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A Rigorous Analysis of Integro-Differential Operators with Non-Singular Kernels

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Abstract: Integro-differential operators with non-singular kernels have been much discussed among fractional calculus researchers. We present a mathematical study to clearly establish the rigorous foundations of this topic. By considering function spaces and mapping results, we show that operators with non-singular kernels can be defined on larger function spaces than operators with singular kernels, as differentiability conditions can be removed. We also discover an analogue of the Sonine invertibility condition, giving two-sided inversion relations between operators with non-singular kernels that are not possible for operators with singular kernels.

Keywords: fractional calculus; non-singular kernels; operational calculus; inversion of operators

MSC: 26A33; 44A40; 47A05

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1. Introduction

The essential idea of fractional calculus is to consider functional operators that can be seen as extensions to non-integer orders of the basic repeated differentiation and integration operators [1]. The first question arising from this idea is: how can such extensions be defined? Interestingly, this question has no unique answer, as there is a rich tapestry of different operators which can be called "fractional derivatives" or "fractional integrals" from one viewpoint or another.

Among these operators, many are expressed as convolutions with some particular kernel function that determines the nature of the operator. The classical choice of kernel function is a fractional power, giving rise to the original Riemann–Liouville fractional calculus as well as other related definitions such as Caputo and Hilfer [2,3]. Other popular choices of kernel function have included Mittag-Leffler functions, products of power and exponential functions, hypergeometric functions, and many others [4,5].

In response to the flood of new definitions using assorted kernel functions, mathematicians have realised that many of these are not so different from each other, and that it makes more sense to study general convolution operators in order to discover mathematical properties [5]. The general operators thus proposed have included convolutions with analytic kernels [6,7], Sonine kernels [8–10], and other types of general kernels [11–13]. Among the studies so far, a particularly interesting direction has been the study of Sonine kernels: these are rigorously posed with suitable function spaces clearly defined, but they do not depend on any explicit real or complex number playing the role of a "fractional order", instead having two kernel functions which satisfy a sort of inversion-type relation making them suitable for defining fractional integral and derivative operators with fundamental theorem of calculus type relations.

A point of debate within the fractional calculus community has concerned the operators with non-singular kernel functions: that is, functions that do not have a singularity (blowup) at the endpoint of the integral. For example, the function $(t-\tau)^{-\alpha}$ has a singularity at $\tau = t$ for positive values of the parameter α, but the function $e^{\alpha(t-\tau)}$ is non-singular. Some scientists have claimed that operators with non-singular kernels are not suitable to be called "fractional" [14,15], while others have claimed that this approach is faulty [16]. We prefer to sidestep this terminology debate, pointing out instead that the operators with non-singular kernels have found real-world applications [17,18] and are therefore also worth studying from the mathematical point of view [19], regardless of whether we call them "fractional" or not. Many studies of integral equations, not necessarily in a fractional setting, have not put any particular emphasis on whether a kernel is singular or non-singular [20,21].

So far, most publications related to such operators with non-singular kernels have been written by engineers and applied scientists who are concerned mainly with their applications. We, as mathematicians, have noticed some interesting mathematical properties of these operators, such as their function spaces and inversion properties, which we believe deserve to be highlighted and discussed. As a detailed mathematical investigation of these operators is currently missing in the literature, we are taking the opportunity to conduct one and present our findings to be used by mathematicians and scientists alike.

Our work is arranged as follows. Section 2 serves to correct a (minor but important) misconception regarding function spaces appearing in the previous literature; Section 3 focuses on how derivative-type operators with non-singular kernels can be rewritten so as not to need differentiability; Section 4 uses an abstract algebra viewpoint to discover a quasi-inverse relation which can serve as an alternative to the Sonine relation; and finally, Section 5 provides a summary and concluding remarks. We note once again that we do not intend to enter the debate on whether or not it is appropriate to include operators with non-singular kernels as "fractional derivatives". This is not our business here: we are simply presenting some objective mathematical facts concerning certain operators and the associated function spaces, and we will allow the results to speak for themselves.

2. A Misconception on Function Spaces

The classical C_α space of Dimovski [22,23] is defined as follows:

$$C_\alpha = \left\{ f : (0,\infty) \to \mathbb{C} \,:\, f(t) = t^p f_1(t),\ p > \alpha,\ f_1 \in C[0,\infty) \right\},$$

where $\alpha \in \mathbb{R}$ is arbitrary.

In his construction of the fractional calculus with Sonine kernels [9] and its generalisation [10], Luchko made use of a function space $C_{-1,0}$ inspired by this C_α. The general space $C_{\alpha,\beta}$ was defined in ([10] Equation (23)), as a natural generalisation of the space $C_{-1,0}$ first defined in ([9] Equation (33)), as follows:

$$C_{\alpha,\beta} = \left\{ f : (0,\infty) \to \mathbb{C} \,:\, f(t) = t^p f_1(t),\ \alpha < p < \beta,\ f_1 \in C[0,\infty) \right\},$$

where $\alpha < \beta$ in $\mathbb{R}$. The purpose of using $C_{-1,0}$ for the fractional operators with Sonine kernels, instead of simply C_{-1} as was done for the classical fractional operators [24], was in order to obtain "an integrable singularity at the point zero" [9]: Reference [10] explicitly states that functions in $C_{-1,0}$ "possess the integrable singularities of the power function type at the origin". We show here that this is not necessarily the case, and that a different function space must be used if it is desired to exclude non-singular functions in $C[0,\infty)$ from the space of kernels.

Theorem 1. *For any $\alpha < \beta$ in $\mathbb{R}$, the space $C_{\alpha,\beta}$ is exactly the space C_α.*

Proof. It is clear that $C_{\alpha,\beta} \subset C_\alpha$. For the converse, let $f \in C_\alpha$ be arbitrary, say $f(t) = t^p f_1(t)$ with $p > \alpha$ and $f_1 \in C[0,\infty)$. If $p < \beta$, then we are done. Otherwise, let $q = p - \beta + \frac{\beta-\alpha}{2} > 0$ and $p' = p - q = \frac{\alpha+\beta}{2}$, so that $\alpha < p' < \beta$ and

$$f(t) = t^p f_1(t) = t^{p'} t^q f_1(t) = t^{p'} g_1(t),$$

where $g_1(t) = t^q f_1(t)$ so that $f_1 \in C[0,\infty) \Rightarrow g_1 \in C[0,\infty)$. Therefore, the function f is in the space $C_{\alpha,\beta}$, which completes the proof. □

Example 1. *The non-singular functions $f(t) = e^t$ and $g(t) = E_\alpha(t^\alpha)$ are in the space $C_{-1,0}$ because they can be expressed as $f(t) = t^{-1/2} f_1(t)$ and $g(t) = t^{-1/2} g_1(t)$ with $f_1, g_1 \in C[0,\infty)$ defined by*

$$f_1(t) = t^{1/2} e^t, \qquad g_1(t) = t^{1/2} E_\alpha(t^\alpha).$$

As a result, we see that the function space used by Luchko [9,10] for the operators with Sonine kernels does not explicitly exclude non-singular kernels as it was intended to. However, the Sonine condition itself excludes them because if k and κ are both in $C[0,\infty)$, then $\kappa * k \in C[0,\infty)$ with $(\kappa * k)(0) = 0$, so $\kappa * k$ cannot be either $\{1\}$ or $\{1\}^n$ ($n \in \mathbb{N}$) for the Sonine [9] or generalised Sonine [10] kernels. So, if κ and k are a pair of Sonine kernels in the space $\mathcal{S}_{-1}$ defined in ([9] Definition 2) or the space $\mathcal{L}_n$ defined in ([10] Definition 2), then at least one of κ and k must have a singularity at zero. For this reason, it seems that the results of Luchko [9,10] on Sonine kernels are not seriously affected by the realisation of our Theorem 1 above.

Is there any way to replace Luchko's definition of the $C_{\alpha,\beta}$ space with another space that successfully excludes all non-singular functions? An obvious choice would be the following set:

$$C_\alpha \backslash C_\beta = \Big\{ f : (0,\infty) \to \mathbb{C} \,:\, f(t) = t^p f_1(t),\ p > \alpha,\ f_1 \in C[0,\infty), \\ \text{and } f \text{ cannot be written as } f(t) = t^q f_2(t),\ q > \beta,\ f_2 \in C[0,\infty) \Big\},$$

but this set is not a vector space because we can always add two functions in $C_\alpha \backslash C_\beta$ to obtain a function in C_β. It is also not closed under convolution (rng multiplication), and indeed we can state a stronger result, as in the following theorem.

Theorem 2. *Let $\alpha \geq -1$, and let A be a subset of C_α that is non-trivial (i.e., contains at least one non-zero function). If A contains only singular functions (in other words, if $A \cap C[0,\infty) = \emptyset$), or more generally if $A \cap C_\beta = \emptyset$ for any fixed $\beta > \alpha$, then A cannot be closed under convolution, and A cannot be an ideal or subrng of C_α.*

Proof. It is well known [25,26] that the convolution of a function in C_α and a function in C_λ is always a function in $C_{\alpha+\lambda+1} \subset C_\alpha \cap C_\lambda$ when $\alpha, \lambda \geq -1$. Therefore, repeatedly taking the convolution of any function in C_α with itself will eventually yield a function in C_β for arbitrarily large $\beta \in \mathbb{R}$. Therefore, any subrng or ideal, or any set closed under multiplication in C_α, must contain elements of C_β for arbitrarily large values of β. Since the C_β spaces are nested, with $C_\beta \subset C_\gamma$ if $\beta \geq \gamma$, this completes the proof. □

3. Equivalent Formulations of Differential Operators with Non-Singular Kernels

Let us consider a non-singular kernel $k \in C[0,\infty)$ and define corresponding derivative-type operators as follows:

$$\left({}^{R}\mathbb{D}_{(k)}f\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t k(t-\tau)f(\tau)\,\mathrm{d}\tau\right), \tag{1}$$

$$\left({}^{C}\mathbb{D}_{(k)}f\right)(t) = \int_0^t k(t-\tau)f'(\tau)\,\mathrm{d}\tau, \tag{2}$$

in both cases for suitable functions f. (We will discuss suitable function spaces for these operators shortly.)

One of the distinguishing features of non-singular kernels, as discussed in ([27] Section 4.1) is the fact that operators such as (1) and (2) with non-singular kernels can be rewritten without any explicit differentiation and without needing any differentiability assumptions on the function f. This is done using, respectively, the fundamental theorem of calculus and the method of integration by parts:

$$\begin{aligned}\left({}^{R}\mathbb{D}_{(k)}f\right)(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t k(t-\tau)f(\tau)\,\mathrm{d}\tau\right) \\ &= \Big[k(t-\tau)f(\tau)\Big]_{\tau=t} + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}k(t-\tau)f(\tau)\,\mathrm{d}\tau \\ &= k(0)f(t) + \int_0^t k'(t-\tau)f(\tau)\,\mathrm{d}\tau;\end{aligned} \tag{3}$$

$$\begin{aligned}\left({}^{C}\mathbb{D}_{(k)}f\right)(t) &= \int_0^t k(t-\tau)f'(\tau)\,\mathrm{d}\tau \\ &= \Big[k(t-\tau)f(\tau)\Big]_{\tau=0}^{\tau=t} - \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}[k(t-\tau)]f(\tau)\,\mathrm{d}\tau \\ &= k(0)f(t) - k(t)f(0) + \int_0^t k'(t-\tau)f(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{4}$$

This manipulation works whenever k and f are both in the space

$$C^1_{-1} = \left\{f : (0,\infty) \to \mathbb{C} \;:\; f \text{ differentiable, } f' \in C_{-1}\right\},$$

but the final expressions (3) and (4) are valid for any $f \in C_{-1}$ or any $f \in C[0,\infty)$, respectively, as long as the fixed kernel function k is in a suitable space such as C^1_{-1}. We state this result formally as follows.

Theorem 3. *If the kernel function k is in the space C_{-1}, then the formulae on the right-hand sides of Equations (1) and (2) give well-defined mappings ${}^{R}\mathbb{D}_{(k)}$ and ${}^{C}\mathbb{D}_{(k)}$ from C^1_{-1} into C_{-1}.*

If the kernel function k is in the space C^1_{-1}, then the formulae on the right-hand sides of Equations (3) and (4) give well-defined mappings

$${}^{R}\mathbb{D}_{(k)} : C_{-1} \longrightarrow C_{-1}$$

and

$${}^{C}\mathbb{D}_{(k)} : C[0,\infty) \subset C_{-1} \longrightarrow C_{-1},$$

and these operators are identical to those defined by (1) and (2) when f is in the subspace C^1_{-1} such that all are defined.

Therefore, assuming that $k \in C^1_{-1}$, the suitable (largest) function spaces on which to define the operators ${}^{R}\mathbb{D}_{(k)}$ and ${}^{C}\mathbb{D}_{(k)}$ are the spaces C_{-1} and $C[0,\infty)$, respectively. In particular, no differentiability conditions are required on f in order to define ${}^{R}\mathbb{D}_{(k)}f$ and ${}^{C}\mathbb{D}_{(k)}f$ as functions in C_{-1}.

Proof. Firstly, when $k \in C_{-1}$ and $f \in C^1_{-1}$, Formula (2) defines a function in C_{-1} since k and f' are in the space C_{-1} (which is closed under convolution), and Formula (1) defines a function in C_{-1} by ([25] Theorem 2.5).

Now let us fix $k \in C^1_{-1}$ and consider expressions (3) and (4). Since $k' \in C_{-1}$, the integral in (3) defines a function in C_{-1} for any $f \in C_{-1}$, while $k(0)$ exists by ([25] Fact (3)), so we are done for (3). The difference between (3) and (4) is only in the term $k(t)f(0)$, so the latter defines a function in C_{-1} whenever $f \in C_{-1}$ and $f(0)$ exists, which is the same as requiring $f \in C[0,\infty) \subset C_{-1}$.

Finally, the equivalence of (1) and (2) with (3) and (4) is clear from the above arguments, using the fundamental theorem of calculus and integration by parts, which are valid when k and f are both differentiable on $(0,\infty)$ with $k(0), f(0)$ existing and $k', f' \in C_{-1}$. □

Remark 1. *Note that the rewriting of Equations (1) and (2) to Equations (3) and (4) is only possible when the kernel k is non-singular, otherwise the $k(0)$ terms in (3) and (4) cannot be meaningful. Thus, our reformulation of Riemann–Liouville and Caputo derivative-type operators, to a form that does not require differentiability conditions, is only possible for operators defined using non-singular kernels.*

Example 2. *Taking the kernel function k to be a power function with exponent greater than one, in the form*

$$k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}, \qquad \alpha > 1,$$

we find that $k \in C^1_{-1}$ since

$$k'(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Rightarrow k' \in C[0,\infty).$$

In this case, the operators ${}^{R}\mathbb{D}_{(k)}$ and ${}^{C}\mathbb{D}_{(k)}$, although denoted as derivatives, are actually the Riemann–Liouville fractional integrals of order $\alpha - 1 > 0$. This reminds us that the Riemann–Liouville fractional integrals of order greater than one are operators with non-singular kernels.

Example 3. *Taking the kernel function k to be a constant times a Mittag-Leffler function, in the form*

$$k(t) = \frac{B(\alpha)}{1-\alpha} E_\alpha\left(\frac{-\alpha}{1-\alpha} t^\alpha\right),$$

with $0 < \alpha < 1$ and B being a suitable function satisfying $B(0) = B(1) = 1$, we certainly have $k \in C^1_{-1}$ since

$$k'(t) = \frac{B(\alpha)}{1-\alpha} \sum_{n=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^n \frac{t^{n\alpha-1}}{\Gamma(n\alpha)},$$

which is $t^{\alpha-1}$ times a continuous function in $C[0,\infty)$. In this case, the operators ${}^{R}\mathbb{D}_{(k)}$ and ${}^{C}\mathbb{D}_{(k)}$ are precisely the Atangana–Baleanu derivatives, respectively of Riemann–Liouville type and Caputo type.

Here, Theorem 3 tells us, as already known [28], that these Atangana–Baleanu derivatives can be defined on larger function spaces than most fractional derivatives, without requiring any differentiability assumptions. The reformulation (4) of the Caputo-type operator gives, in this case, precisely the formula that is called the "modified ABC derivative" in [29].

Example 4. *More generally, taking the kernel function k to be a three-parameter Mittag-Leffler function [30] with the second parameter equal to one, in the form*

$$k(t) = E^{\gamma}_{\alpha,1}(\delta t^\alpha),$$

with $\alpha, \gamma, \delta \in \mathbb{C}$ constant and $\operatorname{Re}\alpha > 0$, we again have $k \in C^1_{-1}$ since

$$k'(t) = \sum_{n=1}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \cdot \frac{t^{n\alpha-1}}{\Gamma(n\alpha)},$$

which is $t^{\alpha-1}$ times a continuous function in $C[0,\infty)$ as in the previous example. In this case, the operators ${}^{R}\mathbb{D}_{(k)}$ and ${}^{C}\mathbb{D}_{(k)}$ are precisely the Prabhakar fractional derivatives with second parameter equal to one, respectively of Riemann–Liouville type and Caputo type. These particular types of Prabhakar operators were already discussed in ([27] Section 4.1) as the non-singular case of Prabhakar fractional calculus, and the only case where Prabhakar derivatives can be defined on large enough function spaces to not need any differentiability assumptions.

The above results can be extended to operators defined using multiple (repeated) derivatives together with convolution integrals. Let us consider the following operators, for $n \in \mathbb{N}$ and for suitable functions k and f:

$$\left({}^{R}_{n}\mathbb{D}_{(k)} f\right)(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_0^t k(t-\tau) f(\tau)\,\mathrm{d}\tau\right), \tag{5}$$

$$\left({}^{C}_{n}\mathbb{D}_{(k)} f\right)(t) = \int_0^t k(t-\tau) f^{(n)}(\tau)\,\mathrm{d}\tau, \tag{6}$$

and try to rewrite Formulas (5) and (6) using the fundamental theorem of calculus and integration by parts as before.

Repeatedly differentiating the integral in (5) and using the fundamental theorem of calculus, we find:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t k(t-\tau) f(\tau)\,\mathrm{d}\tau\right) &= k(0)f(t) + \int_0^t k'(t-\tau) f(\tau)\,\mathrm{d}\tau, \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\int_0^t k(t-\tau) f(\tau)\,\mathrm{d}\tau\right) &= k(0)f'(t) + k'(0)f(t) + \int_0^t k''(t-\tau) f(\tau)\,\mathrm{d}\tau, \\
\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(\int_0^t k(t-\tau) f(\tau)\,\mathrm{d}\tau\right) &= k(0)f''(t) + k'(0)f'(t) + k''(0)f(t) + \int_0^t k'''(t-\tau) f(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

and finally

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_0^t k(t-\tau) f(\tau)\,\mathrm{d}\tau\right) = \sum_{i=0}^{n-1} k^{(i)}(0) f^{(n-1-i)}(t) + \int_0^t k^{(n)}(t-\tau) f(\tau)\,\mathrm{d}\tau, \tag{7}$$

where we note that if $k(0) = k'(0) = \cdots = k^{(n-2)}(0) = 0$, then no differentiation of f needs to take place in the terms outside the integral.

Repeatedly using integration by parts in (6), we find:

$$\begin{aligned}
\int_0^t k(t-\tau) f^{(n)}(\tau)\,\mathrm{d}\tau &= \left[\sum_{i=0}^{n-1} k^{(i)}(t-\tau) f^{(n-1-i)}(\tau)\right]_{\tau=0}^{\tau=t} + \int_0^t k^{(n)}(t-\tau) f(\tau)\,\mathrm{d}\tau \\
&= \sum_{i=0}^{n-1}\left[k^{(i)}(0) f^{(n-1-i)}(t) - k^{(i)}(t) f^{(n-1-i)}(0)\right] + \int_0^t k^{(n)}(t-\tau) f(\tau)\,\mathrm{d}\tau,
\end{aligned} \tag{8}$$

where in this case some derivatives of f are involved in the boundary terms regardless of the nature of k.

Theorem 4. *If the kernel function k is in the space C_{-1}, then the formulae on the right-hand sides of Equations (5) and (6) are well-defined functions in C_{-1} provided that f is such that $k * f \in C^n_{-1}$ (for ${}^{R}_{n}\mathbb{D}_{(k)}$) or $f \in C^n_{-1}$ (for ${}^{C}_{n}\mathbb{D}_{(k)}$), respectively.*

If the kernel function k is in the space C^n_{-1} and $k(0) = k'(0) = \cdots = k^{(n-2)}(0) = 0$, then the formulae on the right-hand sides of Equations (7) and (8) give well-defined mappings

$$^{R}\mathbb{D}_{(k)} : C_{-1} \longrightarrow C_{-1}$$

and

$$^{C}\mathbb{D}_{(k)} : \left\{ f \in C[0,\infty) \ : \ f^{(n-1)}(0) \text{ exists} \right\} \subset C_{-1} \longrightarrow C_{-1},$$

and these operators are identical to those defined by (5) and (6) when f is such that all are defined.

Therefore, assuming that $k \in C^n_{-1}$ and $k(0) = k'(0) = \cdots = k^{(n-2)}(0) = 0$, the suitable (largest) function spaces on which to define the operators ${}^{R}_{n}\mathbb{D}_{(k)}$ and ${}^{C}_{n}\mathbb{D}_{(k)}$ are respectively the space C_{-1} and the subspace of $C[0,\infty)$ consisting of functions $n-1$ times differentiable at 0. In particular, no differentiability conditions are required on f in order to define ${}^{R}_{n}\mathbb{D}_{(k)} f$ as a function in C_{-1}.

Proof. Firstly, it is clear that (5) defines a function in C_{-1} when $k * f \in C^n_{-1}$, and that (6) defines a function in C_{-1} when $k \in C_{-1}$ and $f \in C^n_{-1}$.

If $k \in C^n_{-1}$ and $k(0) = k'(0) = \cdots = k^{(n-2)}(0) = 0$, then $k^{(n)} \in C_{-1}$ and the finite sum in (7) reduces to a single term $k^{(n-1)}(0)f(0)$, so (7) defines a function in C_{-1} for all $f \in C_{-1}$. The difference between (7) and (8) is only in the terms $k(t)f^{(n-1)}(0), \cdots, k^{(n-1)}(t)f(0)$, so the latter defines a function in C_{-1} whenever $f \in C_{-1}$ and $f(0), \cdots, f^{(n-1)}(0)$ exist.

Finally, the equivalence of (5) and (6) with (7) and (8) is clear from the above arguments, using the fundamental theorem of calculus and integration by parts, which are valid when k and f are both C^n on $(0,\infty)$ with derivatives up to $(n-1)$th order existing at 0. □

Example 5. *Taking the kernel function k to be a power function with exponent greater than n, in the form*

$$k(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \qquad \alpha > n,$$

we find that $k \in C^n_{-1}$ with $k(0) = k'(0) = \cdots = k^{(n-2)}(0) = 0$, since

$$k^{(r)}(t) = \frac{t^{\alpha-r}}{\Gamma(\alpha-r+1)}, \qquad r = 1,2,\cdots,n.$$

In this case, the operators ${}^{R}\mathbb{D}_{(k)}$ and ${}^{C}\mathbb{D}_{(k)}$, although notated as derivatives, are actually the Riemann–Liouville fractional integrals of order $\alpha - n > 0$. Once again, as in Example 2, this reminds us that the Riemann–Liouville fractional integrals of order greater than one are operators with non-singular kernels.

Remark 2. *Note that the rewriting of Equations (5) and (6) to Equations (7) and (8) is only possible when the kernel k is non-singular, otherwise the $k(0)$ terms in (7) and (8) cannot be meaningful. It is known [25] that the space C^n_{-1} is contained in $C[0,\infty)$. Thus, once again, our reformulation of Riemann–Liouville and Caputo derivative-type operators, to a form that requires less strong differentiability conditions, is only possible for operators defined using non-singular kernels.*

4. Mikusiński's Operational Calculus for Non-Singular Kernels

In the above, we have considered what happens when an operator of fractional derivative type (a combination of a derivative or repeated derivative operator with a convolution integral operator) is permitted to have a non-singular kernel. A rigorous examination of the required assumptions and function spaces for such operators revealed exactly which differentiability assumptions could be relaxed. However, so far we have not considered any sort of inversion relation, as is commonly seen between derivatives and integrals.

In Luchko's work on Sonine kernels [9,10], he imposed Sonine-type conditions in order to ensure inversion relations between his operators. If two kernels κ and k satisfy

$$\kappa * k = \{1\} \in C_{-1},$$

then the integral operator $\mathbb{I}_{(\kappa)}$ defined by $\mathbb{I}_{(\kappa)} f = \kappa * f$ obeys a fundamental theorem of calculus together with the differential operators ${}^{R}\mathbb{D}_{(k)}$ and ${}^{C}\mathbb{D}_{(k)}$ defined by (1) and (2). Similarly, if κ and k instead satisfy

$$\kappa * k = \{1\}^n \in C_{-1},$$

then the integral operator $\mathbb{I}_{(\kappa)}$ defined by $\mathbb{I}_{(\kappa)} f = \kappa * f$ obeys a fundamental theorem of calculus together with the differential operators ${}^{R}_{n}\mathbb{D}_{(k)}$ and ${}^{C}_{n}\mathbb{D}_{(k)}$ defined by (5) and (6).

But some types of operators with non-singular kernels have meaningful inversion relations between integrals and derivatives even without satisfying a Sonine condition. How can we extend the invertibility condition beyond the Sonine one in order to capture these different behaviours?

To answer this, we need some concepts and results from abstract algebra.

Proposition 1 ([31])**.** *The space $C[0,\infty)$ under the operations of addition and convolution forms a commutative rng (ring without identity) as well as a vector space over $\mathbb{C}$. Its field of fractions $\mathcal{M}$ is both a field and a vector space over $\mathbb{C}$.*

Proof. This is a well-known result in the theory of Mikusiński's operational calculus. □

Theorem 5. *The algebra $\mathcal{A}$ generated by $C[0,\infty)$ and the identity element from $\mathcal{M}$, i.e., the set of elements $\lambda I + f$ with $\lambda \in \mathbb{C}$ and $f \in C[0,\infty)$ (where I is the multiplicative identity from $\mathcal{M}$) is a commutative ring with identity as well as a vector space over $\mathbb{C}$.*

Proof. This is a standard variant of the well-known Dorroh extension [32] that allows every rng to be extended to a ring. Under the operations

$$(\lambda I + f) + (\mu I + g) = (\lambda + \mu)I + (f + g)$$

of addition and

$$\mu(\lambda I + f) = (\lambda\mu)I + (\mu f)$$

of scalar multiplication and

$$(\lambda I + f) * (\mu I + g) = (\lambda\mu)I + (\lambda g + \mu f + f * g)$$

of multiplication, it is immediate that the new space satisfies all the axioms to be both a vector space and a commutative ring with identity. □

Theorem 6. *All elements of $\mathcal{A}\backslash C[0,\infty)$ have multiplicative inverse elements in $\mathcal{A}\backslash C[0,\infty)$.*

Proof. One of the main results of Heatherly and Huffmann ([33] Theorem 2.1) was that every element $f \in C[0,\infty)$ (considered as a commutative rng under addition and convolution) has a quasi-inverse $\widetilde{f} \in C[0,\infty)$, i.e., such that

$$f + \widetilde{f} + f * \widetilde{f} = 0 \in C[0,\infty).$$

Adding $I \in \mathcal{A}$ to both sides of this equation gives

$$(I + f) * (I + \widetilde{f}) = I \in \mathcal{A}.$$

Therefore, for any $\lambda \in \mathbb{C}\backslash\{0\}$ and any $f \in C[0,\infty)$, the element $\lambda I + f \in \mathcal{A}\backslash C[0,\infty)$ satisfies

$$\left(\lambda I + f\right) * \left(\lambda^{-1} I + \lambda^{-1}\widetilde{g}\right) = (I + g) * (I + \widetilde{g}) = I,$$

where $\widetilde{g} \in C[0,\infty)$ is the quasi-inverse of $g := \lambda^{-1} f \in C[0,\infty)$. □

Let us now discuss why the above results are important.

In Mikusiński's operational calculus, a function space is interpreted as an rng, in which functions can be added, subtracted, and multiplied, but not divided, and there is a zero (additive identity) but not a one (multiplicative identity). The construction of the Mikusiński field (the field of fractions of this rng, using a theorem of Titchmarsh to determine that there are no zero divisors) enables division of functions, but at the expense of moving into a more abstract space: dividing two functions does not necessarily yield another function, but an abstract field element, called an "operator" by Mikusiński [31]. In this field, there is a multiplicative identity element, whose convolution with any function yields the same function back again; as a generalised function concept, this identity element is analogous to the Dirac delta from distribution theory.

The full structure of the Mikusiński field is necessary in order to allow inverses of all elements of the starting rng, but it is not necessary if all we need is a multiplicative identity element. Theorem 5 shows that adjoining just one element (and generating a new vector space) is enough to get a commutative ring with a multiplicative identity, and Theorem 6 shows that almost all elements of this ring have inverses: it is not a field, but it is almost as good as being a field.

Furthermore, elements of the ring $\mathcal{A}$ can be understood as operators without needing to resort to abstract "generalised functions". Any element $a = \lambda I + f \in \mathcal{A}$ can be interpreted as an operator from $C[0,\infty)$ to $C[0,\infty)$ in a very natural way: mapping any $g \in C[0,\infty)$ to $a * g = \lambda g + f * g \in C[0,\infty)$. This means that $\mathcal{A}$ is a very natural setting for defining operators if we do not strictly require that everything must have a multiplicative inverse.

Theorem 7 (Quasi-inverse functions give inverse functional operators)**.** *For any non-singular kernel $k \in C[0,\infty) \subset C_{-1}$ and any nonzero constant $\lambda \in \mathbb{C}$, the operator $A : C_{-1} \to C_{-1}$ defined by*

$$(Af)(t) = \lambda f(t) + \int_0^t k(t-\tau) f(\tau)\,\mathrm{d}\tau, \qquad f \in C_{-1},$$

has an inverse operator A^{-1} defined by another function $\kappa \in C[0,\infty) \subset C_{-1}$, namely

$$\left(A^{-1} f\right)(t) = \lambda^{-1} f(t) + \int_0^t \kappa(t-\tau) f(\tau)\,\mathrm{d}\tau, \qquad f \in C_{-1},$$

where $AA^{-1}f = A^{-1}Af = f$ for all $f \in C_{-1}$ and the relation between the kernel functions can be expressed as

$$\lambda^{-1} k + \lambda\kappa + k * \kappa = 0. \tag{9}$$

Proof. This is essentially a restatement of Theorem 6. The operator A is precisely multiplication by the element $\lambda I + k \in \mathcal{A}\backslash C[0,\infty)$, which has an inverse $\lambda^{-1} I + \lambda^{-1}\widetilde{k} \in \mathcal{A}\backslash C[0,\infty)$ by Theorem 6. Defining $\kappa = \lambda^{-1}\widetilde{k} \in C[0,\infty)$, we have the operator A^{-1} as stated being multiplication by $(\lambda I + k)^{-1} \in \mathcal{A}\backslash C[0,\infty)$.

Since $C[0,\infty)$ embeds as a subalgebra in C_{-1}, their fields of fractions similarly embed, and so $\mathcal{A}$ embeds as a subalgebra of the corresponding algebra generated by I and C_{-1}. Therefore, if the product of $\lambda I + k$ and $\lambda^{-1} I + \kappa$ is the identity in $\mathcal{A}$, then it is also the identity in the field of fractions of C_{-1}. So, the product of these two elements with any $f \in C_{-1}$ is again f, since multiplication in C_{-1} is commutative and associative.

The inversion relation $(\lambda I + k)\left(\lambda^{-1} I + \kappa\right) = I$ in $\mathcal{A}$ reduces to $\lambda^{-1} k + \lambda\kappa + k * \kappa = 0$ in $C[0,\infty)$ after subtracting I from both sides. □

Definition 1. *We can say two pairs* (λ, k) *and* (μ, κ)*, each consisting of a constant in* $\mathbb{C}$ *and a function in* $C[0,\infty)$*, are quasi-inverse to each other if* $\mu = \lambda^{-1}$ *and Equation* (9) *is satisfied, i.e., if* $\lambda\mu = 1$ *and* $\mu k + \lambda\kappa + k * \kappa = 0$*. This is a sufficient condition to get well-defined integral operators* $C_{-1} \to C_{-1}$ *that are inverse to each other as in Theorem 7.*

This is essentially an analogue, for non-singular kernels, of the Sonine relation. We have a direct relation between functions κ *and* k*, which can be formulated in terms of their convolutions in a suitable function space, which gives rise to an inversion relation between corresponding operators.*

Remark 3. *Note that the inversion relation given by Theorem 7 is symmetric: the operators* A *and* A^{-1} *are two-sided inverses of each other. This is unlike the usual derivative and integral operators (either classical, or fractional with singular kernels), which have asymmetric inversion relations given by the fundamental theorem of calculus. Thus, we observe an interestingly different structure for operators with non-singular kernels, as previously noted in some other works [27,28,34].*

Example 6. *Taking* k *to be a constant function in Theorem 7, we find the operator* A *to be a linear combination of a function with its integral, the same formulation used for the Caputo–Fabrizio integral:*

$$(Af)(t) = \lambda f(t) + k\int_0^t f(\tau)\,\mathrm{d}\tau, \qquad f \in C_{-1}. \tag{10}$$

How do we find a suitable function κ *satisfying the identity* (9) *when* k *is a constant function? The answer can be easily found using Mikusiński's methodology in the field of fractions* $\mathcal{M}$*:*

$$\begin{aligned}
\lambda^{-1}I + \kappa = \big(\lambda I + k\big)^{-1} &= \frac{\lambda^{-1}I}{I + \lambda^{-1}k\{1\}} \\
&= \lambda^{-1}\sum_{n=0}^{\infty}\big(-\lambda^{-1}k\{1\}\big)^n \\
&= \lambda^{-1}I + \lambda^{-1}\sum_{n=1}^{\infty}(-1)^n\lambda^{-n}k^n \cdot \left\{\frac{t^{n-1}}{(n-1)!}\right\} \\
&= \lambda^{-1}I - \lambda^{-2}k\Big\{\exp\big(-\lambda^{-1}kt\big)\Big\},
\end{aligned}$$

so the kernel function $\kappa \in C[0,\infty)$ *is given by*

$$\kappa(t) = -\frac{k}{\lambda^2}e^{-kt/\lambda}.$$

Thus, we have proved that the quasi-inverse in $C[0,\infty)$ *of a constant function is a constant times an exponential function, and the inverse of the operator* (10) *is given by:*

$$\Big(A^{-1}f\Big)(t) = \frac{1}{\lambda}f(t) + \frac{k}{\lambda^2}\int_0^t e^{k(t-\tau)/\lambda}f(\tau)\,\mathrm{d}\tau, \qquad f \in C_{-1}.$$

In the usual Caputo–Fabrizio notation [34,35], we would write $\lambda = \frac{1-\alpha}{M(\alpha)}$ *and* $k = \frac{\alpha}{M(\alpha)}$*, where* $\alpha \in (0,1)$ *and* $M(\alpha)$ *is a function satisfying* $M(0) = M(1) = 1$*, to find:*

$$\begin{aligned}
(Af)(t) &= \frac{1-\alpha}{M(\alpha)}f(t) + \frac{\alpha}{M(\alpha)}\int_0^t f(\tau)\,\mathrm{d}\tau, \qquad f \in C_{-1}; \\
\Big(A^{-1}f\Big)(t) &= \frac{M(\alpha)}{1-\alpha}f(t) - \frac{\alpha M(\alpha)}{(1-\alpha)^2}\int_0^t e^{-\alpha(t-\tau)/(1-\alpha)}f(\tau)\,\mathrm{d}\tau, \qquad f \in C_{-1}.
\end{aligned}$$

The operator A is the Caputo–Fabrizio integral, while the operator A^{-1} is the corresponding derivative of Riemann–Liouville type [36], as we can see from a quick computation:

$$\left(A^{-1}f\right)(t) = \frac{M(\alpha)}{1-\alpha} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t e^{-\alpha(t-\tau)/(1-\alpha)} f(\tau)\,\mathrm{d}\tau.$$

Note that this operator is a two-sided inverse to the Caputo–Fabrizio integral A, on the space C_{-1} and therefore on any subspace such as $C[0,\infty)$. The corresponding Caputo-type derivative is not an exact algebraic inverse; instead, it has a Newton–Leibniz type formula together with the integral operator, as discussed in detail in [34].

Remark 4. *The Atangana–Baleanu integral is a linear combination of a function with its Riemann–Liouville fractional integral. As the Riemann–Liouville integral is singular, this is not interpretable as an operator in our algebra $\mathcal{A}$ generated from the rng of continuous functions $C[0,\infty)$, and so the Atangana–Baleanu operators are not covered by our Theorem 7. Further work will be necessary to ascertain whether operators like those of Atangana–Baleanu can be covered by an extension of the result of Theorem 7.*

5. Conclusions

In this paper, we have formed a rigorous mathematical study of what happens when fractional integral and derivative operators defined by convolutions (such as the Sonine-type operators) are permitted to have non-singular kernels. After correcting one misapprehension regarding the suitable function spaces for Sonine kernels (which does not significantly affect the work in the literature on the operators with Sonine kernels), we were able to achieve the following results.

- Integro-differential operators analogous to the Riemann–Liouville and Caputo derivatives with orders between 0 and 1, defined by a combination of a convolution integral and a first-order derivative either inside or outside the integral, can be equivalently defined without any differentiability conditions, provided the kernel function k is in the space C^1_{-1}.
- If the first-order derivative is replaced by an nth-order derivative, then the Riemann–Liouville-type operator can be equivalently defined without any differentiability conditions, and the Caputo-type operator with differentiability conditions only at zero, provided the kernel function k is in the space C^n_{-1} with $k^{(r)}(0) = 0$ for $0 \leq r \leq n-2$.
- If an operator can be expressed as a linear combination of the identity operator and a convolution with a non-singular kernel, then it is invertible, with a two-sided inverse operator expressible in the same form. The non-singular kernels $\kappa, k \in C[0,\infty)$ must satisfy a relation of the form $\lambda^{-1}k + \lambda\kappa + k * \kappa = 0$ with $\lambda \in \mathbb{C}\backslash\{0\}$, this being the equivalent of the Sonine condition giving an inversion relation between operators.

These rigorous results provide some solid mathematical foundations for the theory of operators with non-singular kernels, which have been much discussed within the fractional calculus community. It is hoped that having the mathematical facts clearly stated and proved will help to resolve some of the scientific discussions and disputes on the subject of such operators. For a proper understanding of their nature and where they are likely to be applicable, it is always useful to have the mathematical fundamentals clearly laid out.

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