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Convergence Analysis of a New Implicit Iterative Scheme and Its Application to Delay Caputo Fractional Differential Equations

Austine Efut Ofem [1], Mfon Okon Udo [2], Oboyi Joseph [3], Reny George [4,*] and Chukwuka Fernando Chikwe [3]

1 School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban 4001, South Africa
2 Department of Mathematics, Akwa Ibom State University, Ikot Akpaden, Mkpat Enin P.M. Box 1167, Nigeria
3 Department of Mathematics, University of Calabar, Calabar P.M. Box 1115, Nigeria
4 Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
* Correspondence: renygeorge02@yahoo.com

Abstract: This article presents a new three-step implicit iterative method. The proposed method is used to approximate the fixed points of a certain class of pseudocontractive-type operators. Additionally, the strong convergence results of the new iterative procedure are derived. Some examples are constructed to authenticate the assumptions in our main result. At the end, we use our new method to solve a fractional delay differential equation in the sense of Caputo. Our main results improve and generalize the results of many prominent authors in the existing literature.

Keywords: fixed point; implicit scheme; delay Caputo fractional differential equations; Banach space; total asymptotically pseudocontractive mappings

MSC: 05C07; 05C09; 05C31; 05C76; 05C99

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1. Introduction

Let $\mathcal{E}$ be a nonempty closed subset of a Banach space G with a dual G^*. The normalized duality mapping from G into 2^{G^*} is denoted by J and defined by

$$J(t) = \{v^* \in G^* : \langle t, v^* \rangle = \|t\|^2 = \|v^*\|^2\}, \ \forall\, t \in G, \tag{1}$$

where $\langle .,. \rangle$ stands for the generalized duality pairing. In this manuscript, we use j to stand for the single-valued normalized duality. The set of all positive real numbers is denoted by $\Re^+$, the set of all natural numbers is denoted by $\mathbb{N}$ and the set of all the fixed points of a mapping $H : G \to G$ is denoted by $F(H) = \{t \in G : Ht = t\}$.

Most problems in engineering and applied sciences are formulated as functional equations. Such equations can be formulated as fixed-point equations. Operator equations representing phenomena occurring in diverse fields, such as chemical reactions, steady-state temperature distribution, economic theories, epidemics and neutron transport theory, often require adequate and appropriate solutions. Thus, the target of finding a solution to these equations is to locate the fixed point and approximate its value. However, once we are certain of the existence of fixed points of given operators, it is always desirable to develop methods that can be efficiently used to approximate that fixed point. The iterative process is one of the fundamental tools that can be used to locate a fixed point [1,2]. Computing the value of a given fixed point of an operator analytically is quite tedious. Therefore, obtaining an efficient iterative method is required. In the last few years, various authors have introduced numerous iterative schemes that have been utilized widely to estimate the fixed points of operators. The Banach contraction theorem, which is one of the most widely and extensively utilized results, incorporates the Picard iteration method for locating the fixed point.

It was observed that the Picard iterative method cannot approximate the fixed points of mappings that are higher than contraction mappings. In order to overcome this drawback, several authors started introducing various iterative methods (see, e.g., [3–7] and the references therein).

The fixed-point approximation of the class of TAP mappings using iterative methods has been studied by several authors in recent years (see [8–10] and the reference therein).

Over the course of time, due to the advantages of implicit iterative schemes over explicit schemes, many iterative schemes have been developed by several authors for the approximation of the fixed points of different classes of mappings (see, e.g., [3,5,11]).

One of the first iterative methods was studied by Xu and Ori [11] in Hilbert spaces for the common fixed point of nonexpansive mappings. The implicit scheme of Xu and Ori [12] has been studied in diverse directions for the past two decades (see [5,13–18] and the references therein).

In [19], Saluja introduced the following averaging iterative scheme in Hilbert spaces:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} t_{k-1}, \end{cases} \quad k \in \mathbb{N}, \tag{2}$$

where $\{\alpha_k\}$ is a sequence in [0,1], $k = (n-1)N + i$, $i = k(i) \in I = \{1, 2, ..., N\}$, $n = n(k) \in \mathbb{N}$ and $n(k) \to +\infty$ as $k \to +\infty$. The author proved some convergence results of the implicit scheme (2) for the common fixed point of a finite family of strictly AP mappings in the intermediate sense.

In 2021, Ofem and Igbokwe [20] introduced the following two-step implicit iterative scheme for approximating the common fixed point of two total asymptotically pseudocontractive mappings:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} z_k \\ z_k = (1-u_k-v_k-s_k)t_k + u_k t_{k-1} + v_k S_{i(k)}^{n(k)} t_k + s_k t_{k-1}, \end{cases} \quad k \in \mathbb{N}, \tag{3}$$

where $\{\alpha_k\}, \{\beta_k\}, \{u_k\}, \{v_k\}$ and $\{s_k\}$ are sequences in [0,1] such that $u_k + v_k + s_k \leq 1$, $k = (n-1)N + i$, $i = k(i) \in I = \{1, 2, ..., N\}$, $n = n(k) \in \mathbb{N}$ and $n(k) \to +\infty$ as $k \to +\infty$.

Very recently, Ofem et al. [21] introduced the following three-step implicit iterative scheme for approximating the common fixed point of two total asymptotically pseudocontractive mappings:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} w_k, \\ w_k = (1-\beta_k)t_{k-1} + \beta_k R_{i(k)}^{n(k)} z_k, \\ z_k = (1-v_k)t_k + v_k S_{i(k)}^{n(k)} t_k, \end{cases} \quad k \in \mathbb{N}, \tag{4}$$

where $k = (n-1)N + i$, $i = k(i) \in I = \{1, 2, ..., N\}$, $n = n(k) \in \mathbb{N}$ and $n(k) \to +\infty$ as $k \to +\infty$.

On the other hand, for some years now, fractional calculus theory has attracted the attention of several authors in diverse fields. Indeed, it was noticed that fractional derivatives are useful tools for modeling many problems in sciences and engineering (see e.g., [22,23] and the references therein). To gain a better understanding of the models' behavior, different kinds of fractional operators have been constructed. Some of these operators include the Hadamard, Riemann–Liouville, Atangana–Baleanu, Katugampola, Caputo, Caputo–Fabrizio, Atangana–Koca, Atangana–Gomez, Atangana beta-derivative, Atangana bi-order, truncated $\mathbb{M}$–derivative and several others; each of these has some advantages and disadvantages over the others. For instance, Riemann–Liouville fractional operators require the presence of fractional order conditions to solve mathematical models under study, which makes them difficult to utilize. Interestingly, the Caputo fractional operator deals with this drawback and permits one to use the initial conditions with integer-order derivatives that have a clear physical meaning. For the past few decades, many methods have been constructed to solve fractional integro-differential equations, fractional partial differential equations and dynamic systems containing fractional derivatives, such as He's variational iteration method, the Adomian decomposition method, the homotopy analysis method, the homotopy perturbation method and existence and uniqueness results via the monotone method. Another well-known method that can also give the explicit form of the solution is the Laplace transform method, which permits one to transform fractional differential equations into algebraic equations, and, thus, by solving these algebraic equations, one can derive the unknown function via the inverse Laplace transform [24]. In this article, we use an iterative method to estimate the solution to a delay Caputo fractional differential equation.

Motivated by the above results, the aim of this manuscript is to propose a three-step iterative scheme for finite families of three uniformly L-Lipschitzian TAP mappings as follows:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} w_k \\ w_k = (1-\beta_k)t_{k-1} + \beta_k R_{i(k)}^{n(k)} z_k, \\ z_k = (1-u_k-v_k-s_k)t_k + u_k t_{k-1} + v_n S_{i(k)}^{n(k)} t_k + s_k t_{k-1}, \end{cases} \quad k \in \mathbb{N}, \tag{5}$$

where $\{\alpha_k\}, \{\beta_k\}, \{u_k\}, \{v_k\}$ and $\{s_k\}$ are sequences in [0,1] such that $u_k + v_k + s_k \leq 1$, $k = (n-1)N + i$, $i = k(i) \in I = \{1, 2, ..., N\}$, $n = n(k) \in \mathbb{N}$ and $n(k) \to +\infty$ as $k \to +\infty$.

Additionally, by using a different approach, we prove the strong convergence theorem of the new iterative method (5) for the common fixed points of the finite families of three uniformly L-Lipschitzian TAP mappings in Banach spaces. Furthermore, we provide a nontrivial example to validate the assumptions in our main results and also show the efficiency of our new method over some existing methods. Finally, we apply our result to the solution of a delay Caputo fractional differential equation.

Remark 1. *Clearly, if we use $\beta_k = 0$ in (5), we obtain (3), and, if we set $u_k = s_k = 0$ in (5), then we obtain (4). Thus, our new method properly includes the methods considered by Ofem and Igbokwe [20] and Ofem et al. [21]. Again, observe that the proposed method properly contains the corresponding methods in [25–27].*

The article is arranged as follows: In Section 2, we list certain definitions and lemmas that will be helpful in deriving our main results. In Section 3, we prove our main results and also add some corollaries. In Section 4, we numerically show the convergence of our new iterative scheme through some examples. In Section 5, we approximate the solution of a delay Caputo fractional differential equation via a special case of our new iterative method. In Section 6, we discuss the summary of the results obtained in this article.

2. Preliminaries

The following definition, lemmas and proposition are used to obtain our main results.

Definition 1. *Let $H : \mathcal{E} \to \mathcal{E}$ be a mapping. Then, H is called*

(i) Uniformly L-Lipschitzian, if, for all $t, m \in \mathcal{E}$, there exists a constant $L \geq 0$ such that

$$\|H^k t - H^k m\| \leq L\|t - m\|, \quad k \in \mathbb{N}. \tag{6}$$

(ii) *Pseudocontractive, if, for all $t, m \in \mathcal{E}$, there exists $j(t-m) \in J(t-m)$ such that*

$$\langle Ht - Hm, j(t-m)\rangle \leq \|t-m\|^2. \tag{7}$$

(iii) *Asymptotically pseudocontractive if a sequence $\{h_k\} \subset [1,\infty)$ exists with $h_k \to 1$ as $k \to \infty$ such that*

$$\langle H^k t - H^k m, j(t-m)\rangle \leq h_k\|t-m\|^2, \;\; k \in \mathbb{N}, \;\text{and}\;\; t, m \in \mathcal{E}. \tag{8}$$

(iv) *Total asymptotically pseudocontractive mappings in the intermediate sense, if a sequence $\{h_k\} \subset [1,\infty)$ exists with $h_k \to 1$ as $k \to \infty$ and $j(t-m) \in J(t-m)$ such that*

$$\limsup_{k\to\infty} \sup_{(t,m)\in\mathcal{E}} \left(\langle H^k t - H^k m, j(t-m)\rangle - h_k\|t-m\|^2\right) \leq 0. \tag{9}$$

Put

$$\rho_k = \max\left\{0, \sup_{t,m\in\mathcal{E}} \left(\langle H^k t - H^k m, j(t-m)\rangle - h_k\|t-m\|^2\right)\right\}.$$

This implies that $\rho_k \geq 0$, $\rho_k \to 0$ as $k \to \infty$. Thus, (9) gives the inequality:

$$\langle H^k t - H^k m, j(t-m)\rangle \leq h_k\|t-m\|^2 + \rho_k, \;\; k \in \mathbb{N}, \;\; t, m \in \mathcal{E}. \tag{10}$$

This class of mappings was introduced by Qin et al. [28].

(v) *total asymptotically pseudocontractive (TAP) [29], if $\{\mu_k\} \subset [0,+\infty)$ and $\{\xi_k\} \subset [0,+\infty)$ exist with $\mu_k \to 0$ and $\xi_k \to 0$ as $k \to +\infty$ such that*

$$\langle H^k t - H^k m, j(t-m)\rangle \leq \|t-m\|^2 + \mu_k \phi(\|t-m\|) + \xi_k, \tag{11}$$

$\forall k \in \mathbb{N}$ and $t, m \in \mathcal{E}$, where ϕ is a self-map that is continuously strictly increasing as defined on $[+0,\infty)$ with $\phi(0) = 0$.

Remark 2. *Suppose $\phi(t) = t^2$; then, (11) becomes a class of the asymptotically pseudocontractive mappings in the intermediate sense, as follows:*

$$\langle H^k t - H^k m, j(t-m)\rangle \leq (1+\mu_k)\|t-m\|^2 + \xi_k \tag{12}$$

for all $k \in \mathbb{N}$, $t, m \in \mathcal{E}$. Set

$$\rho_k = \max\left\{0, \sup_{t,m\in\mathcal{E}} \left(\langle H^k t - H^k m, j(t-m)\rangle - (1+\mu_k)\|t-m\|^2\right)\right\}.$$

Remark 3. *For $\rho_k = 0, k \in \mathbb{N}$, it is not hard to see that the class of asymptotically pseudocontractive mappings in the intermediate sense reduces to a class of asymptotically pseudocontractive mappings.*

As demonstrated in Remarks 2 and 3, it is clear that the class of TAP mappings is a superclass of the class mappings mentioned above.

Definition 2 ([30])**.** *A family $\{H_i\}_{i=1}^N : \mathcal{E} \to \mathcal{E}$ with $\mathbf{\Gamma} = \cap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy condition (B) on $\mathcal{E}$ if there exists a nondecreasing function $g : [0,+\infty) \to [0,+\infty)$ with $g(0) = 0$ and $g(p) > 0$, for all $0 < p < 1$, such that for all $t \in \mathcal{E}$*

$$\max_{1\leq i\leq N}\{\|t - H_i t\|\} \geq f(d(t, \mathbf{\Gamma})). \tag{13}$$

Lemma 1 ([12]). *Let the normalized duality mapping be defined by $J : G \to 2^{G^*}$. Then, for all $t, m \in \mathcal{E}$, one has*

$$\|t+m\|^2 \leq \|t\|^2 + 2\langle m, j(t+m)\rangle, \ \ \forall j(t+m) \in J(t+m). \tag{14}$$

Lemma 2 ([31]). *Let $\{\vartheta_k\}$, $\{\Lambda_k\}$ and $\{\Omega_k\}$ be nonnegative real number sequences such that:*

$$\vartheta_k \leq (1+\Lambda_k)\vartheta_n + \Omega_k, \ k \in \mathbb{N}. \tag{15}$$

Suppose $\sum_{k=1}^{\infty} \Lambda_k < +\infty$ and $\sum_{k=1}^{+\infty} \Omega_k < +\infty$. Then, $\lim_{k\to+\infty} \vartheta_k$ exists. Furthermore, if $\{\vartheta_k\}$ has a subsequence $\{\vartheta_{k_i}\}$ with $\vartheta_{k_i} \to 0$, it implies $\lim_{k\to+\infty} \vartheta_k = 0$.

3. Main Results

In the sequel, $i \in I = [1, N]$, where $N \in \mathbb{N}$. Now, we prove that (5) is suitable for the convergence of the common fixed points of three continuous TAP mappings. Let H_i be a L_h^i–Lipschitz TAP mapping with sequences $\nu_k^i \in [0, \infty)$ and $\lambda_k^i \in [0, +\infty)$ with $\nu_k^i \to 0$ and $\lambda_k^i \to 0$ as $k \to \infty$. Let R_i be a L_r^i–Lipschitz TAP mapping with the sequences $c_k^i \in [0, +\infty)$ and $d_k^i \in [0, +\infty)$ with $c_n^i \to 0$ and $d_k^i \to 0$ as $k \to +\infty$. Let S_i be a L_s^i–Lipschitz TAP mapping with the sequences $\eta_k^i \in [0, +\infty)$ and $l_k^i \in [0, +\infty)$ with $\eta_k^i \to 0$ and $l_k^i \to 0$ as $k \to +\infty$.

Let $W_k : \mathcal{E} \to \mathcal{E}$ be the mapping defined by

$$\begin{aligned} W_k(t) \ &= \ (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)}\{(1-\beta_k)t_{k-1} \\ &\quad +\beta_k R_{i(k)}^{n(k)}[(1-u_k-v_k-s_k)t + u_k t_{k-1} \\ &\quad +v_k S_{i(k)}^{n(k)} t + s_k H_{i(k)}^{n(k)} t_{k-1}]\}, \ \ k \in \mathbb{N}. \end{aligned} \tag{16}$$

From (16), we have

$$\begin{aligned} \|W_k(t) - W_k(m)\| \ &= \ \alpha_k \|H_{i(k)}^{n(k)}\{(1-\beta_k)t_{k-1} + \beta_k R_{i(k)}^{n(k)}[(1-u_k-v_k-s_k)t \\ &\quad +u_k t_{k-1} + v_k S_{i(k)}^{n(k)} t + s_n H_{i(k)}^{n(k)} t_{k-1}]\} \\ &\quad -H_{i(k)}^{n(k)}\{(1-\beta_k)t_{k-1} + \beta_k R_{i(k)}^{n(k)}[(1-u_k-v_k-s_k)m \\ &\quad +u_k t_{k-1} + v_k S_{i(k)}^{n(k)} m + s_k H_{i(k)}^{n(k)} t_{k-1}]\}\| \\ &\leq \ \alpha_k\beta_k L \|R_{i(k)}^{n(k)}[(1-u_k-v_k-s_k)t + u_k t_{k-1} \\ &\quad +v_k S_{i(k)}^{n(k)} t + s_k H_{i(k)}^{n(k)} t_{k-1}] - R_{i(k)}^{n(k)}[(1-u_k-v_k-s_k)m \\ &\quad +u_k t_{k-1} + v_k S_{i(k)}^{n(k)} m + s_k H_{i(k)}^{n(k)} t_{k-1}]\| \\ &\leq \ \alpha_k\beta_k L^2[(1-u_k-v_k-s_k)\|t-m\| \\ &\quad +v_n k\|S_{i(k)}^{n(k)} t - S_{i(k)}^{n(k)} m\|] \\ &\leq \ \alpha_k\beta_k L^2[(1-u_k-v_k-s_k)\|t-m\| + v_k L\|t-m\|] \\ &= \ \alpha_k\beta_k L^2[(1-u_k+v_k(L-1)-s_k)]\|t-m\|, \end{aligned} \tag{17}$$

for all $t, m \in \mathcal{E}$, where $L = \max\{L_h^1, ..., L_h^N, L_r^1, ..., L_r^N, L_s^1, ..., L_s^N\}$.

If $\alpha_k\beta_k L^2[(1-u_k+v_k(L-1)-s_k)] < 1$, for all $k \in \mathbb{N}$, then, from (17), it follows that the mapping W_k is a contraction. According to the contraction principle, this implies that a unique point $t_k \in \mathcal{E}$ exists such that

$$t_k = W(t_k).$$

This shows that the implicit iteration method (5) is well defined. Thus, we can employ the iterative method (5) to estimate the fixed point of the mapping in Definition 1(v).

Lemma 3. *Let G denote an arbitrary Banach space and $\mathcal{E}$ denote a nonempty closed convex subset of G. Let $H_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_h^i–Lipschitzian TAP mappings with the sequences $\{v_k^i\} \subset [0,\infty)$ and $\{\lambda_k^i\} \subset [0,\infty)$, where $v_k^i \to 0$ and $\lambda_k^i \to 0$ as $k \to \infty$. Let $R_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_r^i–Lipschitzian TAP mappings with the sequences $\{c_k^i\} \subset [0,\infty)$ and $\{d_k^i\} \subset [0,\infty)$, where $c_k^i \to 0$ and $d_k^i \to 0$ as $k \to \infty$, and let $S_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_s^i–Lipschitzian TAP mappings with the sequences $\{\eta_k^i\} \subset [0,\infty)$ and $\{l_k^i\} \subset [0,\infty)$, where $\eta_k^i \to 0$ and $l_k^i \to 0$ as $n \to \infty$, for all $i \in I$. Let $\mu_k = \max\{v_k, c_k, \eta_k\}$, where $v_k = \max\{v_k^i\}$, $c_k = \max\{c_k^i\}$ and $\eta_k = \max\{\eta_k^i\}$. Let $\xi_k = \max\{\lambda_k, d_k, l_k\}$, where $\lambda_k = \max\{\lambda_k^i\}$, $d_k = \max\{d_k^i\}$ and $l_k = \max\{l_k^i\}$. Assume that $\mathbf{\Gamma} = (\bigcap_{i=1}^N F(H_i)) \bigcap (\bigcap_{i=1}^N F(R_i)) \bigcap (\bigcap_{i=1}^N F(S_i)) \neq \emptyset$, and there exist $\mathcal{V}, \mathcal{V}^* > 0$ such that $\phi(e) \le \mathcal{V}^* e^2$ for all $e \ge \mathcal{V}$. Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{u_k\}$, $\{v_k\}$ and $\{s_k\}$ be sequences in [0,1] such that $u_k + v_k + s_k \le 1$, for all $k \in \mathbb{N}$. Let $\{t_k\}$ be the sequence defined by (5). Suppose the following assumptions hold:*

(J_1) $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(J_2) $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$;

(J_3) $\sum_{k=1}^{\infty} \alpha_k \mu_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \xi_k < \infty$;

(J_4) $\sum_{k=1}^{\infty} \alpha_k \beta_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \beta_k u_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \beta_k s_k < \infty$;

(J_5) $\alpha_k \beta_k L^2[(1 - u_k + v_k(L-1) - s_k)] < 1$, *$k \in \mathbb{N}$, where* $L = \max\{L_h^1, ..., L_h^N, L_r^1, ..., L_r^N, L_s^1, ..., L_s^N\}$.

Then, $\lim_{k\to\infty} \|t_k - q\|$ *exists for all* $q \in \mathbf{\Gamma}$.

Proof. Suppose $q \in \mathbf{\Gamma}$. Using (5), we obtain

$$\begin{aligned}
\|z_k - q\| &= \|(1 - u_k - v_k - s_k)(t_k - q) + u_k(t_{k-1} - q) \\
&\quad + v_k(S_{i(k)}^{n(k)} t_k - q) + s_k(H_{i(k)}^{n(k)} t_{k-1} - q)\| \\
&\le (1 - u_k - v_k - s_k)\|t_k - q\| + u_k\|t_{k-1} - q\| \\
&\quad + v_k\|S_{i(k)}^{n(k)} t_k - q\| + s_k\|H_{i(k)}^{n(k)} t_{k-1} - q\| \\
&\le \|t_k - q\| + u_k\|t_{k-1} - q\| \\
&\quad + v_k\|S_{i(k)}^{n(k)} t_k - q\| + s_k\|H_{i(k)}^{n(k)} t_{k-1} - q\| \\
&\le \|t_k - q\| + u_k\|t_{k-1} - q\| \\
&\quad + v_k L\|t_k - q\| + s_k L\|t_{k-1} - q\| \\
&\le (1 + L)\|t_k - q\| + (u_k + s_k L)\|t_{k-1} - q\|.
\end{aligned} \tag{18}$$

Using (5) and (18), we obtain

$$\begin{aligned}
\|w_k - q\| &= \|(1 - \beta_k)t_{k-1} + \beta_k R_{i(k)}^{n(k)} z_k - q\| \\
&\le (1 - \beta_k)\|t_{k-1} - q\| + \beta_k\|R_{i(k)}^{n(k)} z_k - q\| \\
&\le \|t_{k-1} - q\| + \beta_k L\|z_k - q\| \\
&\le \|t_{k-1} - q\| + \beta_k L[(1 + L)\|t_k - q\| + (u_k + s_k L)\|t_{k-1} - q\|] \\
&= [1 + \beta_k L(u_k + s_k L)]\|t_{k-1} - q\| + \beta_k L(1 + L)\|t_k - q\|.
\end{aligned} \tag{19}$$

Now, from (5) and Lemma 1, we obtain

$$\begin{aligned}
\|t_k - q\|^2 &\leq (1-\alpha_K)^2\|t_{k-1} - q\|^2 + 2\alpha_n\langle H_{i(k)}^{n(k)} w_k - q, j(t_k - q)\rangle \\
&= (1-\alpha_k)^2\|t_{k-1} - q\|^2 + 2\alpha_k\langle H_{i(k)}^{n(k)} w_k - H_{i(k)}^{n(k)} t_k + H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle \\
&= (1-\alpha_k)^2\|t_{k-1} - q\|^2 + 2\alpha_k\langle H_{i(k)}^{n(k)} w_k - H_{i(k)}^{n(k)} t_k, j(t_k - q)\rangle \\
&\quad + 2\alpha_k\langle H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle \\
&\leq (1-\alpha_k)^2\|t_{k-1} - q\|^2 + 2\alpha_k\|H_{i(k)}^{n(k)} w_k - H_{i(k)}^{n(k)} t_k\|\|t_k - q\| \\
&\quad + 2\alpha_k\langle H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle \\
&\leq (1-\alpha_k)^2\|t_{k-1} - q\|^2 + 2\alpha_k L\|w_k - t_k\|\|t_k - q\| \\
&\quad + 2\alpha_k\langle H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle.
\end{aligned} \tag{20}$$

According to (5), we have that

$$\begin{aligned}
\|w_k - t_k\| &\leq \beta_k\|R_{i(k)}^{n(k)} z_k - q\| + \beta_k\|t_{k-1} - q\| + \alpha_k\|t_{k-1} - q\| + \alpha_k\|H_{i(k)}^{n(k)} w_k - q\| \\
&\leq \beta_k L[(1+L)\|t_k - q\| + (u_k + s_k L)\|t_{k-1} - q\|] + \beta_k\|t_{k-1} - q\| \\
&\quad + \alpha_k\|t_{k-1} - q\| + \alpha_k L\{[1 + \beta_k L(u_k + s_k L)]\|t_{k-1} - q\| + \beta_k L(1+L)\|t_k - q\|\} \\
&\leq \beta_k L(1+L)\|t_k - q\| + \beta_n L(u_k + s_k L)\|t_{k-1} - q\| + \beta_k\|t_{k-1} - q\| \\
&\quad + \alpha_k\|t_{k-1} - q\| + \alpha_k L[1 + \beta_k L(u_k + s_k L)]\|t_{k-1} - q\| + \alpha_k\beta_k L^2(1+L)\|t_k - q\| \\
&= [\beta_k L(u_k + s_k L) + \alpha_k + \beta_k + \alpha_k L\beta_k L(u_k + s_k L)]\|t_{k-1} - q\| \\
&\quad + [\beta_k L(1+L) + \alpha_k\beta_k L^2(1+L)]\|t_k - q\| \\
&= [\beta_k L(u_k + s_k L)(1 + \alpha L) + \alpha_k(1+L) + \beta_k]\|t_{k-1} - q\| \\
&\quad + [\beta_k L(1+L)(1 + \alpha_k L)]\|t_k - q\| \\
&\leq [\beta_k L(u_k + s_k L)(1+L) + \alpha_k(1+L) + \beta_k]\|t_{k-1} - q\| \\
&\quad + [\beta_k L(1+L)(1+L)]\|t_k - q\| \\
&= \{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}\|t_{k-1} - q\| \\
&\quad + \beta_k L(1+L)^2\|t_k - q\|
\end{aligned} \tag{21}$$

Putting (21) into (20), we have

$$\begin{aligned}
\|t_k - q\|^2 &\leq (1-\alpha_k)^2\|t_{k-1} - p\|^2 + 2\alpha_k L[\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}\|t_{k-1} - q\| \\
&\quad + \beta_k L(1+L)^2\|t_k - q\|]\|t_k - q\| + 2\alpha_k\langle H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle \\
&= (1-\alpha_k)^2\|t_{k-1} - p\|^2 + 2\alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}\|t_{k-1} - q\|\|t_k - q\| \\
&\quad + 2\alpha_k\beta_k L^2(1+L)^2\|t_k - q\|^2 \\
&\quad + 2\alpha_k\langle H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle.
\end{aligned} \tag{22}$$

From a classical analysis, it is well known that

$$\|t_{k-1} - q\|\|t_k - q\| \leq \frac{1}{2}(\|t_{k-1} - q\|^2 + \|t_k - q\|^2). \tag{23}$$

Using (22) and (23), we have

$$\begin{aligned}\|t_k - q\|^2 &\leq (1-\alpha_k)^2\|t_{k-1}-q\|^2 + 2\alpha_n L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\} \\ &\quad \times\frac{1}{2}(\|t_{k-1}-q\|^2 + \|t_k - q\|^2) \\ &\quad +2\alpha_k\beta_k L^2(1+L)^2\|t_k - q\|^2 \\ &\quad +2\alpha_k\langle H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle \\ &\leq [(1-\alpha_k)^2 + \alpha_k L\{[\beta_n L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}]\|t_{k-1} - q\|^2 \\ &\quad +[2\alpha_k\beta_k L^2(1+L)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}]\|t_k - q\|^2 \\ &\quad +2\alpha_k\langle H_{i(k)}^{n(k)} t_k - q, j(t_k - q)\rangle.\end{aligned}$$

Since each H is a total asymptotically pseudocontractive mapping, from (24), we have

$$\begin{aligned}\|t_n - q\|^2 &\leq [(1-\alpha_k)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}]\|t_{k-1} - p\|^2 \\ &\quad +[2\alpha_k\beta_k L^2(1+L)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}]\|t_k - q\|^2 \\ &\quad +2\alpha_k(\|t_k - q\|^2 + \mu_k\phi(\|t_k - q\|) + \xi_k) \\ &= [(1-\alpha_k)^2 + \alpha_k L\{[\beta_n L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}]\|t_{k-1} - q\|^2 \\ &\quad +[2\alpha_k\beta_n L^2(1+L)^2 + \alpha_n L\{[\beta_n L(u_n + s_n L) + \alpha_k](1+L) + \beta_k\} + 2\alpha_k]\|t_n - q\|^2 \\ &\quad + + 2\alpha_k\mu_k\phi(\|t_k - p\|) + 2\alpha_k\xi_k.\end{aligned}$$

Since we know that ϕ is a strictly increasing function, it follows that $\phi(e) \leq \phi(\mathcal{V})$, if $e \leq \mathcal{V}$; $\phi(e) \leq \mathcal{V}^* e^2$, if $e \geq \mathcal{V}$. In either case, we can obtain

$$\phi(e) \leq \phi(\mathcal{V}) + \mathcal{V}^* e^2. \tag{24}$$

Using (24) and (24), we obtain

$$\begin{aligned}\|t_k - q\|^2 &\leq [(1-\alpha_k)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}]\|t_{k-1} - p\|^2 \\ &\quad +[2\alpha_k\beta_k L^2(1+L)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\} \\ &\quad +2\alpha_n]\|t_k - q\|^2 + 2\alpha_k\mu_k\phi(\mathcal{V}) + 2\alpha_k\mathcal{V}^*\mu_k\|t_k - q\|^2 + 2\alpha_k\xi_k \\ &= [(1-\alpha_k)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}]\|t_{k-1} - p\|^2 \\ &\quad +[2\alpha_k\beta_k L^2(1+L)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\} \\ &\quad +2\alpha_k + 2\alpha_k\mathcal{V}^*\mu_k]\|t_k - q\|^2 + 2\alpha_k\mu_k\phi(\mathcal{V}) + 2\alpha_k\xi_k \\ &= H_k\|t_{k-1} - q\|^2 + Q_k\|t_k - q\|^2 + 2\alpha_k\mu_k\phi(\mathcal{M}) + 2\alpha_k\xi_k,\end{aligned} \tag{25}$$

where

$$\begin{aligned}H_k &= (1-\alpha_k)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\}, \\ Q_k &= 2\alpha_k\beta_k L^2(1+L)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\} + 2\alpha_k + 2\alpha_k\mathcal{V}^*\mu_k.\end{aligned}$$

By transposing and simplifying (25), we obtain

$$\begin{aligned}\|t_k - q\|^2 &\leq \frac{H_k}{1-Q_k}\|t_{k-1} - q\|^2 + \frac{2\alpha_k\mu_k\phi(\mathcal{V})}{1-Q_k} + \frac{2\alpha_k\xi_k}{1-\Psi_k} \\ &= \left(1 + \frac{H_k + Q_k - 1}{1-Q_k}\right)\|t_{k-1} - p\|^2 + \frac{2\alpha_k\mu_k\phi(\mathcal{M})}{1-Q_k} \\ &\quad + \frac{2\alpha_k\xi_k}{1-Q_k}.\end{aligned} \tag{26}$$

Observe that

$$\begin{aligned} H_k + Q_k - 1 &= \alpha_k^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\} + 2\alpha_k\beta_k L^2(1+L)^2 \\ &\quad + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\} + 2\alpha_k \mathcal{V}^* \mu_k. \end{aligned} \tag{27}$$

Now, set

$$K_k = H_k + Q_k - 1. \tag{28}$$

Since $\lim_{n\to\infty} \alpha_k = 0$, then, from conditions (J_3)–(J_4), we obtain

$$\begin{aligned} Q_k &= 2\alpha_k\beta_k L^2(1+L)^2 + \alpha_k L\{[\beta_k L(u_k + s_k L) + \alpha_k](1+L) + \beta_k\} \\ &\quad + 2\alpha_k + 2\alpha_k \mathcal{V}^*\mu_k \to 0 \text{ as } k \to \infty. \end{aligned}$$

Therefore, a positive integer k_0 exists such that

$$\frac{1}{2} < 1 - Q_k \leq 1, \ \forall\, k \geq k_0.$$

Thus, using (26) we obtain

$$\begin{aligned} \|t_k - q\|^2 &\leq (1 + 2K_k)\|t_{k-1} - q\|^2 + 4\alpha_k\mu_k\phi(\mathcal{V}) + 4\alpha_k\xi_k \\ &= (1 + \Upsilon_k)\|t_{k-1} - q\|^2 + \varphi_k, , \ \forall\, k \geq k_0, \end{aligned} \tag{29}$$

where

$$\begin{aligned} \Upsilon_k &= 2K_k, \\ \varphi_k &= 4\alpha_k\mu_k\phi(\mathcal{V}) + 4\alpha_k\xi_k. \end{aligned}$$

According to assumptions (J_2)–(J_4), it follows that $\sum_{k=1}^{\infty} \Upsilon_k < \infty$ and $\sum_{k=1}^{\infty} \varphi_k < \infty$. Obviously, from (29), it clear that all the assumptions in Lemma 2 are performed. Hence, $\lim_{k\to\infty} \|t_k - q\|$ exists for all $q \in \mathbf{\Gamma}$. □

Theorem 1. *Let G denote an arbitrary Banach space and $\mathcal{E}$ denote a nonempty closed convex subset of G. Let $H_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_h^i–Lipschitzian TAP mappings with the sequences $\{\nu_k^i\} \subset [0, +\infty)$ and $\{\lambda_k^i\} \subset [0, +\infty)$, where $\nu_k^i \to 0$ and $\lambda_k^i \to 0$ as $k \to +\infty$. Let $R_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_r^i–Lipschitzian TAP mappings with the sequences $\{c_k^i\} \subset [0, +\infty)$ and $\{d_k^i\} \subset [0, +\infty)$, where $c_k^i \to 0$ and $d_k^i \to 0$ as $k \to \infty$, and let $S_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_s^i–Lipschitzian TAP mappings with the sequences $\{\eta_k^i\} \subset [0, +\infty)$ and $\{l_k^i\} \subset [0, +\infty)$, where $\eta_k^i \to 0$ and $l_k^i \to 0$ as $k \to +\infty$, for all $i \in I$. Let $\mu_k = \max\{\nu_k, c_k, \eta_k\}$, where $\nu_k = \max\{\nu_k^i\}$, $c_k = \max\{c_k^i : i \in I\}$ and $\eta_k = \max\{\eta_k^i\}$. Let $\xi_k = \max\{\lambda_k, d_k, l_k\}$, where $\lambda_k = \max\{\lambda_k^i\}$, $d_k = \max\{d_k^i\}$ and $l_k = \max\{l_k^i\}$. Assume that $\mathbf{\Gamma} = (\bigcap_{i=1}^N F(H_i)) \bigcap (\bigcap_{i=1}^N F(R_i)) \bigcap (\bigcap_{i=1}^N F(S_i)) \neq \emptyset$ and there exist $\mathcal{V}, \mathcal{V}^* > 0$ such that $\phi(e) \leq \mathcal{V}^* e^2$ for all $e \geq \mathcal{V}$. Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{u_k\}$, $\{v_k\}$ and $\{s_k\}$ be sequences in [0,1] such that $u_k + v_k + s_k \leq 1$, for all $k \in \mathbb{N}$. Suppose the following assumptions hold:*

(J_1) $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(J_2) $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$;

(J_3) $\sum_{k=1}^{\infty} \alpha_k\mu_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k\xi_k < \infty$;

(J_4) $\sum_{k=1}^{\infty} \alpha_k\beta_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k\beta_k u_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k\beta_k s_k < \infty$;

(J_5) $\alpha_k\beta_k L^2[(1-u_k+v_k(L-1)-s_k)] < 1$, $k\in\mathbb{N}$, *where* $L=\max\{L_h^1,...,L_h^N, L_r^1,...,L_r^N, L_s^1,...,L_s^N\}$.

Then, the sequence $\{t_k\}$*, defined by* (5)*, converges strongly to an element in* $\mathbf{\Gamma}$ *if and only if*

$$\liminf_{k\to\infty} d(t_k,\mathbf{F}) = \mathbf{0}. \tag{30}$$

Proof. Observe that the necessity of condition (30) is trivial.

Now, we prove the sufficiency of Theorem 1. For all $q\in\mathbf{\Gamma}$, then, from (29) in Lemma 2, we have that

$$[d(t_k,\mathbf{\Gamma})]^2 \le (1+\Upsilon_k)[d(t_{k-1},\mathbf{\Gamma})]^2 + \varphi_k, \text{ for all } k\ge k_0. \tag{31}$$

Obviously, from assumptions (J_2)–(J_4), we know that $\sum_{k=1}^{\infty}\Upsilon_k<\infty$ and $\sum_{k=1}^{\infty}\varphi_k<\infty$. According to (31) and Lemma 2, $\lim_{k\to\infty}[d(t_k,,\mathbf{F})]^2$ exists. Furthermore, $\lim_{n\to\infty} d(t_k,\mathbf{\Gamma})$ exists. According to (30), we obtain

$$\lim_{k\to\infty} d(t_k,\mathbf{\Gamma}) = 0. \tag{32}$$

Now, we show that the sequence $\{t_k\}$ is Cauchy in $\mathcal{E}$. Clearly, since $\sum_{k=1}^{\infty}\varphi_k<\infty$, then $1+x\le e^x$ for each $x>0$, and, from (29), we therefore have

$$\|t_k-q\|^2 \le e^{\Upsilon_k}\|t_{k-1}-q\|^2+\varphi_k, \text{ for all } k\ge k_0. \tag{33}$$

For any given positive integers $k,l\ge k_0$, using (33), we obtain

$$\begin{aligned}
\|t_{k+l}-q\|^2 &\le e^{\Upsilon_{k+l}}\|t_{k+l-1}-q\|^2+\varphi_{k+l}\\
&\le e^{\Upsilon_{k+l}}[e^{\Upsilon_{k+l-1}}\|t_{k+l-2}-q\|^2+\varphi_{k+l-1}]+\sigma_{k+l}\\
&\le e^{\Upsilon_{k+l}+\Upsilon_{k+l-1}}\|t_{k+l-2}-q\|^2+\varphi_{k+l-1}]+\varphi_{k+l}\\
&\le \cdots\\
&\le e^{\sum_{i=k+1}^{k+l}\Upsilon_i}\|t_k-q\|^2+e^{\sum_{i=k+2}^{k+l}\Upsilon_i}\sum_{i=k+1}^{k+l}\varphi_i\\
&\le \varrho\|t_k-q\|^2+\varrho\sum_{i=k+1}^{\infty}\psi_i,
\end{aligned} \tag{34}$$

where $\varrho=e^{\sum_{k=1}^{\infty}\Upsilon_k}<\infty$.

Since $\lim_{k\to\infty}\varphi_k<\infty$, then using (32) and for any given $\epsilon>0$, there exists a positive integer $k_1\ge k_0$ such that

$$[d(t_k,\mathbf{\Gamma})]^2<\frac{\epsilon^2}{8(\varrho+1)},\quad \sum_{i=k+1}^{\infty}\varphi_i<\frac{\epsilon^2}{4\varrho}, \text{ for all } k\ge k_1. \tag{35}$$

Therefore, there exists $q_1\in\mathbf{\Gamma}$ such that

$$\|t_k-q_1\|^2<\frac{\epsilon^2}{8(\varrho+1)}, \text{ for all } k\ge k_1. \tag{36}$$

Consequently, for any $k \geq k_1$ and for all $l \geq 1$, we obtain

$$\begin{aligned} \|t_{k+l} - t_k\|^2 &\leq 2(\|t_{k+l} - q_1\|^2 + \|t_n - q_1\|^2) \\ &\leq 2(1+\varrho)\|t_k - q_1\|^2 + 2\varrho \sum_{i=k+1}^{\infty} \varphi_i \\ &< 2 \cdot \frac{\epsilon^2}{4(\varrho+1)}(1+\varrho) + 2\varrho \cdot \frac{\epsilon^2}{4\varrho} \\ &= \epsilon^2, \end{aligned}$$

i.e.,

$$\|t_{k+l} - t_k\| < \epsilon.$$

It follows that the sequence $\{t_k\}$ is Cauchy in $\mathcal{E}$. Since $\mathcal{E}$ is a complete space, we can say that $t_k \to q^* \in \mathcal{E}$.

Next, we show that $q^* \in \mathbf{\Gamma}$. To prove by contradiction, we assume that q^* is not in $\mathbf{\Gamma} = (\bigcap_{i=1}^{N} F(H_i)) \bigcap (\bigcap_{i=1}^{N} F(R_i)) \bigcap (\bigcap_{i=1}^{N} F(S_i)) \neq \emptyset$. Since $\mathbf{\Gamma}$ is a closed subset of G, it follows that $d(q^*, \mathbf{\Gamma}) > 0$. Thus, for all $q^* \in \mathbf{\Gamma}$, we have

$$\|q^* - q\| \leq \|q^* - t_k\| + \|t_k - q\|, \tag{37}$$

which implies that

$$d(q^*, \mathbf{\Gamma}) \leq \|t_k - q^*\| + d(t_k, \mathbf{\Gamma}). \tag{38}$$

Therefore, we have $d(q^*, \mathbf{\Gamma}) = 0$ as $k \to \infty$, which is in contradiction to $d(q^*, \mathbf{\Gamma}) > 0$. Hence, $q^* \in \mathbf{F}$. This completes the proof. □

The following results are obtained directly from Theorem 1:

Corollary 1. *Let G denote an arbitrary Banach space and $\mathcal{E}$ denote a nonempty closed convex subset of G. Let $H_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_h^i–Lipschitzian TAP mappings with the sequences $\{\nu_k^i\} \subset [0, +\infty)$ and $\{\lambda_k^i\} \subset [0, +\infty)$, where $\nu_k^i \to 0$ and $\lambda_k^i \to 0$ as $k \to +\infty$. Let $R_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_r^i–Lipschitzian TAP mappings with the sequences $\{c_k^i\} \subset [0, +\infty)$ and $\{d_k^i\} \subset [0, +\infty)$, where $c_k^i \to 0$ and $d_k^i \to 0$ as $k \to +\infty$, and let $S_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_s^i–Lipschitzian TAP mappings with the sequences $\{\eta_k^i\} \subset [0, +\infty)$ and $\{l_k^i\} \subset [0, +\infty)$, where $\eta_k^i \to 0$ and $l_k^i \to 0$ as $k \to +\infty$, for all $i \in I$. Let $\mu_k = \max\{\nu_k, c_k, \eta_k\}$, where $\nu_k = \max\{\nu_k^i\}$, $c_k = \max\{c_k^i : i \in I\}$ and $\eta_k = \max\{\eta_k^i\}$. Let $\xi_k = \max\{\lambda_k, d_k, l_k\}$, where $\lambda_k = \max\{\lambda_k^i\}$, $d_k = \max\{d_k^i\}$ and $l_k = \max\{l_k^i\}$. Suppose that $\mathbf{\Gamma} = (\bigcap_{i=1}^{N} F(H_i)) \bigcap (\bigcap_{i=1}^{N} F(R_i)) \bigcap (\bigcap_{i=1}^{N} F(S_i)) \neq \emptyset$ and there exist $\mathcal{V}, \mathcal{V}^* > 0$ such that $\phi(e) \leq \mathcal{V}^* e^2$ for all $e \geq \mathcal{V}$. Let $\{\alpha_k\}$, $\{\beta_k\}$ and $\{v_k\}$ be sequences in [0,1], for all $k \in \mathbb{N}$. Suppose the following assumptions hold:*

(J_1) $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(J_2) $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$;

(J_3) $\sum_{k=1}^{\infty} \alpha_k \mu_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \xi_k < \infty$;

(J_4) $\sum_{k=1}^{\infty} \alpha_k \beta_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \beta_k u_k < \infty$,

(J_5) $\alpha_k \beta_k L^2 [v_k(L-1)] < 1$, $k \in \mathbb{N}$, *where* $L = \max\{L_h^1, ..., L_h^N, L_r^1, ..., L_r^N, L_s^1, ..., L_s^N\}$.

Let be $\{t_k\}$ the sequence defined by:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} w_k, \\ w_k = (1-\beta_k)t_{k-1} + \beta_k R_{i(k)}^{n(k)} z_k, \\ z_k = (1-v_k)t_k + v_n S_{i(k)}^{n(k)} t_k, \end{cases} \quad k \in \mathbb{N}. \tag{39}$$

Then, $\{t_k\}$ converges to an element in $\boldsymbol{\Gamma}$ if and only if

$$\liminf_{k\to\infty} d(t_k, \boldsymbol{\Gamma}) = 0. \tag{40}$$

Proof. Put $u_k = s_k = 0$ in Theorem (1). □

Corollary 2. *Let G denote an arbitrary Banach space and $\mathcal{E}$ a nonempty closed convex subset of G. Let $H_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_h^i–Lipschitzian TAP mappings with sequences $\{\nu_k^i\} \subset [0, +\infty)$ and $\{\lambda_k^i\} \subset [0, +\infty)$, where $\nu_k^i \to 0$ and $\lambda_k^i \to 0$ as $k \to +\infty$. Let $\mu_k = \max\{\nu_k^i\}$ and $\xi_k = \max\{\lambda_k^i\}$. Assume that $\boldsymbol{\Gamma} \neq \emptyset$ and there exist $\mathcal{V}, \mathcal{V}^* > 0$ such that $\phi(e) \leq \mathcal{V}^* e^2$ for all $e \geq \mathcal{V}$. Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{u_k\}$, $\{v_k\}$ and $\{s_k\}$ be sequences in [0,1] such that $u_k + v_k + s_k \leq 1$, for all $k \in \mathbb{N}$. Suppose the following assumptions hold:*

(J_1) $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(J_2) $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$;

(J_3) $\sum_{k=1}^{\infty} \alpha_k \mu_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \xi_k < \infty$;

(J_4) $\sum_{k=1}^{\infty} \alpha_k \beta_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \beta_k u_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \beta_k s_k < \infty$;

(J_5) $\alpha_k \beta_k L^2[(1 - u_k + v_k(L-1) - s_k)] < 1$, $k \in \mathbb{N}$, *where* $L = \max\{L_h^1, ..., L_h^N\}$.

Let be $\{t_k\}$ the sequence defined by:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} w_k, \\ w_k = (1-\beta_k)t_{k-1} + \beta_k H_{i(k)}^{n(k)} z_k, \\ z_k = (1-u_k-v_k-s_k)t_k + u_k t_{k-1} + v_n H_{i(k)}^{n(k)} t_k + s_k t_{k-1}, \end{cases} \quad k \in \mathbb{N}. \tag{41}$$

Then, $\{t_k\}$ converges to a unique element in $\boldsymbol{\Gamma}$ if and only if

$$\liminf_{k\to\infty} d(t_k, \boldsymbol{\Gamma}) = 0. \tag{42}$$

Proof. If we set $S_i = R_i = H_i$ in Theorem 1, then the required result follows. □

Corollary 3. *Let G denote an arbitrary Banach space and $\mathcal{E}$ a nonempty closed convex subset of G. Let $H_i : \mathcal{E} \to \mathcal{E}$ be a finite family of uniformly L_h^i–Lipschitzian TAP mappings with sequences $\{\nu_k^i\} \subset [0, +\infty)$ and $\{\lambda_k^i\} \subset [0, +\infty)$, where $\nu_k^i \to 0$ and $\lambda_k^i \to 0$ as $k \to \infty$. Let $\mu_k = \max\{\nu_k^i\}$ and $\xi_k = \max\{\lambda_k^i\}$. Assume that $\boldsymbol{\Gamma} = \bigcap_{i=1}^N F(H_i) \neq \emptyset$ and there exist $\mathcal{V}, \mathcal{V}^* > 0$ such that $\phi(e) \leq \mathcal{V}^* e^2$ for all $e \geq \mathcal{V}$. Let $\{\alpha_k\}$, $\{\beta_k\}$ and $\{v_k\}$ be sequences in [0,1], for all $k \in \mathbb{N}$. Suppose the following assumptions hold:*

(J_1) $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(J_2) $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$;

(J_3) $\sum_{k=1}^{\infty} \alpha_k \mu_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \xi_k < \infty$;

(J_4) $\sum_{k=1}^{\infty} \alpha_k \beta_k < \infty$;

(J_5) $\alpha_k \beta_k L^2[(1 - v_k(L-1)] < 1$, $k \in \mathbb{N}$, *where* $L = \max\{L_t^1, ..., L_t^N\}$.

Let $\{t_k\}$ *be the sequence defined by:*

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} w_k, \\ w_k = (1-\beta_k)t_{k-1} + \beta_k H_{i(k)}^{n(k)} z_k, \\ z_k = (1-v_k)t_k + v_n H_{i(k)}^{n(k)} t_k, \end{cases} \quad k \in \mathbb{N}. \tag{43}$$

Then, $\{t_k\}$ *converges to a unique element in* $\mathbf{\Gamma}$ *if and only if*

$$\liminf_{n\to\infty} d(t_k, \mathbf{\Gamma}) = 0. \tag{44}$$

Proof. Put $u_k = s_k = 0$ in Corollary 2; then, the desired result follows immediately. □

Corollary 4. *Let G denote an arbitrary Banach space and* $\mathcal{E}$ *denote a nonempty closed convex subset of G. Let* $H_i : \mathcal{E} \to \mathcal{E}$ *be a finite family of uniformly* L_h^i*–Lipschitzian TAP mapping with the sequences* $\{\nu_k^i\} \subset [0, +\infty)$ *and* $\{\lambda_k^i\} \subset [0, +\infty)$*, where* $\nu_k^i \to 0$ *and* $\lambda_k^i \to 0$ *as* $k \to +\infty$*. Let* $\mu_k = \max\{\nu_k^i\}$ *and* $\xi_k = \max\{\lambda_k^i\}$*. Assume that* $\mathbf{\Gamma} \neq \emptyset$ *and there exist* $\mathcal{V}, \mathcal{V}^* > 0$ *such that* $\phi(e) \leq \mathcal{V}^* e^2$ *for all* $e \geq \mathcal{V}$*. Let the sequences* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *be in [0,1], for all* $k \in \mathbb{N}$*. Suppose the following assumptions hold:*

(J_1) $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(J_2) $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$;

(J_3) $\sum_{k=1}^{\infty} \alpha_k \mu_k < \infty$, $\sum_{k=1}^{\infty} \alpha_k \xi_k < \infty$;

(J_4) $\sum_{k=1}^{\infty} \alpha_k \beta_k < \infty$;

(J_5) $\alpha_k \beta_k L^2 < 1$, $k \in \mathbb{N}$, *where* $L = \max\{L_t^1, ..., L_t^N\}$.

Let $\{t_k\}$ *be the sequence defined by:*

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} w_k, \\ w_k = (1-\beta_k)t_{k-1} + \beta_k H_{i(k)}^{n(k)} t_k, \end{cases} \quad k \in \mathbb{N}. \tag{45}$$

Then, $\{t_k\}$ *converges to an element in* $\mathbf{\Gamma}$ *if and only if*

$$\liminf_{k\to\infty} d(t_k, \mathbf{\Gamma}) = 0. \tag{46}$$

Proof. Set $v_k = 0$ in Corollary 3. □

Corollary 5. *Let G denote an arbitrary Banach space and* $\mathcal{E}$ *denote a nonempty closed convex subset of G. Let* $H_i : \mathcal{E} \to \mathcal{E}$ *be a finite family of uniformly* L_h^i*–Lipschitzian TAP mappings with the sequences* $\{\nu_k^i\} \subset [0, +\infty)$ *and* $\{\lambda_k^i\} \subset [0, +\infty)$ *, where* $\nu_k^i \to 0$ *and* $\lambda_k^i \to 0$ *as* $k \to +\infty$*. Let* $\mu_k = \max\{\nu_k^i\}$ *and* $\xi_k = \max\{\lambda_k^i\}$*. Suppose* $\mathbf{\Gamma} = \bigcap_{i=1}^N F(H_i) \neq \emptyset$*. Assume that there exist* $\mathcal{V}, \mathcal{V}^* > 0$ *such that* $\phi(e) \leq \mathcal{V}^* e^2$ *for all* $e \geq \mathcal{V}$*. Let* $\{\alpha_k\}$ *be a sequence in [0,1], for all* $k \in \mathbb{N}$*. Suppose that the following assumptions hold:*

(J_1) $\sum\limits_{k=1}^{\infty} \alpha_k = \infty;$

(J_2) $\sum\limits_{k=1}^{\infty} \alpha_k^2 < \infty;$

(J_3) $\sum\limits_{k=1}^{\infty} \alpha_k \mu_k < \infty, \sum\limits_{k=1}^{\infty} \alpha_k \xi_k < \infty.$

Let be $\{t_k\}$ the sequence defined by:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H_{i(k)}^{n(k)} t_{k-1}, \end{cases} \qquad k \in \mathbb{N}. \tag{47}$$

Then, $\{t_k\}$ converges to a unique element in $\mathbf{\Gamma}$ if and only if

$$\liminf_{k\to\infty} d(t_k, \mathbf{\Gamma}) = 0. \tag{48}$$

Proof. If we set $\beta_k = 0$ in Corollary 4, then the required result follows immediately. □

Corollary 6. *Let G and $\mathcal{E}$ be as defined in Lemma (3). Let $H : \mathcal{E} \to \mathcal{E}$ be a L_h–Lipschitzian pseudocontractive mapping. Suppose $\Gamma = F(H) \neq \emptyset$. Let $\{\alpha_k\}$ be a sequence in [0,1], for all $k \in \mathbb{N}$. Suppose the following assumptions hold:*

(J_1) $\sum\limits_{k=1}^{\infty} \alpha_k = \infty;$

(J_2) $\sum\limits_{k=1}^{\infty} \alpha_k^2 < \infty.$

Let $\{t_k\}$ be the sequence defined by:

$$\begin{cases} t_0 \in \mathcal{E}, \\ t_k = (1-\alpha_k)t_{k-1} + \alpha_k H t_{k-1}. \end{cases} \qquad k \in \mathbb{N}, \tag{49}$$

Then, $\{t_k\}$ converges a unique element in Γ if and only if

$$\liminf_{k\to\infty} d(t_k, \Gamma) = 0. \tag{50}$$

Proof. For $k = 1$, set $N = 1$ and $\xi_k = \mu_k = 0$ in Corollary 5. □

Corollaries 1–6 are some of the several results one can derive from Theorem 1.

4. Numerical Example

In this section, we give some numerical examples to support the claims in our main results.

Example 1. *Let $G = (-\infty, +\infty)$ with the usual norm and $\mathcal{E} = [0, +\infty)$. Let $\phi : [0, +\infty) \to [0, +\infty)$ be a strictly increasing function with $\phi(0) = 0$. For $N = 2$, let $\{H_i\}_{i=1}^2, \{R_i\}_{i=1}^2$ and $\{S_i\}_{i=1}^2 : \mathcal{E} \to \mathcal{E}$ be defined by:*

$$\begin{aligned} H_1 p &= \frac{5t}{2(1+t)}, \ t \in [0, +\infty), \\ H_2 p &= \frac{4t}{(1+t)}, \ t \in [0, +\infty), \\ R_1 p &= \frac{2p}{1+2t}, \quad t \in [0, +\infty), \\ R_2 p &= \frac{6t}{1+\alpha t}, \ t \in [0, +\infty) \ \text{and } \alpha \text{ is closing to zero}, \ \forall v \in \mathbb{N}, \end{aligned}$$

$$\begin{aligned} S_1 p &= \frac{t^3}{1+t^2},\ t \in [0,+\infty), \\ S_2 p &= \frac{3t}{4},\ t \in [0,+\infty). \end{aligned}$$

Now, if we define two sequences $\{\xi_k\}, \{\mu_k\} \in \mathbb{R}^+$ *by* $\xi_k = \mu_k = \frac{1}{k}$, $k \in \mathbb{N}$, *then we know that* $\xi_k \to 0$ *and* $\mu_k \to 0$ *as* $k \to +\infty$. *Clearly,* $\{H_i\}_{i=1}^2, \{R_i\}_{i=1}^2$ *and* $\{S_i\}_{i=1}^2$ *are asymptotically pseucontractive mappings with the constant sequence* $\{h_k\} = \{1\}$ *for all* $k \geq 1$ *and also uniformly Lipschitzian mappings on* $[0,+\infty)$, *and, hence, they are total asymptotically pseudocontractive mappings. Obviously,* $\Gamma = (\bigcap_{i=1}^2 F(M_i)) \bigcap (\bigcap_{i=1}^2 F(H_i)) \bigcap (\bigcap_{i=1}^2 F(G_i)) = \{0\} \neq \emptyset$.

Let the control parameters be defined as:

$$\alpha_k = \beta_k = u_k = v_k = s_k = \frac{1}{k+1}, k \in \mathbb{N}.$$

For arbitrary $t_0 \in \mathcal{E}$, *the sequence* $\{t_k\}_{k=1}^{+\infty} \in \mathcal{E}$, *defined by* (5), *converges strongly to the common fixed point of* H_i, R_i *and* S_i $(i = 1, 2)$ *which is* 0.

From the above example, it is easy to see that all of the assumptions in Theorem 1 are fulfilled. Hence, it implies that our results are applicable.

Example 2. *Let* $G = \Re^2$ *and* $\mathcal{E} = \{t = (t_1, t_2) : (t_1, t_2) \in [0,1] \times [0,1]\}$ *be subset of G with the norm* $\|t\| = \|(t_1, t_2)\| = |t_1| + |t_2|$. *For* $N = 1$, *let* $H, R, S : \mathcal{E} \to \mathcal{E}$ *be defined by*

$$H(t_1,t_2) = \begin{cases} \left(\frac{t_1}{2}, \frac{t_2}{2}\right), & \text{if } (t_1,t_2) \in [0,\frac{1}{2}) \times [0,\frac{1}{2}) \\ (0,0), & \text{if } (t_1,t_2) \in [\frac{1}{2},1] \times [\frac{1}{2},1]. \end{cases}$$

$$R(t_1,t_2) = \begin{cases} \left(\frac{t_1}{3}, \frac{t_2}{3}\right), & \text{if } (t_1,t_2) \in [0,\frac{1}{2}) \times [0,\frac{1}{2}) \\ (0,0), & \text{if } (t_1,t_2) \in [\frac{1}{2},1] \times [\frac{1}{2},1]. \end{cases}$$

$$S(t_1,t_2) = \begin{cases} \left(\frac{t_1}{4}, \frac{t_2}{4}\right), & \text{if } (t_1,t_2) \in [0,\frac{1}{2}) \times [0,\frac{1}{2}) \\ (0,0), & \text{if } (t_1,t_2) \in [\frac{1}{2},1] \times [\frac{1}{2},1]. \end{cases}$$

Let ϕ *be a strictly increasing self-function defined on* $[0,\infty)$ *such that* $\phi(0) = 0$. *Now, if we define two sequences* $\{\xi_k\}, \{\mu_k\} \in \Re^+$ *by* $\xi_k = \mu_k = \frac{1}{k}$, $k \in \mathbb{N}$, *then we know that* $\xi_k \to 0$ *and* $\mu_k \to 0$ *as* $k \to +\infty$. *Clearly, the mappings H, R and S are asymptotically pseudocontractive mappings with* $h_k = 1$. *This implies that H, R and S are total asymptotically pseudocontractive mappings. Observe that the common fixed point of H, R and S is (0,0), i.e.,* $\mathbf{\Gamma} = F(H) \cap F(R) \cap F(S) = \{(0,0)\}$.

For the initial values $(0.2, 0.4)$, $(0.1, 0.3)$, $(0.5, 0.7)$ *and parameters* $\alpha_k = \beta_k = u_k = v_k = s_k = \frac{1}{k+1}$, $k \in \mathbb{N}$, *in five iterations, we show that our new iterative method* (5) *converges to* $(0,0)$ *as follows:*

For $k = 1$ *and from* (5), *we have*

$$\begin{aligned} z_1 &= -\frac{1}{2}t_1 + \frac{1}{2}t_0 + \frac{1}{8}t_1 + \frac{1}{2}t_0 = -\frac{3}{8}t_1 + t_0. \\ w_1 &= \frac{1}{2}t_0 + \frac{1}{6}\left[-\frac{3}{8}t_1 + t_0\right] = -\frac{1}{16}t_1 + \frac{2}{3}t_0. \\ t_1 &= \frac{1}{2}t_0 + \frac{1}{4}\left[-\frac{1}{16}t_1 + \frac{2}{3}t_0\right] \implies t_1 = \frac{128}{195}t_0. \end{aligned}$$

For $k = 2$ *and from* (5), *we have*

$$\begin{aligned} z_2 &= \frac{1}{3}t_1 + \frac{1}{12}t_2 + \frac{1}{3}t_1 = \frac{2}{3}t_1 + \frac{1}{12}t_2. \\ w_2 &= \frac{2}{3}t_+ \frac{1}{9}\left[\frac{2}{3}t_1 + \frac{1}{12}t_2\right] = \frac{20}{27}t_1 + \frac{1}{108}t_2. \\ t_2 &= \frac{2}{3}t_1 + \frac{1}{6}\left[\frac{20}{27}t_1 + \frac{1}{108}t_2\right] \implies t_2 = \frac{41408}{52488}t_1. \end{aligned}$$

For $k = 3$ and from (5), we have

$$\begin{aligned} z_3 &= \frac{1}{4}t_3 + \frac{1}{4}t_2 + \frac{1}{16}t_3 + \frac{1}{4}t_2 = \frac{5}{16}t_3 + \frac{1}{2}t_2. \\ w_3 &= \frac{3}{4}t_2 + \frac{1}{12}\left[\frac{5}{16}t_3 + \frac{1}{2}t_2\right] = \frac{19}{24}t_2 + \frac{5}{192}t_3. \\ t_3 &= \frac{3}{4}t_2 + \frac{1}{8}\left[\frac{19}{24}t_2 + \frac{5}{192}t_3\right] \implies t_3 = \frac{250368}{293952}t_2 \end{aligned}$$

For $k = 4$ and from (5), we have

$$\begin{aligned} z_4 &= \frac{2}{5}t_4 + \frac{1}{5}t_3 + \frac{1}{20}t_4 + \frac{1}{5}t_3 = \frac{9}{20}t_4 + \frac{2}{5}t_3. \\ w_4 &= \frac{4}{5}t_3 + \frac{1}{15}\left[\frac{9}{20}t_4 + \frac{2}{5}t_3\right] = \frac{62}{75}t_3 + \frac{3}{100}t_4. \\ t_4 &= \frac{4}{5}t_3 + \frac{1}{10}\left[\frac{62}{75}t_3 + \frac{3}{100}t_4\right] \implies t_4 = \frac{33100}{373875}t_3. \end{aligned}$$

Now, in Table 1, we illustrate the convergence of (5) for various initial values.

Table 1. Convergence of (5) for various initial values.

Step	Initial Value 1	Initial Value 2	Initial Value 3
1	(0.20000, 0.40000)	(0.10000, 0.30000)	(0.50000, 0.70000)
1	(0.13128, 0.26256)	(0.06564, 0.19692)	(0.32821, 0.45949)
2	(0.10357, 0.20713)	(0.05178, 0.15535)	(0.25893, 0.36249)
3	(0.08821, 0.17642)	(0.04410, 0.13232)	(0.22054, 0.30874)
4	(0.07809, 0.15619)	(0.03904, 0.11715)	(0.19525, 0.27333)

Figure 1 shows the convergence of new method for different initial values.

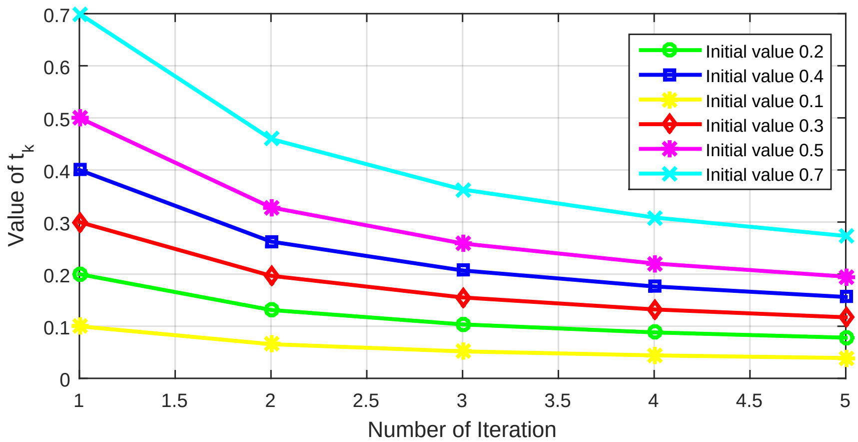

Figure 1. Graph corresponding to Table 1.

5. Application to Delay Caputo Fractional Differential Equations

In [32], Mandelbort noticed that there are several fractional dimension phenomena existing in technology and nature; namely, several physical systems have fractional-order dynamical behaviors because of their chemical properties and special materials. For this, fractional calculus, which is a generalization of the ordinary differentiation and integration to an arbitrary non-integer order, has been applied in various fields of science and engineering, specifically, control systems, electrical engineering, signal processing, viscoelastic mechanics, physics, biology and many others [33,34]. In [35], Richard observed that phenomena of delay exist in many physical processes.

In this section, we consider the following delay Caputo fractional differential equation:

$$ {}^{c}\mathcal{D}t(w) = f(w, t(w), t(w-\rho), \; w \in [e, V], \tag{51} $$

with the initial conditions

$$ t(w) = \varrho(w), \; w \in [e-\eta, e], \tag{52} $$

where $\gamma \in (0,1)$, $\rho > 0$, $V > 0$, $\eta > 0$, $\varrho \in C([e-\rho, e] : \Re^k)$, $t \in \Re^k$ is a continuous mapping, and $f : [e, V] \times \Re^k \times \Re^k \to \Re^k$ is a continuous mapping. We opine that the following assumptions are performed:

(Z_1) There exists a Lipschitz constant $L_f > 0$ such that

$$ \|f(w, t_1, h_1) - f(w, t_2, h_2)\| \leq L_f(\|t_1 - h_1\| + \|t_2 - h_2\|) $$

for each $w \in \Re^+$ and $g_1, h_1, g_2, h_2 \in \Re^k$;

(Z_2) There exists a constant $\delta_L > 0$ such that $\frac{2L}{\delta_L} < 1$.

If $q \in C([e-\rho, V] : \Re^k) \cap C^1([e, V] : \Re^k)$ is a function that satisfies (51) and (52), then q is called the solution to problems (51) and (52). It is shown in [36] that the solution to the following integral equation is equivalent to the solution to problems (51) and (52):

$$ t(w) = \varrho(e) + \frac{1}{\Gamma(\gamma)} \int_e^w (w-\eta)^{(\gamma-1)} f(\eta, t(\eta), t(\eta-\rho)) d\eta, \; \forall \; w \in [e, V], \tag{53} $$

where $t(w) = \varrho(w), \forall \in [e-\eta, e]$. Let the norm $\| \cdot \|_{\delta_L}$ on $C([e-\rho, V] : \Re^k)$ be defined by

$$ \|\varrho\|_{\delta_L} = \frac{\sup \|\varrho(w)\|}{E_\gamma(\delta_L w_\gamma)} \text{ for all } \varrho \in C([e-\rho, e] : \Re^k), \tag{54} $$

where $E_\gamma : \Re \to \Re$ is called the Mittag–Leffler function, which is defined as follows:

$$ E_\gamma(w) = \sum_{k=0}^{\infty} \frac{w_k}{\Gamma(\gamma k + 1)}, \text{ for all } w \in \Re. $$

Obviously, $C([e-\rho, e] : \Re^k, \| \cdot \|_{\delta_L})$ is a Banach space [4].

Under assumption (Z_1), Wang et al. [34] gave the existence and uniqueness results of problems (51) and (52). In this article, we apply the iterative scheme (49) in Corollary 6 to approximate the solution to the delay Caputo fractional differential Equations (51) and (52).

Now, our main result is given here in the following theorem:

Theorem 2. *Let the functions t and ϱ be the same as defined above. Suppose assumptions (Z_1)–(Z_2) are fulfilled. Then, the sequence defined by* (49) *converges to a unique solution of problems* (51) *and* (52)*, denoted as q, in $G = C([e-\rho, V] : \Re^k) \cap C^1([e, V] : \Re^k)$.*

Proof. We define an operator $H : G \to G$ as:

$$Ht(w) = \begin{cases} \varrho(e) + \frac{1}{\Gamma(\gamma)} \int_e^w (w-\eta)^{(\gamma-1)} f(\eta, t(\eta), t(\eta-\rho)) d\eta, & w \in [e, V], \\ \varrho(w), & w \in [e-\eta, e]. \end{cases}$$

Now, we show that $t_k \to q$ as $k \to \infty$. If $w \in [e-\eta, e]$, then it is not hard to see that $t_k \to q$ as $k \to \infty$. Next, if $w \in [e, V]$, then using (49) and assumptions (Z_1)–(Z_2), we have

$$\begin{aligned} \|g_k - q\| &= \|(1-\alpha_k)t_{k-1} + \alpha_k H t_{k-1} - q\| \\ &\leq (1-\alpha_k)\|t_{k-1} - q\| + \alpha_k \|Ht_{k-1} - q\|. \end{aligned} \quad (55)$$

Using the supremum over $[e-\rho, V]$ on both sides of (55), we obtain

$$\begin{aligned} \sup_{w\in[e-\rho,V]} \|g_k - q\| &\leq (1-\alpha_k) \sup_{w\in[e-\rho,V]} \|t_{k-1} - q\| + \alpha_k \sup_{w\in[e-\rho,V]} \|Ht_{k-1} - Hq\| \\ &\leq (1-\alpha_k) \sup_{w\in[e-\rho,V]} \|t_{k-1} - q\| \\ &\quad + \alpha_k \sup_{w\in[e-\rho,V]} \|\frac{1}{\Gamma(\gamma)} \int_e^w (w-\eta)^{(\gamma-1)} f(\eta, t_{k-1}(\eta), t_{k-1}(\eta-\rho)) d\eta \\ &\quad - \frac{1}{\Gamma(\gamma)} \int_e^w (w-\eta)^{(\gamma-1)} f(\eta, q(\eta), q(\eta-\rho)) d\eta\| \\ &\leq (1-\alpha_k) \sup_{w\in[e-\rho,V]} \|t_{k-1} - q\| + \alpha_k \sup_{w\in[e-\rho,V]} \frac{1}{\Gamma(\gamma)} \int_e^w (w-\eta)^{(\gamma-1)} \\ &\quad \times L_f(\|t_{k-1}(\eta) - q(\eta)\| + \|t_{k-1}(\eta-\rho) - q(\eta-\rho)\|) d\eta \\ &\leq (1-\alpha_k) \sup_{w\in[e-\rho,V]} \|t_{k-1} - q\| + \alpha_k \frac{1}{\Gamma(\gamma)} \int_e^w (w-\eta)^{(\gamma-1)} d\eta \times \\ &\quad L_f(\sup_{w\in[e-\rho,V]} \|t_{k-1}(\eta) - q(\eta)\| \\ &\quad + \sup_{w\in[e-\rho,V]} \|t_{k-1}(\eta-\rho) - q(\eta-\rho)\|) \end{aligned} \quad (56)$$

If we divide both sides of (56) by $E_\gamma(\delta_L w_\gamma)$, then we have

$$\begin{aligned} \frac{\sup_{w\in[e-\rho,V]} \|g_k - q\|}{E_\gamma(\delta_L w_\gamma)} &\leq \frac{(1-\alpha_k) \sup_{w\in[e-\rho,V]} \|t_{k-1} - q\|}{E_\gamma(\delta_L w_\gamma)} + \alpha_k \frac{L_f}{\Gamma(\gamma)} \int_e^w (w-\eta)^{(\gamma-1)} d\eta \times \quad (57) \\ &\quad \left(\frac{\sup_{w\in[e-\rho,V]} \|t_{k-1}(\eta) - q(\eta)\|}{E_\gamma(\delta_L w_\gamma)} \right. \\ &\quad \left. + \frac{\sup_{w\in[e-\rho,V]} \|t_{k-1}(\eta-\rho) - q(\eta-\rho)\|}{E_\gamma(\delta_L w_\gamma)} \right). \end{aligned} \quad (58)$$

According to (54), (58) becomes

$$
\begin{aligned}
\|t_k - q\|_{\delta_L} &\leq (1-\alpha_k)\|t_{k-1} - q\|_{\delta_L} + \frac{\alpha_k}{\Gamma(\gamma)}\int_e^w (w-\eta)^{(\gamma-1)}d\eta \times \\
& L_f(\|t_{k-1}(\eta) - q(\eta)\|_{\delta_L} + \|t_{k-1}(\eta-\rho) - q(\eta-\rho)\|_{\delta_L}) \\
&= (1-\alpha_k)\|t_{k-1} - q\|_{\delta_L} + \alpha_k(2L_f)\|t_{k-1} - q\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_e^w (w-\eta)^{(\gamma-1)}d\eta \\
&= (1-\alpha_k)\|t_{k-1} - q\|_{\delta_L} + \frac{\alpha_k(2L_f)}{E_\gamma(\delta_L w_\gamma)}\|t_{k-1} - q\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_e^w (w-\eta)^{(\gamma-1)}E_\gamma(\delta_L w_\gamma)d\eta \\
&= (1-\alpha_k)\|t_{k-1} - q\|_{\delta_L} + \frac{\alpha_k(2L_f)}{E_\gamma(\delta_L w_\gamma)}\|t_{k-1} - q\|_{\delta_L}.{}^{c}\mathcal{I}^{\bigcirc}\left({}^{c}\mathcal{D}\left(\frac{E_\gamma(\delta_L w_\gamma)}{\delta_L}\right)\right) \\
&= (1-\alpha_k)\|t_{k-1} - q\|_{\delta_L} + \frac{\alpha_k(2L_f)}{E_\gamma(\delta_L w_\gamma)}.\frac{E_\gamma(\delta_L w_\gamma)}{\delta_L}\|t_{k-1} - q\|_{\delta_L} \\
&= (1-\alpha_k)\|t_{k-1} - q\|_{\delta_L} + \frac{\alpha_k(2L_f)}{\delta_L}\|t_{k-1} - q\|_{\delta_L}. \qquad (59)
\end{aligned}
$$

Since $\frac{2L_f}{\delta_L} < 1$, we have

$$\|t_k - q\|_{\delta_L} \leq \|t_{k-1} - q\|_{\delta_L}. \qquad (60)$$

If we put $\|t_{k-1} - q\|_{\delta_L} = \psi_k$, then we obtain

$$\psi_{k+1} \leq \psi_k, \ \forall\, k \in \mathbb{N}.$$

Hence, $\{\psi_k\}$ is a monotone decreasing sequence of real numbers. Furthermore, it is a bounded sequence, so we have

$$\lim_{k\to\infty} \psi_k = \inf\{\psi_k\} = 0.$$

Therefore,

$$\lim_{k\to\infty} \|t_k - q\|_{\delta_L} = 0.$$

□

6. Conclusions

In this article, we have introduced an implicit iterative method (5), which properly includes several other existing iterative methods. The class of TAP mappings considered in our main results properly contains the classes of pseudocontractive-type and nonexpansive mappings considered by so many authors in the literature. We provided a nontrivial example to authenticate the conditions in our main result, and, with this example, we study the common fixed point convergence behavior of our new iterative method (5). Finally, we apply our result to the solution of a delay Caputo fractional differential equation. Our results complement, generalize, improve and extend the corresponding results of Osilike and Akuchu [17], Qin et al. [28], Thakur [37], Saluja [19], Chen [13], Xu and Ori [12] and several others in the existing literature.

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