fractal and fractional

Article

New Hadamard Type Inequalities for Modified *h*-Convex Functions

Daniel Breaz [1], Çetin Yildiz [2], Luminiţa-Ioana Cotîrlă [3,*], Gauhar Rahman [4] and Büşra Yergöz [2]

1 Department of Mathematics, Decembrie University of Alba-Iulia, 510009 Alba-Iulia, Romania
2 Department of Mathematics, Kazım Karabekir Education Faculty, Atatürk University, 25240 Erzurum, Turkey
3 Department of Mathematics, Technical University of Cluj-Napoca, 400020 Cluj-Napoca, Romania
4 Department of Mathematics and Statistic, Hazara University, Mansehra 21300, Pakistan
* Correspondence: luminita.cotirla@math.utcluj.ro

Abstract: In this article, we demonstrated various Hermite–Hadamard and Fejér type inequalities for modified h-convex functions. We showed several inequalities for the products of two modified h-convex functions. New identities related to inequalities in various forms are also established for different values of the $h(\varphi_t)$ function. We believe that the approach presented in this paper will inspire more research in this area.

Keywords: Hermite–Hadamard inequality; Fejér inequality; modified h-convex functions; super-additive functions

MSC: 26D15; 26D07; 26A51

check for updates

Citation: Breaz, D.; Yildiz, Ç.; Cotîrlă, L.-I.; Rahman, G.; Yergöz, B. New Hadamard Type Inequalities for Modified *h*-Convex Functions. *Fractal Fract.* **2023**, *7*, 216. https://doi.org/10.3390/fractalfract7030216

Academic Editors: Ricardo Almeida and Mahamadi Warma

Received: 31 January 2023
Revised: 17 February 2023
Accepted: 22 February 2023
Published: 25 February 2023

1. Introduction

Convex functions are different from other function classes in that they are widely used in the areas of mathematics, statistics, optimization theory, and applied sciences. It results from the fact that its specific and practical meaning has a geometric interpretation. It is also one of the fundamentals of inequality theory and has developed into the main motivating element behind a number of inequalities. Convex analysis in the field of inequality theory has proven it to be the most significant and successful use of this notion since the concept of a convex function is beneficial in many fields of mathematical analysis and statistics. With the use of this concept, a number of traditional and analytical inequalities have been established, especially those of the Hermite–Hadamard, Fejér, Hardy, Simpson, and Ostrowski types [1–3].

One of the fundamental theorems of inequality theory is the notion of a convex function as follows:

Definition 1. *On a non-empty interval I on the real line $\mathbb{R}$, define the real function κ. The function κ is said to be convex on I if inequality*

$$\kappa(\varphi_t\delta + (1-\varphi_t)\mu) \le \varphi_t\kappa(\delta) + (1-\varphi_t)\kappa(\mu),$$

holds for all $\delta, \mu \in I$ and $\varphi_t \in [0,1]$.

The Hermite–Hadamard inequality, which is the significant component of the widespread use and great geometrical interpretation of convex functions, has attracted a lot of interest in fundamental mathematics. Due to its numerous applications, especially in the fields of numerical analysis, engineering, physical science, and chemistry, this inequality has attracted the interest of several researchers from around the world. Recent years have seen rapid development in the theory of inequality. Many inequalities can be obtained for convex functions; nevertheless, among those, Hermite–Hadamard's inequality is one of the most widely as well as intensively studied results. It is worth reflecting on the fact that the theories of inequality and convexity are closely related to one another. The idea of inequality is more intriguing as a result of this reality. In recent years, several new extensions, generalizations, and definitions of novel convexity have been given, and parallel developments in the theory of convexity inequality, particularly integral inequalities theory, have been emphasized. The Hermite–Hadamard inequality is formally expressed as follows:

Let $\kappa : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers, $\gamma_1, \gamma_2 \in I$ and $\gamma_1 < \gamma_2$.

$$\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right) \leq \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)d\delta \leq \frac{\kappa(\gamma_1)+\kappa(\gamma_2)}{2}. \tag{1}$$

The inequality in (1) will hold in reverse directions if κ is a concave function. The Hermite–Hadamard inequality, which is based on geometry, gives an upper and lower estimate for the integral mean of any convex function defined on a closed and limited domain, which includes the endpoints and midpoint of the domain of the function. Due to the significance of this inequality, several variations of the Hermite–Hadamard inequality have been examined in the literature for various classes of convexity, including harmonically convex, exponentially convex, s-convex, h-convex, and co-ordinate convex functions [4–19].

In [4], the definition of modified h-convexity and the Hermite–Hadamard inequality for the modified h-convex functions are shown as follows:

Definition 2. *Let $\kappa, h : J \subset \mathbb{R} \to \mathbb{R}$ be positive functions. Then κ is called a modified h-convex function if*

$$\kappa(\varphi_t\delta + (1-\varphi_t)\mu) \leq h(\varphi_t)\kappa(\delta) + (1-h(\varphi_t))\kappa(\mu),$$

$\forall \delta, \mu \in J$, $\varphi_t \in [0,1]$.

Theorem 1. *Let $\kappa : I \to \mathbb{R}$ be modified h-convex function on the interval $[\gamma_1, \gamma_2]$ with $\gamma_1 < \gamma_2$, then we have*

$$\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right) \leq \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)d\delta \leq \kappa(\gamma_1) + \{\kappa(\gamma_2)-\kappa(\gamma_1)\}\int_0^1 h(\varphi_t)d\varphi_t. \tag{2}$$

If $h(\varphi_t) = \varphi_t$ is taken in (2), then the classical Hermite–Hadamard inequality is obtained.

The Hermite–Hadamard inequalities, also known as Hermie–Hadamard–Fejér inequalities (Fejér inequalities), or its weighted versions are the most well-known inequalities relating to the integral mean of a convex function κ.

Theorem 2 ([20])**.** *Let $\kappa : I \to \mathbb{R}$ be convex on I and let $\gamma_1, \gamma_2 \in I$ with $\gamma_1 < \gamma_2$. Then the inequality*

$$\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\int_{\gamma_1}^{\gamma_2}\vartheta(\delta)d\delta \leq \int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \leq \frac{\kappa(\gamma_1)+\kappa(\gamma_2)}{2}\int_{\gamma_1}^{\gamma_2}\vartheta(\delta)d\delta \tag{3}$$

holds, where $\vartheta : [\gamma_1, \gamma_2] \to \mathbb{R}$ is non-negative and symmetric to $\frac{\gamma_1+\gamma_2}{2}$.

We are discussing the Hermite–Hadamard inequality if $\vartheta(\delta) = 1$ is considered in (3). Hermite–Hadamard- and Fejér-type inequalities have received substantial study and application in the last several decades in the fields of numerical analysis, information theory, optimisation theory, special means theory, and approximation theory. Numerous papers and monographs have further information regarding those inequalities. For recent results and generalizations regarding Fejér inequality see [20–25].

Moreover, in [4], Noor et al. proved a Fejér inequality using the modified h-convex functions again.

Lemma 1 (See [4]). *Let κ be modified h-convex function. Then*

$$\kappa(\gamma_1+\gamma_2-\delta) \le \kappa(\gamma_1)+\kappa(\gamma_2)-\kappa(\delta), \quad \forall \delta \in [\gamma_1,\gamma_2],$$

where $\delta = \varphi_t\gamma_1 + (1-\varphi_t)\gamma_2$, $\varphi_t \in [0,1]$.

Definition 3 (See [18]). *A function $h: J \to \mathbb{R}$ is said to be a super-additive function if*

$$h(\delta+\mu) \ge h(\delta)+h(\mu)$$

for all $\delta,\mu \in J$.

Definition 4 (See [19]). *Two functions $\kappa: X \to \mathbb{R}$ and $\vartheta: X \to \mathbb{R}$ are said to be similarly ordered, if*

$$(\kappa(\delta)-\kappa(\mu))(\vartheta(\delta)-\vartheta(\mu)) \ge 0$$

for every $\delta,\mu \in X$.

Theorem 3 (See [26]). *Let $\kappa,\vartheta : [\gamma_1,\gamma_2] \to [0,\infty)$ be convex functions on $[\gamma_1,\gamma_2] \subset \mathbb{R}$, $\gamma_1<\gamma_2$. Then*

$$\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \le \frac{1}{3}M(\gamma_1,\gamma_2)+\frac{1}{6}N(\gamma_1,\gamma_2) \tag{4}$$

and

$$2\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right) \le \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta + \frac{1}{6}M(\gamma_1,\gamma_2)+\frac{1}{3}N(\gamma_1,\gamma_2), \tag{5}$$

where

$$M(\gamma_1,\gamma_2) = \kappa(\gamma_1)\vartheta(\gamma_1)+\kappa(\gamma_2)\vartheta(\gamma_2) \text{ and } N(\gamma_1,\gamma_2) = \kappa(\gamma_1)\vartheta(\gamma_2)+\kappa(\gamma_2)\vartheta(\gamma_1).$$

In this study, in the first part, the basic identities used in the theory of inequality are given. In addition, inequalities and results obtained in the literature related to modified h-convex functions are provided. Modified h-convex function properties and fundamental calculus principles are used in the second section to arrive at conclusions that pertain to both sides of the Hadamard and Fejér inequalities.

Using the characteristics of modified h-convex functions, our aim in this article is to produce novel inequalities. These inequalities are related to the integral of the product of two functions. Furthermore, for different values of the $h(\varphi_t)$ function, new identities related to inequalities in different forms are obtained.

2. Main Results

Theorem 4. *Let $\kappa : [\gamma_1,\gamma_2] \to \mathbb{R}$ be a modified h-convex function on the interval $[\gamma_1,\gamma_2]$ with $\gamma_1<\gamma_2$, $\kappa \in L[\gamma_1,\gamma_2]$ and $\vartheta : [\gamma_1,\gamma_2] \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{\gamma_1+\gamma_2}{2}$, then*

$$\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \le \frac{\kappa(\gamma_1)-\kappa(\gamma_2)}{2}\int_{\gamma_1}^{\gamma_2}\left[h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)+h\left(\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\right)\right]\vartheta(\delta)d\delta + \Xi(\kappa,\vartheta)$$

where

$$\Xi(\kappa,\vartheta) = \kappa(\gamma_2)\int_{\gamma_1}^{\gamma_2}\vartheta(\delta)d\delta.$$

Proof. Since $\kappa : [\gamma_1, \gamma_2] \to \mathbb{R}$ can be the modified h-convex function and $\vartheta : [\gamma_1, \gamma_2] \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{\gamma_1+\gamma_2}{2}$, we can obtain that

$$\begin{aligned}
& \int_{\gamma_1}^{\gamma_2} \kappa(\delta)\vartheta(\delta)d\delta \\
= & \frac{1}{2}\left\{\int_{\gamma_1}^{\gamma_2} \kappa(\delta)\vartheta(\delta)d\delta + \int_{\gamma_1}^{\gamma_2} \kappa(\gamma_1+\gamma_2-\delta)\vartheta(\gamma_1+\gamma_2-\delta)d\delta\right\} \\
= & \frac{1}{2}\int_{\gamma_1}^{\gamma_2} [\kappa(\delta)+\kappa(\gamma_1+\gamma_2-\delta)]\vartheta(\delta)d\delta \\
= & \frac{1}{2}\int_{\gamma_1}^{\gamma_2} \left[\kappa\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\gamma_1+\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\right)+\kappa\left(\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\gamma_1+\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\gamma_2\right)\right]\vartheta(\delta)d\delta \\
\leq & \frac{1}{2}\int_{\gamma_1}^{\gamma_2} \left[h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\kappa(\gamma_1)+\left(1-h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\right)\kappa(\gamma_2)\right. \\
& \left.+h\left(\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\right)\kappa(\gamma_1)+\left(1-h\left(\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\right)\right)\kappa(\gamma_2)\right]\vartheta(\delta)d\delta \\
= & \frac{\kappa(\gamma_1)-\kappa(\gamma_2)}{2}\int_{\gamma_1}^{\gamma_2}\left[h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)+h\left(\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\right)\right]\vartheta(\delta)d\delta+\kappa(\gamma_2)\int_{\gamma_1}^{\gamma_2}\vartheta(\delta)d\delta.
\end{aligned}$$

This completes the proof. □

Remark 1. *In Theorem 4,*

1. *If we take $h(\varphi_t) = \varphi_t$, then we have the right-hand side of the Fejér inequality,*
2. *If we choose $h(\varphi_t) = \varphi_t$ and $\vartheta(\delta) = 1$, then we have the right-hand side of the Hermite–Hadamard inequality.*

Corollary 1. *Let h be a super-additive function. Under the assumptions of Theorem 4, then we have the following inequalities:*

$$\int_{\gamma_1}^{\gamma_2} \kappa(\delta)\vartheta(\delta)d\delta \leq \left[\frac{\kappa(\gamma_1)-\kappa(\gamma_2)}{2}h(1)+\kappa(\gamma_2)\right]\int_{\gamma_1}^{\gamma_2}\vartheta(\delta)d\delta.$$

Also if we take $\vartheta(\delta) = 1$,

$$\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)d\delta \leq \frac{\kappa(\gamma_1)-\kappa(\gamma_2)}{2}h(1)+\kappa(\gamma_2).$$

Theorem 5. *Let $\kappa, \vartheta : [\gamma_1, \gamma_2] \to \mathbb{R}$ be two modified h-convex functions such that κ and ϑ are similarly ordered functions. If $\kappa, \vartheta \in L[\gamma_1, \gamma_2]$, then*

$$\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \leq \kappa(\gamma_2)\vartheta(\gamma_2)+\frac{\kappa(\gamma_1)\vartheta(\gamma_1)-\kappa(\gamma_2)\vartheta(\gamma_2)}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)d\delta.$$

Proof. Since κ and ϑ are modified h-convex functions, then

$$\begin{aligned}
\kappa(\delta)\vartheta(\delta) &= \kappa\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\gamma_1+\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\right)\vartheta\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\gamma_1+\frac{\delta-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\right) \\
&\leq \left[h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\kappa(\gamma_1)+\left(1-h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\right)\kappa(\gamma_2)\right] \\
&\quad\times\left[h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\vartheta(\gamma_1)+\left(1-h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\right)\vartheta(\gamma_2)\right] \\
&= h^2\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\kappa(\gamma_1)\vartheta(\gamma_1) \\
&\quad+h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\left(1-h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\right)\{\kappa(\gamma_1)\vartheta(\gamma_2)+\kappa(\gamma_2)\vartheta(\gamma_1)\} \\
&\quad+\left(1-h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\right)^2\kappa(\gamma_2)\vartheta(\gamma_2).
\end{aligned}$$

By using the similarly ordered properties of κ and ϑ, we have

$$\begin{aligned}
\kappa(\delta)\vartheta(\delta) &\leq h^2\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\kappa(\gamma_1)\vartheta(\gamma_1) \\
&\quad+h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\left(1-h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\right)\{\kappa(\gamma_1)\vartheta(\gamma_1)+\kappa(\gamma_2)\vartheta(\gamma_2)\} \\
&\quad+\left(1-h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right)\right)^2\kappa(\gamma_2)\vartheta(\gamma_2) \\
&= \kappa(\gamma_2)\vartheta(\gamma_2)+\{\kappa(\gamma_1)\vartheta(\gamma_1)-\kappa(\gamma_2)\vartheta(\gamma_2)\}h\left(\frac{\gamma_2-\delta}{\gamma_2-\gamma_1}\right).
\end{aligned}$$

Integrating the above inequality with respect to $\delta \in [\gamma_1, \gamma_2]$, we obtain the required result. □

Corollary 2. *In Theorem 5,*

1. *If we take $h(\varphi_t) = \varphi_t$, then we have*

$$\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \leq \frac{\kappa(\gamma_1)\vartheta(\gamma_1)+\kappa(\gamma_2)\vartheta(\gamma_2)}{2} = \frac{M(\gamma_1,\gamma_2)}{2},$$

2. *If we choose $h(\varphi_t) = 1$, then*

$$\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \leq \kappa(\gamma_1)\vartheta(\gamma_1),$$

3. *If we choose $h(\varphi_t) = (\varphi_t)^s$, we obtain*

$$\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \leq \frac{\kappa(\gamma_1)\vartheta(\gamma_1)+s\kappa(\gamma_2)\vartheta(\gamma_2)}{s+1}.$$

Theorem 6. *Let κ and ϑ be two modified h-convex functions such that κ and ϑ are similarly ordered functions. If $h\left(\frac{1}{2}\right) \neq \frac{1}{2}$, then we have*

$$\begin{aligned}
&\frac{1}{\left(1-2h\left(\frac{1}{2}\right)\right)^2}\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right)-Q(\kappa,\vartheta,h,\delta) \\
\leq\ &\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta,
\end{aligned}$$

where

$$\begin{aligned} Q(\kappa,\vartheta,h,\delta) &= \frac{2h^2\left(\frac{1}{2}\right)}{\left(1-2h\left(\frac{1}{2}\right)\right)^2}M(\gamma_1,\gamma_2)+\frac{h\left(\frac{1}{2}\right)}{(\gamma_2-\gamma_1)\left(1-2h\left(\frac{1}{2}\right)\right)} \\ &\quad \times\left[\{\kappa(\gamma_1)+\kappa(\gamma_2)\}\int_{\gamma_1}^{\gamma_2}\vartheta(\delta)d\delta+\{\vartheta(\gamma_1)+\vartheta(\gamma_2)\}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)d\delta\right] \end{aligned}$$

and

$$M(\gamma_1,\gamma_2)=\kappa(\gamma_1)\vartheta(\gamma_1)+\kappa(\gamma_2)\vartheta(\gamma_2).$$

Proof. Since κ and ϑ are modified h-convex functions, then we get

$$\begin{aligned} \kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right) &= \kappa\left(\frac{\gamma_1+\gamma_2+\delta-\delta}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2+\delta-\delta}{2}\right) \\ &\le \left[h\left(\frac{1}{2}\right)\kappa(\gamma_1+\gamma_2-\delta)+\left(1-h\left(\frac{1}{2}\right)\right)\kappa(\delta)\right] \\ &\quad \times\left[h\left(\frac{1}{2}\right)\vartheta(\gamma_1+\gamma_2-\delta)+\left(1-h\left(\frac{1}{2}\right)\right)\vartheta(\delta)\right]. \end{aligned}$$

Using the Lemma 1 and κ, ϑ are similarly ordered functions, then

$$\begin{aligned} &\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right) \\ \le\ & \left[h\left(\frac{1}{2}\right)\{\kappa(\gamma_1)+\kappa(\gamma_2)\}+\left(1-2h\left(\frac{1}{2}\right)\right)\kappa(\delta)\right] \\ & \times\left[h\left(\frac{1}{2}\right)\{\vartheta(\gamma_1)+\vartheta(\gamma_2)\}+\left(1-2h\left(\frac{1}{2}\right)\right)\vartheta(\delta)\right] \\ =\ & h^2\left(\frac{1}{2}\right)\{\kappa(\gamma_1)\vartheta(\gamma_1)+\kappa(\gamma_1)\vartheta(\gamma_2)+\kappa(\gamma_2)\vartheta(\gamma_1)+\kappa(\gamma_2)\vartheta(\gamma_2)\} \\ & +h\left(\frac{1}{2}\right)\left(1-2h\left(\frac{1}{2}\right)\right)[\{\kappa(\gamma_1)+\kappa(\gamma_2)\}\vartheta(\delta)+\{\vartheta(\gamma_1)+\vartheta(\gamma_2)\}\kappa(\delta)] \\ & +\left(1-2h\left(\frac{1}{2}\right)\right)^2\kappa(\delta)\vartheta(\delta) \\ \le\ & h^2\left(\frac{1}{2}\right)2M(\gamma_1,\gamma_2) \\ & +h\left(\frac{1}{2}\right)\left(1-2h\left(\frac{1}{2}\right)\right)[\{\kappa(\gamma_1)+\kappa(\gamma_2)\}\vartheta(\delta)+\{\vartheta(\gamma_1)+\vartheta(\gamma_2)\}\kappa(\delta)] \\ & +\left(1-2h\left(\frac{1}{2}\right)\right)^2\kappa(\delta)\vartheta(\delta). \end{aligned}$$

Integrating the above inequality with respect to $\delta\in[\gamma_1,\gamma_2]$, we obtain the required result. □

Corollary 3. *In Theorem 6, if we choose $h(\varphi_t)=1$, then*

$$\begin{aligned} &\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right)-2M(\gamma_1,\gamma_2)+\frac{1}{\gamma_2-\gamma_1}\gamma(\kappa,\vartheta) \\ \le\ & \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta, \end{aligned}$$

where

$$\gamma(\kappa,\vartheta)=\{\kappa(\gamma_1)+\kappa(\gamma_2)\}\int_{\gamma_1}^{\gamma_2}\vartheta(\delta)d\delta+\{\vartheta(\gamma_1)+\vartheta(\gamma_2)\}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)d\delta.$$

In addition to the above inequality, if $\vartheta(\delta) = 1$*, then*

$$\frac{1}{\gamma_2 - \gamma_1} \int_{\gamma_1}^{\gamma_2} \kappa(\delta) d\delta \leq [\kappa(\gamma_1) + \kappa(\gamma_2)] - \kappa\left(\frac{\gamma_1 + \gamma_2}{2}\right).$$

Remark 2. *From Theorems 5 and 6, we obtain the following identities:*

$$\begin{aligned} & \frac{1}{\left(1 - 2h\left(\frac{1}{2}\right)\right)^2} \kappa\left(\frac{\gamma_1 + \gamma_2}{2}\right) \vartheta\left(\frac{\gamma_1 + \gamma_2}{2}\right) - Q(\kappa, \vartheta, h, \delta) \\ \leq \quad & \frac{1}{\gamma_2 - \gamma_1} \int_{\gamma_1}^{\gamma_2} \kappa(\delta) \vartheta(\delta) d\delta \\ \leq \quad & \kappa(\gamma_2)\vartheta(\gamma_2) + \frac{\kappa(\gamma_1)\vartheta(\gamma_1) - \kappa(\gamma_2)\vartheta(\gamma_2)}{\gamma_2 - \gamma_1} \int_{\gamma_1}^{\gamma_2} h\left(\frac{\gamma_2 - \delta}{\gamma_2 - \gamma_1}\right) d\delta. \end{aligned}$$

Theorem 7. *Let* κ *and* ϑ *be two modified h-convex functions such that* κ *and* ϑ *are similarly ordered functions. Then we have the following inequality:*

$$\begin{aligned} & \kappa\left(\frac{\gamma_1 + \gamma_2}{2}\right) \vartheta\left(\frac{\gamma_1 + \gamma_2}{2}\right) - h\left(\frac{1}{2}\right) \left[1 - h\left(\frac{1}{2}\right)\right] \psi(\kappa, \vartheta, \varphi_t) \\ \leq \quad & \left[1 - h\left(\frac{1}{2}\right)\right]^2 \frac{2}{\gamma_2 - \gamma_1} \int_{\gamma_1}^{\frac{\gamma_1 + \gamma_2}{2}} \kappa(\delta)\vartheta(\delta) d\delta + \left[h\left(\frac{1}{2}\right)\right]^2 \frac{2}{\gamma_2 - \gamma_1} \int_{\frac{\gamma_1 + \gamma_2}{2}}^{\gamma_2} \kappa(\delta)\vartheta(\delta) d\delta, \end{aligned}$$

where

$$\psi(\kappa, \vartheta, \varphi_t) = M(\gamma_1, \gamma_2) \int_0^1 \left[1 - h\left(\frac{\varphi_t}{2}\right) + \left(h\left(\frac{\varphi_t}{2}\right)\right)^2\right] d\varphi_t.$$

and

$$M(\gamma_1, \gamma_2) = \kappa(\gamma_1)\vartheta(\gamma_1) + \kappa(\gamma_2)\vartheta(\gamma_2).$$

Proof. From κ and ϑ are the modified h-convexity functions, then we can obtain

$$\begin{aligned} & \kappa\left(\frac{\gamma_1 + \gamma_2}{2}\right) \vartheta\left(\frac{\gamma_1 + \gamma_2}{2}\right) \\ = \quad & \kappa\left(\frac{\frac{2-\varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2 + \frac{\varphi_t}{2}\gamma_1 + \frac{2-\varphi_t}{2}\gamma_2}{2}\right) \vartheta\left(\frac{\frac{2-\varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2 + \frac{\varphi_t}{2}\gamma_1 + \frac{2-\varphi_t}{2}\gamma_2}{2}\right) \\ \leq \quad & \left\{ \left[1 - h\left(\frac{1}{2}\right)\right] \kappa\left(\frac{2 - \varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2\right) + h\left(\frac{1}{2}\right) \kappa\left(\frac{\varphi_t}{2}\gamma_1 + \frac{2 - \varphi_t}{2}\gamma_2\right) \right\} \\ & \times \left\{ \left[1 - h\left(\frac{1}{2}\right)\right] \vartheta\left(\frac{2 - \varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2\right) + h\left(\frac{1}{2}\right) \vartheta\left(\frac{\varphi_t}{2}\gamma_1 + \frac{2 - \varphi_t}{2}\gamma_2\right) \right\} \\ = \quad & \left[1 - h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{2 - \varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2\right) \vartheta\left(\frac{2 - \varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2\right) \\ & + \left[h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{\varphi_t}{2}\gamma_1 + \frac{2 - \varphi_t}{2}\gamma_2\right) \vartheta\left(\frac{\varphi_t}{2}\gamma_1 + \frac{2 - \varphi_t}{2}\gamma_2\right) \\ & + h\left(\frac{1}{2}\right) \left[1 - h\left(\frac{1}{2}\right)\right] \left\{ \kappa\left(\frac{2 - \varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2\right) \vartheta\left(\frac{\varphi_t}{2}\gamma_1 + \frac{2 - \varphi_t}{2}\gamma_2\right) \right. \\ & \qquad \left. + \kappa\left(\frac{\varphi_t}{2}\gamma_1 + \frac{2 - \varphi_t}{2}\gamma_2\right) \vartheta\left(\frac{2 - \varphi_t}{2}\gamma_1 + \frac{\varphi_t}{2}\gamma_2\right) \right\}. \end{aligned}$$

Again using the definition of modified h-convex function, we can obtain

$$\begin{aligned}
& \kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right) \\
\leq\ & \left[1-h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{2-\varphi_t}{2}\gamma_1+\frac{\varphi_t}{2}\gamma_2\right)\vartheta\left(\frac{2-\varphi_t}{2}\gamma_1+\frac{\varphi_t}{2}\gamma_2\right) \\
& +\left[h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{\varphi_t}{2}\gamma_1+\frac{2-\varphi_t}{2}\gamma_2\right)\vartheta\left(\frac{\varphi_t}{2}\gamma_1+\frac{2-\varphi_t}{2}\gamma_2\right) \\
& +h\left(\frac{1}{2}\right)\left[1-h\left(\frac{1}{2}\right)\right] \\
& \quad \times\left\{\left(\left[1-h\left(\frac{\varphi_t}{2}\right)\right]\kappa(\gamma_1)+h\left(\frac{\varphi_t}{2}\right)\kappa(\gamma_2)\right)\left(h\left(\frac{\varphi_t}{2}\right)\vartheta(\gamma_1)+\left[1-h\left(\frac{\varphi_t}{2}\right)\right]\vartheta(\gamma_2)\right)\right. \\
& \quad\quad \left.+\left(h\left(\frac{\varphi_t}{2}\right)\kappa(\gamma_1)+\left[1-h\left(\frac{\varphi_t}{2}\right)\right]\kappa(\gamma_2)\right)\left(\left[1-h\left(\frac{\varphi_t}{2}\right)\right]\vartheta(\gamma_1)+h\left(\frac{\varphi_t}{2}\right)\vartheta(\gamma_2)\right)\right\} \\
=\ & \left[1-h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{2-\varphi_t}{2}\gamma_1+\frac{\varphi_t}{2}\gamma_2\right)\vartheta\left(\frac{2-\varphi_t}{2}\gamma_1+\frac{\varphi_t}{2}\gamma_2\right) \\
& +\left[h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{\varphi_t}{2}\gamma_1+\frac{2-\varphi_t}{2}\gamma_2\right)\vartheta\left(\frac{\varphi_t}{2}\gamma_1+\frac{2-\varphi_t}{2}\gamma_2\right) \\
& +h\left(\frac{1}{2}\right)\left[1-h\left(\frac{1}{2}\right)\right] \\
& \quad \times\left\{h\left(\frac{\varphi_t}{2}\right)\left[1-h\left(\frac{\varphi_t}{2}\right)\right](2\kappa(\gamma_1)\vartheta(\gamma_1)+2\kappa(\gamma_2)\vartheta(\gamma_2))\right. \\
& \quad\quad \left.+\left[\left(1-h\left(\frac{\varphi_t}{2}\right)\right)^2+\left(h\left(\frac{\varphi_t}{2}\right)\right)^2\right](\kappa(\gamma_1)\vartheta(\gamma_2)+\kappa(\gamma_2)\vartheta(\gamma_1))\right\} \\
\leq\ & \left[1-h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{2-\varphi_t}{2}\gamma_1+\frac{\varphi_t}{2}\gamma_2\right)\vartheta\left(\frac{2-\varphi_t}{2}\gamma_1+\frac{\varphi_t}{2}\gamma_2\right) \\
& +\left[h\left(\frac{1}{2}\right)\right]^2 \kappa\left(\frac{\varphi_t}{2}\gamma_1+\frac{2-\varphi_t}{2}\gamma_2\right)\vartheta\left(\frac{\varphi_t}{2}\gamma_1+\frac{2-\varphi_t}{2}\gamma_2\right) \\
& +h\left(\frac{1}{2}\right)\left[1-h\left(\frac{1}{2}\right)\right]\left[1-h\left(\frac{\varphi_t}{2}\right)+\left(h\left(\frac{\varphi_t}{2}\right)\right)^2\right]\{\kappa(\gamma_1)\vartheta(\gamma_1)+\kappa(\gamma_2)\vartheta(\gamma_2)\}.
\end{aligned}$$

In the inequality mentioned above, we obtain the preferred result by integrating both sides with regard to $\varphi_t \in [0,1]$. □

Theorem 8. *Let κ be modified h_1-convex function, ϑ be the modified h_2-convex function and κ and ϑ similarly ordered functions. $\kappa\vartheta \in L[\gamma_1,\gamma_2]$ and $h_1h_2 \in L[0,1]$, then*

$$\begin{aligned}
& \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \\
\leq\ & [M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)]\int_0^1 h_1(\varphi_t)h_2(\varphi_t)d\varphi_t+\Phi(\kappa,\vartheta,h_1,h_2) \\
\leq\ & \Phi(\kappa,\vartheta,h_1,h_2)
\end{aligned}$$

where

$$\begin{aligned}
\Phi(\kappa,\vartheta,h_1,h_2) =\ & \kappa(\gamma_2)\vartheta(\gamma_2)\int_0^1[1-h_1(\varphi_t)-h_2(\varphi_t)]d\varphi_t \\
& +\kappa(\gamma_1)\vartheta(\gamma_2)\int_0^1 h_1(\varphi_t)d\varphi_t+\kappa(\gamma_2)\vartheta(\gamma_1)\int_0^1 h_2(\varphi_t)d\varphi_t
\end{aligned}$$

and

$$M(\gamma_1,\gamma_2) = \kappa(\gamma_1)\vartheta(\gamma_1) + \kappa(\gamma_2)\vartheta(\gamma_2),\ N(\gamma_1,\gamma_2) = \kappa(\gamma_1)\vartheta(\gamma_2) + \kappa(\gamma_2)\vartheta(\gamma_1).$$

Proof. Since κ and ϑ are modified h_1-convex and h_2-convex functions, respectively, we obtain

$$\begin{aligned} \kappa(\varphi_t\gamma_1 + (1-\varphi_t)\gamma_2) &\leq h_1(\varphi_t)\kappa(\gamma_1) + (1-h_1(\varphi_t))\kappa(\gamma_2) \\ \vartheta(\varphi_t\gamma_1 + (1-\varphi_t)\gamma_2) &\leq h_2(\varphi_t)\vartheta(\gamma_1) + (1-h_2(\varphi_t))\vartheta(\gamma_2), \end{aligned}$$

for all $\varphi_t \in [0,1]$. Hence,

$$\begin{aligned} & \kappa(\varphi_t\gamma_1 + (1-\varphi_t)\gamma_2)\vartheta(\varphi_t\gamma_1 + (1-\varphi_t)\gamma_2) \\ \leq\ & h_1(\varphi_t)h_2(\varphi_t)\kappa(\gamma_1)\vartheta(\gamma_1) + h_1(\varphi_t)(1-h_2(\varphi_t))\kappa(\gamma_1)\vartheta(\gamma_2) \\ & +(1-h_1(\varphi_t))h_2(\varphi_t)\kappa(\gamma_2)\vartheta(\gamma_1) + (1-h_1(\varphi_t))(1-h_2(\varphi_t))\kappa(\gamma_2)\vartheta(\gamma_2) \\ =\ & \kappa(\gamma_2)\vartheta(\gamma_2) + h_1(\varphi_t)h_2(\varphi_t)\{M(\gamma_1,\gamma_2) - N(\gamma_1,\gamma_2)\} \\ & +h_1(\varphi_t)\{\kappa(\gamma_1)\vartheta(\gamma_2) - \kappa(\gamma_2)\vartheta(\gamma_2)\} + h_2(\varphi_t)\{\kappa(\gamma_2)\vartheta(\gamma_1) - \kappa(\gamma_2)\vartheta(\gamma_2)\}. \end{aligned}$$

By integrating both sides with respect to φ_t over the interval $[0,1]$, we obtain the required result.

Since κ and ϑ are similarly ordered functions, we can write $N(\gamma_1,\gamma_2) \leq M(\gamma_1,\gamma_2)$. Therefore, the second inequality follows easily. □

Remark 3. *In Theorem 8, if we take $h_1(\varphi_t) = h_2(\varphi_t) = \varphi_t$, then we obtain the inequality (4).*

Theorem 9. *Let κ be a modified h_1-convex function and ϑ be a modified h_2-convex function, κ and ϑ are similarly ordered functions. $\kappa\vartheta \in L[\gamma_1,\gamma_2]$ and $h_1h_2 \in L[0,1]$, then we have*

$$\begin{aligned} & \frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right) - \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta \\ \leq\ & \frac{1}{2}\left\{[M(\gamma_1,\gamma_2) - N(\gamma_1,\gamma_2)]\int_0^1[2h_1(\varphi_t)h_2(\varphi_t) - h_1(\varphi_t) - h_2(\varphi_t)]d\varphi_t - N(\gamma_1,\gamma_2)\right\} \\ & +\frac{1}{2h_2\left(\frac{1}{2}\right)}\left\{[M(\gamma_1,\gamma_2) - N(\gamma_1,\gamma_2)]\int_0^1[h_2(\varphi_t) - 2h_1(\varphi_t)h_2(\varphi_t)]d\varphi_t\right. \\ & \qquad \left. +[\kappa(\gamma_1)\vartheta(\gamma_1) - \kappa(\gamma_2)\vartheta(\gamma_1)]\int_0^1[2h_1(\varphi_t) - 1]d\varphi_t\right\} \\ & +\frac{1}{2h_1\left(\frac{1}{2}\right)}\left\{[M(\gamma_1,\gamma_2) - N(\gamma_1,\gamma_2)]\int_0^1[h_1(\varphi_t) - 2h_1(\varphi_t)h_2(\varphi_t)]d\varphi_t\right. \\ & \qquad \left. +[\kappa(\gamma_1)\vartheta(\gamma_1) - \kappa(\gamma_1)\vartheta(\gamma_2)]\int_0^1[2h_2(\varphi_t) - 1]d\varphi_t\right\} \\ & +\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left\{[M(\gamma_1,\gamma_2) - N(\gamma_1,\gamma_2)]\int_0^1 h_1(\varphi_t)h_2(\varphi_t)d\varphi_t\right. \\ & \qquad + [\kappa(\gamma_2)\vartheta(\gamma_1) - \kappa(\gamma_1)\vartheta(\gamma_1)]\int_0^1 h_1(\varphi_t)d\varphi_t \\ & \qquad \left. +[\kappa(\gamma_1)\vartheta(\gamma_2) - \kappa(\gamma_1)\vartheta(\gamma_1)]\int_0^1 h_2(\varphi_t)d\varphi_t + \kappa(\gamma_1)\vartheta(\gamma_1)\right\} \end{aligned}$$

where

$$M(\gamma_1,\gamma_2) = \kappa(\gamma_1)\vartheta(\gamma_1) + \kappa(\gamma_2)\vartheta(\gamma_2),\ N(\gamma_1,\gamma_2) = \kappa(\gamma_1)\vartheta(\gamma_2) + \kappa(\gamma_2)\vartheta(\gamma_1).$$

Proof. Let κ and ϑ be modified h_1-convex and h_2-convex functions, respectively. Then we obtain

$$\begin{aligned}
& \kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right) \\
= & \kappa\left(\frac{\gamma_1\varphi_t+(1-\varphi_t)\gamma_2}{2}+\frac{(1-\varphi_t)\gamma_1+\varphi_t\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1\varphi_t+(1-\varphi_t)\gamma_2}{2}+\frac{(1-\varphi_t)\gamma_1+\varphi_t\gamma_2}{2}\right) \\
\leq & \left\{h_1\left(\frac{1}{2}\right)\kappa(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2)+\left(1-h_1\left(\frac{1}{2}\right)\right)\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\right\} \\
& \times\left\{h_2\left(\frac{1}{2}\right)\vartheta(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2)+\left(1-h_2\left(\frac{1}{2}\right)\right)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\right\} \\
= & h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\{\kappa(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2)\vartheta(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2) \\
& \quad +\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\} \\
& -h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\{\kappa(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2) \\
& \quad +\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2)\} \\
& +h_1\left(\frac{1}{2}\right)\{\kappa(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2) \\
& \quad -\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\} \\
& +h_2\left(\frac{1}{2}\right)\{\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2) \\
& \quad -\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\} \\
& +\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2).
\end{aligned}$$

Again, using the definition of modified h_1-convex and h_2-convex functions, we have

$$\begin{aligned}
& \kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right) \\
\leq & h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\{\kappa(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2)\vartheta(\gamma_1\varphi_t+(1-\varphi_t)\gamma_2) \\
& \quad +\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\} \\
& -h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\{[h_1(\varphi_t)+h_2(\varphi_t)-2h_1(\varphi_t)h_2(\varphi_t)][M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)]+N(\gamma_1,\gamma_2)\} \\
& +h_1\left(\frac{1}{2}\right)\{[h_2(\varphi_t)-2h_1(\varphi_t)h_2(\varphi_t)][M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)] \\
& \quad +[2h_1(\varphi_t)-1][\kappa(\gamma_1)\vartheta(\gamma_1)-\kappa(\gamma_2)\vartheta(\gamma_1)]\} \\
& +h_2\left(\frac{1}{2}\right)\{[h_1(\varphi_t)-2h_1(\varphi_t)h_2(\varphi_t)][M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)] \\
& \quad +[2h_2(\varphi_t)-1][\kappa(\gamma_1)\vartheta(\gamma_1)-\kappa(\gamma_1)\vartheta(\gamma_2)]\} \\
& +\kappa((1-\varphi_t)\gamma_1+\varphi_t\gamma_2)\vartheta((1-\varphi_t)\gamma_1+\varphi_t\gamma_2).
\end{aligned}$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\kappa\left(\frac{\gamma_1+\gamma_2}{2}\right)\vartheta\left(\frac{\gamma_1+\gamma_2}{2}\right)-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\kappa(\delta)\vartheta(\delta)d\delta\\
\leq\ &\frac{1}{2}\left\{[M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)]\int_0^1[2h_1(\varphi_t)h_2(\varphi_t)-h_1(\varphi_t)-h_2(\varphi_t)]d\varphi_t-N(\gamma_1,\gamma_2)\right\}\\
&+\frac{1}{2h_2\left(\frac{1}{2}\right)}\left\{[M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)]\int_0^1[h_2(\varphi_t)-2h_1(\varphi_t)h_2(\varphi_t)]d\varphi_t\right.\\
&\qquad\left.+[\kappa(\gamma_1)\vartheta(\gamma_1)-\kappa(\gamma_2)\vartheta(\gamma_1)]\int_0^1[2h_1(\varphi_t)-1]d\varphi_t\right\}\\
&+\frac{1}{2h_1\left(\frac{1}{2}\right)}\left\{[M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)]\int_0^1[h_1(\varphi_t)-2h_1(\varphi_t)h_2(\varphi_t)]d\varphi_t\right.\\
&\qquad\left.+[\kappa(\gamma_1)\vartheta(\gamma_1)-\kappa(\gamma_1)\vartheta(\gamma_2)]\int_0^1[2h_2(\varphi_t)-1]d\varphi_t\right\}\\
&+\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left\{[M(\gamma_1,\gamma_2)-N(\gamma_1,\gamma_2)]\int_0^1h_1(\varphi_t)h_2(\varphi_t)d\varphi_t\right.\\
&\qquad+[\kappa(\gamma_2)\vartheta(\gamma_1)-\kappa(\gamma_1)\vartheta(\gamma_1)]\int_0^1h_1(\varphi_t)d\varphi_t\\
&\qquad\left.+[\kappa(\gamma_1)\vartheta(\gamma_2)-\kappa(\gamma_1)\vartheta(\gamma_1)]\int_0^1h_2(\varphi_t)d\varphi_t+\kappa(\gamma_1)\vartheta(\gamma_1)\right\}.
\end{aligned}$$

The proof is completed. □

Remark 4. *In Theorem 9, if we take $h_1(\varphi_t)=h_2(\varphi_t)=\varphi_t$, then we obtain the inequality (5).*

3. Conclusions

In this study, the properties of modified h-convex functions are investigated. By using the properties of modified h-convex functions, new integral inequalities of the Hermite–Hadamard and Fejér types, well known in the literature, are obtained. The properties of the modified h-convex functions are used in this study to encourage the development of numerous Hermite–Hadamard inequalities.

Author Contributions: Conceptualization, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; methodology, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; software, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; validation, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; formal analysis, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; investigation, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; resources, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; data curation, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; writing—original draft preparation, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; writing—review and editing, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; visualization, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; supervision, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; project administration, Ç.Y., L.-I.C., G.R., B.Y. and D.B.; funding acquisition, D.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Nasir, J.; Qaisar, S.; Butt, S.I.; Qayyum, A. Some Ostrowski type inequalities for mappings whose second derivatives are preinvex function via fractional integral operator. *AIMS Math.* **2022**, *7*, 3303–3320. [CrossRef]
2. Butt, S.I.; Umar, M.; Rashid, S.; Akdemir, A.O.; Chu, Y.M. New Hermite–Jensen–Mercer-type inequalities via k-fractional integrals. *Adv. Diff. Equ.* **2020**, *1*, 635. [CrossRef]

3. Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Kodamasingh, B.; Nonlaopon, K.; Abualnaja, K.M. Interval valued Hadamard-Fejér and Pachpatte Type inequalities pertaining to a new fractional integral operator with exponential kernel. *AIMS Math.* **2022**, *7*, 15041–15063. [CrossRef]
4. Noor, M.A.; Noor K.I.; Awan, M.U. Hermite-Hadamard inequalities for modified h-convex functions. *TJMM* **2014**, *6*, 171–180.
5. Varošanec, S. On h-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
6. Li, P.Y.; Yan, Q.; Chu, Y.M.; Mukhtar, S.; Waheed, S. On some fractional integral inequalities for generalized strongly modified h-convex function. *AIMS Math.* **2020**, *5*, 6620–6638. [CrossRef]
7. Feng, B.; Ghafoor, M.; Chu, Y.M.; Qureshi, M.I.; Feng, X.; Yao, C.; Qiao, X. Hermite-Hadamard and Jensen's type inequalities for modified (p, h)-convex functions. *AIMS Math.* **2020**, *5*, 6959–6971. [CrossRef]
8. Wang, X.; Saleem, M.S.; Aslam, K.N.; Wu, X.; Zhou, T. On Caputo–Fabrizio Fractional Integral Inequalities of Hermite–Hadamard Type for Modified h-convex Functions. *J. Math.* **2020**, *2020*, 8829140. [CrossRef]
9. Özdemir, M.E.; Gürbüz, M.; Yildiz, C. Inequalities for mappings whose second derivatives are quasi-convex or h-convex functions. *Miskolc Math. Notes* **2014**, *15*, 635–649. [CrossRef]
10. Mitrinovic, D.S.; Pecaric J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993.
11. Zhao, T.; Saleem, M.S.; Nazeer, W.; Bashir, I., Hussain, I. On generalized strongly modified h-convex functions. *J. Inequalities Appl.* **2020**, *2020*, 11. [CrossRef]
12. Sarıkaya, M.Z.; Sağlam, A.; Yıldırım, H. On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal* **2008**, *2*, 335–341. [CrossRef]
13. Zhao, D.; Zhao, G.; Ye, G.; Liu, W.; Dragomir, S.S. On Hermite–Hadamard-Type Inequalities for Coordinated h-convex Interval-Valued Functions. *Mathematics* **2021**, *9*, 2352. [CrossRef]
14. Bombardelli, M.; Varošanec, S. Properties of convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **2009**, *58*, 1869–1877. [CrossRef]
15. İşcan, İ. New refinements for integral and sum forms of Hölder inequality. *J. Inequalities Appl.* **2019**, *2019*, 304. [CrossRef]
16. Kadakal, M.; İşcan, İ.; Kadakal H.; Bekar, K. On improvements of some integral inequalities. *Honam Math. J.* **2021**, *43*, 441–452. [CrossRef]
17. Xiao, Z.G.; Zhang, Z.H.; Wu, Y.D. On weight Hermite-Hadamard inequalities. *App. Math. Comp.* **2011**, *218*, 1147–1152. [CrossRef]
18. Alzer, H. A superadditive property of Hadamard's gamma function. *Abh. Math. Semin. Univ. Hambg.* **2009**, *79*, 11–23. [CrossRef]
19. Skala, H.J. On the characterization of certain similarly ordered super-additive functions. *Proc. Am. Math. Soc.* **1998**, *126*, 1349–1353. [CrossRef]
20. Fejér, L. Uberdie Fourierreihen, II. *Math. Naturwiss Anz. Ungar. Akad. Wiss Hung.* **1906**, *24*, 369–390. (In Hungarian)
21. Sarikaya, M.Z. On new Hermite Hadamard Fejér type integral inequalities. *Stud. Univ. Babes-Bolyai Math.* **2015**, *57*, 377–386.
22. Kalsoom, H.; Vivas-Cortez, M.; Amer Latif, M.; Ahmad, H. Weighted Midpoint Hermite-Hadamard-Fejér Type Inequalities in Fractional Calculus for Harmonically Convex Functions. *Fractal Fract.* **2021**, *5*, 252. [CrossRef]
23. Khan, M.B.; Macías-Díaz, J.E.; Treanţă, S.; Soliman, M.S. Some Fejér-Type Inequalities for Generalized Interval-Valued Convex Functions. *Mathematics* **2022**, *10*, 3851. [CrossRef]
24. Vivas-Cortez, M.; Kórus P.; Napoles-Valdes, J.E. Some generalized Hermite–Hadamard–Fejér inequality for convex functions. *Adv. Diff. Equa.* **2021**, *1*, 1–11. [CrossRef]
25. Set, E.; İşcan, İ.; Sarikaya M.Z.; Özdemir, M.E. On new inequalities of Hermite–Hadamard–Fejér type for convex functions via fractional integrals. *Appl. Math. Comput.* **2015**, *259*, 875–881. [CrossRef]
26. Pachpatte, B.G. On some inequalities for convex functions. *RGMIA Res. Rep. Coll.* **2003**, *6*, 1–9.