - \chi \nabla \cdot (c \nabla v) + \mu c - \nu c^2, & \text{in } \mathbb{T}^3 \times (0, \infty), \\ v_t + \mathbf{u} \cdot \nabla v = \Delta v - cv, & \text{in } \mathbb{T}^3 \times (0, \infty), \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta \mathbf{u} - \nabla P + c \nabla \phi + g, \quad \nabla \cdot u = 0, & \text{in } \mathbb{T}^3 \times (0, \infty), \end{cases} \tag{8}$$

on a three-dimensional periodic torus $\mathbb{T}^3$. They investigated the global existence of weak solutions of (8) in the case of a weaker diffusion, and after some waiting time, the weak solutions in fact become smooth and converge to the semi-trivial steady state $(\frac{\mu}{\nu}, 0, 0)$.

Inspired by [21,49], we investigated a fractional chemotaxis Navier–Stokes system with matrix-valued sensitivities and attractive–repulsive signals on a two-dimensional periodic torus $\mathbb{T}^2$ in the present paper. Compared with [49], we considered an attraction–repulsion chemotaxis phenomenon, where the attractive–repulsive signals are produced by the cells themselves and degrade at a constant rate. Since the periodic torus is a region without boundary, in comparison with [21,25], we did not need to consider the nonlinear boundary problem arising from the matrix-valued sensitivities $S_1(x, c, v)$ and $S_2(x, c, w)$. In addition, inspired by [49], some estimates of the solution can also be obtained, which plays a crucial role in proving the global existence of the solution by using the semigroup method. It is worth noting that, in comparison with [21,50], because of the existence of fractional diffusion term, when proving the boundedness of $\|c\|_{L^p(\mathbb{T}^2)}$, the fractional diffusion term $-p \int_{\mathbb{T}^2} c^{p-1}(-\Delta)^\alpha c$ cannot directly control the chemotaxis terms $-p \int_{\mathbb{T}^2} c^{p-1} \nabla \cdot (c S_1(x, c, v) \nabla v)$ and $+p \int_{\mathbb{T}^2} c^{p-1} \nabla \cdot (c S_2(x, c, w) \nabla w)$. Therefore, we used Lemma 4 for $(-\Delta)^\alpha c$ to handle the fractional diffusion term. Moreover, since $S_1(x, c, v)$ and $S_2(x, c, w)$ are matrix-valued sensitivity functions, we needed to deal with the chemotaxis terms by Parseval's identity and Kato–Ponce's commutator estimates, so that the chemotaxis terms can be controlled by the fractional diffusion term, thus obtaining the inequalities (87) and (88). Then, from some estimates in Lemmas 1 and 2, we arrived at the existence of the global solution of (1). Finally, we supposed that $\mu = 0$, then we have $\|c\|_{L^\infty(\mathbb{T}^2)} \to 0$, $\|v\|_{L^\infty(\mathbb{T}^2)} \to 0$, $\|w\|_{L^\infty(\mathbb{T}^2)} \to 0$ as $t \to \infty$. Meanwhile, when $\mu > 0$ and ν is sufficiently large, we have $\|c\|_{L^\infty(\mathbb{T}^2)} \to \frac{\mu}{\nu}$, $\|v\|_{L^\infty(\mathbb{T}^2)} \to \frac{\alpha_1}{\beta_1}\frac{\mu}{\nu}$, $\|w\|_{L^\infty(\mathbb{T}^2)} \to \frac{\alpha_2}{\beta_2}\frac{\mu}{\nu}$ as $t \to \infty$. Additionally, we found that, if $\int_0^\infty \|g\|^2_{L^2(\mathbb{T}^2)} dt < \infty$, in either case, $\mathbf{u}$ satisfies $\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)} \to 0$ as $t \to \infty$. Here, we list our main conclusions.

Theorem 1. *Assume that* $\frac{1}{2} < \alpha < 1$, $\mu \geq 0$, *and* $\nu > 0$. *Suppose that (2)–(5) hold. Then, System (1) possesses a global classical solution* $(c, v, w, \mathbf{u})$:

$$\begin{cases} c \in L^\infty(\mathbb{T}^2 \times [0,\infty)) \cap C^{2\alpha,1}(\mathbb{T}^2 \times (0,\infty)), \\ v \in L^\infty(\mathbb{T}^2 \times [0,\infty)) \cap C^{2,1}(\mathbb{T}^2 \times (0,\infty)), \\ w \in L^\infty(\mathbb{T}^2 \times [0,\infty)) \cap C^{2,1}(\mathbb{T}^2 \times (0,\infty)), \\ \mathbf{u} \in L^\infty(\mathbb{T}^2 \times [0,\infty)) \cap C^{2,1}(\mathbb{T}^2 \times (0,\infty)), \\ P \in C^{1,0}(\mathbb{T}^2 \times (0,\infty)) \end{cases}$$

in $\mathbb{T}^2 \times (0,\infty)$.

Theorem 2. *Assume that* $\frac{1}{2} < \alpha < 1$. *Suppose that (2)–(5) hold. Then, the solution of System (1) has*

$$\|c\|_{L^\infty(\mathbb{T}^2)} \to 0,\ \|v\|_{L^\infty(\mathbb{T}^2)} \to 0,\ \|w\|_{L^\infty(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty,$$

for $\mu = 0$ *and* $\nu > 0$.

Theorem 3. *Assume that* $\frac{1}{2} < \alpha < 1$. *Suppose that (2)–(5) hold. Then, the solution of System (1) has*

$$\|c\|_{L^\infty(\mathbb{T}^2)} \to \frac{\mu}{\nu},\ \|v\|_{L^\infty(\mathbb{T}^2)} \to \frac{\alpha_1}{\beta_1}\frac{\mu}{\nu},\ \|w\|_{L^\infty(\mathbb{T}^2)} \to \frac{\alpha_2}{\beta_2}\frac{\mu}{\nu} \text{ as } t \to \infty,$$

for $\mu > 0$ *and* $\nu > \frac{B}{4}$, *where B is a constant that is already defined in (99).*

Theorem 4. *Assume that* $\frac{1}{2} < \alpha < 1$. *Suppose that (2)–(5) hold. If g fulfills*

$$\int_0^\infty \|g\|^2_{L^2(\mathbb{T}^2)} ds < \infty, \tag{9}$$

then $\mathbf{u}$ *satisfies*

$$\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty,$$

for $\mu \geq 0$ *and* $\nu > \frac{B}{4}$, *where B is a constant that is already defined in (99).*

This paper is organized as follows. Several useful lemmas are introduced in Section 2. In Section 3, by means of the Banach fixed-point theorem and regularity results, we arrive at the local existence and uniqueness of the classical solution. In Section 4, we obtain some estimates of c, v, w, and $\mathbf{u}$. Meanwhile, in Section 5, the existence of a global classical solution is studied. Finally, in Section 6, we prove the stabilization with some certain coefficient conditions.

2. Preliminary

In this section, some lemmas that will be crucial in the following proofs are introduced. First, we show some important inequalities, which were proven in [51].

Lemma 1 ([51])**.** *Suppose that* $\{T_t^\alpha(x)\}_{t\geq 0}$ *is the analytic semigroup generated by* $-(-\Delta)^\alpha - I$ *on* $L^p(\mathbb{T}^2)$. *Then, for every* $p \in [1,\infty)$, $q \in [p,\infty]$ *and* $r \in [1,\infty]$, *and there exist constants* C_1 *and* C_2 *depending on* α, p, *and* q *only, which have the following properties:*

$$\|T_t^\alpha(x)w\|_{L^q(\mathbb{T}^2)} \leq C_1 e^{-t} t^{-\frac{1}{\alpha}(\frac{1}{p}-\frac{1}{q})} \|w\|_{L^p(\mathbb{T}^2)} \ \text{ for all } t > 0 \tag{10}$$

and

$$\|T_t^\alpha(x)\nabla \cdot w\|_{L^q(\mathbb{T}^2)} \leq C_2 e^{-t} t^{-\frac{1}{2\alpha}(1+\frac{2}{p}-\frac{2}{q})} \|w\|_{L^p(\mathbb{T}^2)} \ \text{ for all } t > 0. \tag{11}$$

Furthermore, we can obtain that

$$\|T_t^\alpha(x)w\|_{L^r(\mathbb{T}^2)} \le C_1 e^{-t}\|w\|_{L^r(\mathbb{T}^2)} \;\; for\ all\ t>0 \tag{12}$$

and

$$\|T_t^\alpha(x)\nabla\cdot w\|_{L^r(\mathbb{T}^2)} \le C_2 e^{-t} t^{-\frac{1}{2\alpha}}\|w\|_{L^r(\mathbb{T}^2)} \;\; for\ all\ t>0. \tag{13}$$

Proof. Since $\{T_t^\alpha(x)\}_{t\ge 0}$ is a semigroup generated by $-(-\Delta)^\alpha - I$ on $L^p(\mathbb{T}^2)$, we define $K_t^\alpha(x)$ as the fractional periodic heat kernel and note

$$K_t^\alpha(x) = \sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}} K^\alpha((x+n)t^{\frac{1}{2\alpha}}) \;\text{ and }\; K^\alpha(x) = (2\pi)^{-2}\int_{\mathbb{T}^2} e^{ix\cdot\xi} e^{-|\xi|^{2\alpha}} d\xi.$$

Therefore, we can obtain

$$T_t^\alpha(x)w = (K_t^\alpha(x) * w)(x) = (2\pi)^{-2} e^{-t}\int_{\mathbb{T}^2} K_t^\alpha(x-y)w(y)dy \tag{14}$$

and

$$\begin{aligned} T_t^\alpha(x)\partial_{x_i} w &= (2\pi)^{-2} e^{-t}\int_{\mathbb{T}^2} K_t^\alpha(y)\partial_{x_i} w(x-y)dy \\ &= (2\pi)^{-2} e^{-t}\int_{\mathbb{T}^2} \partial_{y_i} K_t^\alpha(y) w(x-y)dy. \end{aligned} \tag{15}$$

By the $L^p - L^q$ estimates for the convolution product, we have

$$\|T_t^\alpha(x)w\|_{L^q(\mathbb{T}^2)} \le e^{-t}\|K_t^\alpha(y)\|_{L^r(\mathbb{T}^2)}\|w\|_{L^p(\mathbb{T}^2)} \tag{16}$$

and

$$\|T_t^\alpha(x)\partial_{x_i} w\|_{L^q(\mathbb{T}^2)} \le e^{-t}\|\partial_{y_i} K_t^\alpha(y)\|_{L^r(\mathbb{T}^2)}\|w\|_{L^p(\mathbb{T}^2)}, \tag{17}$$

where $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$. We recall Inequalities (2.3), (2.4), and (2.5) of [51], and we find

$$\begin{aligned} \|K_t^\alpha(y)\|_{L^r(\mathbb{T}^2)} &= \Big(\int_{\mathbb{T}^2} |\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}} K^\alpha\big((y+n)t^{-\frac{1}{2\alpha}}\big)|^r dy\Big)^{\frac{1}{r}} \\ &= \Big(\int_{\mathbb{T}^2} |\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}} K^\alpha\big(x+nt^{-\frac{1}{2\alpha}}\big)|^r t^{\frac{1}{\alpha}} dx\Big)^{\frac{1}{r}} \\ &\le Ct^{-\frac{1}{\alpha}(1-\frac{1}{r})}\Big(\int_{\mathbb{T}^2} |\sum_{n\in\mathbb{Z}^2} (1+|x+nt^{-\frac{1}{2\alpha}}|)^{-2-2\alpha}|^r dx\Big)^{\frac{1}{r}} \\ &\le Ct^{-\frac{1}{2\alpha}(1-\frac{1}{r})}\Big(\int_{\mathbb{T}^2} |\sum_{n\in\mathbb{Z}^2} 1+|x+nt^{-\frac{1}{2\alpha}}|^{-2-2\alpha}|^r dx\Big)^{\frac{1}{r}} \\ &\le C_1 t^{-\frac{1}{\alpha}(1-\frac{1}{r})} \end{aligned} \tag{18}$$

and

$$\begin{aligned}\|\partial_{y_i}K_t^{\alpha}(y)\|_{L^r(\mathbb{T}^2)} &= \Big(\int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}}\partial_{y_i}K^{\alpha}\big((y+n)t^{-\frac{1}{2\alpha}}\big)|^r dy\Big)^{\frac{1}{r}}\\ &= \Big(\int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}}t^{-\frac{1}{2\alpha}}(\partial_{y_i}K^{\alpha})\big(x+nt^{-\frac{1}{2\alpha}}\big)|^r t^{\frac{1}{\alpha}}dx\Big)^{\frac{1}{r}}\\ &\le Ct^{-\frac{1}{2\alpha}-\frac{1}{\alpha}(1-\frac{1}{r})}\Big(\int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2}(1+|x+nt^{-\frac{1}{2\alpha}}|)^{-3}|^r dx\Big)^{\frac{1}{r}}\\ &\le Ct^{-\frac{1}{2\alpha}-\frac{1}{\alpha}(1-\frac{1}{r})}\Big(\int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2}1+|x+nt^{-\frac{1}{2\alpha}}|^{-3}|^r dx\Big)^{\frac{1}{r}}\\ &\le C_2t^{-\frac{1}{2\alpha}-\frac{1}{\alpha}(1-\frac{1}{r})}.\end{aligned} \tag{19}$$

Substituting the above equations into (16) and (17), we discover (10) and (11).

For special cases, we also have

$$\begin{aligned}\|K_t^{\alpha}(y)\|_{L^1(\mathbb{T}^2)} &= \int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}}K^{\alpha}\big((y+n)t^{-\frac{1}{2\alpha}}\big)|dy\\ &= \int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}}K^{\alpha}\big(x+nt^{-\frac{1}{2\alpha}}\big)|t^{\frac{1}{\alpha}}dx\\ &\le C\int_{\mathbb{T}^2}\sum_{n\in\mathbb{Z}^2}(1+|x+nt^{-\frac{1}{2\alpha}}|)^{-2-2\alpha}dx\\ &\le C\int_{\mathbb{T}^2}\sum_{n\in\mathbb{Z}^2}1+|x+nt^{-\frac{1}{2\alpha}}|^{-2-2\alpha}dx\\ &\le C_1\end{aligned} \tag{20}$$

and

$$\begin{aligned}\|\partial_{y_i}K_t^{\alpha}(y)\|_{L^1(\mathbb{T}^2)} &= \int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}}\partial_{y_i}K^{\alpha}\big((y+n)t^{-\frac{1}{2\alpha}}\big)|dy\\ &= \int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2} t^{-\frac{1}{\alpha}}t^{-\frac{1}{2\alpha}}(\partial_{y_i}K^{\alpha})\big(x+nt^{-\frac{1}{2\alpha}}\big)|t^{\frac{1}{\alpha}}dx\\ &\le Ct^{-\frac{1}{2\alpha}}\int_{\mathbb{T}^2}|\sum_{n\in\mathbb{Z}^2}(1+|x+nt^{-\frac{1}{2\alpha}}|)^{-3}dx\\ &\le Ct^{-\frac{1}{2\alpha}}\int_{\mathbb{T}^2}\sum_{n\in\mathbb{Z}^2}1+|x+nt^{-\frac{1}{2\alpha}}|^{-3}dx\\ &\le C_2t^{-\frac{1}{2\alpha}}.\end{aligned} \tag{21}$$

Hence, combining these inequalities with (16) and (17) implies (12) and (13). □

Lemma 2 ([51])**.** *Suppose that $\{T_t(x)\}_{t\ge 0}$ is the analytic semigroup generated by $\Delta - I$ on $L^p(\mathbb{T}^2)$. Then, for any $p\in[1,\infty)$, $q\in[p,\infty]$ and $r\in[1,\infty]$, and there exist constants C_3 and C_4 depending on α, p, and q only, which have the following properties:*

$$\|T_t(x)w\|_{L^q(\mathbb{T}^2)} \le C_3e^{-t}t^{-(\frac{1}{p}-\frac{1}{q})}\|w\|_{L^p(\mathbb{T}^2)} \ \ \textit{for all } t>0 \tag{22}$$

and

$$\|T_t(x)\nabla\cdot w\|_{L^q(\mathbb{T}^2)} \le C_4e^{-t}t^{-\frac{1}{2}(1+\frac{2}{p}-\frac{2}{q})}\|w\|_{L^p(\mathbb{T}^2)} \ \ \textit{for all } t>0. \tag{23}$$

Furthermore, we can obtain that

$$\|T_t(x)w\|_{L^r(\mathbb{T}^2)} \le C_3e^{-t}\|w\|_{L^r(\mathbb{T}^2)} \ \ \textit{for all } t>0 \tag{24}$$

and

$$\|T_t(x)\nabla \cdot w\|_{L^r(\mathbb{T}^2)} \le C_4 e^{-t} t^{-\frac{1}{2}} \|w\|_{L^r(\mathbb{T}^2)} \text{ for all } t > 0. \tag{25}$$

Proof. We can complete the proof for Lemma 2 by taking $\alpha = 1$ in the proof of Lemma 1. □

Now, we give some lemmas used to prove some estimates in Section 4.

Lemma 3 (Gagliardo–Nirenberg interpolation inequality [52]). *Suppose that $1 \le p, q, r \le \infty$ and $0 \le k, m \le l$. $\theta \in [0, 1]$ and k, m, l satisfy*

$$\frac{m}{2} - \frac{1}{p} = \theta\left(\frac{k}{2} - \frac{1}{q}\right) + (1-\theta)\left(\frac{l}{2} - \frac{1}{r}\right).$$

Then, we have that

$$\|\partial^m w\|_{L^p(\mathbb{T}^2)} \le C\|\partial^k w\|^{\theta}_{L^q(\mathbb{T}^2)} \|\partial^l w\|^{1-\theta}_{L^r(\mathbb{T}^2)} + C\|\partial^k w\|_{L^q(\mathbb{T}^2)}. \tag{26}$$

Proof. This lemma can be referenced to Remarks 5 and 7 in [52]. To avoid confusion, we will denote a general positive constant as C in the present paper. □

Lemma 4 (A pointwise inequality for $(-\Delta)^\alpha$ [53]). *Let $0 \le \alpha \le 1$. Then, we can obtain*

$$\varphi'(g(x))(-\Delta)^\alpha g(x) \ge (-\Delta)^\alpha(\varphi(g))(x),$$

where $\varphi \in C^1(\mathbb{R})$ is a convex function.

Proof. The proof of Lemma 4 can be referenced to [53] (Theorem 2.1). □

Lemma 5 (Kato–Ponce's commutator estimates [54]). *Suppose that $\alpha > 0$ and $1 < p < \infty$, then we have that*

$$\|\Lambda^\alpha(fg)\|_{L^p(\mathbb{T}^2)} \le C\|f\|_{L^{p_1}(\mathbb{T}^2)}\|\Lambda^\alpha g\|_{L^{p_2}(\mathbb{T}^2)} + C\|\Lambda^\alpha f\|_{L^{p_3}(\mathbb{T}^2)}\|g\|_{L^{p_4}(\mathbb{T}^2)}, \tag{27}$$

with $1 < p_1, p_4 \le \infty$ and $1 < p_2, p_3 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Proof. Kenig, Ponce, and Vega in Lemma 2.10 of [54] proved the above lemma with Λ being replaced by $J = (1-\Delta)^{\frac{1}{2}}$ and the homogeneous $H^{s,p}$ spaces being replaced by non-homogeneous ones. □

Lemma 6 ([55]). *Suppose that $y(t) \in [0, \infty)$ is absolutely continuous and there exists a nonnegative function $j \in L^1_{loc}([0, T))$ satisfying*

$$\int_t^{t+s} j(\tau)d\tau \le e$$

for any $t \in [0, T-s)$ such that

$$y'(t) + dy(t) \le j(t)$$

for a.e. $t \in (0, T)$. Then,

$$y(t) \le \max\left\{y(0) + e, \frac{e}{ds} + 2e\right\}$$

for all $t \in (0, T)$, where we assumed that $T > 0$, $s \in (0, T)$, $d > 0$, and $e > 0$.

Proof. The proof of Lemma 6 can be obtained in Lemma 3.4 of [55] with $s = 1$. □

Finally, we introduce fractional Fisher information, which is key to studying the asymptotic behavior.

Lemma 7 (Fractional Fisher information [40])**.** *Suppose that $N \geq 1$, $w \geq 0$ is a smooth, given function and $0 < \alpha < 1$ is a fixed constant. Suppose that $\Gamma(z) : \mathbb{R}^+ \to \mathbb{R}$ is an increasing C^1-function such that $\Gamma'(z) \geq \frac{C^*}{z} \geq 0$, where C^* is a constant. Then,*

$$\int_{\mathbb{T}^N} |\Lambda^\alpha w|^2 dx \leq C(\alpha, N, \Gamma) \|w\|_{L^\infty(\mathbb{T}^N)} \int_{\mathbb{T}^N} \Gamma(w(x)) \Lambda^{2\alpha} w(x) dx. \tag{28}$$

Proof. We can find the proof of Lemma 7 in Section 4.1 of [40]. □

3. Local Existence and Uniqueness

Now, the local existence and uniqueness of the classical solution is proven.

Lemma 8. *Assume that $\frac{1}{2} < \alpha < 1$, $\mu \geq 0$, and $\nu > 0$. Suppose that (2)–(5) hold. Then, there exist a time $T_{max} \in (0, \infty]$ and a unique classical solution $(c, v, w, \mathbf{u})$ such that*

$$\begin{cases} 0 < c \in L^\infty(\mathbb{T}^2 \times [0, T_{max})) \cap C^{2\alpha,1}(\mathbb{T}^2 \times (0, T_{max})), \\ 0 \leq v \in L^\infty(\mathbb{T}^2 \times [0, T_{max})) \cap C^{2,1}(\mathbb{T}^2 \times (0, T_{max})), \\ 0 \leq w \in L^\infty(\mathbb{T}^2 \times [0, T_{max})) \cap C^{2,1}(\mathbb{T}^2 \times (0, T_{max})), \\ \mathbf{u} \in L^\infty(\mathbb{T}^2 \times [0, T_{max})) \cap C^{2,1}(\mathbb{T}^2 \times (0, T_{max})), \end{cases}$$

together with $P \in C^{1,0}(\mathbb{T}^2 \times [0, T_{max}))$. In addition, we have $T_{max} = \infty$, or

$$\limsup_{t \nearrow T_{max}} \left(\|c\|_{L^\infty(\mathbb{T}^2)} + \|v\|_{W^{1,\infty}(\mathbb{T}^2)} + \|w\|_{W^{1,\infty}(\mathbb{T}^2)} + \|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)} \right) = \infty.$$

Proof. Part 1. Existence of mild solution: First, we prove the existence of the mild solution of (1) with nonnegative initial data $(c_0, v_0, w_0) \in L^\infty(\mathbb{T}^2) \times W^{1,\infty}(\mathbb{T}^2) \times W^{1,\infty}(\mathbb{T}^2)$ and $\mathbf{u}_0 \in L^\infty(\mathbb{T}^2)$, then there exists $(c(t), v(t), w(t), \mathbf{u}(t))$ satisfying

$$\begin{cases} c(t) = T_t^\alpha c_0 - \int_0^t T_{t-\tau}^\alpha \nabla \cdot (\mathbf{u}c + cS_1(x, c, v)\nabla v - cS_2(x, c, w)\nabla w) d\tau \\ + \int_0^t T_{t-\tau}^\alpha \big((\mu + 1)c - \nu c^2\big) d\tau, \\ v(t) = T_t v_0 - \int_0^t T_{t-\tau}(\mathbf{u} \cdot \nabla v) d\tau + \int_0^t T_{t-\tau}\big(\alpha_1 c - (\beta_1 - 1)v\big) d\tau, \\ w(t) = T_t w_0 - \int_0^t T_{t-\tau}(\mathbf{u} \cdot \nabla w) d\tau + \int_0^t T_{t-\tau}\big(\alpha_2 c - (\beta_2 - 1)w\big) d\tau, \\ \mathbf{u}(t) = T_t \mathbf{u}_0 - \int_0^t T_{t-\tau} \mathcal{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) d\tau + \int_0^t T_{t-\tau} \mathcal{P}(\mathbf{u}) d\tau + \int_0^t T_{t-\tau} \mathcal{P}(c \nabla \phi + g) d\tau, \end{cases}$$

where, for every vector $a, b \in \mathbb{R}^2$, we represent the matrix $(a_i b_j)_{i,j=1,2}$ by $a \otimes b$. Fix $(c_0, v_0, w_0, \mathbf{u}_0)$ satisfying (5), and let $R > 0$, $T \in (0, 1) \cap (0, T_{max})$ and

$$Y := L^\infty((0, T); L^\infty(\mathbb{T}^2) \times W^{1,\infty}(\mathbb{T}^2) \times W^{1,\infty}(\mathbb{T}^2) \times L^\infty(\mathbb{T}^2)),$$

which is the Banach space. We define the closed set:

$$\begin{aligned} \mathbf{X} := \Big\{ (c, v, w, \mathbf{u}) \in Y \mid \|c\|_{L^\infty(\mathbb{T}^2)} + \|v\|_{W^{1,\infty}(\mathbb{T}^2)} \\ + \|w\|_{W^{1,\infty}(\mathbb{T}^2)} + \|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)} \leq R, \text{ for a.e. } t \in (0, T) \Big\}, \end{aligned}$$

and a mapping $\Psi = (\Phi_1, \Psi_2, \Psi_3, \Psi_4)$ on $\mathbf{X}$:

$$\begin{cases} \Psi_1(c,v,w,\mathbf{u}) = T_t^\alpha c_0 - \int_0^t T_{t-\tau}^\alpha \nabla \cdot (\mathbf{u}c + cS_1(x,c,v)\nabla v - cS_2(x,c,w)\nabla w)d\tau \\ + \int_0^t T_{t-\tau}^\alpha \big((\mu+1)c - \nu c^2\big)d\tau, \\ \Psi_2(c,v,w,\mathbf{u}) = T_t v_0 - \int_0^t T_{t-\tau}(\mathbf{u}\cdot\nabla v)d\tau + \int_0^t T_{t-\tau}\big(\alpha_1 c - (\beta_1 - 1)v\big)d\tau, \\ \Psi_3(c,v,w,\mathbf{u}) = T_t w_0 - \int_0^t T_{t-\tau}(\mathbf{u}\cdot\nabla w)d\tau + \int_0^t T_{t-\tau}\big(\alpha_2 c - (\beta_2 - 1)w\big)d\tau, \\ \Psi_4(c,v,w,\mathbf{u}) = T_t\mathbf{u}_0 - \int_0^t T_{t-\tau}\mathcal{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})d\tau + \int_0^t T_{t-\tau}\mathcal{P}(\mathbf{u} + c\nabla\phi + g)d\tau \end{cases}$$

for $(c,v,w,\mathbf{u}) \in \mathbf{X}$ and $t \in (0,T)$.

Now, we prove Ψ maps $\mathbf{X}$ into itself. Employing the spatio-temporal estimates of the analytic semigroup $\{T_t^\alpha(x)\}_{t\geq 0}$ in Lemma 1 and (3), we obtain that

$$\begin{aligned} &\|\Psi_1(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)} \\ \leq& \|T_t^\alpha c_0\|_{L^\infty(\mathbb{T}^2)} + \int_0^t \|T_{t-\tau}^\alpha \nabla\cdot(\mathbf{u}c)\|_{L^\infty(\mathbb{T}^2)}d\tau + \int_0^t \|T_{t-\tau}^\alpha\nabla\cdot(cS_1(x,c,v)\nabla v)\|_{L^\infty(\mathbb{T}^2)}d\tau \\ &+ \int_0^t \|T_{t-\tau}^\alpha\nabla\cdot(cS_2(x,c,w)\nabla w)\|_{L^\infty(\mathbb{T}^2)}d\tau + \int_0^t \|T_{t-\tau}^\alpha\big((\mu+1)c - \nu c^2\big)\|_{L^\infty(\mathbb{T}^2)}d\tau \\ \leq& C_1 e^{-t}\|c_0\|_{L^\infty(\mathbb{T}^2)} + C_2\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}}\|\mathbf{u}c\|_{L^\infty(\mathbb{T}^2)}d\tau \\ &+ C_2\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}}(\|cS_1(x,c,v)\nabla v\|_{L^\infty(\mathbb{T}^2)} + \|cS_2(x,c,w)\nabla w\|_{L^\infty(\mathbb{T}^2)})d\tau \\ &+ C_1\int_0^t e^{-(t-\tau)}\|(\mu+1)c - \nu c^2\|_{L^\infty(\mathbb{T}^2)}d\tau \\ \leq& C_1\|c_0\|_{L^\infty(\mathbb{T}^2)} + C_2\int_0^t (t-\tau)^{-\frac{1}{2\alpha}}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\|c\|_{L^\infty(\mathbb{T}^2)} \\ &+ C_2\int_0^t (t-\tau)^{-\frac{1}{2\alpha}}(C_{S_1}\|c\|_{L^\infty(\mathbb{T}^2)}\|\nabla v\|_{L^\infty(\mathbb{T}^2)} + C_{S_2}\|c\|_{L^\infty(\mathbb{T}^2)}\|\nabla w\|_{L^\infty(\mathbb{T}^2)})d\tau \\ &+ C_1\int_0^t \big((\mu+1)\|c\|_{L^\infty(\mathbb{T}^2)} + \nu\|c\|^2_{L^\infty(\mathbb{T}^2)}\big)d\tau \\ \leq& C_1\|c_0\|_{L^\infty(\mathbb{T}^2)} + \frac{2\alpha}{2\alpha-1}C_2(1 + C_{S_1} + C_{S_2})R^2T^{1-\frac{1}{2\alpha}} + C_1\big((\mu+1)R + \nu R^2\big)T. \end{aligned} \tag{29}$$

It follows from Lemma 2 that

$$\begin{aligned} &\|\Psi_2(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)} \\ \leq& \|T_t v_0\|_{L^\infty(\mathbb{T}^2)} + \int_0^t \|T_{t-\tau}(\mathbf{u}\cdot\nabla v)\|_{L^\infty(\mathbb{T}^2)}d\tau + \int_0^t \|T_{t-\tau}\big(\alpha_1 c - (\beta_1-1)v\big)\|_{L^\infty(\mathbb{T}^2)}d\tau \\ \leq& C_3 e^{-t}\|v_0\|_{L^\infty(\mathbb{T}^2)} + C_3\int_0^t e^{-(t-\tau)}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\|\nabla v\|_{L^\infty(\mathbb{T}^2)}d\tau \\ &+ C_3\alpha_1\int_0^t e^{-(t-\tau)}\|c\|_{L^\infty(\mathbb{T}^2)}d\tau + C_3\int_0^t e^{-(t-\tau)}(\beta_1+1)\|v\|_{L^\infty(\mathbb{T}^2)}d\tau \\ \leq& C_3\|v_0\|_{L^q(\mathbb{T}^2)} + C_3\big(R^2 + \alpha_1 R + (\beta_1+1)R\big)T \end{aligned} \tag{30}$$

and

$$
\begin{aligned}
&\|\nabla\Psi_2(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}\\
\leq&\|T_t\nabla v_0\|_{L^\infty(\mathbb{T}^2)}+\int_0^t\|T_{t-\tau}\nabla(\mathbf{u}\cdot\nabla v)\|_{L^\infty(\mathbb{T}^2)}d\tau+\int_0^t\|T_{t-\tau}\nabla\big(\alpha_1 c-(\beta_1-1)v\big)\|_{L^\infty(\mathbb{T}^2)}d\tau\\
\leq&C_4e^{-t}\|\nabla v_0\|_{L^\infty(\mathbb{T}^2)}+C_4\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\|\nabla v\|_{L^\infty(\mathbb{T}^2)}d\tau\\
&+C_4\alpha_1\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|c\|_{L^\infty(\mathbb{T}^2)}d\tau+C_4(\beta+1)\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|v\|_{L^\infty(\mathbb{T}^2)}d\tau\\
\leq&C_4\|\nabla v_0\|_{L^\infty(\mathbb{T}^2)}+2C_4\big(R^2+\alpha_1R+(\beta_1+1)R\big)T^{\frac{1}{2}}.
\end{aligned}
\tag{31}
$$

By using the same method for $\|\Psi_3(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}$ and $\|\nabla\Psi_3(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}$, we deduce that

$$
\|\Psi_3(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}\leq C_3\|w_0\|_{L^\infty(\mathbb{T}^2)}+C_3\big(R^2+\alpha_2R+(\beta_2+1)R\big)T \tag{32}
$$

and

$$
\|\nabla\Psi_3(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}\leq C_4\|\nabla w_0\|_{L^\infty(\mathbb{T}^2)}+2C_4\big(R^2+\alpha_2R+(\beta_2+1)R\big)T^{\frac{1}{2}}. \tag{33}
$$

Similarly, since the Helmholtz projection $\mathcal{P}$ is a bounded linear operator, we can find $C(\mathcal{P})$ such that

$$
\begin{aligned}
&\|\Psi_4(c,v,w,\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}\\
\leq&\|T_t\mathbf{u}_0\|_{L^\infty(\mathbb{T}^2)}+\int_0^t\|T_{t-\tau}\mathcal{P}\nabla(\mathbf{u}\otimes\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}d\tau+\int_0^t\|T_{t-\tau}\mathcal{P}(\mathbf{u}+c\nabla\phi+g)\|_{L^\infty(\mathbb{T}^2)}d\tau\\
\leq&C_3e^{-t}\|\mathbf{u}_0\|_{L^\infty(\mathbb{T}^2)}+C_4C(\mathcal{P})\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\mathbf{u}\|^2_{L^\infty(\mathbb{T}^2)}d\tau\\
&+C_3C(\mathcal{P})\int_0^t e^{-(t-\tau)}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}d\tau\\
&+C_3C(\mathcal{P})\int_0^t e^{-(t-\tau)}\big(\|c\|_{L^\infty(\mathbb{T}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}+\|g\|_{L^\infty(\mathbb{T}^2)}\big)d\tau\\
\leq&C_3\|\mathbf{u}_0\|_{L^\infty(\mathbb{T}^2)}+2C_4C(\mathcal{P})R^2T^{\frac{1}{2}}+C_3C(\mathcal{P})\big(R+R\|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}+\|g\|_{L^\infty(\mathbb{T}^2)}\big)T.
\end{aligned}
\tag{34}
$$

Combining (29)–(33) with (34), if we choose $R>0$ large enough and pick $T>0$ sufficiently small, we can obtain that Ψ maps $\mathbf{X}$ into itself, where R depends on $\|c_0\|_{L^\infty(\mathbb{T}^2)}$, $\|v_0\|_{W^{1,\infty}(\mathbb{T}^2)}$, $\|w_0\|_{W^{1,\infty}(\mathbb{T}^2)}$, and $\|\mathbf{u}_0\|_{L^\infty(\mathbb{T}^2)}$.

Next, we show that Ψ is a contraction on $\mathbf{X}$. Let us take $(c_1,v_1,w_1,\mathbf{u}_1),(c_2,v_2,w_2,\mathbf{u}_2)\in\mathbf{X}$ and denote

$$
\|c_1-c_2\|_{L^\infty(\mathbb{T}^2)}+\|v_1-v_2\|_{W^{1,\infty}(\mathbb{T}^2)}+\|w_1-w_2\|_{W^{1,\infty}(\mathbb{T}^2)}+\|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)}
$$

by $\|(c_1,v_1,w_1,\mathbf{u}_1),(c_2,v_2,w_2,\mathbf{u}_2)\|_{\mathbf{X}}$.

Recalling once again the spatio-temporal estimates of the analytic semigroup $\{T_t^\alpha(x)\}_{t\geq0}$ in Lemma 1 and arguments involving the Lipschitz continuity of S_1 and S_2 on $[0,R]\times[0,R]$, for some positive constants C_5 and C_6, we have that

$$
\begin{aligned}
&\|\Psi_1(c_1, v_1, w_1, \mathbf{u}_1) - \Psi_1(c_2, v_2, w_2, \mathbf{u}_2)\|_{L^\infty(\mathbb{T}^2)} \\
\leq &\int_0^t \|T_{t-\tau}^\alpha \nabla \cdot \big((\mathbf{u}_1 - \mathbf{u}_2)c_1 + \mathbf{u}_2(c_1 - c_2)\big)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ \int_0^t \|T_{t-\tau}^\alpha \nabla \cdot \big((c_1 - c_2)S_1(x, c_1, v_1)\nabla v_1 + c_2 S_1(x, c_2, v_2)\nabla(v_1 - v_2)\big)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ \int_0^t \|T_{t-\tau}^\alpha \nabla \cdot \big((c_1 - c_2)S_2(x, c_1, w_1)\nabla w_1 + c_2 S_2(x, c_2, w_2)\nabla(w_1 - w_2)\big)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ \int_0^t \|T_{t-\tau}^\alpha \nabla \cdot \big(c_2(S_1(x, c_1, v_1) - S_1(x, c_2, v_2))\nabla v_1\big)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ \int_0^t \|T_{t-\tau}^\alpha \nabla \cdot \big(c_2(S_2(x, c_1, w_1) - S_2(x, c_2, w_2))\nabla w_1\big)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ \int_0^t \|T_{t-\tau}^\alpha \big((\mu + 1)(c_1 - c_2) - \nu(c_1 - c_2)(c_1 + c_2)\big)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
\leq &C_2 \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}} (\|(\mathbf{u}_1 - \mathbf{u}_2)c_1\|_{L^\infty(\mathbb{T}^2)} + \|\mathbf{u}_2(c_1 - c_2)\|_{L^\infty(\mathbb{T}^2)}) d\tau \\
&+ C_2 C_{S_1} \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}} (\|(c_1 - c_2)\nabla v_1\|_{L^\infty(\mathbb{T}^2)} + \|c_2 \nabla(v_1 - v_2)\|_{L^\infty(\mathbb{T}^2)}) d\tau \\
&+ C_2 C_{S_2} \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}} (\|(c_1 - c_2)\nabla w_1\|_{L^\infty(\mathbb{T}^2)} + \|c_2 \nabla(w_1 - w_2)\|_{L^\infty(\mathbb{T}^2)}) d\tau \\
&+ C_2 C_5 \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}} \|c_2\big(|c_1 - c_2| + |v_1 - v_2|\big)\nabla v_1\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ C_2 C_6 \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}} \|c_2\big(|c_1 - c_2| + |w_1 - w_2|\big)\nabla w_1\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ C_1 \int_0^t e^{-(t-\tau)} \|(\mu + 1)(c_1 - c_2) - \nu(c_1 - c_2)(c_1 + c_2)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
\leq &C_2 R \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \Big(\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)} + \|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)}\Big) d\tau \\
&+ C_2 R C_{S_1} \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \Big(\|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)} + \|\nabla(v_1 - v_2)\|_{L^\infty(\mathbb{T}^2)}\Big) d\tau \\
&+ C_2 R C_{S_2} \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \Big(\|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)} + \|\nabla(w_1 - w_2)\|_{L^\infty(\mathbb{T}^2)}\Big) d\tau \\
&+ C_2 C_5 R^2 \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}} (\|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)} + \|v_1 - v_2\|_{L^\infty(\mathbb{T}^2)}) d\tau \\
&+ C_2 C_6 R^2 \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}} (\|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)} + \|w_1 - w_2\|_{L^\infty(\mathbb{T}^2)}) d\tau \\
&+ C_1 \int_0^t e^{-(t-\tau)} \Big((\mu + 1)\|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)} + 2\nu R \|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)}\Big) d\tau \\
\leq &\Big(\frac{2\alpha}{2\alpha - 1} C_2(1 + C_{S_1} + C_{S_2} + C_5 R + C_6 R) R T^{1-\frac{1}{2\alpha}} + C_1(\mu + 1 + 2\nu R)T\Big) \\
&\|(c_1, v_1, w_1, \mathbf{u}_1), (c_2, v_2, w_2, \mathbf{u}_2)\|_{\mathbf{X}}.
\end{aligned}
$$

By Lemma 2, we can obtain that

$$
\begin{aligned}
&\|\Psi_2(c_1, v_1, w_1, \mathbf{u}_1) - \Psi_2(c_2, v_2, w_2, \mathbf{u}_2)\|_{L^\infty(\mathbb{T}^2)} \\
\leq &\int_0^t \|T_{t-\tau}\Big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla v_1 + \mathbf{u}_2 \cdot \nabla(v_1 - v_2) + \alpha_1(c_1 - c_2) - (\beta_1 - 1)(v_1 - v_2)\Big)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
\leq &C_3 \int_0^t e^{-(t-\tau)} \Big(\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)} \|\nabla v_1\|_{L^\infty(\mathbb{T}^2)} + \|\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)} \|\nabla(v_1 - v_2)\|_{L^\infty(\mathbb{T}^2)} \\
&+ \alpha_1 \|c_1 - c_2\|_{L^\infty(\mathbb{T}^2)} + (\beta_1 + 1)\|v_1 - v_2\|_{L^\infty(\mathbb{T}^2)}\Big) d\tau \\
\leq &C_3(R + \alpha_1 + \beta_1 + 1)T \|(c_1, v_1, w_1, \mathbf{u}_1), (c_2, v_2, w_2, \mathbf{u}_2)\|_{\mathbf{X}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\nabla(\Psi_2(c_1,v_1,w_1,\mathbf{u}_1)-\Psi_2(c_2,v_2,w_2,\mathbf{u}_2))\|_{L^\infty(\mathbb{T}^2)}\\
\le&\int_0^t \|T_{t-\tau}\nabla\Big((\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla v_1+\mathbf{u}_2\cdot\nabla(v_1-v_2)+\alpha_1(c_1-c_2)\\
&-(\beta_1-1)(v_1-v_2)\Big)\|_{L^\infty(\mathbb{T}^2)}d\tau\\
\le&C_4\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\Big(\|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)}\|\nabla v_1\|_{L^\infty(\mathbb{T}^2)}+\|\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)}\|\nabla(v_1-v_2)\|_{L^\infty(\mathbb{T}^2)}\\
&+\alpha_1\|c_1-c_2\|_{L^\infty(\mathbb{T}^2)}+\beta_1\|v_1-v_2\|_{L^\infty(\mathbb{T}^2)}\Big)d\tau\\
\le&2C_4(R+\alpha_1+\beta_1+1)T^{\frac{1}{2}}\|(c_1,v_1,w_1,\mathbf{u}_1),(c_2,v_2,w_2,\mathbf{u}_2)\|_{\mathbf{X}}.
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
&\|\Psi_3(c_1,v_1,w_1,\mathbf{u}_1)-\Psi_3(c_2,v_2,w_2,\mathbf{u}_2)\|_{L^\infty(\mathbb{T}^2)}\\
\le&C_3(R+\alpha_2+\beta_2+1)T\|(c_1,v_1,w_1,\mathbf{u}_1),(c_2,v_2,w_2,\mathbf{u}_2)\|_{\mathbf{X}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\nabla(\Psi_3(c_1,v_1,w_1,\mathbf{u}_1)-\Psi_3(c_2,v_2,w_2,\mathbf{u}_2))\|_{L^\infty(\mathbb{T}^2)}\\
\le&2C_4(R+\alpha_2+\beta_2+1)T^{\frac{1}{2}}\|(c_1,v_1,w_1,\mathbf{u}_1),(c_2,v_2,w_2,\mathbf{u}_2)\|_{\mathbf{X}},
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\|\Psi_4(c_1,v_1,w_1,\mathbf{u}_1)-\Psi_4(c_2,v_2,w_2,\mathbf{u}_2)\|_{L^\infty(\mathbb{T}^2)}\\
\le&\int_0^t\|T_{t-\tau}\mathcal{P}\nabla\cdot\big((\mathbf{u}_1-\mathbf{u}_2)\otimes\mathbf{u}_1+\mathbf{u}_2\otimes(\mathbf{u}_1-\mathbf{u}_2)\big)\|_{L^\infty(\mathbb{T}^2)}d\tau\\
&+\int_0^t\|T_{t-\tau}\mathcal{P}\big((\mathbf{u}_1-\mathbf{u}_2)+(c_1-c_2)\nabla\phi\big)\|_{L^\infty(\mathbb{T}^2)}d\tau\\
\le&C_4C(\mathcal{P})\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\Big(\|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)}\|\mathbf{u}_1\|_{L^\infty(\mathbb{T}^2)}\\
&+\|\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)}\|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)}\Big)d\tau\\
&+C_3C(\mathcal{P})\int_0^t e^{-(t-\tau)}\Big(\|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty(\mathbb{T}^2)}+\|c_1-c_2\|_{L^\infty(\mathbb{T}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}\Big)d\tau\\
\le&(2C_4RT^{\frac{1}{2}}+C_3\|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}T)C(\mathcal{P})\|(c_1,v_1,w_1,\mathbf{u}_1),(c_2,v_2,w_2,\mathbf{u}_2)\|_{\mathbf{X}}.
\end{aligned}
$$

We selected T so small that Ψ is a contraction on $\mathbf{X}$. We can then apply Banach's fixed-point theorem to find a unique mild solution $(c,v,w,\mathbf{u})\in\mathbf{X}$ of the problem (1) existing on the time interval $[0,T]$.

Part 2. Regularity: In this part, the mild solution $(c(t),v(t),w(t),\mathbf{u}(t))$ of (1) on $[0,T_{max})$ obtained in **Part 1** is a classical solution of (1) proven on $[0,T_{max})$. First, according to the Stokes semigroup [56,57] and the standard regularity theory for the parabolic equation, we know that

$$v,w,\mathbf{u}\in C^{2,1}(\mathbb{T}^2\times(0,T)).$$

Then, as for the smoothness of c, c is affected by the fractional Laplacian operator, so it cannot be obtained directly by the standard regularity theory for the parabolic equation. We note that

$$
\begin{aligned}
&c_t+(-\Delta)^\alpha c+\underbrace{(\mathbf{u}+S_1(x,c,v)\nabla v-S_2(x,c,w)\nabla w)}_{B(x,t)}\cdot\nabla c\\
=&\underbrace{(-\nabla S_1(x,c,v)\nabla v+\nabla S_2(x,c,w)\nabla w)c-S_1(x,c,v)c\Delta v+S_2(x,c,w)c\Delta w+\mu c-\nu c^2}_{f(x,t)}.
\end{aligned}
$$

Recalling (2), the regularity of v, w, and $\mathbf{u}$, and the boundedness of c, we arrive at $B(x,t)$, $f(x,t) \in L^\infty(\mathbb{T}^2 \times [0,T))$. Then, we can use Theorem 3.2 of [44] to obtain $c(x,t) \in C^{\theta,\frac{\theta}{2\alpha}}(\mathbb{T}^2 \times [0,T))$. Thus, combining this with Theorem 3.3 of [44], we have $c \in C^{2\alpha+\theta,1+\frac{\theta}{2\alpha}}$ $(\mathbb{T}^2 \times [0,T))$. Furthermore, it was proven in [58] that there exists a smooth function P such that Problem (3) has a classical solution $(c,v,w,\mathbf{u},P)$ in $\mathbb{T}^2 \times (0,T)$.

Part 3. Nonnegative: Now, the classical solution (c,v,w) being nonnegative is proven. We assumed that (c,v,w) solves System (1) classically in $\mathbb{T}^2 \times (0,T)$. Let

$$\min_{x\in\mathbb{T}^2} c(x,t) = c(x_{1_t},t),\ \min_{x\in\mathbb{T}^2} v(x,t) = v(x_{2_t},t),\ \min_{x\in\mathbb{T}^2} w(x,t) = w(x_{3_t},t).$$

Using the proof of Theorem 1 of [33] and Theorem 4.1 of [59], evaluating the first equation of (1) at the minimum point of c, and using the kernel expression for $(-\Delta)^\alpha$, we deduce that

$$\begin{aligned}\frac{d}{dt}c(x_{1_t},t) \geq &-(-\Delta)^\alpha c(x_{1_t},t) - c(x_{1_t},t)S_1(x_{1_t},c,v)\cdot\Delta v(x_{1_t},t)\\ &+ c(x_{1_t},t)S_2(x_{1_t},c,w)\cdot\Delta w(x_{1_t},t)\\ &+ (-\nabla S_1(x_{1_t},c,v)\nabla v(x_{1_t},t) + \nabla S_2(x_{1_t},c,w)\nabla w(x_{1_t},t))c(x_{1_t},t)\\ &+ \mu c(x_{1_t},t) - \nu c^2(x_{1_t},t)\\ \geq &- c(x_{1_t},t)S_1(x_{1_t},c,v)\cdot\Delta v(x_{1_t},t) + c(x_{1_t},t)S_2(x_{1_t},c,w)\cdot\Delta w(x_{1_t},t)\\ &+ (-\nabla S_1(x_{1_t},c,v)\nabla v(x_{1_t},t) + \nabla S_2(x_{1_t},c,w)\nabla w(x_{1_t},t))c(x_{1_t},t)\\ &+ \mu c(x_{1_t},t) - \nu c^2(x_{1_t},t).\end{aligned}$$

By a comparison argument, we have

$$\begin{aligned}c(x_{1_t},t) \geq c_0(x)exp\Big\{\int_0^t \mu - S_1(x_\tau,c,v)\cdot\Delta v(x_\tau,\tau) + S_2(x_\tau,c,w)\cdot\Delta w(x_\tau,\tau) - \nu c(x_\tau,\tau)\\ - \nabla S_1(x_\tau,c,v)\nabla v(x_\tau,\tau) + \nabla S_2(x_\tau,c,w)\nabla w(x_\tau,\tau)d\tau\Big\}.\end{aligned}$$

By the positive of $c_0(x)$, we can obtain $c(x,t) > 0$. Moreover, the maximum principle of the parabolic equation ensures $v(x,t) \geq v(x_{2_t},t) \geq 0$ and $w(x,t) \geq w(x_{3_t},t) \geq 0$. □

4. A Priori Estimates

In this section, in order to obtain the uniform boundedness of the L^p-norms of c, we made a series of a priori estimates for the components of the solution. These a priori estimates not only help to prove the global existence of the solution, but also play a key role in the study of asymptotic stability.

First, we show the following basic, but important inequalities.

Lemma 9. *There is C such that the solution $(c,v,w,\mathbf{u})$ of System (1) fulfills*

$$\int_{\mathbb{T}^2} c \leq C \qquad \text{for all } t \in (0,T_{max}) \tag{35}$$

and

$$\int_t^{t+s}\int_{\mathbb{T}^2} c^2 \leq C \qquad \text{for all } t \in (0,T_{max}-s), \tag{36}$$

where $s := \min\{1,\frac{1}{2}T_{max}\}$.

Proof. Integrating the first equation of System (1) over $\mathbb{T}^2$ and employing $\nabla\cdot\mathbf{u} = 0$ and the Cauchy–Schwarz inequality, we have that

$$\frac{d}{dt}\int_{\mathbb{T}^2} c = \mu\int_{\mathbb{T}^2} c - \nu\int_{\mathbb{T}^2} c^2 \leq \mu\int_{\mathbb{T}^2} c - \frac{\nu}{|\mathbb{T}^2|}\Big(\int_{\mathbb{T}^2} c\Big)^2 \tag{37}$$

for all $t \in (0, T_{max})$. By a straightforward ordinary differential equation comparison argument, we can obtain (35).

Furthermore, integrating (37) in time yields

$$\nu \int_t^{t+s} \int_{\mathbb{T}^2} c^2 = \mu \int_t^{t+s} \int_{\mathbb{T}^2} c - \int_{\mathbb{T}^2} c(t+s) + \int_{\mathbb{T}^2} c(t) \le C$$

for all $t \in (0, T_{max} - s)$. Then, the proof of Lemma 9 is complete. □

With the help of Lemma 9, we can obtain the following boundedness estimates for $\mathbf{u}$.

Lemma 10. *There is C such that the solution $(c, v, w, \mathbf{u})$ of System (1) fulfills*

$$\int_{\mathbb{T}^2} |\mathbf{u}|^2 \le C \qquad \text{for all } t \in (0, T_{max}), \tag{38}$$

$$\int_t^{t+s} \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 \le C \qquad \text{for all } t \in (0, T_{max} - s) \tag{39}$$

and

$$\int_t^{t+s} \int_{\mathbb{T}^2} |\mathbf{u}|^4 \le C \qquad \text{for all } t \in (0, T_{max} - s), \tag{40}$$

where $s := \min\{1, \frac{1}{2} T_{max}\}$.

Proof. We multiply the third equation of System (1) by $\mathbf{u}$, integrate over $\mathbb{T}^2$, and integrate by parts, yielding

$$\int_{\mathbb{T}^2} \mathbf{u}_t \cdot \mathbf{u} - \frac{1}{2} \int_{\mathbb{T}^2} \mathbf{u}^2 \nabla \cdot \mathbf{u} = - \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 + \int_{\mathbb{T}^2} c \nabla \phi \cdot \mathbf{u} + \int_{\mathbb{T}^2} g \cdot \mathbf{u}.$$

Using $\nabla \cdot \mathbf{u} = 0$, we deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} |\mathbf{u}|^2 + \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 = \int_{\mathbb{T}^2} c \nabla \phi \cdot \mathbf{u} + \int_{\mathbb{T}^2} g \cdot \mathbf{u}. \tag{41}$$

Combining Hölder's inequality with Sobolev's inequality, we know

$$\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} \le C \|\mathbf{u}\|_{L^6(\mathbb{T}^2)} \le C \|\nabla \mathbf{u}\|_{L^{\frac{3}{2}}(\mathbb{T}^2)} \le C \|\nabla \mathbf{u}\|_{L^2(\mathbb{T}^2)}. \tag{42}$$

Applying the above inequality, Hölder's inequality, and Young's inequality to (41), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} |\mathbf{u}|^2 + \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 &\le \|\nabla \phi\|_{L^\infty(\mathbb{T}^2)} \|c\|_{L^2(\mathbb{T}^2)} \|\mathbf{u}\|_{L^2(\mathbb{T}^2)} + \|g\|_{L^2(\mathbb{T}^2)} \|\mathbf{u}\|_{L^2(\mathbb{T}^2)} \\ &\le \frac{1}{4} \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{T}^2)} + C(\|c\|^2_{L^2(\mathbb{T}^2)} + \|g\|^2_{L^2(\mathbb{T}^2)}) \end{aligned} \tag{43}$$

for all $t \in (0, T_{max})$, which means

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} |\mathbf{u}|^2 + \frac{1}{2} \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 + \frac{1}{4C} \int_{\mathbb{T}^2} |\mathbf{u}|^2 \le C(\|c\|^2_{L^2(\mathbb{T}^2)} + \|g\|^2_{L^2(\mathbb{T}^2)}). \tag{44}$$

By dropping $\frac{1}{2} \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2$ in (44), letting $y(t) := \int_{\mathbb{T}^2} |\mathbf{u}|^2$ and $j(t) := C(\int_{\mathbb{T}^2} |c|^2 + \int_{\mathbb{T}^2} |g|^2)$, and combining this with Lemma 9 and (4), there is $C_5 > 0$ such that

$$\int_t^{t+s} j(\tau) d\tau \le C_5 := C(1 + |\mathbb{T}^2| \|g\|^2_{L^\infty(\mathbb{T}^2 \times [0,\infty))}) \tag{45}$$

for all $t \in (0, T_{max} - s)$. Applying Lemma 6, we have

$$\int_{\mathbb{T}^2} |\mathbf{u}|^2 \leq \max\{\int_{\mathbb{T}^2} |\mathbf{u}_0|^2 + C_5, \frac{C_5}{s} + 2C_5\} \tag{46}$$

for all $t \in (0, T_{max} - s)$, which means (38).

Integrating (44) in time and employing (4), (35), and (38), we can obtain (39). Finally, in conjunction with (38) and (39), the Gagliardo–Nirenberg interpolation inequality shows that

$$\int_t^{t+s} \int_{\mathbb{T}^2} |\mathbf{u}|^4 = \int_t^{t+s} \|\mathbf{u}\|^4_{L^4(\mathbb{T}^2)} \leq C \int_t^{t+s} \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{T}^2)} \|\mathbf{u}\|^2_{L^2(\mathbb{T}^2)} + \|\mathbf{u}\|^4_{L^2(\mathbb{T}^2)} \leq C$$

for all $t \in (0, T_{max} - s)$. Thus, we are finished proving Lemma 10. □

Lemma 11. *There is $C > 0$ such that the solution $(c, v, w, \mathbf{u})$ of System (1) fulfills*

$$\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 \leq C \qquad \textit{for all } t \in (0, T_{max}). \tag{47}$$

Proof. Applying the Helmholtz projector $\mathcal{P}$ to the third equation of System (1), we arrive at

$$\mathbf{u}_t + \mathcal{A}\mathbf{u} = -\mathcal{P}[(\mathbf{u} \cdot \nabla)\mathbf{u}] + \mathcal{P}(c\nabla\phi) + \mathcal{P}(g), \tag{48}$$

where $\mathcal{A} := -\mathcal{P}\Delta$ denotes the realization of the Stokes operator in the solenoidal subspace $L^2_\sigma(\mathbb{T}^2)$ of $L^2(\mathbb{T}^2)$. Multiplying (48) by $\mathcal{A}\mathbf{u}$ and integration by parts and combining the orthogonal projection property of $\mathcal{P}$ [58] (Lemma 2.5.2) with Young's inequality yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} |\mathcal{A}^{\frac{1}{2}}\mathbf{u}|^2 + \int_{\mathbb{T}^2} |\mathcal{A}\mathbf{u}|^2 &= -\int_{\mathbb{T}^2} \mathcal{P}[(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathcal{A}\mathbf{u} + \int_{\mathbb{T}^2} \mathcal{P}(c\nabla\phi) \cdot \mathcal{A}\mathbf{u} + \int_{\mathbb{T}^2} \mathcal{P}(g) \cdot \mathcal{A}\mathbf{u} \\ &\leq \frac{3}{4}\int_{\mathbb{T}^2} |\mathcal{A}\mathbf{u}|^2 + \int_{\mathbb{T}^2} |(\mathbf{u} \cdot \nabla)\mathbf{u}|^2 + \|\nabla\phi\|^2_{L^\infty(\mathbb{T}^2)} \int_{\mathbb{T}^2} c^2 + \int_{\mathbb{T}^2} |g|^2. \end{aligned} \tag{49}$$

It follows from Hölder's inequality, the Gagliardo–Nirenberg inequality, Lemma 10, and Young's inequality that we arrive at

$$\begin{aligned} \int_{\mathbb{T}^2} |(\mathbf{u} \cdot \nabla)\mathbf{u}|^2 &\leq \|\mathbf{u}\|^2_{L^\infty(\mathbb{T}^2)} \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{T}^2)} \\ &\leq C\|\mathcal{A}\mathbf{u}\|_{L^2(\mathbb{T}^2)} \|\mathbf{u}\|_{L^2(\mathbb{T}^2)} \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{T}^2)} + C\|\mathbf{u}\|^2_{L^2(\mathbb{T}^2)} \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{T}^2)} \\ &\leq C\|\mathcal{A}\mathbf{u}\|_{L^2(\mathbb{T}^2)} \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{T}^2)} + C\|\nabla \mathbf{u}\|^2_{L^2(\mathbb{T}^2)} \\ &\leq \frac{1}{4}\int_{\mathbb{T}^2} |\mathcal{A}\mathbf{u}|^2 + C\Big[\Big(\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2\Big)^2 + \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2\Big]. \end{aligned} \tag{50}$$

Moreover, combining this with [58] (Lemma 2.2.1),

$$\int_{\mathbb{T}^2} |\mathcal{A}^{\frac{1}{2}}\mathbf{u}|^2 = \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2.$$

Thus, (49) becomes

$$\frac{d}{dt}\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 \leq C\Big[\Big(\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2\Big)^2 + \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 + \int_{\mathbb{T}^2} c^2 + \int_{\mathbb{T}^2} |g|^2\Big]. \tag{51}$$

Let $y(t) := \int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2$, $j_1(t) := C\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2$ and $j_2(t) := C(\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 + \int_{\mathbb{T}^2} c^2 + \int_{\mathbb{T}^2} |g|^2)$, then (51) becomes

$$y'(t) = j_1(t)y(t) + j_2(t) \tag{52}$$

for all $t \in (0, T_{max})$. Lemma 9, Lemma 10, and (4) show that there are $C_6, C > 0$ such that

$$\int_t^{t+s} h_1(s)ds = C\int_t^{t+s}\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 \le C_6 \tag{53}$$

and

$$\int_t^{t+s} h_2(s)ds = C\int_t^{t+s}\Big(\int_{\mathbb{T}^2} |\nabla \mathbf{u}|^2 + \int_{\mathbb{T}^2} c^2 + \int_{\mathbb{T}^2} |g|^2\Big) \le C \tag{54}$$

for all $t \in (0, T_{max} - s)$ with $s := \min\{1, \frac{1}{2}T_{max}\}$. Furthermore, (53) entails that for given $t \in (0, T_{max})$, there exists $t_0 = t_0(t) \in (t - s, t) \cap [0, \infty)$ such that

$$\int_{\mathbb{T}^2} |\nabla \mathbf{u}(t_0)|^2 \le \max\Big\{\int_{\mathbb{T}^2} |\nabla \mathbf{u}_0|^2, \frac{C_6}{s}\Big\}. \tag{55}$$

Thus, integrating (52) over (t_0, t) yields

$$y(t) \le y(t_0)e^{\int_{t_0}^t j_1(\tau)d\tau} + \int_{t_0}^t e^{\int_\tau^t j_1(\sigma)d\sigma} j_2(\tau)d\tau \le C, \tag{56}$$

which implies (47). □

Next, we study some estimates of the higher derivatives of v and w.

Lemma 12. *There exists C such that the solution $(c, v, w, \mathbf{u})$ of System (1) fulfills*

$$\int_{\mathbb{T}^2} v^2 \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{57}$$

and

$$\int_{\mathbb{T}^2} w^2 \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{58}$$

as well as

$$\int_t^{t+\tau}\int_{\mathbb{T}^2} |\nabla v|^2 \le C \qquad \textit{for all } t \in (0, T_{max} - \tau) \tag{59}$$

and

$$\int_t^{t+\tau}\int_{\mathbb{T}^2} |\nabla w|^2 \le C \qquad \textit{for all } t \in (0, T_{max} - \tau), \tag{60}$$

where $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. Applying v to the second equation of System (1), integrating over $\mathbb{T}^2$ and integration by parts yield

$$\int_{\mathbb{T}^2} vv_t + \int_{\mathbb{T}^2} v(\mathbf{u}\cdot\nabla v) = -\int_{\mathbb{T}^2} |\nabla v|^2 + \alpha_1\int_{\mathbb{T}^2} cv - \beta_1\int_{\mathbb{T}^2} v^2.$$

By Young's inequality, we infer from

$$\int_{\mathbb{T}^2} v(\mathbf{u}\cdot\nabla v) = \frac{1}{2}\int_{\mathbb{T}^2} \mathbf{u}\cdot\nabla v^2 = -\frac{1}{2}\int_{\mathbb{T}^2} v^2\nabla\cdot\mathbf{u} = 0$$

to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} v^2 + \int_{\mathbb{T}^2} |\nabla v|^2 + \beta_1\int_{\mathbb{T}^2} v^2 = \alpha_1\int_{\mathbb{T}^2} cv \le \frac{\alpha_1^2}{2\beta_1}\int_{\mathbb{T}^2} c^2 + \frac{\beta_1}{2}\int_{\mathbb{T}^2} v^2.$$

Therefore, we have that

$$\frac{d}{dt}\int_{\mathbb{T}^2} v^2 + 2\int_{\mathbb{T}^2} |\nabla v|^2 + \beta_1\int_{\mathbb{T}^2} v^2 \le \frac{\alpha_1^2}{\beta_1}\int_{\mathbb{T}^2} c^2. \tag{61}$$

Let $y(t) := \int_{\mathbb{T}^2} v^2$ and $j(t) = \frac{\alpha_1^2}{\beta_1} \int_{\mathbb{T}^2} c^2$; by dropping $2\int_{\mathbb{T}^2} |\nabla v|^2$ in (61),

$$y'(t) + \beta_1 y(t) \le j(t)$$

for all $t \in (0, T_{max})$. By Lemma 6, moreover, (36) shows that there exists $C_7 > 0$ such that

$$\int_t^{t+s} h(\tau) = \frac{\alpha_1^2}{\beta_1} \int_t^{t+s} \int_{\mathbb{T}^2} c^2 \le \frac{\alpha_1^2}{\beta_1} C_7$$

for all $t \in (0, T_{max} - s)$. Thus, we have

$$\int_{\mathbb{T}^2} v^2 = y(t) \le \max\Big\{ \int_{\mathbb{T}^2} v_0^2 + \frac{\alpha_1^2}{\beta_1} C_7, \frac{\alpha_1^2}{\beta_1^2 s} C_7 + \frac{2\alpha_1^2}{\beta_1} C_7 \Big\}$$

for all $t \in (0, T_{max})$, which, in turn, yields (57). Integrating (61) with respect to time and making use of (36) and (57), we can immediately arrive at (59).

Similarly, it is easy to obtain

$$\int_{\mathbb{T}^2} w^2 \le \max\Big\{ \int_{\mathbb{T}^2} w_0^2 + \frac{\alpha_2^2}{\beta_2} C_7, \frac{\alpha_2^2}{\beta_2^2 s} C_7 + \frac{2\alpha_2^2}{\beta_2} C_7 \Big\}$$

for all $t \in (0, T_{max})$ and (60). The proof of Lemma 12 is complete. □

Lemma 13. *There exists C such that the solution $(c, v, w, \mathbf{u})$ of System (1) fulfills*

$$\int_{\mathbb{T}^2} |\nabla v|^2 \le C \quad \text{for all } t \in (0, T_{max}) \tag{62}$$

and

$$\int_{\mathbb{T}^2} |\nabla w|^2 \le C \quad \text{for all } t \in (0, T_{max}) \tag{63}$$

as well as

$$\int_t^{t+\tau} \int_{\mathbb{T}^2} |\Delta v|^2 \le C \quad \text{for all } t \in (0, T_{max} - \tau) \tag{64}$$

and

$$\int_t^{t+\tau} \int_{\mathbb{T}^2} |\Delta w|^2 \le C \quad \text{for all } t \in (0, T_{max} - \tau), \tag{65}$$

where $\tau = \min\{1, \frac{1}{2} T_{max}\}$.

Proof. Test the second equation of System (1) by $-\Delta v$. Using integration by parts, Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg inequality, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} |\nabla v|^2 + \int_{\mathbb{T}^2} |\Delta v|^2 \\
=& -\beta_1 \int_{\mathbb{T}^2} |\nabla v|^2 - \alpha_1 \int_{\mathbb{T}^2} c\Delta v + \int_{\mathbb{T}^2} (\mathbf{u} \cdot \nabla v)\Delta v \\
\le& \frac{1}{4}\int_{\mathbb{T}^2} |\Delta v|^2 + \alpha_1^2 \int_{\mathbb{T}^2} c^2 + \|\mathbf{u}\|_{L^4(\mathbb{T}^2)} \|\nabla v\|_{L^4(\mathbb{T}^2)} \|\Delta v\|_{L^2(\mathbb{T}^2)} \\
\le& \frac{1}{4}\int_{\mathbb{T}^2} |\Delta v|^2 + \alpha_1^2 \int_{\mathbb{T}^2} c^2 + C\|\mathbf{u}\|_{L^4(\mathbb{T}^2)} \Big(\|\Delta v\|_{L^2(\mathbb{T}^2)}^{\frac{1}{2}} \|\nabla v\|_{L^2(\mathbb{T}^2)}^{\frac{1}{2}} + \|\nabla v\|_{L^2(\mathbb{T}^2)} \Big) \|\Delta v\|_{L^2(\mathbb{T}^2)} \\
\le& \frac{1}{4}\int_{\mathbb{T}^2} |\Delta v|^2 + \alpha_1^2 \int_{\mathbb{T}^2} c^2 + C\|\mathbf{u}\|_{L^4(\mathbb{T}^2)} \|\Delta v\|_{L^2(\mathbb{T}^2)}^{\frac{3}{2}} \|\nabla v\|_{L^2(\mathbb{T}^2)}^{\frac{1}{2}} \\
&+ C\|\mathbf{u}\|_{L^4(\mathbb{T}^2)} \|\nabla v\|_{L^2(\mathbb{T}^2)} \|\Delta v\|_{L^2(\mathbb{T}^2)} \\
\le& \frac{1}{4}\int_{\mathbb{T}^2} |\Delta v|^2 + \alpha_1^2 \int_{\mathbb{T}^2} c^2 + \frac{1}{4}\|\Delta v\|_{L^2(\mathbb{T}^2)}^2 + C\|\mathbf{u}\|_{L^4(\mathbb{T}^2)}^4 \|\nabla v\|_{L^2(\mathbb{T}^2)}^2 + C\|\mathbf{u}\|_{L^4(\mathbb{T}^2)}^2 \|\nabla v\|_{L^2(\mathbb{T}^2)}^2 \\
\le& \frac{1}{2}\int_{\mathbb{T}^2} |\Delta v|^2 + \alpha_1^2 \int_{\mathbb{T}^2} c^2 + \int_{\mathbb{T}^2} |\nabla v|^2 \Big[C\int_{\mathbb{T}^2} |\mathbf{u}|^4 + C\Big(\int_{\mathbb{T}^2} |\mathbf{u}|^4\Big)^{\frac{1}{2}} \Big].
\end{aligned} \tag{66}$$

Thus, (66) can be written as

$$\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla v|^2+\int_{\mathbb{T}^2}|\Delta v|^2\le\int_{\mathbb{T}^2}|\nabla v|^2\Big[2C\int_{\mathbb{T}^2}|\mathbf{u}|^4+2C\Big(\int_{\mathbb{T}^2}|\mathbf{u}|^4\Big)^{\frac{1}{2}}\Big]+2\alpha_1^2\int_{\mathbb{T}^2}c^2. \quad (67)$$

Let $y(t):=\int_{\mathbb{T}^2}|\nabla v|^2$, $j_3(t)=2C\int_{\mathbb{T}^2}|\mathbf{u}|^4+2C(\int_{\mathbb{T}^2}|\mathbf{u}|^4)^{\frac{1}{2}}$ and $j_4(t):=2\alpha_1^2\int_{\mathbb{T}^2}c^2$, then (67) becomes

$$y'(t)+\int_{\mathbb{T}^2}|\Delta v|^2\le y(t)j_3(t)+j_4(t). \quad (68)$$

We invoke Lemma 9 and Lemma 10 and find $C_8, C_9>0$ such that

$$\int_t^{t+s}j_3(s)\le C_8 \text{ and } \int_t^{t+s}j_4(s)\le 2\alpha_1^2C_9 \quad (69)$$

for all $t\in(0,T_{max}-s)$. Meanwhile, Lemma 12 shows that there is $C_{10}>0$ such that

$$\int_t^{t+s}\int_{\mathbb{T}^2}|\nabla v|^2\le C_{10} \quad (70)$$

for all $t\in(0,T_{max}-s)$. Thus, for given $t\in(0,T_{max})$, we can employ (70) to choose $t_0=t_0(t)\in(t-s,t)\cap[0,\infty)$ satisfying

$$y(t_0)=\int_{\mathbb{T}^2}|\nabla v(t_0)|^2\le C_{11}:=\max\Big\{\int_{\mathbb{T}^2}|\nabla v_0|^2,\frac{C_{10}}{s}\Big\}. \quad (71)$$

Dropping $\int_{\mathbb{T}^2}|\Delta v|^2$ of (68) and integrating over (t_0,t), we arrive at

$$y(t)\le y(t_0)e^{\int_{t_0}^t h_3(\tau)}+\int_{t_0}^t e^{\int_\tau^t j_3(\sigma)}j_4(\tau), \quad (72)$$

which, in light of (69)–(72), implies that

$$y(t):=\int_{\mathbb{T}^2}|\nabla v|^2\le C_{11}e^{C_8}+C_9e^{C_8},$$

which implies (62). Integrating (68) in time and once more using (62), (69), and (70), we can deduce the inequality (64). Furthermore, by employing the same method for w, we can obtain the inequalities (63) and (65). The proof of Lemma 13 is complete. □

At the end of this section, according to the above results, by an inductive method, we arrive at the uniform boundedness of $\|c\|_{L^p(\mathbb{T}^2)}$. To be precise, we used Lemma 4 for $(-\Delta)^\alpha c$ to handle $-p\int_{\mathbb{T}^2}c^{p-1}(-\Delta)^\alpha c$. When dealing with the chemotaxis terms, due to the influence of S_i $(i=1,2)$, we need to use Kato–Ponce's commutator estimates and Parseval's identity to handle $\|\Lambda^{1-\alpha}(S_1(x,c,v)\nabla v)\|_{L^{p_4}(\mathbb{T}^2)}$ and $\|\Lambda^{1-\alpha}(S_2(x,c,w)\nabla w)\|_{L^{p_4}(\mathbb{T}^2)}$. Then, the chemotaxis terms can be controlled by the fractional diffusion term; thus, we can obtain (89).

Lemma 14. *Suppose that $p\ge 2$. Then, there are $M>0$ and $C=C(p,M)>0$ such that the solution $(c,v,w,\mathbf{u})$ of System (1) fulfills*

$$\int_t^{t+\tau}\int_{\mathbb{T}^2}c^p\le M \quad \textit{for all } t\in(0,T_{max}-\tau), \quad (73)$$

then we have that

$$\int_{\mathbb{T}^2}c^p\le C \quad \textit{for all } t\in(0,T_{max}) \quad (74)$$

and

$$\int_t^{t+\tau}\int_{\mathbb{T}^2}c^{p+1}\le C \quad \textit{for all } t\in(0,T_{max}-\tau), \quad (75)$$

where $\tau = \min\{1, \frac{1}{2}T_{max}\}$.

Proof. Multiplying the first equation of System (1) by pc^{p-1} and integrating over $\mathbb{T}^2$, we have

$$\begin{aligned} p\int_{\mathbb{T}^2} c^{p-1}c_t + \nu p\int_{\mathbb{T}^2} c^{p+1} =& -p\int_{\mathbb{T}^2} c^{p-1}(-\Delta)^\alpha c + \mu p\int_{\mathbb{T}^2} c^p - p\int_{\mathbb{T}^2} c^{p-1}\mathbf{u}\cdot\nabla c \\ &- \int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(cS_1(x,c,v)\nabla v) \\ &+ \int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(cS_2(x,c,w)\nabla w) \\ =& \mu p\int_{\mathbb{T}^2} c^p + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4, \end{aligned} \tag{76}$$

where

$$\mathrm{I}_1 = -p\int_{\mathbb{T}^2} c^{p-1}(-\Delta)^\alpha c,\ \mathrm{I}_2 = -p\int_{\mathbb{T}^2} c^{p-1}\mathbf{u}\cdot\nabla c$$

and

$$\mathrm{I}_3 = -\int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(cS_1(x,c,v)\nabla v),\ \mathrm{I}_4 = \int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(cS_2(x,c,w)\nabla w).$$

For the term I_1, by Lemma 4, we have that

$$\begin{aligned} \mathrm{I}_1 =& -p\int_{\mathbb{T}^2} c^{p-1}(-\Delta)^\alpha c \\ =& -p\int_{\mathbb{T}^2} c^{\frac{p}{2}}c^{\frac{p}{2}-1}(-\Delta)^\alpha c \\ \le& -2\int_{\mathbb{T}^2} c^{\frac{p}{2}}(-\Delta)^\alpha c^{\frac{p}{2}} \\ \le& -2\int_{\mathbb{T}^2} |(-\Delta)^{\frac{\alpha}{2}}c^{\frac{p}{2}}|^2. \end{aligned} \tag{77}$$

Since $\mathbf{u}$ is solenoidal, for the term I_2, we have that

$$\mathrm{I}_2 = -p\int_{\mathbb{T}^2} c^{p-1}\mathbf{u}\cdot\nabla c = -\int_{\mathbb{T}^2}\mathbf{u}\cdot\nabla c^p = \int_{\mathbb{T}^2} c^p\nabla\cdot\mathbf{u} = 0. \tag{78}$$

To be able to deal with I_3 and I_4, first, let us introduce the Riesz transform. We define

$$\widehat{\mathcal{R}_j f(\xi)} = -i\frac{\xi_j}{|\xi|}\widehat{f}(\xi) \quad j = 1,2$$

and write

$$\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2).$$

Recall [37,60]; we arrive at

$$\mathcal{R}\Lambda = -\nabla.$$

If $1 < p < \infty$, we know the boundedness of the Riesz transform $\mathcal{R} : L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$. Combining this with integration by parts and Hölder's inequality, we can obtain that

$$
\begin{aligned}
\mathrm{I}_3 &= -\int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(cS_1(x,c,v)\nabla v) \\
&= \int_{\mathbb{T}^2} p(p-1)c^{p-1}\nabla c S_1(x,c,v)\nabla v \\
&= \int_{\mathbb{T}^2} p(p-1)c^{\frac{p}{2}}c^{\frac{p}{2}-1}\nabla c S_1(x,c,v)\nabla v \\
&= 2\int_{\mathbb{T}^2}(p-1)c^{\frac{p}{2}}\nabla c^{\frac{p}{2}}\cdot S_1(x,c,v)\nabla v \\
&= -2\int_{\mathbb{T}^2}(p-1)c^{\frac{p}{2}}(\mathcal{R}\Lambda^{1-\alpha}\Lambda^{\alpha}c^{\frac{p}{2}})\cdot S_1(x,c,v)\nabla v \\
&= -2\int_{\mathbb{T}^2}(p-1)\mathcal{R}\Lambda^{\alpha}c^{\frac{p}{2}}\Lambda^{1-\alpha}(c^{\frac{p}{2}}S_1(x,c,v)\nabla v) \\
&\leq 2(p-1)\|\Lambda^{\alpha}c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|\Lambda^{1-\alpha}(c^{\frac{p}{2}}S_1(x,c,v)\nabla v)\|_{L^2(\mathbb{T}^2)}.
\end{aligned}
\tag{79}
$$

Using Kato–Ponce's commutator estimates (27) to deal with $\|\Lambda^{1-\alpha}(c^{\frac{p}{2}}S_1(x,c,v)\ \nabla v)\|_{L^2(\mathbb{T}^2)}$, we can find $\frac{1}{2}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$ such that

$$
\begin{aligned}
&\|\Lambda^{1-\alpha}(c^{\frac{p}{2}}S_1(x,c,v)\nabla v)\|_{L^2(\mathbb{T}^2)} \\
\leq& C(\|\Lambda^{1-\alpha}c^{\frac{p}{2}}\|_{L^{p_1}(\mathbb{T}^2)}\|S_1(x,c,v)\nabla v\|_{L^{p_2}(\mathbb{T}^2)}+\|c^{\frac{p}{2}}\|_{L^{p_3}(\mathbb{T}^2)}\|\Lambda^{1-\alpha}(S_1(x,c,v)\nabla v)\|_{L^{p_4}(\mathbb{T}^2)}) \\
=& C(\mathrm{II}_1\mathrm{II}_2+\mathrm{II}_3\mathrm{II}_4),
\end{aligned}
\tag{80}
$$

where

$$\mathrm{II}_1=\|\Lambda^{1-\alpha}c^{\frac{p}{2}}\|_{L^{p_1}(\mathbb{T}^2)},\ \mathrm{II}_2=\|S_1(x,c,v)\nabla v\|_{L^{p_2}(\mathbb{T}^2)}$$

and

$$\mathrm{II}_3=\|c^{\frac{p}{2}}\|_{L^{p_3}(\mathbb{T}^2)},\ \mathrm{II}_4=\|\Lambda^{1-\alpha}(S_1(x,c,v)\nabla v)\|_{L^{p_4}(\mathbb{T}^2)}.$$

For the terms II_1 and II_3, by the Gagliardo–Nirenberg interpolation inequality (26), we discover

$$\mathrm{II}_1=\|\Lambda^{1-\alpha}c^{\frac{p}{2}}\|_{L^{p_1}(\mathbb{T}^2)}\leq C\|\Lambda^{\alpha}c^{\frac{p}{2}}\|^{a_1}_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|^{1-a_1}_{L^2(\mathbb{T}^2)}+C\|c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}, \tag{81}$$

where $a_1\in(0,1)$, $1-\frac{\alpha}{2}-\frac{1}{p_1}=\frac{\alpha}{2}a_1$ and

$$\mathrm{II}_3=\|c^{\frac{p}{2}}\|_{L^{p_3}(\mathbb{T}^2)}\leq C\|\Lambda^{\alpha}c^{\frac{p}{2}}\|^{a_3}_{L^2(\mathbb{T}^3)}\|c^{\frac{p}{2}}\|^{1-a_3}_{L^2(\mathbb{T}^2)}+C\|c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}, \tag{82}$$

where $a_3\in(0,\alpha)$ and $\frac{1}{2}-\frac{1}{p_3}=\frac{\alpha}{2}a_3$. For the term II_2, using the Gagliardo–Nirenberg inequality twice, (26), (62), and (3), we have that

$$
\begin{aligned}
\mathrm{II}_2 &= \|S_1(x,c,v)\nabla v\|_{L^{p_2}(\mathbb{T}^2)} \\
&\leq C_{S_1}\|\nabla v\|_{L^{p_2}(\mathbb{T}^2)} \\
&\leq C_{S_1}C\|\nabla v\|^{a_2}_{L^2(\mathbb{T}^2)}\|\Lambda^{2-a_1}v\|^{1-a_2}_{L^2(\mathbb{T}^2)}+C_{S_1}C\|\nabla v\|_{L^2(\mathbb{T}^2)} \\
&\leq C_{S_1}C\|\nabla v\|^{a_2}_{L^2(\mathbb{T}^2)}\big(C\|\nabla v\|^{a_1}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-a_1}_{L^2(\mathbb{T}^2)}+C\|\nabla v\|_{L^2(\mathbb{T}^2)}\big)^{1-a_2}+C_{S_1}C\|\nabla v\|_{L^2(\mathbb{T}^2)} \\
&\leq C_{S_1}(C\|\Delta v\|^{(1-a_1)(1-a_2)}_{L^2(\mathbb{T}^2)}+C),
\end{aligned}
\tag{83}
$$

where $a_2\in(0,1)$, $\frac{1}{2}-\frac{1}{p_2}=\frac{(1-a_1)(1-a_2)}{2}$. For the term II_4, using again Kato–Ponce's commutator estimates (27) and (3), we can find $\frac{1}{p_4}=\frac{1}{p_4'}+\frac{1}{p_4''}$ such that

$$
\begin{aligned}
\mathrm{II}_4 &= \|\Lambda^{1-\alpha}(S_1(x,c,v)\nabla v)\|_{L^{p_4}(\mathbb{T}^2)} \\
&\le C(C_{S_1}\|\Lambda^{1-\alpha}\nabla v\|_{L^{p_4}(\mathbb{T}^2)} + \|\nabla v\|_{L^{p_4'}(\mathbb{T}^2)}\|\Lambda^{1-\alpha}S_1(x,c,v)\|_{L^{p_4''}(\mathbb{T}^2)}) \\
&\le C(C_{S_1}\|\Lambda^{1-\alpha}\nabla v\|_{L^{p_4}(\mathbb{T}^2)} + \|\nabla v\|_{L^{p_4'}(\mathbb{T}^2)}\|\Lambda^{1-\alpha}S_1(x,c,v)\|_{L^{p_4''}(\mathbb{T}^2)}) \\
&= C(C_{S_1}\mathrm{III}_1 + \mathrm{III}_2\mathrm{III}_3).
\end{aligned} \tag{84}
$$

For the term III_1,

$$
\begin{aligned}
\mathrm{III}_1 &= \|\Lambda^{1-\alpha}\nabla v\|_{L^{p_4}(\mathbb{T}^2)} \\
&\le C\|\nabla v\|^{a_4}_{L^2(\mathbb{T}^2)}\|\Lambda^{2-a_3}v\|^{1-a_4}_{L^2(\mathbb{T}^2)} + C\|\nabla v\|_{L^2(\mathbb{T}^2)} \\
&\le C\|\nabla v\|^{a_4}_{L^2(\mathbb{T}^2)}\left(C\|\nabla v\|^{a_3}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-a_3}_{L^2(\mathbb{T}^2)} + C\|\nabla v\|_{L^2(\mathbb{T}^2)}\right)^{1-a_4} + C\|\nabla v\|_{L^2(\mathbb{T}^2)} \\
&\le C\|\Delta v\|^{(1-a_3)(1-a_4)}_{L^2(\mathbb{T}^2)} + C,
\end{aligned} \tag{85}
$$

where $a_4 \in (0,1)$, $1 - \frac{\alpha}{2} - \frac{1}{p_4} = \frac{(1-a_3)(1-a_4)}{2}$. For the term III_2, employ the Gagliardo–Nirenberg inequality (26) and Young's inequality:

$$
\begin{aligned}
\mathrm{III}_2 &= \|\nabla v\|_{L^{p_4'}(\mathbb{T}^2)} \\
&\le C\|\nabla v\|^{a_5}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-a_5}_{L^2(\mathbb{T}^2)} + \|\nabla v\|_{L^2(\mathbb{T}^2)} \\
&\le C\|\Delta v\|^{1-a_5}_{L^2(\mathbb{T}^2)} + C
\end{aligned} \tag{86}
$$

where $a_5 = \frac{2}{p_4'}$. For the term III_3, by the Gagliardo–Nirenberg inequality (26) and Parseval's identity, then there exist $C, a_6 \in (0,1)$, and $\rho \in (0, 3\alpha - 1)$ such that $\frac{1}{2} - \frac{\alpha}{2} - \frac{1}{p_4''} = a_6\frac{\rho-\alpha}{2}$ and

$$
\begin{aligned}
\mathrm{III}_3 =&\left\|\Lambda^{1-\alpha}S_1(x,c,v)\right\|_{L^{p_4''}(\mathbb{T}^2)} \\
\le& C\left\|\Lambda^{1-\alpha+\rho}S_1(x,c,v)\right\|^{a_6}_{L^2(\mathbb{T}^2)}\left\|S_1(x,c,v)\right\|^{1-a_6}_{L^\infty(\mathbb{T}^2)} + C\left\|S_1(x,c,v)\right\|_{L^\infty(\mathbb{T}^2)} \\
=& C(2\pi)^{\frac{a_6}{2}}\left\||\xi|^{1-\alpha+\rho}\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}S_1(y,c,v)e^{-i\xi\cdot y}dy\right\|^{a_6}_{L^2(\mathbb{T}^2)}C^{1-a_6}_{S_1} + C_{S_1}C \\
\le& C_{S_1}C\||\xi|^{1-\alpha+\rho}\|^{a_6}_{L^2(\mathbb{T}^2)} + C_{S_1}C \\
\le& C_{S_1}C.
\end{aligned}
$$

Combining this with (84), (85), and (86), we can find C such that

$$
\mathrm{II}_4 = C_{S_1}(C\|\Delta v\|^{(1-a_3)(1-a_4)}_{L^2(\mathbb{T}^2)} + C\|\Delta v\|^{(1-a_5)}_{L^2(\mathbb{T}^2)} + C).
$$

Inserting (81)–(84) into (80) and employing Young's inequality, (79) becomes

$$\begin{aligned}
\mathrm{I}_3 =& -\int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(nS_1(x,c,v)\nabla v)\\
\leq& 2C_{S_1}(p-1)C\big(\|\Lambda^\alpha c^{\frac{p}{2}}\|^{1+a_1}_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|^{1-a_1}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{(1-a_1)(1-a_2)}_{L^2(\mathbb{T}^2)} + \|\Lambda^\alpha c^{\frac{p}{2}}\|^{1+a_1}_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|^{1-a_1}_{L^2(\mathbb{T}^2)}\\
&+\|\Lambda^\alpha c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|\Delta v\|^{(1-a_1)(1-a_2)}_{L^2(\mathbb{T}^2)} + \|\Lambda^\alpha c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\\
&+\|\Lambda^\alpha c^{\frac{p}{2}}\|^{1+a_3}_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|^{1-a_3}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{(1-a_3)(1-a_4)}_{L^2(\mathbb{T}^2)} + \|\Lambda^\alpha c^{\frac{p}{2}}\|^{1+a_3}_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|^{1-a_3}_{L^2(\mathbb{T}^2)}\\
&+\|\Lambda^\alpha c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|\Delta v\|^{(1-a_3)(1-a_4)}_{L^2(\mathbb{T}^2)} + \|\Lambda^\alpha c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\\
&+\|\Lambda^\alpha c^{\frac{p}{2}}\|^{1+a_3}_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|^{1-a_3}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{(1-a_5)}_{L^2(\mathbb{T}^2)} + \|\Lambda^\alpha c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|c^{\frac{p}{2}}\|_{L^2(\mathbb{T}^2)}\|\Delta v\|^{(1-a_5)}_{L^2(\mathbb{T}^2)}\big)\\
\leq& \|\Lambda^\alpha c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)} + C\Big(\|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)} + \|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}\|\Delta v\|^{2(1-a_2)}_{L^2(\mathbb{T}^2)} + \|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}\|\Delta v\|^{2(1-a_1)(1-a_2)}_{L^2(\mathbb{T}^2)}\\
&+\|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}\|\Delta v\|^{2(1-a_4)}_{L^2(\mathbb{T}^2)} + \|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}\|\Delta v\|^{2(1-a_3)(1-a_4)}_{L^2(\mathbb{T}^2)} + \|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}\|\Delta v\|^{2(1-a_5)}_{L^2(\mathbb{T}^2)}\Big).
\end{aligned}$$

Let $\theta := \max\{2(1-a_2), 2(1-a_4), 2(1-a_5)\} \in (0,2)$; we have that

$$\begin{aligned}
\mathrm{I}_3 =& -\int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(cS_1(x,c,v)\nabla v)\\
\leq& \|\Lambda^\alpha c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)} + C\big(\|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)} + \|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}\|\Delta v\|^{\theta}_{L^2(\mathbb{T}^2)}\big).
\end{aligned}\tag{87}$$

By using the same method for I_4, we can deduce that

$$\begin{aligned}
\mathrm{I}_4 =& \int_{\mathbb{T}^2} pc^{p-1}\nabla\cdot(cS_2(x,c,w)\nabla w)\\
\leq& \|\Lambda^\alpha c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)} + C\big(\|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)} + \|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}\|\Delta w\|^{\theta}_{L^2(\mathbb{T}^2)}\big).
\end{aligned}\tag{88}$$

Combining (77), (78), and (87) with (88) and (76) turns into

$$\frac{d}{dt}\int_{\mathbb{T}^2} c^p + \nu p\int_{\mathbb{T}^2} c^{p+1} \leq C\big(\|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)} + \|c^{\frac{p}{2}}\|^2_{L^2(\mathbb{T}^2)}(\|\Delta v\|^{\theta}_{L^2(\mathbb{T}^2)} + \|\Delta w\|^{\theta}_{L^2(\mathbb{T}^2)})\big). \tag{89}$$

By (73), we have that, for given $t \in (0, T_{max})$, we can pick $t_1 = t_1(t) \in (t-s, t) \cap [0, \infty)$, then

$$\int_{\mathbb{T}^2} c^p(t_1) \leq \max\Big\{\int_{\mathbb{T}^2} c_0^p, \frac{M}{s}\Big\}.$$

Thus, dropping $\nu p\int_{\mathbb{T}^2} c^{p+1}$ of (89) and integrating over (t_1, t) yield

$$\int_{\mathbb{T}^2} c^p \leq \int_{\mathbb{T}^2} c^p(t_1)e^{\int_{t_1}^t C(1+\|\Delta v\|^{\theta}_{L^2(\mathbb{T}^2)}+\|\Delta w\|^{\theta}_{L^2(\mathbb{T}^2)})} \leq C,$$

which implies (74). Furthermore, integrating (89) in time and employing (74), we can obtain (75). The proof of Lemma 14 is complete. □

Lemma 15. *For any $p \geq 1$, we can find $C = C(p) > 0$ such that*

$$\int_{\mathbb{T}^2} c^p \leq C \quad \textit{for all } t \in (0, T_{max}).$$

Proof. This result was obtained from Lemma 14 and Lemma 9. □

5. Global Existence

Through Lemma 8, the local existence and uniqueness of the classical solution of Problem (1) is obtained on the interval $[0, T_{\max})$. Combining this with a priori estimates in Section 3, we prove the global existence of the classical solution (Theorem 1).

Proof of Theorem 1. First, we claim that, if T_{max} is finite, then

$$\limsup_{t\nearrow T_{max}} \left(\|c\|_{L^\infty(\mathbb{T}^2)} + \|v\|_{W^{1,\infty}(\mathbb{T}^2)} + \|w\|_{W^{1,\infty}(\mathbb{T}^2)} + \|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\right) \le C. \tag{90}$$

To check this, we used a priori estimates in the previous lemmas and the estimates in Lemmas 1 and 2; recalling the Gagliardo–Nirenberg inequality, we have that, for $2 \le r \le \infty$,

$$\|\mathbf{u}\|_{L^r(\mathbb{R}^2)} \le C\|\mathbf{u}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{r}}\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^2)}^{1-\frac{2}{r}} \le C.$$

Let $p > 2$; we can obtain that

$$\begin{aligned}
&\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\\
\le&\|T_t\mathbf{u}_0\|_{L^\infty(\mathbb{T}^2)} + \int_0^t \|T_{t-\tau}\mathcal{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})\|_{L^\infty(\mathbb{T}^2)}d\tau + \int_0^t \|T_{t-\tau}\mathcal{P}(\mathbf{u}+c\nabla\phi+g)\|_{L^\infty(\mathbb{T}^2)}d\tau\\
\le&C_3e^{-t}\|\mathbf{u}_0\|_{L^\infty(\mathbb{T}^2)} + C_4C(\mathcal{P})\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{5}{6}}\|\mathbf{u}\|_{L^6(\mathbb{T}^2)}^2d\tau\\
&+ C_3C(\mathcal{P})\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}(\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} + \|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}\|c\|_{L^2(\mathbb{T}^2)} + \|g\|_{L^\infty(\mathbb{T}^2)})d\tau\\
\le&C_3\|\mathbf{u}_0\|_{L^\infty(\mathbb{T}^2)} + 6C_4C(\mathcal{P})CT_{max}^{\frac{1}{6}}\|\mathbf{u}\|_{L^2(\mathbb{T}^2)}^{\frac{2}{3}}\|\nabla\mathbf{u}\|_{L^2(\mathbb{T}^2)}^{\frac{4}{3}}\\
&+ C_3C(\mathcal{P})T_{max}^{\frac{1}{2}}(\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} + \|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}\|c\|_{L^2(\mathbb{T}^2)} + \|g\|_{L^\infty(\mathbb{T}^2)})\\
\le&C
\end{aligned}$$

and

$$\begin{aligned}
&\|v\|_{L^\infty(\mathbb{T}^2)}\\
\le&\|T_tv_0\|_{L^\infty(\mathbb{T}^2)} + \int_0^t \|T_{t-\tau}\mathbf{u}\cdot\nabla v\|_{L^\infty(\mathbb{T}^2)}d\tau + \alpha_1\int_0^t \|T_{t-\tau}c\|_{L^\infty(\mathbb{T}^2)}\\
&+ (\beta_1+1)\int_0^t \|T_{t-\tau}v\|_{L^\infty(\mathbb{T}^2)}d\tau\\
\le&C_3e^{-t}\|v_0\|_{L^\infty(\mathbb{T}^2)} + C_4\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\|\nabla v\|_{L^2(\mathbb{T}^2)}d\tau\\
&+ C_3\alpha_1\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{p}}\|c\|_{L^p(\mathbb{T}^2)} + C_3(\beta_1+1)\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|v\|_{L^2(\mathbb{T}^2)}d\tau\\
\le&C_3\|v_0\|_{L^\infty(\mathbb{T}^2)} + 2C_4T_{max}^{\frac{1}{2}}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\|v\|_{L^2(\mathbb{T}^2)} + C_3\frac{p}{p-1}T_{max}^{1-\frac{1}{p}}\alpha_1\|c\|_{L^p(\mathbb{T}^2)}\\
&+ 2T_{max}^{\frac{1}{2}}C_3(\beta_1+1)\|v\|_{L^2(\mathbb{T}^2)}\\
\le&C.
\end{aligned}$$

Next, we fix $q > 2$ and, moreover, $q_1, q_2 \in (q,\infty)$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Let $\theta = 1 - \frac{2}{q_2}$; we thereby obtain

$$
\begin{aligned}
&\|\nabla v\|_{L^\infty(\mathbb{T}^2)} \\
\leq &\|T_t \nabla v_0\|_{L^\infty(\mathbb{T}^2)} + \int_0^t \|T_{t-\tau}\nabla(\mathbf{u}\cdot\nabla v)\|_{L^\infty(\mathbb{T}^2)} + \alpha_1 \|T_{t-\tau}\nabla c\|_{L^\infty(\mathbb{T}^2)} \\
&+ (\beta_1+1)\|T_{t-\tau}\nabla v\|_{L^\infty(\mathbb{T}^2)} d\tau \\
\leq &C_3 e^{-t}\|\nabla v_0\|_{L^\infty(\mathbb{T}^2)} + C_4\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}-\frac{1}{q}}\|\mathbf{u}\|_{L^{q_1}(\mathbb{T}^2)}\|\nabla v\|_{L^{q_2}(\mathbb{T}^2)} d\tau \\
&+ C_4\alpha_1 \int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}-\frac{1}{p}}\|c\|_{L^p(\mathbb{T}^2)} d\tau + C_4(\beta_1+1)\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|v\|_{L^\infty(\mathbb{T}^2)} d\tau \\
\leq &C_3\|\nabla v_0\|_{L^\infty(\mathbb{T}^2)} + \frac{2p}{p-2} C_4 T_{max}^{\frac{1}{2}-\frac{1}{p}}\|c\|_{L^p(\mathbb{T}^2)} + 2C_4(\beta_1+1)T_{max}^{\frac{1}{2}}\|v\|_{L^\infty(\mathbb{T}^2)} \\
&+ C_4 C\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2}-\frac{1}{q}}\|\mathbf{u}\|_{L^{q_1}(\mathbb{T}^2)}(\|\nabla v\|^{\theta}_{L^\infty(\mathbb{T}^2)}\|v\|^{1-\theta}_{L^\infty(\mathbb{T}^2)} + \|v\|_{L^\infty(\mathbb{T}^2)}) d\tau.
\end{aligned} \tag{91}
$$

Let $T \in (0, T_{max})$ and $M' := \sup_{t\in(0,T)} \|\nabla v\|_{L^\infty(\mathbb{T}^2)}$. We see from (91) that

$$
M' \leq C + CM'^{\theta},
$$

with some $C > 0$. Since $\theta < 1$, by Young's inequality, we have

$$
\|\nabla v\|_{L^\infty(\mathbb{T}^2)} \leq C
$$

Similarly, by using the same method for w, we can obtain that

$$
\|w\|_{L^\infty(\mathbb{T}^2)} \leq C \text{ and } \|\nabla w\|_{L^\infty(\mathbb{T}^2)} \leq C.
$$

Finally, combining the above three inequalities with Lemma 1, we have that

$$
\begin{aligned}
&\|c\|_{L^\infty(\mathbb{T}^2)} \\
\leq &\|T_t^\alpha c_0\|_{L^\infty(\mathbb{T}^2)} + \int_0^t \|T^\alpha_{t-\tau}\nabla\cdot(\mathbf{u}c + cS_1(x,c,v)\nabla v - cS_2(x,c,w)\nabla w)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
&+ \int_0^t \|T^\alpha_{t-\tau}((\mu+1)c - \nu c^2)\|_{L^\infty(\mathbb{T}^2)} d\tau \\
\leq &C_1 e^{-t}\|c_0\|_{L^\infty(\mathbb{T}^2)} + C_1\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{\alpha p}}((\mu+1)\|c\|_{L^p(\mathbb{T}^2)} + \nu\|c^2\|_{L^p(\mathbb{T}^2)}) d\tau \\
&+ C_2\int_0^t e^{-(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}(1+\frac{2}{p})}\|\mathbf{u}c + cS_1(x,c,v)\nabla v - cS_2(x,c,w)\nabla w\|_{L^p(\mathbb{T}^2)} d\tau \\
\leq &C_1\|c_0\|_{L^\infty(\mathbb{T}^2)} + \frac{2\alpha p}{2\alpha p - p - 2} C_2 T_{max}^{1-\frac{p+2}{2\alpha p}}(\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)}\|c\|_{L^p(\mathbb{T}^2)} + C_{S_1}\|c\|_{L^p(\mathbb{T}^2)}\|\nabla v\|_{L^\infty(\mathbb{T}^2)} \\
&+ C_{S_2}\|c\|_{L^p(\mathbb{T}^2)}\|\nabla w\|_{L^\infty(\mathbb{T}^2)}) + \frac{\alpha p}{\alpha p - 1} C_1 T_{max}^{1-\frac{1}{\alpha p}}((\mu+1)\|c\|_{L^p(\mathbb{T}^2)} + \nu\|c\|^2_{L^{2p}(\mathbb{T}^2)}) \\
\leq &C.
\end{aligned}
$$

This proves (90). Combining this claim with Lemma 8, we can obtain $T_{max} = \infty$. The proof of Theorem 1 is complete. □

6. Stabilization

In this section, the asymptotic behavior of the global bounded classical solution of (1) is studied. In the case $\mu = 0$, we discovered that the solutions c, v, and w converge to a zero equilibrium, which was shown in [49]. In the case $\mu > 0$, if ν is large enough, the solutions c, v, and w converge to a non-zero equilibrium. The key idea of our method is to employ the Lyapunov functional, the form of which was inspired by [61]. Specifically, we dealt with the fractional diffusion term $\int_{\mathbb{T}^2} c_* \frac{(-\Delta)^\alpha c}{c}$ by Lemma 7 (fractional Fisher information) and uses the Riesz transform to handle the chemotaxis terms $\int_{\mathbb{T}^2} c_* \frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c}$ and $\int_{\mathbb{T}^2} c_* \frac{\nabla\cdot(cS_2(x,c,w)\nabla w)}{c}$; we can obtain Inequalities (104), (105), and (106). Moreover, because of the influence of S_i $(i = 1, 2)$, we needed to employ Kato–Ponce's commutator estimates and Parseval's identity to handle $\|\Lambda^{1-\alpha} \frac{S_1(x,c,v)\nabla v}{c}\|_{L^2(\mathbb{T}^2)}$ and $\|\Lambda^{1-\alpha} \frac{S_2(x,c,w)\nabla w}{c}\|_{L^2(\mathbb{T}^2)}$. Thus, we can obtain (111) and (112). Finally, when $\int_0^\infty \|g\|^2_{L^2(\mathbb{T}^2)} d\tau < \infty$, then we arrive at $\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)} \to 0$ as $t \to \infty$ in either case.

6.1. Stability of c, v, and w in the Case $\mu = 0$

In this subsection, the asymptotic stability of the solution to System (1) is studied in the case $\mu = 0$.

Lemma 16. *Suppose that $\mu = 0$ in System (1). Then, there is $C > 0$ such that the solution $(c, v, w, \mathbf{u})$ of System (1) fulfills*

$$\int_{\mathbb{T}^2} c(x,t) \le \frac{C}{t+1} \qquad \text{for all } t > 0, \tag{92}$$

$$\int_{\mathbb{T}^2} v(x,t) \le \frac{C}{t+1} \qquad \text{for all } t > 0, \tag{93}$$

$$\int_{\mathbb{T}^2} w(x,t) \le \frac{C}{t+1} \qquad \text{for all } t > 0 \tag{94}$$

and

$$\int_0^\infty \int_{\mathbb{T}^2} c(x,t)^2 \le C. \tag{95}$$

Proof. From (37) in the proof process of Lemma 9, we obtain that

$$\frac{d}{dt} \int_{\mathbb{T}^2} c = -\nu \int_{\mathbb{T}^2} c^2 \tag{96}$$

for all $t > 0$. Accordingly, the Cauchy–Schwarz inequality ensures that

$$\frac{d}{dt} \int_{\mathbb{T}^2} c = -\nu \int_{\mathbb{T}^2} c^2 \le -\frac{\nu}{|\mathbb{T}^2|} \Big(\int_{\mathbb{T}^2} c \Big)^2$$

for all $t > 0$. Thus, we have that

$$\int_{\mathbb{T}^2} c \le \frac{\int_{\mathbb{T}^2} c_0}{1 + \frac{\nu t}{|\mathbb{T}^2|} \int_{\mathbb{T}^2} c_0} \le \frac{C_{12}}{t+1} \tag{97}$$

for all $t > 0$, where $C_{12} := \max\{\int_{\mathbb{T}^2} c_0, \frac{|\mathbb{T}^2|}{\nu}\}$, whereupon a time integration in (96) yields (95).

Suppose that $y(t) := \int_{\mathbb{T}^2} v(x,t)$ for $t > 0$. We integrate the second equation of System (1) over $\mathbb{T}^2$ and use (97) to estimate

$$\begin{aligned} y'(t) &= -\beta_1 y(t) + \alpha_1 \int_{\mathbb{T}^2} c(x,t) \\ &\le -\beta_1 y(t) + \frac{\alpha_1 C_{12}}{t+1} \end{aligned}$$

for all $t > 0$. Let $C_{13} := \max\{\frac{2\int_{\mathbb{T}^2} v_0}{\alpha_1}, \frac{4C_{12}}{2\beta_1 - 1}\}$ and $z(t) := \frac{\alpha_1 C_{13}}{t+2}$ for all $t > 0$. Then, we have that $z(0) = \frac{\alpha_1 C_{13}}{2} \ge \int_{\mathbb{T}^2} v_0 = y(0)$. Moreover,

$$
\begin{aligned}
z'(t) + \beta_1 z(t) - \frac{\alpha_1 C_{12}}{t+1} &= -\frac{\alpha_1 C_{13}}{(t+2)^2} + \frac{\alpha_1 \beta_1 C_{13}}{t+2} - \frac{\alpha_1 C_{12}}{t+1} \\
&= \frac{\alpha_1 C_{13}}{t+2}\Big(\beta_1 - \frac{1}{t+2} - \frac{C_{12}}{C_{13}}\frac{t+2}{t+1}\Big) \\
&\geq \frac{\alpha_1 C_{13}}{t+2}\Big(\beta_1 - \frac{1}{2} - \frac{2C_{12}}{C_{13}}\Big) \\
&\geq 0
\end{aligned}
$$

for all $t > 0$. We once more use the comparison principle to see that $z(t) \geq y(t)$ for all $t > 0$, which means (93). The inequality (94) can be derived similarly. □

Proof of Theorem 2. Now, we prove the theorem by contradiction. Let $\|c\|_{L^\infty(\mathbb{T}^2)} \to 0$ as $t \to \infty$ fail. Then, there exist $\varepsilon_0 > 0$ and $t_j \to \infty$, for all $j \geq N$, such that

$$\|c(t_j)\|_{L^\infty(\mathbb{T}^2)} \geq \varepsilon_0.$$

Recall Theorem 1; we have that $\|c\|_{L^\infty(\mathbb{T}^2)} \leq C$, so we can choose a function $c_\infty \in L^\infty(\mathbb{T}^2)$ and a subsequence $\{t_{j_k}\}_{k\in\mathbb{N}}$ of $\{t_j\}_{j\in\mathbb{N}}$ such that

$$\|c(t_{j_k})\|_{L^\infty(\mathbb{T}^2)} \to c_\infty \text{ as } t_{j_k} \to \infty.$$

From (92), we have

$$\|c\|_{L^1(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty.$$

Thus, $c_\infty = 0$ and $\|c(t_{j_k})\|_{L^\infty(\mathbb{T}^2)} \to 0$ as $t_{j_k} \to \infty$, which contradicts the former assumption. Therefore, the theorem is valid. Similarly, for the asymptotic stability of v and w, we have that

$$\|v\|_{L^\infty(\mathbb{T}^2)} \to 0 \text{ and } \|w\|_{L^\infty(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty.$$

□

6.2. Stability of c, v, and w in the Case $\mu > 0$

Lemma 17. *Suppose that $\mu > 0$ and $\nu > \frac{B}{4}$, B is a constant that is already defined in (99), and $(c, v, w, \mathbf{u})$ is a global bounded classical solution of System (1) with the initial condition fulfilling (5). Then, there exist B_1, $B_2 > 0$, and $C_0 > 0$ such that the function E_{c_*,B_1,B_2} defined by*

$$
\begin{aligned}
E_{c_*,B_1,B_2}(t) := &\int_{\mathbb{T}^2}\Big(c - c_* - c_* \ln\frac{c}{c_*}\Big) + \frac{B_1}{2}\int_{\mathbb{T}^2}\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big)^2 + \frac{B_1}{2}\int_{\mathbb{T}^2}\Big(w - \frac{\alpha_2}{\beta_2}c_*\Big)^2 \\
&+ \frac{B_2}{2}\int_{\mathbb{T}^2}|\nabla v|^2 + \frac{B_2}{2}\int_{\mathbb{T}^2}|\nabla w|^2
\end{aligned}
$$

for all $t > 0$, satisfies

$$
\begin{aligned}
\frac{d}{dt}E_{c_*,B_1,B_2}(t) + C_0\Big\{&\int_{\mathbb{T}^2}|\Lambda^\alpha c|^2 + \int_{\mathbb{T}^2}|\nabla v|^2 + \int_{\mathbb{T}^2}|\nabla w|^2 + \int_{\mathbb{T}^2}|\Delta v|^2 + \int_{\mathbb{T}^2}|\Delta w|^2 \\
&+ \int_{\mathbb{T}^2}\Big(c - c_*\Big)^2 + \int_{\mathbb{T}^2}\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big)^2 + \int_{\mathbb{T}^2}\Big(w - \frac{\alpha_2}{\beta_2}c_*\Big)^2\Big\} \leq 0
\end{aligned} \tag{98}
$$

for all $t > 0$, with $c_* = \frac{\mu}{\nu}$.

Proof. According to $\nu > \frac{B}{4}$, where

$$B := \frac{c_*^2(C_{S_1}^2 + C_{S_2}^2) + B_2^2(\alpha_1^2 + \alpha_2^2)\varepsilon}{2c_*} C_* C(\alpha, \Gamma)\|c\|_{L^\infty(\mathbb{T}^2)} \tag{99}$$

which enables us to fix some $B_1, B_2 > 0$ that simultaneously fulfill

$$4\nu > B_1 > B, \quad \min\{B_1, B_2\} > B, \tag{100}$$

and take $\eta > 0$ such that

$$\nu > B_1\Big(\frac{\alpha_1^2}{4(\beta_1-\eta)} + \frac{\alpha_2^2}{4(\beta_2-\eta)}\Big) > \frac{B_1}{4}, \tag{101}$$

where C_* and ε are some positive constants to be determined later. With this value of B_1 and B_2 fixed henceforth, we denote

$$A_1(t) := \int_{\mathbb{T}^2}\Big(c - c_* - c_*\ln\frac{c}{c_*}\Big),\ A_2(t) := \frac{B_1}{2}\int_{\mathbb{T}^2}\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big)^2$$

and

$$A_3(t) := \frac{B_1}{2}\int_{\mathbb{T}^2}\Big(w - \frac{\alpha_2}{\beta_2}c_*\Big)^2,\ A_4(t) := \frac{B_2}{2}\int_{\mathbb{T}^2}|\nabla v|^2,\ A_5(t) := \frac{B_2}{2}\int_{\mathbb{T}^2}|\nabla w|^2$$

for all $t > 0$, and we write

$$E_{c_*,B_1,B_2}(t) := A_1(t) + A_2(t) + A_3(t) + A_4(t) + A_5(t)$$

for all $t > 0$. Taking the derivative of $E_{c_*,B_1,B_2}(t)$ in time, using integration by parts, $\nabla\cdot\mathbf{u} = 0$, and Young's inequality, we discover

$$\begin{aligned}\frac{d}{dt}A_1(t) &= \int_{\mathbb{T}^2}\Big(c_t - \frac{c_*}{c}c_t\Big)\\ &= \int_{\mathbb{T}^2}\mu c - \nu c^2 + c_*\frac{(-\Delta)^\alpha c}{c} + c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c} - c_*\frac{\nabla\cdot(cS_2(x,c,w)\nabla w)}{c}\\ &\quad - c_*(\mu - \nu c) + c_*\frac{\mathbf{u}\cdot\nabla c}{c}\\ &= \int_{\mathbb{T}^2} -\nu(c-c_*)^2 + c_*\frac{(-\Delta)^\alpha c}{c} + c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c}\\ &\quad - c_*\frac{\nabla\cdot(cS_2(x,c,w)\nabla w)}{c}\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}A_2(t) &= \int_{\mathbb{T}^2} B_1\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big)v_t\\ &= \int_{\mathbb{T}^2} B_1\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big)\Big(\Delta v + \alpha_1 c - \beta_1 v - \mathbf{u}\cdot\nabla v\Big)\\ &= \int_{\mathbb{T}^2}\alpha_1 B_1\Big(c - c_*\Big)\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big) - \beta_1 B_1\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big)^2 - B_1|\nabla v|^2\end{aligned}$$

as well as

$$\begin{aligned}\frac{d}{dt}A_4(t) &= \int_{\mathbb{T}^2} B_2\nabla v\cdot\nabla v_t\\ &= \int_{\mathbb{T}^2} B_2\nabla v\cdot\nabla\Big(\Delta v + \alpha_1 c - \beta_1 v - \mathbf{u}\cdot\nabla v\Big)\\ &= \int_{\mathbb{T}^2} -B_2|\Delta v|^2 + B_2\alpha_1\nabla v\cdot\nabla c - B_2\beta_1|\nabla v|^2 + B_2\mathbf{u}\cdot\nabla(\frac{|\nabla v|^2}{2}).\end{aligned}$$

for all $t > 0$. Similarly,

$$\frac{d}{dt}A_3(t) = \int_{\mathbb{T}^2} \alpha_2 B_1\Big(c - c_*\Big)\Big(v - \frac{\alpha_2}{\beta_2}c_*\Big) - \beta_2 B_1\Big(w - \frac{\alpha_2}{\beta_2}c_*\Big)^2 - B_1|\nabla w|^2$$

and

$$\frac{d}{dt}A_5(t) = \int_{\mathbb{T}^2} -B_2|\Delta w|^2 + B_2\alpha_2\nabla w\cdot\nabla c - B_2\beta_2|\nabla w|^2 + B_2\mathbf{u}\cdot\nabla(\frac{|\nabla w|^2}{2})$$

for all $t > 0$. Therefore, we arrive at

$$\begin{aligned}
\frac{d}{dt}E_{c_*,B_1,B_2}(t) = &\int_{\mathbb{T}^2} -\nu(c-c_*)^2 + c_*\frac{(-\Delta)^\alpha c}{c} + c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c} \\
&- c_*\frac{\nabla\cdot(cS_2(x,c,w)\nabla w)}{c} \\
&+ \int_{\mathbb{T}^2} \alpha_1 B_1\Big(c-c_*\Big)\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big) - \beta_1 B_1\Big(v - \frac{\alpha_1}{\beta_1}c_*\Big)^2 - B_1|\nabla v|^2 \\
&+ \int_{\mathbb{T}^2} \alpha_2 B_1\Big(c-c_*\Big)\Big(w - \frac{\alpha_2}{\beta_2}c_*\Big) - \beta_2 B_1\Big(w - \frac{\alpha_2}{\beta_2}c_*\Big)^2 - B_1|\nabla w|^2 \\
&+ \int_{\mathbb{T}^2} -B_2|\Delta v|^2 + B_2\alpha_1\nabla v\cdot\nabla c - B_2\beta_1|\nabla v|^2 + B_2\mathbf{u}\cdot\nabla(\frac{|\nabla v|^2}{2}) \\
&+ \int_{\mathbb{T}^2} -B_2|\Delta w|^2 + B_2\alpha_2\nabla w\cdot\nabla c - B_2\beta_2|\nabla w|^2 + B_2\mathbf{u}\cdot\nabla(\frac{|\nabla w|^2}{2}) \\
\le &\int_{\mathbb{T}^2} -XAX^T + c_*\frac{(-\Delta)^\alpha c}{c} + c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c} \\
&- c_*\frac{\nabla\cdot(cS_2(x,c,w)\nabla w)}{c} + \int_{\mathbb{T}^2} -B_1|\nabla v|^2 - B_1|\nabla w|^2 \\
&- B_2|\Delta v|^2 - B_2|\Delta w|^2 + B_2\alpha_1\nabla v\cdot\nabla c + B_2\alpha_2\nabla w\cdot\nabla c
\end{aligned} \tag{102}$$

for all $t > 0$, where X is vector functions defined as

$$X := (c - c_*, v - \frac{\alpha_1}{\beta_1}c_*, w - \frac{\alpha_2}{\beta_2}c_*)$$

and the constant matrix A is given by

$$A := \begin{vmatrix} \nu & -\frac{\alpha_1 B_1}{2} & -\frac{\alpha_2 B_1}{2} \\ -\frac{\alpha_1 B_1}{2} & \beta_1 B_1 & 0 \\ -\frac{\alpha_2 B_1}{2} & 0 & \beta_2 B_1 \end{vmatrix}.$$

Now, the important thing is to prove that A is positive definite. If we prove that, then we can obtain

$$XAX^T \ge \varepsilon|X|^2$$

for some $\varepsilon > 0$ and all $x \in \mathbb{T}^2$, $t > 0$. Thus, focusing attention on the desired definiteness properties, we first calculate the three principal minors of A to obtain that

$$M_1 := |\nu| > 0, \quad M_2 := \begin{vmatrix} \nu & -\frac{\alpha_1 B_1}{2} \\ -\frac{\alpha_1 B_1}{2} & \beta_1 B_1 \end{vmatrix} = \beta_1 B_1(\nu - B_1\frac{\alpha_1^2}{4\beta_1}) > 0$$

and

$$M_3 := \begin{vmatrix} \nu & -\frac{\alpha_1 B_1}{2} & -\frac{\alpha_2 B_1}{2} \\ -\frac{\alpha_1 B_1}{2} & \beta_1 B_1 & 0 \\ -\frac{\alpha_2 B_1}{2} & 0 & \beta_2 B_1 \end{vmatrix} = B_1^2\beta_1\beta_2\Big[\nu - B_1(\frac{\alpha_1^2}{4\beta_1} + \frac{\alpha_2^2}{4\beta_2})\Big] > 0,$$

because (101). Sylvester's criterion guarantees that, indeed, A is positive definite. Therefore, we can obtain

$$\begin{aligned}\frac{d}{dt}E_{c_*,B_1,B_2}(t) \leq \int_{\mathbb{T}^2} -\varepsilon|X|^2 + c_*\frac{(-\Delta)^\alpha c}{c} + c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c} \\ - c_*\frac{\nabla\cdot(cS_2(x,c,w)\nabla w)}{c} + \int_{\mathbb{T}^2} -B_1|\nabla v|^2 - B_1|\nabla w|^2 \\ - B_2|\Delta v|^2 - B_2|\Delta w|^2 + B_2\alpha_1\nabla v\cdot\nabla c + B_2\alpha_2\nabla w\cdot\nabla c\end{aligned}$$

for all $t > 0$. We recall the weak maximum principle of the parabolic equation, which implies

$$\min_{x\in\mathbb{T}^2,t>0} c(x,t) \geq \min_{x\in\mathbb{T}^2} c(x,0) > 0. \tag{103}$$

Then, in order to obtain (98), we have to estimate

$$\int_{\mathbb{T}^2} c_*\frac{(-\Delta)^\alpha c}{c}, \quad \int_{\mathbb{T}^2} c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c}, \quad -\int_{\mathbb{T}^2} c_*\frac{\nabla\cdot(cS_2(x,c,w)\nabla w)}{c}$$

and

$$\int_{\mathbb{T}^2} B_2\alpha_1\nabla v\cdot\nabla c, \quad \int_{\mathbb{T}^2} B_2\alpha_2\nabla w\cdot\nabla c.$$

For the term $\int_{\mathbb{T}^2} c_*\frac{(-\Delta)^\alpha c}{c}$, using Lemma 7 with $\Gamma(c) = -\frac{1}{c}$, we have that

$$\int_{\mathbb{T}^2} c_*\frac{(-\Delta)^\alpha c}{c} = -\int_{\mathbb{T}^2} c_*\Lambda^{2\alpha}c(-\frac{1}{c}) \leq -\frac{c_*}{C(a,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}\int_{\mathbb{T}^2}|\Lambda^\alpha c|^2. \tag{104}$$

For the term $\int_{\mathbb{T}^2} c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c}$, by the boundedness of the Riesz transform $\mathcal{R} : L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$ for $1 < p < \infty$, integration by parts, Hölder's inequality, and (3), we can discover

$$\begin{aligned}\int_{\mathbb{T}^2} c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c} &= c_*\int_{\mathbb{T}^2}\frac{\nabla c\cdot S_1(x,c,v)\nabla v}{c} \\ &= -c_*\int_{\mathbb{T}^2}\frac{\mathcal{R}\Lambda^{1-\alpha}\Lambda^\alpha c\cdot S_1(x,c,v)\nabla v}{c} \\ &= -c_*\int_{\mathbb{T}^2}\mathcal{R}\Lambda^\alpha c\Lambda^{1-\alpha}(\frac{S_1(x,c,v)\nabla v}{c}) \\ &\leq c_*\|\Lambda^\alpha c\|_{L^2(\mathbb{T}^2)}\Big\|\Lambda^{1-\alpha}\frac{S_1(x,c,v)\nabla v}{c}\Big\|_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{105}$$

Using Lemma 5, we see

$$\begin{aligned}\Big\|\Lambda^{1-\alpha}(\frac{S_1(x,c,v)\nabla v}{c})\Big\|_{L^2(\mathbb{T}^2)} \leq & CC_{S_1}\|\Lambda^{1-\alpha}\nabla v\|_{L^2(\mathbb{T}^2)}\Big\|\frac{1}{c}\Big\|_{L^\infty(\mathbb{T}^2)} \\ &+ C\|\nabla v\|_{L^{q_1}(\mathbb{T}^2)}\Big\|\Lambda^{1-\alpha}(\frac{S_1(x,c,v)}{c})\Big\|_{L^{q_2}(\mathbb{T}^2)}\end{aligned} \tag{106}$$

with $\frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2}$. We invoke the Gagliardo–Nirenberg inequality (26) to deal with the term $\|\Lambda^{1-\alpha}\nabla v\|_{L^2(\mathbb{T}^2)}$, then

$$\|\Lambda^{1-\alpha}\nabla v\|_{L^2(\mathbb{T}^2)} \leq C\|\nabla v\|^{\alpha}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-\alpha}_{L^2(\mathbb{T}^2)} + C\|\nabla v\|_{L^2(\mathbb{T}^2)}. \tag{107}$$

For the term $\|\frac{1}{c}\|_{L^\infty(\mathbb{T}^2)}$, recalling (103), we know that

$$\frac{1}{c} \leq \frac{1}{\min_{x\in\mathbb{T}^2,t>0} c(x,t)} \leq \frac{1}{\min_{x\in\mathbb{T}^2} c(x,0)}.$$

Thus,

$$\left\|\frac{1}{c}\right\|_{L^\infty(\mathbb{T}^2)} \le C. \tag{108}$$

By the Gagliardo–Nirenberg inequality (26), to handle the term $\|\nabla v\|_{L^{q_1}(\mathbb{T}^2)}$, we arrive at

$$\|\nabla v\|_{L^{q_1}(\mathbb{T}^2)} \le C\|\nabla v\|^{\frac{2}{q_1}}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-\frac{2}{q_1}}_{L^2(\mathbb{T}^2)} + C\|\nabla v\|_{L^2(\mathbb{T}^2)}. \tag{109}$$

For the term $\|\Lambda^{1-\alpha}\frac{S_1(x,c,v)}{c}\|_{L^{q_2}(\mathbb{T}^2)}$, by the Gagliardo–Nirenberg inequality (26) and Parseval's identity, then there exist $a_2 \in (0,1)$ and $\beta_0 \in (0, 3\alpha - 1)$ such that $\frac{1}{2} - \frac{\alpha}{2} - \frac{1}{q_2} = a_2\frac{\beta_0-\alpha}{2}$ and

$$\begin{aligned}
&\left\|\Lambda^{1-\alpha}\left(\frac{S_1(x,c,v)}{c}\right)\right\|_{L^{q_2}(\mathbb{T}^2)}\\
\le& C_{S_1}^{1-a_2}C\left\|\Lambda^{1-\alpha+\beta_0}\left(\frac{S_1(x,c,v)}{c}\right)\right\|^{a_2}_{L^2(\mathbb{T}^2)}\left\|\frac{1}{c}\right\|^{1-a_2}_{L^\infty(\mathbb{T}^2)} + C_{S_1}C\left\|\frac{1}{c}\right\|_{L^\infty(\mathbb{T}^2)}\\
=& C_{S_1}^{1-a_2}C(2\pi)^{\frac{a_2}{2}}\left\||\xi|^{1-\alpha+\beta_0}\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\left(\frac{S_1(y,c,v)}{c(y,t)}\right)e^{-i\xi\cdot y}dy\right\|^{a_2}_{L^2(\mathbb{T}^2)}\left\|\frac{1}{c}\right\|^{1-a_2}_{L^\infty(\mathbb{T}^2)}\\
&+ C_{S_1}C\left\|\frac{1}{c}\right\|_{L^\infty(\mathbb{T}^2)}\\
\le& C_{S_1}C\||\xi|^{1-\alpha+\beta_0}\|^{a_2}_{L^2(\mathbb{T}^2)}\left\|\frac{1}{c}\right\|_{L^\infty(\mathbb{T}^2)} + CC_{S_1}\left\|\frac{1}{c}\right\|_{L^\infty(\mathbb{T}^2)}\\
\le& C_{S_1}C.
\end{aligned} \tag{110}$$

Combining (107)–(110) with (106), using Young's inequality, and choosing $C_{14} > 0$,(105) turns into

$$\begin{aligned}
\int_{\mathbb{T}^2} c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c} \le& C_{S_1}c_*C_{14}\|\Lambda^\alpha c\|_{L^2(\mathbb{T}^2)} \times \Big(\|\nabla v\|^\alpha_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-\alpha}_{L^2(\mathbb{T}^2)}\\
&+ 2\|\nabla v\|_{L^2(\mathbb{T}^2)} + \|\nabla v\|^{\frac{2}{q_1}}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-\frac{2}{q_1}}_{L^2(\mathbb{T}^2)}\Big)\\
\le& \frac{c_*}{8C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}\|\Lambda^\alpha c\|^2_{L^2(\mathbb{T}^2)} + \frac{c_*C^2_{S_1}C^2_{14}C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}{2}\\
&\times\Big(\|\nabla v\|^\alpha_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-\alpha}_{L^2(\mathbb{T}^2)} + 2\|\nabla v\|_{L^2(\mathbb{T}^2)}\\
&+ \|\nabla v\|^{\frac{2}{q_1}}_{L^2(\mathbb{T}^2)}\|\Delta v\|^{1-\frac{2}{q_1}}_{L^2(\mathbb{T}^2)}\Big)^2.
\end{aligned}$$

Due to Young's inequality, we can conclude the existence of $C_{15} > 0$ such that

$$\begin{aligned}
\int_{\mathbb{T}^2} c_*\frac{\nabla\cdot(cS_1(x,c,v)\nabla v)}{c} \le& \frac{c_*}{8C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}\|\Lambda^\alpha c\|^2_{L^2(\mathbb{T}^2)}\\
&+ \frac{c_*C_{15}C^2_{S_1}C^2_{14}C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}{2}\Big(\|\nabla v\|^2_{L^2(\mathbb{T}^2)} + \|\Delta v\|^2_{L^2(\mathbb{T}^2)}\Big)
\end{aligned} \tag{111}$$

Similarly, it is possible to find $C_{16} > 0$ and $C_{17} > 0$ such that

$$\begin{aligned}
\int_{\mathbb{T}^2} -c_*\frac{\nabla\cdot(cS_2(x,c,w)\nabla v)}{c} \le& \frac{c_*}{8C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}\|\Lambda^\alpha c\|^2_{L^2(\mathbb{T}^2)}\\
&+ \frac{c_*C_{17}C^2_{S_2}C^2_{16}C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}{2}\\
&\times\Big(\|\nabla w\|^2_{L^2(\mathbb{T}^2)} + \|\Delta w\|^2_{L^2(\mathbb{T}^2)}\Big).
\end{aligned} \tag{112}$$

By using the same method for the term $\int_{\mathbb{T}^2} B_2\alpha_1 \nabla v \cdot \nabla c$, Young's inequality provides $C_{18} > 0$ and $1 > \varepsilon > 0$ such that

$$\begin{aligned}\int_{\mathbb{T}^2} B_2\alpha_1 \nabla v \cdot \nabla c =& B_2\alpha_1 \int_{\mathbb{T}^2} \mathcal{R}\Lambda^{1-\alpha}\Lambda^{\alpha} c \cdot \nabla v \\ =& B_2\alpha_1 \int_{\mathbb{T}^2} \mathcal{R}\Lambda^{\alpha} c \Lambda^{1-\alpha} \nabla v \\ \leq& B_2\alpha_1 \|\Lambda^{\alpha} c\|_{L^2(\mathbb{T}^2)} \|\Lambda^{1-\alpha}\nabla v\|_{L^2(\mathbb{T}^2)} \\ \leq& B_2\alpha_1 C \|\Lambda^{\alpha} c\|_{L^2(\mathbb{T}^2)} (\|\nabla v\|^{\alpha}_{L^2(\mathbb{T}^2)} \|\Delta v\|^{1-\alpha}_{L^2(\mathbb{T}^2)} + \|\nabla v\|_{L^2(\mathbb{T}^2)}) \\ \leq& \frac{c_*}{8C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}} \|\Lambda^{\alpha} c\|^2_{L^2(\mathbb{T}^2)} + \frac{C_{18}B_2^2\alpha_1^2 C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}{2c_*} \\ & \times \left(\|\nabla v\|^2_{L^2(\mathbb{T}^2)} + \varepsilon\|\Delta v\|^2_{L^2(\mathbb{T}^2)}\right)\end{aligned} \tag{113}$$

Similarly, we can find $C_{19} > 0$ and $1 > \varepsilon > 0$ such that

$$\begin{aligned}\int_{\mathbb{T}^2} B_2\alpha_2 \nabla v \cdot \nabla w \leq& \frac{c_*}{8C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}} \|\Lambda^{\alpha} c\|^2_{L^2(\mathbb{T}^2)} + \frac{C_{19}B_2^2\alpha_2^2 C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}{2c_*} \\ & \times \left(\|\nabla w\|^2_{L^2(\mathbb{T}^2)} + \varepsilon\|\Delta w\|^2_{L^2(\mathbb{T}^2)}\right)\end{aligned} \tag{114}$$

Inserting (104) and (111)–(114) into (102), we can find that

$$C_* := \max\{C_{14}^2, C_{15}, C_{16}^2, C_{17}, C_{18}, C_{19}\} > 0$$

yields

$$\begin{aligned}\frac{d}{dt}E_{c_*,B_1,B_2}(t) \leq& \int_{\mathbb{T}^2} -\varepsilon|X|^2 - \frac{c_*}{2C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}}\|\Lambda^{\alpha} c\|^2_{L^2(\mathbb{T}^2)} \\ & - \left(B_1 - \frac{c_*^2 C_{S_1}^2 + B_2^2\alpha_1^2}{2c_*} C_* C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}\right)\|\nabla v\|^2_{L^2(\mathbb{T}^2)} \\ & - \left(B_2 - \frac{c_*^2 C_{S_1}^2 + B_2^2\alpha_1^2\varepsilon}{2c_*} C_* C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}\right)\|\Delta v\|^2_{L^2(\mathbb{T}^2)} \\ & - \left(B_1 - \frac{c_*^2 C_{S_2}^2 + B_2^2\alpha_2^2}{2c_*} C_* C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}\right)\|\nabla w\|^2_{L^2(\mathbb{T}^2)} \\ & - \left(B_2 - \frac{c_*^2 C_{S_2}^2 + B_2^2\alpha_2^2\varepsilon}{2c_*} C_* C(\alpha,\Gamma)\|c\|_{L^\infty(\mathbb{T}^2)}\right)\|\Delta w\|^2_{L^2(\mathbb{T}^2)},\end{aligned}$$

where ε guarantees that B_2 exists. Therefore, since (100), we can find $C_0 > 0$ such that

$$\begin{aligned}\frac{d}{dt}E_{c_*,B_1,B_2}(t) + C_0\Big\{ \int_{\mathbb{T}^2} |\Lambda^{\alpha} c|^2 + \int_{\mathbb{T}^2} |\nabla v|^2 + \int_{\mathbb{T}^2} |\nabla w|^2 + \int_{\mathbb{T}^2} |\Delta v|^2 + \int_{\mathbb{T}^2} |\Delta w|^2 \\ + \int_{\mathbb{T}^2} \left(c - c_*\right)^2 + \int_{\mathbb{T}^2} \left(v - \frac{\alpha_1}{\beta_1}c_*\right)^2 + \int_{\mathbb{T}^2} \left(w - \frac{\alpha_2}{\beta_2}c_*\right)^2 \Big\} \leq 0\end{aligned}$$

for all $t > 0$. □

Lemma 18. *Suppose that $\mu > 0$ and $\nu > \frac{B}{4}$, B is a constant that is already defined in (99), and $(c, v, w, \mathbf{u})$ is a global bounded classical solution of System (1) with the initial condition satisfying (5). Then, there is a constant $C > 0$ such that*

$$\int_0^\infty \int_{\mathbb{T}^2} \left(c - c_*\right)^2 + \int_0^\infty \int_{\mathbb{T}^2} \left(v - \frac{\alpha_1}{\beta_1}c_*\right)^2 + \int_0^\infty \int_{\mathbb{T}^2} \left(w - \frac{\alpha_2}{\beta_2}c_*\right)^2 \leq C. \tag{115}$$

Proof. It is well known that $A_1(s) := s - c_* - c_* \ln \frac{s}{c_*}$ is convex and $A_1(s) \geq A_1(c_*) = 0$ for all $s > 0$. Thus, we have $E_{c_*,B_1,B_2}(c, v, w) \geq E_{c_*,B_1,B_2}(c_*, \frac{\alpha_1}{\beta_1}c_*, \frac{\alpha_2}{\beta_2}c_*) = 0$. Integrating (98) in time and recalling (64) and (65) and the nonnegativity of $E_{c_*,B_1,B_2}(t)$, we can immediately obtain (115). □

Proof of Theorem 3. We define

$$F^*(t) := \int_{\mathbb{T}^2} \left(c - c_*\right)^2 + \int_{\mathbb{T}^2} \left(v - \frac{\alpha_1}{\beta_1}c_*\right)^2 + \int_{\mathbb{T}^2} \left(w - \frac{\alpha_2}{\beta_2}c_*\right)^2 \Big\}.$$

From Lemma 18, we know

$$\frac{d}{dt}E_{c_*,B_1,B_2}(t) \leq -C_0 F^*(t).$$

The nonnegativity of $E_{c_*,B_1,B_2}(t)$ shows that

$$\int_1^\infty F^*(t)dt \leq \frac{E_{c_*,B_1,B_2}(t)(1)}{C_0} < \infty.$$

According to $F^*(t)$ being uniformly continuous for $t \in (1, \infty)$ (Theorem 1), we obtain $F^*(t) \to 0$ as $t \to \infty$. Combining this with the Gagliardo–Nirenberg inequality yields Theorem 3. □

6.3. Stability of $\mathbf{u}$

In the last subsection, combining the convergence of c, v, and w with (9) with the external force g, we can obtain the convergence of $\mathbf{u}$.

Lemma 19. *Let $\frac{1}{2} < \alpha < 1$ and $\mu \geq 0$. Suppose that (2)– (5) hold. If ν satisfies $\nu > \frac{B}{4}$, B is a constant that is already defined in (99), and g satisfies (9). Then, we arrive at*

$$\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty.$$

Proof. Going back to (41), then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} |\mathbf{u}|^2 &\leq \int_{\mathbb{T}^2} (c - \frac{\mu_+}{\nu})\nabla\phi \cdot \mathbf{u} + \|\mathbf{u}\|_{L^2(\mathbb{T}^2)}\|g\|_{L^2(\mathbb{T}^2)} \\ &\leq \|c - \frac{\mu_+}{\nu}\|_{L^2(\mathbb{T}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} + \|\mathbf{u}\|_{L^2(\mathbb{T}^2)}\|g\|_{L^2(\mathbb{T}^2)}, \end{aligned} \tag{116}$$

for $t > 0$. Multiplying (116) by e^t, integrating over (t_0, t), and employing Hölder's inequality, we discover

$$\begin{aligned} &e^t \int_{\mathbb{T}^2} |\mathbf{u}(x,t)|^2 dx - e^{t_0}\int_{\mathbb{T}^2} |\mathbf{u}(x,t_0)|^2 dx \\ \leq &\int_{t_0}^t e^s \|c - \frac{\mu_+}{\nu}\|_{L^2(\mathbb{T}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} + \int_{t_0}^t e^s\|\mathbf{u}\|_{L^2(\mathbb{T}^2)}\|g\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{117}$$

Further, (117) implies

$$\int_{\mathbb{T}^2} |\mathbf{u}(x,t)|^2 dx \leq e^{-(t-t_0)}\int_{\mathbb{T}^2} |\mathbf{u}(x,t_0)|^2 dx + \int_{t_0}^t e^{-(t-\tau)}h(\tau)d\tau, \tag{118}$$

where

$$h(\tau) = \sup_{\tau\in(t_0,t)} \Big(\|c - \frac{\mu_+}{\nu}\|_{L^2(\mathbb{T}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{T}^2)}\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} + \|\mathbf{u}\|_{L^2(\mathbb{T}^2)}\|g\|_{L^2(\mathbb{T}^2)}\Big).$$

By Lemma 16 and Lemma 18, we know that

$$\int_t^{t+1} \left\| c(x,\tau) - \frac{\mu_+}{\nu} \right\|^2_{L^2(\mathbb{T}^2)} d\tau \to 0 \text{ as } t \to \infty.$$

Combining this with (9), for all $t > t_0$ and any $\delta > 0$, we deduce that

$$\int_t^{t+1} h(\tau) d\tau \leq \frac{\delta}{2}(1 - e^{-\frac{1}{C_P^2}}).$$

Thus, Lemma 3.4 of [61] shows that

$$\int_{t_0}^{t} e^{-(t-\tau)} h(\tau) d\tau \leq \frac{\delta(1 - e^{-\frac{1}{C_P^2}})}{2(1 - e^{-\frac{1}{C_P^2}})} = \frac{\delta}{2}. \tag{119}$$

Furthermore, it follows from (38) that we can find $t_1 > t_0$ large enough, such that

$$(\sup_{t>0} \|\mathbf{u}\|_{L^2(\mathbb{T}^2)})^2 e^{-(t-t_0)} \leq \frac{\delta}{2}$$

for all $t > t_1$, and thus,

$$e^{-(t-t_0)} \int_{\mathbb{T}^2} |\mathbf{u}(x,t_0)|^2 dx \leq \frac{\delta}{2}. \tag{120}$$

Inserting (119) and (120) into (118), for all $t > t_1$, we can obtain that

$$\int_{\mathbb{T}^2} |\mathbf{u}(x,t)|^2 dx \leq \delta.$$

□

Proof of Theorem 4. Suppose that $\|\mathbf{u}(t)\|_{L^\infty(\mathbb{T}^2)} \nrightarrow 0$ as $t \to \infty$. Then, there exist $\varepsilon_0 > 0$ and $t_j \to \infty$ such that

$$\mathbf{u}\|(t_j)\|_{L^\infty(\mathbb{T}^2)} \geq \varepsilon_0$$

for all $j \geq N$. Recall Theorem 1; we know $\|\mathbf{u}\|_{L^\infty(\mathbb{T}^2)} \leq C$, so we can choose a function $\mathbf{u}_\infty \in L^\infty(\mathbb{T}^2)$ and a subsequence $\{t_{j_k}\}_{k\in\mathbb{N}}$ of $\{t_j\}_{j\in\mathbb{N}}$ such that

$$\|\mathbf{u}(t_{j_k})\|_{L^\infty(\mathbb{T}^2)} \to \mathbf{u}_\infty \text{ as } t_{j_k} \to \infty.$$

From (19), we know that

$$\|\mathbf{u}(t)\|_{L^2(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty.$$

Thus,$\mathbf{u}_\infty = 0$ and $\|\mathbf{u}(t_{j_k})\|_{L^\infty(\mathbb{T}^2)} \to 0$ as $t_{j_k} \to \infty$, which contradicts the former supposition. Thus, we can obtain

$$\|\mathbf{u}(t)\|_{L^\infty(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty.$$

□

7. Conclusions

We considered the global boundedness and large time behavior of a fractional chemotaxis Navier–Stokes system with matrix-valued sensitivities and attractive–repulsive signals on a two-dimensional periodic torus $\mathbb{T}^2$. When the cell density may proliferate following a logistic law and the diffusion of cells is fractional Laplace diffusion, the attractive–repulsive signals are produced by the cells themselves and degrade at a constant rate, and the cells and chemical substances are transported by an incompressible viscous fluid under the influence of a force due to the aggregation of cells. Our results showed that the global bounded solution of the system converges to the constant steady state. In addition, we are inspiring further researchers working in the fractional chemotaxis system and drawing the attention of the interested readers towards recent articles (see [47,49,51,62]). In conclusion, we suggest the recently published article by Lei et al. [62], who pointed out the fact that, from the results for the global existence of classical solutions to a coupled chemotaxis Navier–Stokes system with a logistic source and a fractional diffusion, the classical solutions in fact converge to the constant steady state. In addition, we trust that this paper will stimulate a number of researchers to extend this idea for some chemotaxis Navier–Stokes system with matrix-valued sensitivities and fractional diffusion without a logistic source.

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