fractal and fractional

Article

On the Existence Results for a Mixed Hybrid Fractional Differential Equations of Sequential Type

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Abstract: In this article, we study the existence of a solution to the mixed hybrid fractional differential equations of sequential type with nonlocal integral hybrid boundary conditions. The main results are established with the aid of Darbo's fixed point theorem and Hausdorff's measure of noncompactness method. The stability of the proposed fractional differential equation is also investigated using the Ulam–Hyer technique. In addition, an applied example that supports the theoretical results reached through this study is included.

Keywords: existence; fixed point theorems; stability; fractional differential equations

MSC: 26A33; 34B15; 34B18

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Citation: Arab, M.; Awadalla, M.; Manigandan, M.; Abuasbeh, K.; Mahmudov, N.I.; Nandha Gopal, T. On the Existence Results for a Mixed Hybrid Fractional Differential Equations of Sequential Type. *Fractal Fract.* **2023**, *7*, 229. https://doi.org/10.3390/fractalfract7030229

Academic Editors: Mouffak Benchohra, Gaston M. N'Guérékata, Abdelkrim Salim and Ravi P. Agarwal

Received: 27 January 2023
Revised: 22 February 2023
Accepted: 1 March 2023
Published: 4 March 2023

1. Introduction

Fractional calculus is a well-established subject with applications in many fields, such as electro-chemistry, economics, electromagnetics, physical sciences, and medicine. The idea of fractional calculus is to replace the natural numbers in the derivative order with rational ones. In domains such as visco-elasticity, statistical physics, optics, signal processing, control, defense, electrical circuits, and astronomy, fractional differential equations have become extremely prevalent.

When studying any kind of differential equation, researchers are interested in the existence and unity of solutions and their qualitative properties; these properties include the stability of solutions. Stability was, and still is, the most important question in the theory of dynamic systems, as the study of stability was first conducted in mechanics due to the urgent need to study the balance of a system, and questions of stability increased the motivation to introduce new mathematical concepts to engineering, especially control engineering. The fixed-point theorem is not a single theorem, but belongs to a large family of fixed-point theorems that relate to different mathematical fields. The researchers' efforts focused on several theorems from this family to study the stability criteria for functional differential equations, including Banach's fixed point theorem, Schauder's fixed point theorem, and Krasnoselskii's fixed point theorem. We recommend monographs [1–5] and the recently mentioned papers [6–16]. The majority of research on FDEs is based on fractional derivatives of the R-L and Caputo types; see [9,17]. Several studies have been conducted to investigate how stability concepts such as the Mittag–Leffler function, exponential, and Lyapunov stability apply to various types of dynamic systems. Ulam and Hyers, on the other hand, identified a previously unknown type of stability, known as Ulam-stability [18]. Hyer's type of stability study significantly contributes to our understanding of chemical processes and fluid movement, as well as semiconductors, population dynamics, heat conduction, and elasticity. While others have reported results using other types of stability, Ulam's group designed and implemented a type of stability for ordinary, fractional differential, and difference equations; see [19]. Differential equations are found to be of great utility in systems and stochastic processes. They can be applied in sweeping processes, granular systems, nonlinear dynamics of wheeled vehicles, control problems, etc. The details of pressing issues in the stochastic process, control, differential games, optimization, and their applications can be found in [20], in which the authors studied the existence and uniqueness of the following subject to the following boundary conditions:

$$
\begin{aligned}
{}^{CH}\mathcal{D}^{P}\left(\frac{\mathfrak{u}(\xi)}{\mathfrak{z}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right) &= \mathfrak{w}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi)),\ \ \xi \in [1,e], P \in (1,2],\\
{}^{CH}\mathcal{D}^{Q}\left(\frac{\mathfrak{v}(\xi)}{\mathfrak{z}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right) &= \mathfrak{w}_2(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi)),\ \ \xi \in [1,e], Q \in (1,2],
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\mathfrak{u}(\xi)}{\mathfrak{z}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right)_{\xi=1} &=0,\\
{}^{CH}\mathcal{D}\left(\frac{\mathfrak{u}(\xi)}{\mathfrak{z}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right)_{\xi=e} &=\lambda_1 {}^{CH}\mathcal{D}\left(\frac{\mathfrak{u}(\xi)}{\mathfrak{z}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right)_{\xi=\eta_1}\\
\left(\frac{\mathfrak{v}(\xi)}{\mathfrak{z}_2(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right)_{\xi=1} &=0,\\
{}^{CH}\mathcal{D}\left(\frac{\mathfrak{v}(\xi)}{\mathfrak{z}_2(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right)_{\xi=e} &=\lambda_2 {}^{CH}\mathcal{D}\left(\frac{\mathfrak{v}(\xi)}{\mathfrak{z}_2(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right)_{\xi=\eta_2},
\end{aligned}
$$

where ${}^{CH}\mathcal{D}^{\gamma}, \gamma = \{P, Q\}$ is the Caputo—Hadamard fractional derivative of order $1 < \gamma \leq 2$, $\lambda_1, \lambda_2 \in [0,1)$, and $\eta_1, \eta_2 \in (1,e)$.

In [21], the authors studied the existence and uniqueness of the following system of mixed hybrid fractional differential equations

$$
\begin{cases}
\left({}^{C}\mathcal{D}_{1-}\, {}^{RL}\mathcal{D}^{r}_{0+}\right)\left(\frac{\mathfrak{u}(\xi)}{\mathfrak{z}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right) = \mathfrak{w}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi)),\ \ \xi \in [0,1], 1 < q \leq 2, 0 < r \leq 1,\\
\left({}^{C}\mathcal{D}_{1-}\, {}^{RL}\mathcal{D}^{p}_{0+}\right)\left(\frac{\mathfrak{v}(\xi)}{\mathfrak{z}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi))}\right) = \mathfrak{w}_1(\xi,\mathfrak{u}(\xi),\mathfrak{v}(\xi)),\ \ 0 < r \leq 1,\\
\mathfrak{u}(\xi) = \mathfrak{u}'(\xi) = 0,\ \ \mathfrak{u}(1) = \delta\mathfrak{u}(\zeta),\ \delta \in \mathcal{E},\ \zeta \in (0,1),\\
\mathfrak{v}(\xi) = \mathfrak{v}'(\xi) = 0,\ \ \mathfrak{v}(1) = \epsilon\mathfrak{v}(\xi),\ \epsilon \in \mathcal{E},\ \xi \in (0,1).
\end{cases}
$$

In [22] 2022, the authors investigated the existence of the solution for the following hybrid fractional differential equations

$$
\begin{aligned}
{}^{H}\mathcal{D}^{\mu,\nu;\Psi}_{0+}\left[\frac{y(\xi)}{f(\xi,y(\xi))}\right] &= g(\xi,y(\xi)),\ a.e\ t \in (0,\mathcal{T}],\\
(\Phi(\xi) - \Phi(0))^{1-\zeta} y(\xi)|_{t=0} &= y_0 \in \mathcal{E},
\end{aligned}
$$

where $0 < \mu < 1, 0 \leq \nu \leq 1$, $\zeta = \mu + \nu(1-\mu)$, ${}^{H}\mathcal{D}^{\mu,\nu;\Psi}_{0+}(\cdot)$ is the Φ-Hilfer fractional derivative of order μ and type $\nu, f \in \mathcal{C}(\mathcal{J} \times \mathcal{E}, \mathcal{E}\backslash\{0\})$ is bounded, $\mathcal{J} = [0,\mathcal{T}]$ and $g \in \mathscr{C}(\mathcal{J} \times \mathcal{E}, \mathcal{E}) = \{h|$ the map $\omega \to h(\tau,\omega)$ is continuous for each τ and the map $\tau \to h(\tau,\omega)$ is measurable for each $\omega\}$.

In [23], the authors studied the existence of solutions for a class of boundary value problems for nonlinear fractional hybrid differential equations involving a generalized Hilfer fractional derivative

$$
{}^{\alpha}\mathcal{D}^{\vartheta,r}_{a^+}\left(\frac{y(\xi)}{f(\xi,y(\xi))}\right) = \vartheta(\xi,x(\xi)),\ \tau \in (a,b].
$$

$$
c_1\left({}^{\alpha}\mathbb{J}^{1-\zeta}_{a^+}\left(\frac{y(\xi)}{f(\xi,y(\xi))}\right)\right)(a^+) + c_2\left({}^{\alpha}\mathbb{J}^{1-\zeta}_{a^+}\left(\frac{y(\xi)}{f(\xi,y(\xi))}\right)\right)(b) = c_3,
$$

where ${}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}$ and ${}^{\alpha}\mathbb{J}_{a^+}^{1-\zeta}$, respectively, denote the generalized Hilfer derivative operator of order $\vartheta \in (0,1)$ and type $r \in [0,1]$, and generalized fractional integral of order $1-\zeta$, $(\zeta = \vartheta + r - \vartheta r)$, $c_1, c_2, c_3 \in \mathcal{E}, c_1 + c_2 \neq 0, f \in \mathcal{C}([a,b] \times \mathcal{E}, \mathcal{E}) \backslash \{0\})$ and $\vartheta \in \mathcal{C}([a,b] \times \mathcal{E}, \mathcal{E})$.

In the present work, we use Darbo's fixed point theorem and Hausdorff's measure of noncompactness method to investigate the existence results for the following FDE

$$\begin{cases} {}^{c}\mathcal{D}^{\beta}({}^{c}\mathcal{D}^{\varpi} + \lambda)\left(\dfrac{\varepsilon_1(\xi)}{\kappa_1(\xi, \varepsilon_1(\xi))}\right) = \mathcal{F}(\xi, \varepsilon_1(\xi)) \;\; a.e.\, \xi \in \mathcal{J} = [0,1] \\ \dfrac{\varepsilon_1(0)}{\kappa_1(0, \varepsilon_1(0))} + \mu \dfrac{\varepsilon_1(1)}{\kappa_1(1, \varepsilon_1(1))} = \varrho_1 \displaystyle\int_0^1 \mathfrak{g}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1, \\ {}^{c}\mathcal{D}^{\varpi}\left(\dfrac{\varepsilon_1(0)}{\kappa_1(0, \varepsilon_1(0))}\right) + \varphi_1 {}^{c}\mathcal{D}^{\varpi}\left(\dfrac{\varepsilon_1(1)}{\kappa_1(1, \varepsilon_1(1))}\right) = \varrho_2 \displaystyle\int_0^1 \mathfrak{h}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1, \\ {}^{c}\mathcal{D}^{2\varpi}\left(\dfrac{\varepsilon_1(0)}{\kappa_1(0, \varepsilon_1(0))}\right) + \varphi_1 {}^{c}\mathcal{D}^{2\varpi}\left(\dfrac{\varepsilon_1(1)}{\kappa_1(1, \varepsilon_1(1))}\right) = \varrho_3 \displaystyle\int_0^1 \mathfrak{k}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1, \end{cases} \tag{1}$$

where $0 < \varpi < 1, 1 < \beta \leq 2, \lambda, \varphi_1, \varrho_i, \in \mathcal{E}^*$ for $i = 1,2,3$, with $\varphi_1 = -1$. ${}^{c}\mathcal{D}^{\varpi}, {}^{c}\mathcal{D}^{\beta}$ are the Caputo's fractional derivatives, and $\kappa_1 : [0,1] \times \mathcal{E} \to \mathcal{E}\{0\}, \omega \in \mathcal{C}([0,1] \times \mathcal{E}, \mathcal{E})$ and $\mathfrak{g}, \mathfrak{h}, \mathfrak{k} : [0,1] \times \mathcal{E} \to \mathcal{E}$ are a given continuous functions.

By a solution of the problem (1), we mean a function $\varepsilon_1 \in \mathcal{C}(\mathcal{J}, \mathcal{E})$, such that

(i) The function $\xi \rightarrowtail \left(\frac{\varepsilon_1}{\kappa_1(\xi, \varepsilon_1)}\right)$ is continuous for each $\varepsilon_1 \in \mathcal{E}$, and
(ii) ε_1 satisfies the equations in (1).

The originality of this study lies in the use of Darbo's fixed point theory, which is an important but rarely used theory in the literature. In addition, it verifies the existence of the solution to a nonlinear sequential fractional differential equation of the hybrid type and hybrid boundary conditions. Moreover, the stability of the solutions to this equation was verified using the Ulam–Hyres technique, establishing the relevence of this work.

The rest of the article is as follows: Section 2 presents the basic definitions, lemmas, and theorems that underpin our main conclusions. In Section 3, we provide solutions to the given fractional differential Equation (1) using Darbo's fixed point theorem. Section 4 looks at the Ulam–Hyers stability of the provided fractional differential Equation (1). In Section 5, an example is provided to further clarify the study's finding. In Section 6, the conclusion and future works are introduced.

2. Preliminaries

In this section, we state the most important definitions, lemmas, and theorems that are necessary to obtain our main results. In addition, we introduce some useful notations that make our result less complicated. We finish this section with an auxiliary lemma that provides a solution to our proposed fractional differential equation.

Denote the Banach space of all continuous function by: $\mathcal{C}([0,\mathcal{T}], \mathcal{E})$ with the norm

$$||\varepsilon_1||_\infty = \sup_{\xi \in \mathcal{J}} ||\mathcal{J}(\xi)||.$$

Let $\mathcal{L}^1([0,\mathcal{T}])$ represent the space of Bochner integrable functions $\varepsilon_1 : [0,\mathcal{T}] \to \mathcal{E}$, with the norm

$$||\varepsilon_1||_1 = \int_0^{\mathcal{T}} ||\varepsilon_1(\xi)|| d\xi.$$

Definition 1 ([24]). *Let $\mathcal{E}$ be a Banach space and $\mathcal{V}_2$ a bounded subsets of $\mathcal{E}$. Then, the Hausdorff measurable of non-compactness of $\mathcal{V}_2$ is defined by*

$\chi(\mathcal{V}_2) = \inf\{\xi > 0 : \mathcal{V}_2$ *has a finite cover by balls of radius* $\xi\}$.

To discuss the problem in this paper, we need the following lemmas.

Lemma 1 ([24]). *Let $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{E}$ be bounded. Then, HMNC has the following properties:*

(1) $\mathcal{V}_1 \subset \mathcal{V}_2 \Rightarrow \chi(\mathcal{V}_1) \leq \chi(\mathcal{V}_2)$;
(2) $\chi(\mathcal{V}_1) = 0 \Leftrightarrow \mathcal{V}_1$ *is a relatively compact;*
(3) $\chi(\mathcal{V}_1 \cup \mathcal{V}_2) = \max\{\chi(\mathcal{V}_1), \chi(\mathcal{V}_2)\}$;
(4) $\chi(\mathcal{V}_1) = \chi(\bar{\mathcal{V}_1}) = \chi(conv(\mathcal{V}_1))$, *where* $\bar{\mathcal{V}_1}$ *and* $\mathcal{V}_1$ *represent the closure and the convex hull of* $\mathcal{V}_1$*, respectively;*
(5) $\chi(\mathcal{A} + \mathcal{B}) \leq \chi(\mathcal{V}_1) + \chi(\mathcal{V}_2)$, *where* $(\mathcal{V}_1) + (\mathcal{V}_2) = \{u + v : u \in \mathcal{V}_1, v \in \mathcal{V}_2\}$;
(6) $\chi(\varpi\mathcal{V}_1) \leq |\varpi|\chi(\mathcal{V}_1) \; \forall \, \varpi \in \mathbb{R}$,

Lemma 2 ([24]). *If* $\mathcal{F} \subseteq \mathcal{C}([0, \mathcal{T}], \mathcal{E})$ *is bounded and equi-continuous, then* $\chi(\mathcal{F}(\xi))$ *is continuous on* $[0, \mathcal{T}]$ *and*

$$\chi(\mathcal{F}) = \sup_{\xi \in [0,\mathcal{T}]} \chi(\mathcal{F}(\xi)). \tag{2}$$

The set $\mathcal{V}_2 \subset \mathcal{L}([0, \mathcal{T}], \mathcal{E})$ *is (uniformly) bounded if* $\exists \, \sigma \in \mathcal{L}^1([0, \mathcal{T}], \mathbb{R}^+)$*, such that*

$$\|u(\iota_1)\| \leq \sigma(\iota_1), \quad \forall \; \varepsilon_1 \in \mathcal{V}_2. \tag{3}$$

Lemma 3 ([25]). *If* $\{\varepsilon_{1n}\}_{n=1}^{\infty} \subset \mathcal{L}^1([0, \mathcal{T}], \mathcal{E})$ *is integrable (uniformly), then* $\chi(\{\varepsilon_{1n}\}_{n=1}^{\infty})$ *is measurable, and*

$$\chi\left(\left\{\int_0^t \varepsilon_{1n}(\iota_1) d\iota_1\right\}_{n=1}^{\infty}\right) \leq \int_0^t \chi\big(\{\varepsilon_{1n}(\iota_1)\}_{n=1}^{\infty}\big) d\iota_1. \tag{4}$$

Lemma 4 ([26]). *If a set* $\mathcal{F}$ *is bounded, then* $\forall \, \delta$, $\exists \, \{\varepsilon_{1n}\}_{n=1}^{\infty} \subset \mathcal{F}$*, such that*

$$\chi\mathcal{F} \leq 2\chi\big(\{\varepsilon_{1n}(\iota_1)\}_{n=1}^{\infty}\big) + \delta. \tag{5}$$

Definition 2 ([27]). *A function* $\mathscr{J} : [c, d] \times \mathcal{E} \to \mathcal{E}$ *satisfies the carathéodory conditions, if the following can be satisfied*

- $\mathscr{J}(\xi, \varepsilon_1)$ *is continuous w.r.t.* ξ *for* $\varepsilon_1 \in \mathcal{E} \; \forall \, \xi \in [c, d]$.
- $\mathscr{J}(\xi, \varepsilon_1)$ *is measurable w.r.t.* ξ *for* $\varepsilon_1 \in \mathcal{E}$;

Definition 3 ([28]). *The function* $\mathscr{J} : \Omega \subset \mathcal{E} \to \mathcal{E}$ *is a* χ*-contraction, if* $\exists k$, $0 < k < 1$ *such that*

$$\chi(\mathscr{J}(\mathcal{V}_1)) \leq k\chi\mathcal{V}_1, \tag{6}$$

for all bounded $\mathcal{V}_1 \subset \Omega$ *.*

Next, we state the most important theory on which the results of this work are based. This is called the fixed point theory of "Darbo and Sadovskii" [24,29].

Theorem 1. *Let* Ω *be a nonempty, bounded, closed and convex subset of a Banach space* $\mathcal{E}$*, and let* $\mathscr{J} : \Omega \to \Omega$ *be a continuous operator. If* $\mathscr{J}$ *is a* χ*-contraction, then* $\mathscr{J}$ *has at least one fixed point.*

Definition 4 ([1]). *The RL fractional integral of order* $\varrho > 0$ *for a function* $\mathcal{J}_1 : [0, +\infty) \to \mathcal{R}$ *is defined as*

$$\mathcal{I}_{0+}^{\varrho} \mathcal{J}_1(\xi) = \frac{1}{\Gamma(\varrho)} \int_0^{\xi} (\xi - \iota_1)^{\varrho - 1} \mathcal{J}_1(\iota_1) d\iota_1.$$

Definition 5 ([1]). *The Caputo derivative of order* $\varrho > 0$ *for a function* $\mathcal{J}_1 : [0, +\infty) \to \mathcal{R}$ *is written as*

$$\mathcal{D}_{0+}^{\varrho} \mathcal{J}_1(\xi) = \frac{1}{\Gamma(n - \varrho)} \int_0^{\xi} (\xi - \iota_1)^{n - \varrho - 1} \mathcal{J}_1^{(n)}(\iota_1) d\iota_1,$$

where $n = [\varrho] + 1$, $[\varrho]$ is an integral part of ϱ.

Lemma 5. *Let $\varrho > 0$. Then, the differential equation $\mathcal{D}^{\varrho}_{0+}\mathcal{J}_1(\xi) = 0$ has the solution*

$$\mathcal{J}_1(\xi) = c_0 + c_1\xi + c_1\xi^2 + \cdots + c_{n-1}\xi^{n-1},$$

and

$$\mathcal{I}^{\varrho}_{0+}\mathcal{D}^{\varrho}_{0+}\mathcal{J}_1(\xi) = \mathcal{J}_1(\xi) + c_0 + c_1\xi + c_1\xi^2 + \cdots + c_{n-1}\xi^{n-1},$$

where $c_i \in \mathbb{R}$ and $i = 1, 2\cdots, n = [\varrho] + 1$.

Lemma 6. *Assume that hypothesis $(\mathcal{H}_0)$ holds. Then, for any $\mathfrak{y} \in \mathcal{L}^1(\mathcal{J};\mathcal{E})$. The function $\varepsilon_1 \in \mathcal{C}(\mathcal{J};\mathcal{E})$ is a solution to the problem*

$$\begin{cases} {}^{c}\mathcal{D}^{\beta}\left({}^{c}\mathcal{D}^{\varpi} + \lambda\right)\left(\dfrac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right) = \mathfrak{y}(\xi) \ \ a.e.\ \xi \in \mathcal{J} = [0,1] \\ \dfrac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))} + \varphi_1\dfrac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))} = \varrho_1\displaystyle\int_0^1 \mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1, \\ {}^{c}\mathcal{D}^{\varpi}\left(\dfrac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))}\right) + \varphi_1{}^{c}\mathcal{D}^{\varpi}\left(\dfrac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))}\right) = \varrho_2\displaystyle\int_0^1 \mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1, \\ {}^{c}\mathcal{D}^{2\varpi}\left(\dfrac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))}\right) + \varphi_1{}^{c}\mathcal{D}^{2\varpi}\left(\dfrac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))}\right) = \varrho_3\displaystyle\int_0^1 \mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1, \end{cases} \tag{7}$$

if, and only if, ε_1 and satisfies the hybrid equation:

$$\begin{aligned} \varepsilon_1(\xi) =& \kappa_1(\xi,\varepsilon_1(\xi))\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1 + \mathcal{A}_1(\xi)\int_0^1 \mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1 \\ &+\mathcal{A}_2(\xi)\int_0^1 \mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1 + \mathcal{A}_3(\xi)\int_0^1 \mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1 \\ &+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1 + \frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1 \\ &+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1 \\ &-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\Bigg] - \frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi}(\xi-\iota_1)^{\varpi-1}\varepsilon_1(\iota_1)d\iota_1, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mathcal{A}_1 &= \tfrac{\xi^{\varpi}\varrho_2(1-\lambda\Gamma(2-\varpi))}{\Gamma(\varpi+1)(1+\varphi_1)} + \tfrac{\xi^{\varpi+1}\lambda\varrho_2\Gamma(2-\varpi)}{\Gamma(2+\varpi)\varphi_1} \\ &\quad+\left(\tfrac{\varphi_1}{(1+\varphi_1)^2\Gamma(\varpi+1)} - \tfrac{1}{(1+\varphi_1)\Gamma(\varpi+2)}\right)\Gamma(2-\varpi)\lambda\varrho_2 - \tfrac{\varphi_1\varrho_2}{(1+\varphi_1)^2\Gamma(\varpi+1)} \\ \mathcal{A}_2 &= \tfrac{\xi^{\varpi}\lambda\varrho_1}{\Gamma(\varpi+1)(1+\varphi_1)} - \tfrac{\varrho_1}{1+\varphi_1} + \tfrac{\varphi_1\lambda\varrho_1}{(1+\varphi_1)^2\Gamma(\varpi+1)}, \\ \mathcal{A}_3 &= -\tfrac{\xi^{\varpi}\Gamma(2-\varpi)\varrho_3}{\Gamma(\varpi+1)(1+\varphi_1)} + \tfrac{\xi^{\varpi+1}\Gamma(2-\varpi)\varrho_3}{\Gamma(\varpi+2)\varphi_1} + \tfrac{\varphi_1\Gamma(2-\varpi)\varrho_3}{(1+\varphi_1)^2\Gamma(\varpi+1)} - \tfrac{\Gamma(2-\varpi)\varrho_3}{(1+\varphi_1)\Gamma(\varpi+2)}, \\ \mathcal{A}_4 &= \tfrac{\xi^{\varpi}\Gamma(2-\varpi)\varphi_1}{\Gamma(\varpi+1)(1+\varphi_1)} - \tfrac{\xi^{\varpi+1}\Gamma(2-\varpi)}{\Gamma(\varpi+2)} \\ &\quad+\Gamma(2-\varpi)\left(\tfrac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+2)} - \tfrac{\varphi_1^2}{(1+\varphi_1)^2\Gamma(\varpi+1)}\right), \\ \mathcal{A}_5 &= -\tfrac{\xi^2\varphi_1}{\Gamma(\varpi+1)(1+\varphi_1)} - \tfrac{\varphi_1^2}{(1+\varphi_1)^2\Gamma(\varpi+1)}. \end{aligned}$$

Proof. Using Lemma , we obtain

$$\left({}^{c}\mathcal{D}^{\varpi} + \lambda\right)\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right) = \mathcal{I}^{\beta}\mathfrak{y}(\xi) + \mathfrak{a}_0 + \mathfrak{a}_1\xi,$$

$$
\begin{aligned}
({}^{c}\mathcal{D}^{\varpi})\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right) =& \mathcal{I}^{\beta}\mathfrak{y}(\xi) + \mathfrak{a}_0 + \mathfrak{a}_1\xi - \lambda\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right),\\
\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))} =& \mathcal{I}^{\varpi+\beta}\mathfrak{y}(\xi) + \mathcal{I}^{\varpi}\mathfrak{a}_0 + \mathcal{I}^{\varpi}\mathfrak{a}_1\xi - \mathcal{I}^{\varpi}\lambda\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right) + \mathfrak{a}_2,
\end{aligned}
$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_0 \in \mathbb{R}$.

According to the condition:
${}^{c}\mathcal{D}^{2\varpi}\left(\frac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))}\right) + \varphi_1^{c}\mathcal{D}^{2\varpi}\left(\frac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))}\right) = \varrho_3\int_0^1 \mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1$ we find that:

$$
\begin{aligned}
\mathfrak{a}_1 =& \Gamma(2-\varpi)\left(\frac{\varrho_3}{\varphi_1}\int_0^1 \mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1 + \frac{\lambda\varrho_2}{\varphi_1}\int_0^1 \mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\right.\\
&\left. - \frac{1}{\Gamma(\beta-\varpi)}\int_0^1 (1-\iota_1)^{\beta-\varpi}\mathfrak{y}(\iota_1)d\iota_1\right).
\end{aligned}
$$

Using the fact that ${}^{c}\mathcal{D}^{\varpi}\left(\frac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))}\right) + \varphi_1^{c}\mathcal{D}^{\varpi}\left(\frac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))}\right) = \varrho_2\int_0^1 \mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1$, and $\frac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))} + \varphi_1\frac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))} = \varrho_1\int_0^1 \mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1$

$$
\begin{aligned}
\mathfrak{a}_0 =& \frac{-\Gamma(2-\varpi)\varrho_3}{1+\varphi_1}\int_0^1 \mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1 + \frac{(1-\lambda\Gamma(2-\varpi))\varrho_2}{1+\varphi_1}\int_0^1 \mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+ \frac{\lambda\varrho_1}{1+\varphi_1}\int_0^1 \mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+ \frac{\Gamma(2-\varpi)\varphi_1}{(1+\varphi_1)\Gamma(\beta-\varpi)}\int_0^1 (1-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1\\
&- \frac{\varphi_1}{(1+\varphi_1)\Gamma(\beta)}\int_0^1 (1-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{a}_2 =& \tfrac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^{\xi}(1-\iota_1)^{\varpi-1}\varepsilon_1(\iota_1)d\iota_1 + \left(\tfrac{\varrho_1}{1+\varphi_1} - \tfrac{\varphi_1\lambda\varrho_1}{\Gamma(\varpi+1)(1+\varphi_1)^2}\right)\int_0^1 \mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&- \tfrac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1 (1-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+ \tfrac{\varphi_1^2}{(1+\varphi_1)^2\Gamma(\beta)\Gamma(\varpi+1)}\int_0^1 (1-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+ \tfrac{\Gamma(2-\varpi)}{\Gamma(\beta-\varpi)}\left(\tfrac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+2)} - \tfrac{\varphi_1^2}{(1+\varphi_1)^2\Gamma(\varpi+1)}\right)\int_0^1 (1-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+ \left[\left(\tfrac{\varphi_1^2}{(1+\varphi_1)^2\Gamma(\varpi+1)} - \tfrac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+2)}\right)\tfrac{\Gamma(2-\varpi)\lambda\varrho_2}{\varphi_1}\right.\\
&\left. - \tfrac{\varphi_1\varrho_2}{(1+\varphi_1)^2\Gamma(\varpi+1)}\right]\int_0^1 \mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1.
\end{aligned}
$$

Substituting the value of $\mathfrak{a}_0$, $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{b}_0$, $\mathfrak{b}_1$, and $\mathfrak{b}_2$ we obtain

$$
\begin{aligned}
\varepsilon_1(\xi) =& \kappa_1(\xi,\varepsilon_1(\xi))\left[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1 + \mathcal{A}_1(\xi)\int_0^1 \mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\right.\\
&+ \mathcal{A}_2(\xi)\int_0^1 \mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1 + \mathcal{A}_3(\xi)\int_0^1 \mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+ \frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1 (1-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+ \frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1 (1-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1 + \frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1 (1-\iota_1)^{\varpi-1}\varepsilon_1(\iota_1)d\iota_1\\
&\left. - \frac{\varphi_1}{(1+\varphi_1)(\varpi+\beta)}\int_0^1 (1-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\right] - \frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi}(\xi-\iota_1)^{\varpi-1}\varepsilon_1(\iota_1)d\iota_1.
\end{aligned}
$$

□

3. Existence Results via DFPT

To set our main results, we introduce the following assumptions

$(\mathcal{A}_1)$ The function $\mathcal{J}_1 : [0, \mathcal{T}] \times \mathcal{E} \to \mathcal{E}$ satisfies carthéodory conditions.
$(\mathcal{A}_2)$ There exists a function $\Psi \in \mathcal{L}^{\infty}([0, \mathcal{T}], \mathbb{R}_+)$, which shows that

$$||\mathcal{J}_1(\xi, \varepsilon_1(\xi))|| \leq \Psi(\xi)(1 + ||\varepsilon_1||), \forall\, \varepsilon_1 \in \mathcal{C}([0, \mathcal{T}], \mathcal{E})$$

$(\mathcal{A}_3)$ Assume $\mathcal{W} \subset \mathcal{E}$ is any bounded set $\forall\, \xi \in [0, \mathcal{T}]$; then,

$$\chi(\mathcal{J}_1(\xi, \mathcal{W})) \leq \Psi(\xi)\chi(\mathcal{W}).$$

For easy computation, we let

$$\begin{aligned}\mathcal{R} :=& \kappa_1 \left\{ \frac{\mathcal{J}_1}{\Gamma(\varpi + \beta + 1)} + \mathcal{A}_1||\mathfrak{h}|| + \mathcal{A}_2||\mathfrak{g}|| + \mathcal{A}_3||\mathfrak{k}|| + \frac{\mathcal{A}_4||\mathcal{J}_1||}{\Gamma(\beta - \varpi + 1)} + \frac{\mathcal{A}_5||\mathcal{J}_1||}{\Gamma(\beta + 1)} + \right.\\ & \left. \frac{|\varphi_1|||\lambda||}{(|1 + \varphi_1|)\Gamma(\varpi + \beta + 1)} + \frac{|\varphi_1 \lambda|}{(|1 + \varphi_1|)\Gamma(\varpi + 1)}||\mathfrak{y}|| \right\} + \frac{|\lambda|}{\Gamma|(\varpi + 1)|}||\varepsilon_1||,\end{aligned}$$

Theorem 2. *Assume that the assumptions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ *hold true, and let* $\mathcal{R}_\Psi = ||\Psi||\mathcal{R}$. *If*

$$4\mathcal{R}_\Psi < 1, \tag{9}$$

then the BVP (1) has at least one solution, defined on [0,$\mathcal{T}$*].*

Proof. Consider the operator $\mathcal{H} : \mathcal{C}([0, \mathcal{T}], \mathcal{E}) \to \mathcal{C}([0, \mathcal{T}], \mathcal{E})$

$$\begin{aligned}(\mathcal{H}\varepsilon_1)(\xi) =& \kappa_1(\xi, \varepsilon_1(\xi)) \left[\frac{1}{\Gamma(\varpi + \beta)} \int_0^{\xi} (\xi - \iota_1)^{\varpi + \beta - 1} \mathfrak{y}(\iota_1) d\iota_1 \right.\\ & + \mathcal{A}_1(\xi) \int_0^1 \mathfrak{h}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1 \\ & + \mathcal{A}_2(\xi) \int_0^1 \mathfrak{g}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1 \\ & + \mathcal{A}_3(\xi) \int_0^1 \mathfrak{k}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1 + \frac{\mathcal{A}_4(\xi)}{\Gamma(\beta - \varpi)} \int_0^1 (\xi - \iota_1)^{\beta - \varpi - 1} \mathfrak{y}(\iota_1) d\iota_1 \\ & + \frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)} \int_0^1 (\xi - \iota_1)^{\beta - 1} \mathfrak{y}(\iota_1) d\iota_1 + \frac{\varphi_1 \lambda}{(1 + \varphi_1)\Gamma(\varpi)} \int_0^1 (\xi - \iota_1)^{\beta - 1} \mathfrak{y}(\iota_1) d\iota_1 \\ & \left. - \frac{\varphi_1}{(1 + \varphi_1)\Gamma(\varpi + \beta)} \int_0^1 (\xi - \iota_1)^{\varpi + \beta - 1} \mathfrak{y}(\iota_1) d\iota_1 \right] \\ & - \frac{\lambda}{\Gamma(\varpi)} \int_0^{\xi} (\xi - \iota_1)^{\varpi - 1} \varepsilon_1(\iota_1) d\iota_1,\end{aligned}$$

The operator $\mathcal{H}$ is well-defined as a result of $\mathcal{A}_1$ and $\mathcal{A}_2$. Therefore, (1) is equivalent to the following operator equation.

$$\varepsilon_1 = \mathcal{H}\varepsilon_1. \tag{10}$$

Subsequently, showing the existence of fixed point for (10) is equivalent to the existence of a solution for (8).

Let

$$\mathcal{B}_\epsilon = \{\varepsilon_1 \in \mathcal{C}([0, \mathcal{T}], \mathcal{E}) : ||\varepsilon_1||_\infty \leq \epsilon\}$$

be a closed convex set with $\epsilon > 0$, such that

$$\epsilon \geq \frac{\mathcal{R}_\Psi}{1-\mathcal{R}_\Psi}.$$

The applicability of the DFPT will be shown in four steps

Step 1. We show that HB $\mathcal{HB}_\epsilon \subset \mathcal{B}_\epsilon$; using $(\mathcal{A}_2)$, we have

$$\begin{aligned}
|\mathcal{H}\varepsilon_1(\xi)| \leq & \kappa_1\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^\xi(\xi-\iota_1)^{\varpi+\beta-1}\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1 \\
&+\mathcal{A}_1(\xi)\int_0^1\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1 \\
&+\mathcal{A}_2(\xi)\int_0^1\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1 \\
&+\mathcal{A}_3(\xi)\int_0^1\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1 \\
&+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1 \\
&+\frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1 \\
&+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1 \\
&-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1\Bigg] \\
&-\frac{\lambda}{\Gamma(\varpi)}\int_0^\xi(\xi-\iota_1)^{\varpi-1}\Psi(\iota_1)(1+||\varepsilon_1(\iota_1)||)d\iota_1,
\end{aligned}$$

$$\begin{aligned}
\leq \quad & ||\Psi||(1+||\varepsilon_1||)\Big\{\tfrac{\mathcal{J}_1}{\Gamma(\varpi+\beta+1)}+\mathcal{A}_1||\mathfrak{h}||+\mathcal{A}_2||\mathfrak{g}||+\mathcal{A}_3||\mathfrak{k}||+\tfrac{\mathcal{A}_4||\mathcal{J}_1||}{\Gamma(\beta-\varpi+1)}+\tfrac{\mathcal{A}_5||\mathcal{J}_1||}{\Gamma(\beta+1)}+ \\
& \tfrac{|\varphi_1|||\lambda||}{(|1+\varphi_1|)\Gamma(\varpi+\beta+1)}+\tfrac{|\varphi_1\lambda|}{(|1+\varphi_1|)\Gamma(\varpi+1)}||\mathfrak{y}||\Big\}+\tfrac{|\lambda|}{\Gamma|(\varpi+1)|}||\varepsilon_1||.
\end{aligned}$$

$$\begin{aligned}
||\mathcal{H}\varepsilon_1|| \leq & ||\Psi||(1+||\varepsilon_1||)\mathcal{R} \\
\leq & (1+\epsilon)\mathcal{R}_\Psi \\
\leq & \epsilon.
\end{aligned}$$

Thus, $||\mathcal{H}\varepsilon_1|| \leq \epsilon$. That is $\mathcal{HB}_\epsilon \subset \mathcal{B}_\epsilon$.

Step 2: The operator $\mathcal{H}$ is continuous. Let $\{\varepsilon_{1n}\}$ be a sequence in $\mathcal{B}_\epsilon$, such that $\varepsilon_{1n} \to \varepsilon_1$ as $n \to \infty$. Then, $\mathcal{J}_1(\iota_1, \varepsilon_{1n}(\iota_1)) \to \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1))$ as $n \to \infty$, as a sequel of the Carathéodory continuity of $\mathcal{J}_1$. $(\mathcal{A}_2)$ implies

$$\begin{aligned}
||\mathcal{H}\varepsilon_{1n}(\xi)-\mathcal{H}\varepsilon_1(\xi)|| \leq & \kappa_1||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))|| \\
& \times\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^\xi(\xi-\iota_1)^{\varpi+\beta-1}||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1 \\
& +||\mathcal{A}_1(\xi)||\int_0^1||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1 \\
& +||\mathcal{A}_2(\xi)||\int_0^1||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1 \\
& +||\mathcal{A}_3(\xi)||\int_0^1||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1 \\
& +\frac{||\mathcal{A}_4(\xi)||}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{||\mathcal{A}_5(\xi)||}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1\\
&+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1\\
&-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1\Bigg]\\
&-\frac{\lambda}{\Gamma(\varpi)}\int_0^\xi(\xi-\iota_1)^{\varpi-1}||\mathcal{J}_1(\iota_1,\varepsilon_{1n}(\iota_1))-\mathcal{J}_1(\iota_1,\varepsilon_1(\iota_1))||d\iota_1,\\
\leq&\mathcal{R}||\mathcal{J}_1(\cdot,\varepsilon_{1n}(\cdot))-\mathcal{J}_1(\cdot,\varepsilon_1(\cdot))||.
\end{aligned}
$$

Using the Lebesgue dominated convergence theorem, it is obvious that $||\mathcal{H}\mathcal{J}_{1n}(\xi)-\mathcal{H}\varepsilon_1(\xi)||\to 0$ as $n\to\infty$, $\forall\ \xi\in[0,\mathcal{T}]$; consequently, we have

$$
||\mathcal{H}\varepsilon_{1n}-\mathcal{H}\varepsilon_1||\to 0 \ as\ n\to\infty.
$$

Step 3. The operator $\mathcal{H}$ is equicontinuous. For any $0<\xi_1<\xi_2<\mathcal{T}$ and $\varepsilon_1\in\mathcal{B}_\epsilon$, we can obtain

$$
\begin{aligned}
||\mathcal{H}(\varepsilon_1)(\xi_2)-\mathcal{H}(\varepsilon_1)(\xi_1)||\leq&\Bigg|\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi_2}(\xi_2-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&-\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi_1}(\xi_1-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+|\mathcal{A}_1(\xi_2)-\mathcal{A}_1(\xi_1)|\int_0^1\mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+|\mathcal{A}_2(\xi_2)-\mathcal{A}_2(\xi_1)|\int_0^1\mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+|\mathcal{A}_3(\xi_2)-\mathcal{A}_3(\xi_1)|\int_0^1\mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+\frac{|\mathcal{A}_4(\xi_2)-\mathcal{A}_4(\xi_1)|}{\Gamma(\beta-\varpi)}\int_0^1(1-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+\frac{|\mathcal{A}_5(\xi_2)-\mathcal{A}_5(\xi_1)|}{\Gamma(\beta)}\int_0^1(1-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1\Bigg]\Bigg|,
\end{aligned}
$$

$$
\begin{aligned}
||\mathcal{H}(\varepsilon_1)(\xi_2)-\mathcal{H}(\varepsilon_1)(\xi_1)||\leq&\Bigg|\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_{\xi_1}^{\xi_2}\mathfrak{y}(\iota_1)d\iota_1\\
&+|\mathcal{A}_1(\xi_2)-\mathcal{A}_1(\xi_1)|\int_0^1\mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+|\mathcal{A}_2(\xi_2)-\mathcal{A}_2(\xi_1)|\int_0^1\mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+|\mathcal{A}_3(\xi_2)-\mathcal{A}_3(\xi_1)|\int_0^1\mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+\frac{|\mathcal{A}_4(\xi_2)-\mathcal{A}_4(\xi_1)|}{\Gamma(\beta-\varpi)}\int_0^1(1-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+\frac{|\mathcal{A}_5(\xi_2)-\mathcal{A}_5(\xi_1)|}{\Gamma(\beta)}\int_0^1(1-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1\Bigg]\Bigg|,
\end{aligned}
$$

as $\xi_1\to\xi_2$ is the RHS of the above approaches to zero and is free of $\varepsilon_1\in\mathcal{B}_\epsilon$. Hence, operator $\mathcal{H}$ is bounded and equicontinuous.

Step.4. We show that $\mathcal{H}$ is a χ-contraction on $\mathcal{B}_\epsilon$. For all bounded subsets $\mathcal{W} \subset \mathcal{B}_\epsilon$ and $\delta > 0$. With the aid of Lemma 4 and the properties of χ, $\exists \{\varepsilon_{1k}\}_{k=1}^{\infty} \subset \mathcal{W}$ such that

$$
\begin{aligned}
\chi(\mathcal{H}\mathcal{W}(\xi)) \leq & 2\chi\Bigg\{\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1 \\
&+\mathcal{A}_1(\xi)\int_0^1 \mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1 \\
&+\mathcal{A}_2(\xi)\int_0^1 \mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1 \\
&+\mathcal{A}_3(\xi)\int_0^1 \mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1 \\
&+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1 \\
&+\frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1 \\
&+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1 \\
&-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1\Bigg] \\
&-\frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi}(\xi-\iota_1)^{\varpi-1}\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty})d\iota_1,\Bigg\}+\delta.
\end{aligned}
$$

The properties of χ, $(\mathcal{A}_3)$, and Lemma 3 can be used to obtain

$$
\begin{aligned}
\chi(\mathcal{H}\mathcal{W}(\xi)) \leq & 4\Bigg\{\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\mathcal{A}_1(\xi)\int_0^1 \chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\mathcal{A}_2(\xi)\int_0^1 \chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\mathcal{A}_3(\xi)\int_0^1 \chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1\Bigg] \\
&-\frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi}(\xi-\iota_1)^{\varpi-1}\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1,\Bigg\}+\delta,
\end{aligned}
$$

$$
\begin{aligned}
\leq & 4\Bigg\{\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\mathcal{A}_1(\xi)\int_0^1 \Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1 \\
&+\mathcal{A}_2(\xi)\int_0^1 \Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1
\end{aligned}
$$

$$+\mathcal{A}_3(\xi)\int_0^1 \Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1$$
$$+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1 (\xi-\iota_1)^{\beta-\varpi-1}\Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1$$
$$+\frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1 (\xi-\iota_1)^{\beta-1}\Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1$$
$$+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1 (\xi-\iota_1)^{\beta-1}\Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1$$
$$-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1 (\xi-\iota_1)^{\varpi+\beta-1}\Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1\Bigg]$$
$$-\frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi} (\xi-\iota_1)^{\varpi-1}\Psi(\iota_1)\chi(\mathcal{J}_1(\iota_1,\{\varepsilon_{1k}(\iota_1)\}_{k=1}^{\infty}))d\iota_1,\Bigg\}+\delta.$$

$$\leq 4\mathcal{R}_{\Psi}\chi(\mathcal{R}_{\epsilon})+\delta,\ \forall\,\delta>0. \tag{11}$$

Then

$$\chi(\mathcal{H}(\mathcal{W}))=\sup_{\xi\in[0,\mathcal{T}]}\chi(\mathcal{H}\mathcal{W}(\xi))\leq 4\mathcal{R}_{\Psi}\chi(\mathcal{R}_{\epsilon})$$

Using Theorem 2, we conclude the existence of a fixed point for the operator equation given by (10). This completes the proof. □

4. Stability Results

Let $\vartheta>0$ and $\Theta:[0,\mathcal{T}]\to[0,\infty]$ be a continuous function. We consider the following inequalities:

$$|{}^{c}\mathcal{D}^{\beta}({}^{c}\mathcal{D}^{\varpi}+\lambda)\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right)-\mathcal{J}_1(\xi,\varepsilon_1(\xi))|\leq\vartheta,\ \ \xi\in[0,\mathcal{T}], \tag{12}$$
$$|{}^{c}\mathcal{D}^{\beta}({}^{c}\mathcal{D}^{\varpi}+\lambda)\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right)-\mathcal{J}_1(\xi,\varepsilon_1(\xi))|\leq\vartheta\Theta(\xi),\ \ \xi\in[0,\mathcal{T}], \tag{13}$$

Definition 6 ([30]). *Problem (1) is U-H stable if* $\exists\,\mathcal{M}>0$, *such that* $\forall\,\vartheta>0$, $\forall\,\varepsilon_1\in\mathcal{C}$ *of the inequality (12),* $\exists$ *solution* ${\varepsilon_1}^*\in\mathcal{C}$ *of problem (1) with*

$$|\varepsilon_1(\xi)-{\varepsilon_1}^*(\xi)|\leq\mathcal{M}\vartheta,\ \ \xi\in[0,\mathcal{T}]. \tag{14}$$

Definition 7 ([30]). *Problem (1) is generalized U-H stable if* $\exists\,\Theta_{\mathcal{J}_1}\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ *and* $\Theta_{\mathcal{J}_1}(0)=0$, *such that,* $\forall\,\varepsilon_1\in\mathcal{C}$ *of the inequality (13),* $\exists$ *a solution* ${\varepsilon_1}^*\in\mathcal{C}$ *of problem (1) with*

$$|\varepsilon_1(\xi)-{\varepsilon_1}^*(\xi)|\leq\Theta_{\mathcal{J}_1}(\vartheta),\ \ \xi\in[0,\mathcal{T}]. \tag{15}$$

Remark 1 ([30]). *A function* $\varepsilon_1\in\mathcal{C}$ *is a solution of the equality (14)* $\Longleftrightarrow\exists$ *a function* $\mathcal{Z}\in\mathcal{C}$, *such that*

(1) $|\mathcal{Z}(\xi)|\leq\vartheta,\ \ \xi\in[0,\mathcal{T}]$,
(2) ${}^{c}\mathcal{D}^{\beta}({}^{c}\mathcal{D}^{\varpi}+\lambda)\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right)=\mathcal{J}_1(\xi,\varepsilon_1(\xi))+\mathcal{Z}(\xi),\ \ \xi\in[0,\mathcal{T}]$.

Lemma 7. *Let* $1\leq\varrho\leq 2$. *If a function* $\varepsilon_1\in\mathcal{C}$ *is a solution to the inequality, then* ε_1 *is a solution of the following integral inequality:*

$$|\varepsilon_1(\xi)-\mathfrak{E}_{\varepsilon_1}|\leq\mathcal{R}\vartheta, \tag{16}$$

where

$$\begin{aligned}\mathfrak{E}_{\varepsilon_1} =&\kappa_1(\xi,\varepsilon_1(\xi))\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+\mathcal{A}_1(\xi)\int_0^1\mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+\mathcal{A}_2(\xi)\int_0^1\mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+\mathcal{A}_3(\xi)\int_0^1\mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+\frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\Bigg]\\
&-\frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi}(\xi-\iota_1)^{\varpi-1}\varepsilon_1(\iota_1)d\iota_1,\end{aligned}$$

Proof. Using Remark 1,

$$\begin{aligned}\varepsilon_1(\xi) =&\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+\mathcal{A}_1(\xi)\int_0^1\mathfrak{h}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+\mathcal{A}_2(\xi)\int_0^1\mathfrak{g}(\iota_1,\varepsilon_1(\iota_1))d\iota_1\\
&+\mathcal{A}_3(\xi)\int_0^1\mathfrak{k}(\iota_1,\varepsilon_1(\iota_1))d\iota_1+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+\frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\Bigg]\\
&-\frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi}(\xi-\iota_1)^{\varpi-1}\varepsilon_1(\iota_1)d\iota_1,\\
&+\Bigg[\frac{1}{\Gamma(\varpi+\beta)}\int_0^{\xi}(\xi-\iota_1)^{\varpi+\beta-1}\mathfrak{y}(\iota_1)d\iota_1\\
&+\mathcal{A}_1(\xi)\int_0^1\mathcal{Z}(\iota_1)d\iota_1\\
&+\mathcal{A}_2(\xi)\int_0^1\mathcal{Z}(\iota_1)d\iota_1\\
&+\mathcal{A}_3(\xi)\int_0^1\mathcal{Z}(\iota_1)d\iota_1+\frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)}\int_0^1(\xi-\iota_1)^{\beta-\varpi-1}\mathcal{Z}(\iota_1)d\iota_1\\
&+\frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathcal{Z}(\iota_1)d\iota_1+\frac{\varphi_1\lambda}{(1+\varphi_1)\Gamma(\varpi)}\int_0^1(\xi-\iota_1)^{\beta-1}\mathcal{Z}(\iota_1)d\iota_1\\
&-\frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)}\int_0^1(\xi-\iota_1)^{\varpi+\beta-1}\mathcal{Z}(\iota_1)d\iota_1\Bigg]\\
&-\frac{\lambda}{\Gamma(\varpi)}\int_0^{\xi}(\xi-\iota_1)^{\varpi-1}\mathcal{Z}(\iota_1)d\iota_1,\end{aligned}\tag{17}$$

implies

$$
\begin{aligned}
|\varepsilon_1(\xi) - \mathfrak{E}_{\varepsilon_1}| =& \Bigg| \Bigg[\frac{1}{\Gamma(\varpi+\beta)} \int_0^{\xi} (\xi-\iota_1)^{\varpi+\beta-1} \mathfrak{y}(\iota_1) d\iota_1 \\
&+ \mathcal{A}_1(\xi) \int_0^1 \mathcal{Z}(\iota_1) d\iota_1 \\
&+ \mathcal{A}_2(\xi) \int_0^1 \mathcal{Z}(\iota_1) d\iota_1 \\
&+ \mathcal{A}_3(\xi) \int_0^1 \mathcal{Z}(\iota_1) d\iota_1 + \frac{\mathcal{A}_4(\xi)}{\Gamma(\beta-\varpi)} \int_0^1 (\xi-\iota_1)^{\beta-\varpi-1} \mathcal{Z}(\iota_1) d\iota_1 \\
&+ \frac{\mathcal{A}_5(\xi)}{\Gamma(\beta)} \int_0^1 (\xi-\iota_1)^{\beta-1} \mathcal{Z}(\iota_1) d\iota_1 + \frac{\varphi_1 \lambda}{(1+\varphi_1)\Gamma(\varpi)} \int_0^1 (\xi-\iota_1)^{\beta-1} \mathcal{Z}(\iota_1) d\iota_1 \\
&- \frac{\varphi_1}{(1+\varphi_1)\Gamma(\varpi+\beta)} \int_0^1 (\xi-\iota_1)^{\varpi+\beta-1} \mathcal{Z}(\iota_1) d\iota_1 \Bigg] \\
&- \frac{\lambda}{\Gamma(\varpi)} \int_0^{\xi} (\xi-\iota_1)^{\varpi-1} \mathcal{Z}(\iota_1) d\iota_1, \Bigg| \\
&\leq \vartheta \Bigg[\Bigg\{ \frac{\mathcal{J}_1}{\Gamma(\varpi+\beta+1)} + \mathcal{A}_1||\mathfrak{h}|| + \mathcal{A}_2||\mathfrak{g}|| + \mathcal{A}_3||\mathfrak{k}|| + \frac{\mathcal{A}_4||\mathcal{J}_1||}{\Gamma(\beta-\varpi+1)} + \frac{\mathcal{A}_5||\mathcal{J}_1||}{\Gamma(\beta+1)} \\
&+ \frac{|\varphi_1|||\lambda||}{(|1+\varphi_1|)\Gamma(\varpi+\beta+1)} + \frac{|\varphi_1\lambda|}{(|1+\varphi_1|)\Gamma(\varpi+1)} ||\mathfrak{y}|| \Bigg\} + \frac{|\lambda|}{\Gamma|(\varpi+1)|} ||\varepsilon_1|| \Bigg] \\
&\leq \mathcal{R}\vartheta.
\end{aligned} \tag{18}
$$

□

We now state the main theorem as follows:

Theorem 3. *Assume that $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are satisfied with $\mathcal{R}_\Psi < 1$. Then, problem (1) is U-H and is generalized as U-H stable.*

Proof. Suppose that $\varepsilon_1 \in \mathcal{C}$ is a solution of inequality (14) and $\varepsilon_1{}^* \in \mathcal{C}$ is a unique solution of problem (1). Then, it follows from Lemma 7 that

$$
\begin{aligned}
|\varepsilon_1(\xi) - \varepsilon_1{}^*(\xi)| =& \Bigg| \Bigg[\frac{1}{\Gamma(\varpi+\beta)} \int_0^{\xi} (\xi-\iota_1)^{\varpi+\beta-1} \mathfrak{y}(\iota_1) d\iota_1 \\
&+ |\mathcal{A}_1(\xi)| \int_0^1 \mathfrak{h}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1 \\
&+ |\mathcal{A}_2(\xi)| \int_0^1 \mathfrak{g}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1 \\
&+ |\mathcal{A}_3(\xi)| \int_0^1 \mathfrak{k}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1 + \frac{|\mathcal{A}_4(\xi)|}{\Gamma(\beta-\varpi)} \int_0^1 (\xi-\iota_1)^{\beta-\varpi-1} \mathfrak{y}(\iota_1) d\iota_1 \\
&+ \frac{|\mathcal{A}_5(\xi)|}{\Gamma(\beta)} \int_0^1 (\xi-\iota_1)^{\beta-1} \mathfrak{y}(\iota_1) d\iota_1 + \frac{|\varphi_1\lambda|}{(1+\varphi_1)\Gamma(\varpi)} \int_0^1 (\xi-\iota_1)^{\beta-1} \mathfrak{y}(\iota_1) d\iota_1 \\
&- \frac{|\varphi_1|}{(1+\varphi_1)\Gamma(\varpi+\beta)} \int_0^1 (\xi-\iota_1)^{\varpi+\beta-1} \mathfrak{y}(\iota_1) d\iota_1 \Bigg] \\
&- \frac{|\lambda|}{\Gamma(\varpi)} \int_0^{\xi} (\xi-\iota_1)^{\varpi-1} \varepsilon_1(\iota_1) d\iota_1 \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq& |\varepsilon_1(\xi) - \mathfrak{E}_{\varepsilon_1}| + \Bigg| \Bigg[\frac{1}{\Gamma(\varpi+\beta)} \int_0^{\xi} (\xi-\iota_1)^{\varpi+\beta-1} \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, \varepsilon_1{}^*(\iota_1)) d\iota_1 \\
&+ |\mathcal{A}_1(\xi)| \int_0^1 \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, \varepsilon_1{}^*(\iota_1)) d\iota_1
\end{aligned}
$$

$$
\begin{aligned}
&+|\mathcal{A}_2(\xi)| \int_0^1 \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, {\varepsilon_1}^*(\iota_1)) d\iota_1 \\
&+|\mathcal{A}_3(\xi)| \int_0^1 \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, {\varepsilon_1}^*(\iota_1)) d\iota_1 \\
&+\frac{|\mathcal{A}_4(\xi)|}{\Gamma(\beta-\varpi)} \int_0^1 (\xi-\iota_1)^{\beta-\varpi-1} \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, {\varepsilon_1}^*(\iota_1)) d\iota_1 \\
&+\frac{|\mathcal{A}_5(\xi)|}{\Gamma(\beta)} \int_0^1 (\xi-\iota_1)^{\beta-1} \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, {\varepsilon_1}^*(\iota_1)) d\iota_1 \\
&+\frac{|\varphi_1 \lambda|}{(1+\varphi_1)\Gamma(\varpi)} \int_0^1 (\xi-\iota_1)^{\beta-1} \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, {\varepsilon_1}^*(\iota_1)) d\iota_1 \\
&-\frac{|\varphi_1|}{(1+\varphi_1)\Gamma(\varpi+\beta)} \int_0^1 (\xi-\iota_1)^{\varpi+\beta-1} \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, {\varepsilon_1}^*(\iota_1)) d\iota_1 \Bigg] \\
&-\frac{|\lambda|}{\Gamma(\varpi)} \int_0^\xi (\xi-\iota_1)^{\varpi-1} \mathcal{J}_1(\iota_1, \varepsilon_1(\iota_1)) - \mathcal{J}_1(\iota_1, {\varepsilon_1}^*(\iota_1)) d\iota_1 \Bigg| \\
&\leq \mathcal{R}\vartheta + \Psi\mathcal{R}|\varepsilon_1(\xi) - {\varepsilon_1}^*(\xi)|,
\end{aligned}
$$

which implies

$$||\varepsilon_1(\xi) - {\varepsilon_1}^*|| \leq \mathcal{M}\vartheta,$$

where

$$\mathcal{M} = \frac{\mathcal{R}}{1-\Psi\mathcal{R}} > 0.$$

Hence, we conclude that problem (1) is U-H stable. In addition, by denoting $\Theta_{\mathcal{J}_1}(\vartheta) = \mathcal{M}\vartheta$, such that $\Theta_{\mathcal{J}_1}(0) = 0$, problem (1) is generalized U-H stable. □

5. Example

Example 1. *Denote the Banach space of real sequences by* $\mathcal{C}_0 = \{\varepsilon_1 = (\varepsilon_{11}, \varepsilon_{12}, \cdots) : \varepsilon_{1n} \to 0 (n \to \infty)\}$*, with the norm*

$$||\varepsilon_1||_\infty = \sup_{n\geq 1} |\varepsilon_{1n}|$$

Consider the following BVP

$$
\begin{cases}
{}^{\mathfrak{c}}\mathcal{D}^{\beta}\left({}^{\mathfrak{c}}\mathcal{D}^{\varpi} + \lambda\right)\left(\dfrac{\varepsilon_1(\xi)}{\kappa_1(\xi, \varepsilon_1(\xi))}\right) = \mathcal{J}_1(\xi, \varepsilon_1(\xi)) \ \ a.e.\ \xi \in \mathcal{J} = [0,1] \\
\dfrac{\varepsilon_1(0)}{\kappa_1(0, \varepsilon_1(0))} + \varphi_1 \dfrac{\varepsilon_1(1)}{\kappa_1(1, \varepsilon_1(1))} = \varrho_1 \displaystyle\int_0^1 \mathfrak{g}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1, \\
{}^{\mathfrak{c}}\mathcal{D}^{\varpi}\left(\dfrac{\varepsilon_1(0)}{\kappa_1(0, \varepsilon_1(0))}\right) + \varphi_1 {}^{\mathfrak{c}}\mathcal{D}^{\varpi}\left(\dfrac{\varepsilon_1(1)}{\kappa_1(1, \varepsilon_1(1))}\right) = \varrho_2 \displaystyle\int_0^1 \mathfrak{h}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1, \\
{}^{\mathfrak{c}}\mathcal{D}^{2\varpi}\left(\dfrac{\varepsilon_1(0)}{\kappa_1(0, \varepsilon_1(0))}\right) + \varphi_1 {}^{\mathfrak{c}}\mathcal{D}^{2\varpi}\left(\dfrac{\varepsilon_1(1)}{\kappa_1(1, \varepsilon_1(1))}\right) = \varrho_3 \displaystyle\int_0^1 \mathfrak{k}(\iota_1, \varepsilon_1(\iota_1)) d\iota_1,
\end{cases}
\tag{19}
$$

where Here $\varpi = 1/3, \beta = 4/3, \lambda = 1/300, \varphi_1 = 1, \varrho_1 = \varrho_2 = \varrho_3 = 1/200$ *and*

$$\mathcal{R} < 1.354.$$

Here $\mathcal{J}_1 : [0,1] \times \mathcal{C}_0 \to \mathcal{C}_0$ *given by*

$$\mathcal{J}_1(\xi, \varepsilon_1) = \left\{ \frac{1}{\sqrt{\xi^2 + 49}} \left(\frac{\xi \sin \varepsilon_1}{49} + e^{\xi} \cos \xi \right) \right\}_{n\geq 1}, for\ \xi \in [0,1], \ \ \varepsilon_1 = \{\varepsilon_{1n}\}_{n\geq 1} \in \mathcal{C}_0.$$

It is clear that condition $(\mathcal{A}_1)$ *holds, and as*

$$\begin{aligned} ||\mathcal{J}_1(\xi, \varepsilon_1)|| &= \left\| \frac{1}{\sqrt{\xi^2+49}} \left(\frac{\xi \sin \varepsilon_1}{49} + e^{\xi} \cos \xi \right) \right\| \\ &\leq \frac{1}{\sqrt{\xi^2+49}} (1 + ||\varepsilon_1||) \\ &= \Psi(\xi)(1 + ||\varepsilon_1||). \end{aligned}$$

Therefore, $(\mathcal{A}_2)$ *satisfied, with*

$$\Psi(\xi) = \frac{1}{\sqrt{\xi^2+49}}, \ \xi \in [0,1].$$

And the bounded set $\mathcal{J}_1 \subset \mathcal{C}_0$*, we have*

$$\chi(\mathcal{J}_1(\xi, \mathcal{J}_1)) \leq \frac{1}{\sqrt{\xi^2+49}} \chi(\mathcal{J}_1), \ \ \forall \xi \in [0,1].$$

So, $(\mathcal{A}_3)$ *holds true. Indeed,* $4\mathcal{R}_\Psi = 0.76593$ *and* $(1+\epsilon)\mathcal{R}_\Psi = \epsilon$*. Thus,*

$$\epsilon \geq \frac{\mathcal{R}_\Psi}{1-\mathcal{R}_\Psi} = \frac{0.15471}{1-0.19148} = \frac{0.19148}{0.80852} \approx 0.236827784.$$

Then ϵ *can be chosen as* $\epsilon = 0.2 > 0.183$*. Consequently, all conditions of Theorem 2 hold true. This yields the existence of a solution* $\varepsilon_1 \in \mathcal{C}([0,1], \mathcal{C}_0)$ *for the problem (1).*

6. Conclusions

We discussed the existence results for a mixed hybrid fractional differential equation of sequential type with nonlocal integral hybrid boundary conditions. The main results are established with the aid of Darbo's fixed point theorem and Hausdorff's measure of noncompactness method. Using standard functional analysis, we showed Ulam–Hyers stability. Our results in this configuration are novel, and add to the body of knowledge on the theory of fractional differential equations. For future work, we suggest using other types of fractional derivative operator, such as the generalized Hilfer fractional derivative. Anyone interested in the subject can also investigate the existence and uniqueness of solutions to the coupled or tripled systems using several fixed points theorems, such as Banach contraction, mapping principle, Leray–Schuader's alternative, and Mönch's fixed point theorem.

Author Contributions: Methodology, M.A. (Muath Awadalla); Writing—original draft, M.A. (Muath Awadalla) and M.M.; Writing—review & editing, M.A. (Meraa Arab), K.A., N.I.M. and T.N.G.; Supervision, M.A. (Muath Awadalla); Project administration, M.A. (Muath Awadalla). All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. 2785].

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare no conflict of interest.

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