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Cauchy–Dirichlet Problem to Semilinear Multi-Term Fractional Differential Equations

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Abstract: In this paper, we analyze the well-posedness of the Cauchy–Dirichlet problem to an integro-differential equation on a multidimensional domain $\Omega \subset \mathbb{R}^n$ in the unknown $u = u(x,t)$, $\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \mathbf{D}_t^{\nu_1}(\varrho_1 u) - \mathcal{L}_1 u - \int_0^t \mathcal{K}(t-s)\mathcal{L}_2 u(x,s)ds = f(x,t) + g(u)$, $0 < \nu_1 < \nu_0 < 1$, where $\mathbf{D}_t^{\nu_i}$ are the Caputo fractional derivatives, $\varrho_i = \varrho_i(x,t)$ with $\varrho_0 \geq \mu_0 > 0$, and $\mathcal{L}_i$ are uniform elliptic operators with time-dependent smooth coefficients. The principal feature of this equation is related to the integro-differential operator $\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \mathbf{D}_t^{\nu_1}(\varrho_1 u)$, which (under certain assumption on the coefficients) can be rewritten in the form of a generalized fractional derivative with a non-positive kernel. A particular case of this equation describes oxygen delivery through capillaries to tissue. First, under proper requirements on the given data in the linear model and certain relations between ν_0 and ν_1, we derive a priori estimates of a solution in Sobolev–Slobodeckii spaces that gives rise to providing the Hölder regularity of the solution. Exploiting these estimates and constructing appropriate approximate solutions, we prove the global strong solvability to the corresponding linear initial-boundary value problem. Finally, obtaining a priori estimates in the fractional Hölder classes and assuming additional conditions on the coefficients ϱ_0 and ϱ_1 and the nonlinearity $g(u)$, the global one-valued classical solvability to the nonlinear model is claimed with the continuation argument method.

Keywords: a priori estimates; Caputo derivatives; nonlinear oxygen subdiffusion; global classical solvability

MSC: Primary 35R11; 35B45; Secondary 35B655; 26A33; 35Q92

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1. Introduction

Fractional calculus is an effective tool to model the complex nonlinear phenomena (indicated as anomalous) arising in continuum mechanics, thermodynamics, medicine, biology and so on (see, for example, [1–8] and also references therein). Features of anomalous diffusion contain history dependence (memory term), long-range (or nonlocal) correlation in time and heavy-tail characteristics, while its signature is that the mean square displacement of the diffusion species $\langle \Delta x \rangle^2$ scales as a nonlinear power law in time, i.e., $\langle \Delta x \rangle^2 \approx t^\nu$, $\nu > 0$, $\nu \neq 1$. If the *anomalous diffusion exponent* ν belongs to the interval $(0,1)$, the underlying diffusion process is called subdiffusive. The constitutive relation of the viscoelastic material and the anomalous diffusion are successfully described by single-, multi-term or distributed order fractional ordinary or partial differential equations (**FODE** or **FPDE**) and by general integro-differential equations with a generalized fractional derivative:

$$\mathcal{D}_t^{\mathcal{N}} u(x,t) = \frac{\partial}{\partial t}\int_0^t \mathcal{N}(t-\tau)u(x,\tau)d\tau - \mathcal{N}(t)u(x,0), \quad t > 0, \tag{1}$$

where $\mathcal{N}(t)$ is a non-negative locally integrable kernel.

Specifying the kernel $\mathcal{N}(t)$ in (1) gives rise to different types of fractional derivatives. In particular, the Caputo fractional derivative $\mathbf{D}_t^\nu$ of order $\nu \in (0,1)$ is recovered via (1) for the power-law memory kernel $\mathcal{N}(t) = \frac{t^{-\nu}}{\Gamma(1-\nu)}$, with $\Gamma(\cdot)$ being the Euler Gamma function. The distributed order memory kernel

$$\mathcal{N} = \int_0^t \frac{(t-\tau)^{-\nu}}{\Gamma(1-\nu)} p(\nu) d\nu, \tag{2}$$

where $p(\cdot)$ is a non-negative weight function, reduces (1) to the fractional derivative of the distributed order, and corresponding **FPDE**s or **FODE**s of a distributed order. An important particular case of such equations is the diffusion equation with multi-term time-fractional derivatives with respect to time

$$\sum_{i=0}^{M} q_i \mathbf{D}_t^{\nu_i}, \quad 1 > \nu_0 > \nu_1 > \ldots > \nu_M > 0, \quad q_i \geq 0, \tag{3}$$

which is the main focus of this paper. Indeed, to reduce (1) with (2) to the multi-term fractional derivatives, the weight function in (2) is taken in the form of a finite linear combination of the Dirac delta functions with non-negative weight coefficients.

It is worth noting that the order of the corresponding fractional differential equations is defined with *the anomalous diffusion exponent*. In order to derive fractional differential equations from physical laws, one can exploit two different ways. The first approach is related to modeling continuous time random walk processes at the micro-level and taking a continuous limit at the macro-level [9]. The second method is appealed to conservative laws and specific constitutive relations with memory [1,4,10].

In this paper, motivated by the mathematical model for oxygen delivery through capillaries described in [4,5], we focus on the study of the initial-boundary value problem to semilinear diffusion equations with multi-term fractional derivatives, where some coefficients q_i may be non-negative.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with a smooth boundary $\partial\Omega$, and for any fixed $T > 0$, denote

$$\Omega_T = \Omega \times (0,T), \qquad \partial\Omega_T = \partial\Omega \times [0,T].$$

For $0 < \nu_1 < \nu_0 < 1$, we discuss the following non-autonomous multi-term subdiffusion equation with memory terms in the unknown function $u = u(x,t) : \Omega_T \to \mathbb{R}$,

$$\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \mathbf{D}_t^{\nu_1}(\varrho_1 u) - \mathcal{L}_1 u - \mathcal{K} * \mathcal{L}_2 u = f(x,t) + g(u), \tag{4}$$

supplemented with the initial condition

$$u(x,0) = u_0(x) \quad \text{in} \quad \bar{\Omega}, \tag{5}$$

and subject to the Dirichlet boundary condition **(DBC)**

$$u(x,t) = \psi(x,t), \tag{6}$$

where $\varrho_0 > 0$, ϱ_1, ψ, u_0, f, g and the memory kernel $\mathcal{K}$ are prescribed in Section 3.

The symbol $*$ stands for the time-convolution product on $(0,t)$, i.e.,

$$(\mathfrak{h}_1 * \mathfrak{h}_2)(t) = \int_0^t \mathfrak{h}_1(t-s)\mathfrak{h}_2(s) ds,$$

while $\mathbf{D}_t^\theta$ denotes the Caputo fractional derivative of the order $\theta \in (0,1]$ with respect to time t, defined as

$$\mathbf{D}_t^{\theta} u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\theta)} \frac{\partial}{\partial t} \int\limits_0^t \frac{u(x,s)-u(x,0)}{(t-s)^{\theta}} ds, & \text{if} \quad \theta \in (0,1), \\ \frac{\partial u}{\partial t}(x,t), & \text{if} \quad \theta = 1. \end{cases}$$

There is an equivalent definition

$$\mathbf{D}_t^{\theta} u(x,t) = \frac{1}{\Gamma(1-\theta)} \int\limits_0^t (t-s)^{-\theta} \frac{\partial u}{\partial s}(x,s) ds, \quad \theta \in (0,1),$$

if u is an absolutely continuous function. As for operators $\mathcal{L}_i$, they are the linear elliptic operators of the second order with time-dependent coefficients (written in divergence form), that is

$$\mathcal{L}_1 = \mathcal{L}_0 + \sum_{i=1}^{n} a_i(x,t) \frac{\partial}{\partial x_i} + a_0(x,t),$$
$$\mathcal{L}_2 = \mathcal{L}_0 + \sum_{i=1}^{n} b_i(x,t) \frac{\partial}{\partial x_i} + b_0(x,t),$$

where we put

$$\mathcal{L}_0 = \sum_{ij=1}^{n} \frac{\partial}{\partial x_i} \Big(a_{ij}(x,t) \frac{\partial}{\partial x_j} \Big).$$

For the mathematical treatment of single-term time-fractional diffusion equations with and without memory terms (i.e., subdiffusion equations similar to (4) with $\varrho_0 = 1$, $\varrho_1 = 0$ and either $\mathcal{K} \neq 0$ or $\mathcal{K} = 0$), which have been extensively studied (analytically and numerically) for the last few decades, we refer the reader to [11–23]. The diffusion equation with the general integro-differential operator (1) is analyzed in [7,8] (see also the references therein). The Cauchy problem for this equation on an unbounded space domain is discussed in [24]. Exploiting the Fourier method, well-posedness and a maximum principle for the initial-boundary value problem to the subdiffusion equation with multi-term fractional derivatives (3) with the positive constant coefficients q_i are studied in [25]. In [26], a solution to an initial-boundary value problem is formally represented by Fourier series and the multivariate Mittag–Leffler function. However, the authors do not provide the proof of the convergence of these series. This gap in the case of the multi-term time-fractional diffusion equation with positive constant coefficients q_i was filled in [27]. Initial-boundary value problems to equations with operator (3) where coefficients $q_i = q_i(x)$ (i.e., x- dependent) are discussed in [28]. The semilinear equation with the general fractional derivative (1) is analyzed in [29], where the uniqueness and the local/global existence are proved by means of the Schauder fixed-point theorem. Finally, we quote [3–6,30,31], where some analytical and numerical solutions were constructed to the corresponding initial-boundary value problems to the evolution equation with the operator (3).

The main distinction of equation (4) from the equations in the aforementioned previous works is related to the multi-term fractional derivatives: $\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \mathbf{D}_t^{\nu_1}(\varrho_1 u)$, which can be rewritten in the form of (1) with the kernel $\mathcal{N}$ being either a negative function or a function alternating in sign. Indeed, choosing

$$\varrho_1 = \varrho_1(x) \geq \varrho_0 = \varrho_0(x) > 0,$$

and appealing to Lemma 4 in [14], we end up with the equality

$$\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \mathbf{D}_t^{\nu_1}(\varrho_1 u) = \frac{\partial}{\partial t}[\mathcal{N} * (u - u(x,0))],$$

where the kernel

$$\mathcal{N} = \varrho_0(x)\frac{t^{-\nu_0}}{\Gamma(1-\nu_0)} - \varrho_1(x)\frac{t^{-\nu_1}}{\Gamma(1-\nu_1)}$$

is *negative* for $t > e^{-\gamma}$ (γ is the Euler–Mascheroni constant). It is worth noting that the non-negativity of the kernel $\mathcal{N}$ plays a crucial role in the previous investigations of **FPDEs** and related initial/initial-boundary value problems. This assumption is removed in our research. Moreover, equation (4) contains fractional derivatives calculated from the product of two functions: the desired solution u and the prescribed coefficients ϱ_i. The last peculiarity provides additional difficulties to study since the typical Leibniz rule does not work in the case of fractional derivatives, i.e.,

$$\mathbf{D}_t^{\theta}(uv) \neq u\mathbf{D}_t^{\theta}v + v\mathbf{D}_t^{\theta}u.$$

To the author's best knowledge, there are only two papers [32,33] in the published literature addressing the solvability of initial-boundary value problems to the equation similar to (4). Indeed, the first result concerning to existence and uniqueness of global classical solutions to the linear version (i.e., $g(u) \equiv 0$) of the non-autonomous equation (4) with alternating in sign $\varrho_1 = \varrho_1(x,t)$ and $\varrho_0 = \varrho_0(x,t)$ subjected to various types of boundary conditions was presented in [32]. However, solvability in the smooth classes (fractional Hölder spaces) requires stronger assumptions on the right-hand sides in the corresponding problems. Thus, our first goal of this art is to fill this gap, providing the well-posedness to the linear version of (4)–(6) under weaker requirements on the given functions. Namely, assuming that u_0, f, ψ belong to the proper fractional Sobolev spaces, we prove the one-to-one strong solvability in the class $W^{\nu_0,p}((0,T),L^p(\Omega)) \cap L^p((0,T),W^{2,p}(\Omega))$, $p \geq \max(n + \frac{2}{\nu_0}; \frac{1}{\nu_0-\nu_1})$ of (4)–(6) with $g(u) \equiv 0$. On this route, the main ingredient is a priori estimates in the fractional Sobolev spaces, which give rise to the Hölder regularity of a solution. Moreover, we establish similar results to the $(M+1)-$term fractional equations:

$$\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \sum_{i=1}^{M+1}\mathbf{D}_t^{\nu_i}(\varrho_i u) - \mathcal{L}_1 u - \mathcal{K} * \mathcal{L}_2 u = f,$$

$$\varrho_0\mathbf{D}_t^{\nu_0}u - \sum_{i=1}^{M+1}\varrho_i\mathbf{D}_t^{\nu_i}u - \mathcal{L}_1 u - \mathcal{K} * \mathcal{L}_2 u = f$$

with $\varrho_i = \varrho_i(x,t)$, $0 < \nu_i < \nu_0 < 1$.

The second novelty of this paper is related to the well-posedness of the nonlinear Cauchy–Dirichlet problem (4)–(6), i.e., $g(u) \neq 0$. Indeed, in [33], this nonlinear model was analyzed in the case of a one-dimensional space domain and only time-dependent coefficients $\varrho_i = \varrho_i(t)$, $i = 0,1$. Therefore, the second achievement of this art is the extension of the result of [33] to the case of semilinear equation (4) with coefficients ϱ_i depending on the space and time variables and stated in a multidimensional domain Ω. It worth noting that, compared to [33], the analyzed model in the multidimensional case will require $\mathcal{C}^1$-regularity on the memory kernel $\mathcal{K}$. Namely, if g is locally Lipschitz, then the main point to study the global classical solvability is searching a priori estimates for the solution u, and in turn the bound for $\|u\|_{\mathcal{C}(\bar{\Omega}_T)}$. In the one-dimensional case, the Sobolev embedding theorem provides the inequality

$$\|u\|_{\mathcal{C}(\bar{\Omega}_T)} \leq C, \tag{7}$$

exploiting only the bound of $\|u\|_{\mathcal{C}((0,T),W^{1,2}(\Omega))}$. This trick cannot be drawn in the multidimensional case, where bound (7) is eventually reached via the following iterative inequalities:

$$\|u\|_{\mathcal{C}([0,T],L_p(\Omega))} \leq Cp^{1/p}\|u(\cdot,t)\|_{\mathcal{C}([0,T],L_{p/2}(\Omega))}, \quad p > 1.$$

To this end, we first rewrite equation (4) in a suitable form, where the memory term does not contain the principal part of the operator $\mathcal{L}_2$ (i.e., $a_{ij} \equiv 0$). Then, we exploit the integral iterative technique from [18]. At the same time, as a side effect, the term $\bar{\mathcal{K}}' * u$

appears in the equation. Here, $\bar{\mathcal{K}}$ is the conjugate kernel, having the same properties of $\mathcal{K}$. This explains the requirement of a smoother kernel in the multidimensional case.

Outline of the Paper

The paper is organized as follows: in Section 2, we introduce the notation and the functional setting. The main assumptions are given in Section 3. The principal results, Theorems 1–2 and Lemma 1, are stated in Section 4. Theorem 1 is related to a priori estimates of the solution u in $W^{\nu_0,p}((0,T),L^p(\Omega))\cap L^p((0,T),W^{2,p}(\Omega))$ and in $\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_T)$ in the case of the linear version of (4)–(6), while Theorem 2 concerns the global classical solvability of the corresponding nonlinear problem. The existence and the uniqueness of a strong solution to (4)–(6) with $g(u)\equiv 0$ are stated in Lemma 1. It is worth noting that this claim is a simple consequence of Theorem 1 and the results related to the one-to-one classical solvability established in our previous work [32], so we give the proof of this lemma in Section 3. Some definitions and some auxiliary results from fractional calculus, playing a key role in this art, are given in Section 5. The proof of Theorem 1 is carried out in Section 6. Here, exploiting so-called one variant of a Leibniz rule to Caputo derivatives, $\mathbf{D}_t^{\nu_0}(\varrho_0 u)$ and $\mathbf{D}_t^{\nu_1}(\varrho_1 u)$, and following the approach from Section 5 in [18], we rewrite equation (4) in an appropriate form, where the principal part of the integro-differential operator $\mathbf{D}_t^{\nu_0}(\varrho_0 u)-\mathbf{D}_t^{\nu_1}(\varrho_1 u)$ is represented as $\varrho_0(x,t)\mathbf{D}_t^{\nu_0}u-\varrho_1(x,t)\mathbf{D}_t^{\nu_1}u$; the leading part of the operator $\mathcal{L}_2$ (as we wrote above) is not involved in the memory term. After that, in Section 6.1, we first obtain a priori estimates in the fractional Sobolev spaces for a small time interval and then discuss how these estimates can be extended to the whole time interval. In Section 6.2, collecting the obtained estimates in the space $W^{\nu_0,p}((0,T),L^p(\Omega))\cap L^p((0,T),W^{2,p}(\Omega))$ with results in [23], we evaluate the Hölder seminorms of the solution u. In particular, in the case of homogeneous initial and boundary conditions, this estimate reads as

$$\|u\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_T)}\leq C\|f\|_{L_p(\Omega_T)}.$$

Finally, Section 7 is devoted to the verification of Theorem 2. The main tool in our arguments is the continuation method related to the study of a family of auxiliary problems depending on a parameter $\lambda\in[0,1]$. On this route, one has to obtain a priori estimates for the solutions which are independent of λ (see Section 7.1). The key bound is the estimate of $\|u\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_T)}$, produced via integral iteration techniques adapted to the case of multi-term fractional derivatives.

2. Functional Spaces and Notation

Throughout this art, C will be a generic positive constant, depending only on the given data of the model. We will perform our study in the fractional Hölder spaces. To this end, we take two arbitrary (but fixed) quantities

$$\alpha\in(0,1)\quad\text{and}\quad\nu\in(0,1).$$

Let l be any non-negative integer, and $(\mathbf{X},\|\cdot\|_{\mathbf{X}})$ be any Banach space. For any $p\geq 1$, $s\geq 0$, we consider the usual spaces

$$\mathcal{C}^s([0,T],\mathbf{X}),\quad \mathcal{C}^{l+\alpha}(\bar{\Omega}),\quad W^{s,p}(\Omega),\quad L_p(\Omega),\quad W^{s,p}((0,T),\mathbf{X}).$$

Recall that for non-integer s, $W^{s,p}$ is called the Sobolev–Slobodeckii space (for its definition and properties see Chapter 1 in [34], and Chapter 1 in [35]).

Denoting for $\beta\in(0,1)$

$$\langle v\rangle^{(\beta)}_{x,\Omega_T}=\sup\Big\{\frac{|v(x_1,t)-v(x_2,t)|}{|x_1-x_2|^{\beta}}:\quad x_2\neq x_1,\quad x_1,x_2\in\bar{\Omega},\quad t\in[0,T]\Big\},$$
$$\langle v\rangle^{(\beta)}_{t,\Omega_T}=\sup\Big\{\frac{|v(x,t_1)-v(x,t_2)|}{|t_1-t_2|^{\beta}}:\quad t_2\neq t_1,\quad x\in\bar{\Omega},\quad t_1,t_2\in[0,T]\Big\},$$

we assert the following definition.

Definition 1. *A function $v=v(x,t)$ belongs to the class $\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\nu}(\bar{\Omega}_T)$, for $l=0,1,2$, if the function v and its corresponding derivatives are continuous and the norms are finite:*

$$\|v\|_{\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\nu}(\bar{\Omega}_T)}=\begin{cases}\|v\|_{\mathcal{C}([0,T],\mathcal{C}^{l+\alpha}(\bar{\Omega}))}+\sum_{|j|=0}^{l}\langle D_x^j v\rangle^{(\frac{l+\alpha-|j|}{2}\nu)}_{t,\Omega_T}, & l=0,1,\\[2ex] \|v\|_{\mathcal{C}([0,T],\mathcal{C}^{2+\alpha}(\bar{\Omega}))}+\|\mathbf{D}_t^{\nu}v\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\nu}(\bar{\Omega}_T)}+\sum_{|j|=1}^{2}\langle D_x^j v\rangle^{(\frac{2+\alpha-|j|}{2}\nu)}_{t,\Omega_T}, & l=2.\end{cases}$$

In a similar way, we introduce the space $\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\nu}(\partial\Omega_T)$, for $l=0,1,2$.

The properties of these spaces were discussed in Section 2 [18]. It is worth noting that, if $\nu=1$, the class $\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\nu}$ boils down to the usual parabolic Hölder space $H^{l+\alpha,\frac{l+\alpha}{2}}$ (see (1.10)–(1.12) in [36]).

Finally, we will say that a function v defined in Ω_T belongs to $\mathfrak{H}_p^{s_1,s_2}(\Omega_T)$ with $p>1$ and $s_1,s_2\geq 0$, if $v\in W^{s_1,p}((0,T),L_p(\Omega))\cap L_p((0,T),W^{s_2,p}(\Omega))$, and the norm here below is finite

$$\|v\|_{\mathfrak{H}_p^{s_1,s_2}(\Omega_T)}=\|v\|_{W^{s_1,p}((0,T),L_p(\Omega))}+\|v\|_{L_p((0,T),W^{s_2,p}(\Omega))}.$$

The space $\mathfrak{H}_p^{s_1,s_2}(\partial\Omega_T)$ is defined in a similar manner.

3. General Hypothesis

First, we state our general assumptions on the given data in the model. To this end, denoting

$$\omega_{1-\theta}(t)=\frac{t^{-\theta}}{\Gamma(1-\theta)},\quad \theta>0,$$

we designate the positive values ν^* and T^*, $\nu^*\leq 1$, such that the difference

$$\omega_{1-\nu_0}(t)-\omega_{1-\nu_1}(t)$$

is non-negative for all $t\in[0,T^*]$ and $0<\nu_1<\nu_0\leq\nu^*\leq 1$.

We notice that the existence of these values is provided by Lemma 4 [14], and besides, some numerical examples of ν^* and T^* are discussed in Remark 3.2 [33].

H1 (Conditions on the fractional order of the derivatives). *We assume that*

$$\nu_0\in(0,\nu^*)\quad\text{and}\quad \nu_1\in\Big(0,\frac{\nu_0(2-\alpha)}{2}\Big).$$

H2 (Ellipticity conditions). *There are positive constants $0<\mu_1<\mu_2$ and μ_0, μ_3 such that*

$$\mu_1|\xi|^2\leq\sum_{ij=1}^{n}a_{ij}(x,t)\xi_i\xi_j\leq\mu_2|\xi|^2,$$

for any $(x,t,\xi)\in\bar{\Omega}_T\times\mathbb{R}^n$, and

$$\varrho_0(x,t)\geq\mu_0>0,\quad |\varrho_1|\geq\mu_3>0$$

for each $(x,t)\in\bar{\Omega}_T$.

H3 (Conditions on the coefficients). *For $i, j = 1, \ldots, n$,*

$$a_0(x,t), b_0(x,t) \in \mathcal{C}^{\alpha, \frac{\alpha\nu_0}{2}}(\bar{\Omega}_T), \quad a_{ij}(x,t), a_i(x,t), b_i(x,t) \in \mathcal{C}^{1+\alpha, \frac{(1+\alpha)\nu_0}{2}}(\bar{\Omega}_T),$$
$$\varrho_0, \varrho_1 \in \mathcal{C}^{\gamma_0}([0,T], \mathcal{C}^1(\bar{\Omega})), \quad \gamma_0 > \max\Big\{1, \frac{\nu_0(2+\alpha)}{2}\Big\}.$$

H4 (Conditions on the given functions).

$$\mathcal{K}(t) \in \mathcal{C}^1([0,T]), \quad u_0(x) \in C^{2+\alpha}(\bar{\Omega}),$$
$$f(x,t) \in \mathcal{C}^{\alpha, \frac{\alpha\nu_0}{2}}(\bar{\Omega}_T), \quad \psi(x,t) \in \mathcal{C}^{2+\alpha, \frac{2+\alpha}{2}\nu_0}(\partial\Omega_T).$$

H5 (Compatibility conditions). *The following compatibility conditions hold for each $x \in \partial\Omega$ and $t = 0$,*

$$\psi(x,0) = u_0(x) \quad \text{and}$$
$$\mathbf{D}_t^{\nu_0}(\varrho_0\psi)|_{t=0} - \mathbf{D}_t^{\nu_1}(\varrho_1\psi)|_{t=0} = \mathcal{L}_1 u_0(x)|_{t=0} + f(x,0) + g(u_0).$$

H6 (Conditions on the nonlinearity). *We assume that the function g satisfies the local Lipschitz condition, i.e., for every $\rho > 0$ there exists a positive constant C_ρ such that*

$$|g(u_1) - g(u_2)| \leq C_\rho |u_1 - u_2|$$

for any $u_1, u_2 \in [-\rho, \rho]$.

Moreover, there is a positive constant L such that

$$|g(u)| \leq L(1 + |u|)$$

for any $u \in \mathbb{R}$.

H7 (Conditions on the sign of the coefficient ϱ_1). *We require that the function ϱ_1 retains its sing in $\bar{\Omega}_T$, i.e.,*

$$\text{either} \quad sgn\, \varrho_1(x,t) = 1 \quad \text{for all} \quad (x,t) \in \bar{\Omega}_T$$
$$\text{or} \quad sgn\, \varrho_1(x,t) = -1 \quad \text{for all} \quad (x,t) \in \bar{\Omega}_T.$$

If $\varrho_1(x,t)$ is positive in $\bar{\Omega}_T$, we additionally assume that $\varrho_1 \in \mathcal{C}^{\gamma_0}([0,T], \mathcal{C}^{1+\alpha}(\bar{\Omega}))$ and the relation holds

$$\varrho_0(x,t) = \varrho(x,t) + \varrho_1(x,t)$$

with a positive function ϱ having the same regularity as the function ϱ_1.

Moreover, we require that for each $(x,t) \in \bar{\Omega}_T$

$$\frac{\partial}{\partial t}\Big(\frac{\varrho_0}{\varrho_1}\Big) \geq 0 \quad \text{if} \quad \varrho_1 \quad \text{is negative,} \quad \text{and} \quad \frac{\partial}{\partial t}\Big(\frac{\varrho_0}{\varrho}\Big) \leq 0 \quad \text{if} \quad \varrho_1 \quad \text{is positive.}$$

*It is worth noting that the assumption **H7** on the sign of the function ϱ_1 is needed only in the case of the nonlinear model (see Theorem 2), while the analysis of the linear model requires only the regularity of this function stated in **H3**.*

Remark 1. *It is apparent that the simplest example of functions ϱ_1, ϱ_0 satisfying assumption **H7** is*

$$\varrho_0 \equiv C_0 \quad \textit{and} \quad \varrho_1 \equiv C_1,$$

where $C_0 > 0$ and C_1 are given constants, and C_1 is negative if ϱ_1 is negative, while in the case of positive ϱ_1 ($\varrho = C_0 - C_1$), the constant C_1 is related to C_0 via the relation $0 < C_1 < C_0$.

Remark 2. *Thanks to Lemma 4.1 in [17], for any* $u \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\partial\Omega_T)$*, the equality*

$$(\mathcal{K} * \mathcal{L}_2 u)(x,0) = 0$$

holds for any $x \in \partial\Omega$*. That explains the absence of the memory term* $(\mathcal{K} * \mathcal{L}_2 u)$ *in the compatibility condition* ***H5****.*

4. Main Results

Now, we are ready to state our first main result related to the a priori estimates of a solution to problem (4)–(6) in the case of the linear equation which will be a significant point in the analysis of the nonlinear model as well as in the study of the existence of a strong solution to the corresponding linear problem.

Theorem 1. *Let* $g(u) \equiv 0$*,* $n \geq 2$*,* $p > \max\{n + \frac{2}{\nu_0}; \frac{1}{\nu_0 - \nu_1}\}$*,* $\nu_0 \in (0,1)$*; and let* ν_1 *satisfy* ***H1****. We require that assumptions* ***H2–H5*** *hold. Then the classical solution* $u \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)$ *of problem (4)–(6) satisfies the estimate*

$$\begin{aligned} &\|u\|_{\mathfrak{H}_p^{\nu_0,2}(\Omega_T)} + \|u\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_T)} + \|\mathbf{D}_t^{\nu_1} u\|_{L_p(\Omega_T)} \\ &\leq C\{\|f\|_{L_p(\Omega_T)} + \|u_0\|_{W^{2-\frac{2}{p\nu_0},p}(\Omega)} + \|\psi\|_{\mathfrak{H}_p^{\nu_0(1-\frac{1}{2p}),2-\frac{1}{p}}(\partial\Omega_T)}\}. \end{aligned} \tag{8}$$

Here, the generic constant C is independent of the right-hand sides in (4)–(6).

Remark 3. *In this art, we do not discuss the existence of the classical solution to the linear variant of problem (4)–(6), i.e., if* $g(u) \equiv 0$*. This issue was studied in Theorem 4.1 [32].*

Our next result is related to the global solvability of the semilinear problem (4)–(6).

Theorem 2. *Let* $\partial\Omega \in \mathcal{C}^{2+\alpha}$*,* $T > 0$ *be arbitrarily fixed,* ν_1 *satisfy* ***H1****, and let* $\nu_0 \in (0,1)$ *if* ϱ_1 *is negative, while* ν_0 *meets requirement* ***H1*** *if the function* ϱ_1 *is positive. Then, under assumptions* ***H2–H7****, problem (4)–(6) has a unique classical solution* $u = u(x,t)$ *in* Ω_T *possessing regularity*

$$u \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T), \quad \mathbf{D}_t^{\nu_1} u \in \mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_T).$$

Remark 4. *The arguments of Section 7 in the case of negative* ϱ_1 *provide that the results of Theorem 2 hold in the case of more general assumptions on the nonlinearity* $g(u)$*. Namely,*

$$g \in \mathcal{C}^1(\mathbb{R}), \quad |g(u)| \leq L_1(1 + |u|^\delta), \quad g(u)u \leq L_2 - L_3|u|^{\delta+1}, \quad g'(u) \leq L_4$$

with non-negative constants L_i *and* δ*.*

Finally, we assert the result which is a simple consequence of Theorem 1 and is related to the existence of a strong solution of linear version of (4)–(6).

Lemma 1. *Let* $g(u) \equiv 0$*,* $\psi \equiv 0$*,* $\partial\Omega \in C^{2+\alpha}$*, parameters* p, n, ν_0, ν_1 *satisfy the conditions of Theorem 1, and let assumptions* ***H2–H3*** *hold. Moreover, we assume that the kernel* $\mathcal{K}$ *meets requirement* ***H4****; compatibility condition* ***H5*** *is fulfilled on* $\partial\Omega$*, and*

$$f \in L_p(\Omega_T) \cap W^{s_1,r}((0,T), W^{s_2,r}(\Omega)), \quad u_0 \in W^{2-\frac{2}{p\nu_0},p}(\Omega) \cap W^{2+s_2,r}(\Omega), \tag{9}$$

where $r \geq n+1$*,* $s_1 \in (r^{-1}, 1)$*,* $s_2 \in ((n+1)r^{-1}, 1)$*. Then for any fixed* $T > 0$*, the linear initial-boundary value problem (4)–(6) admits a unique strong solution in the class* $\mathfrak{H}_p^{\nu_0,2}(\Omega_T)$*, satisfying estimate (8).*

Indeed, in order to verify this statement, it is enough to construct an approximate solution u_n via Theorem 4.1 in [32] (see Lemma 3 here as a reformulated result). Then, exploiting uniform estimates (8) for u_n and passing to the limit via standard arguments, we obtain a strong solution to (4)–(6) satisfying the regularity established by Lemma 1. Finally, estimate (8) provides the uniqueness of this strong solution.

Remark 5. *We notice that some assumptions in Lemma 1 are not typical to the theory of the existence of strong solutions. Namely, it is related to the second equality in **H5** and to the requirement of greater regularity (see (9)) of the right-hand sides f, u_0 than is demanded in estimate (8). The occurrence of these conditions is explained with our approach, which deals with obtaining the strong solution as a limit of the corresponding approximated smooth solutions u_n given by Lemma 3. Recall that, in order to the smooth solutions exist in the indicated classes, both compatibility conditions in **H5** are needed (see Theorem 4.1 [32]). Clearly, the additional regularity of the functions f, u_0 provides the fulfillment of the second equality in **H5**.*

Remark 6. *Our assumption on the kernel $\mathcal{K}$ admits the case $\mathcal{K} \equiv 0$, which means that the multi-term subdiffusion equation*

$$\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \mathbf{D}_t^{\nu_1}(\varrho_1 u) - \mathcal{L}_1 u = f(x,t) + g(u)$$

fits in our analysis and is described by Theorems 1–2 and Lemma 1.

Remark 7. *Actually, with an inessential modification in the arguments, the results of Theorem 1 and Lemma 1 hold for the $(M+1)$-term fractional equations:*

$$\mathbf{D}_t^{\nu_0}(\varrho_0 u) - \sum_{i=1}^{M} \mathbf{D}_t^{\nu_i}(\varrho_i u) - \mathcal{L}_1 u - \int_0^t \mathcal{K}(t-s)\mathcal{L}_2 u(x,s)ds = f(x,t),$$

$$\varrho_0(x,t)\mathbf{D}_t^{\nu_0} u - \sum_{i=1}^{M} \varrho_i(x,t)\mathbf{D}_t^{\nu_i} u - \mathcal{L}_1 u - \int_0^t \mathcal{K}(t-s)\mathcal{L}_2 u(x,s)ds = f(x,t).$$

In the case of the last equation, the regularity of the functions ϱ_i can be relaxed, namely, we assume that $\varrho_i \in \mathcal{C}^{\alpha,\alpha\nu_0/2}(\bar{\Omega}_T)$. The details are left to the interested readers.

The remaining of the paper is devoted to the proof of Theorems 1 and 2.

5. Technical Results

In this Section we present some properties of fractional derivatives and integrals as well as several technical results that will be used in this art. First, we begin with some definitions of fractional derivatives and integrals.

Throughout this work, for any $\theta > 0$, we denote (as we wrote before)

$$\omega_\theta(t) = \frac{t^{\theta-1}}{\Gamma(\theta)}, \tag{10}$$

and define the fractional Riemann–Liouville integral and the derivative of the order θ, respectively, of a function $v(\cdot,t)$ with respect to time t as

$$I_t^\theta v(\cdot,t) = (\omega_\theta * v)(\cdot,t), \quad \partial_t^\theta v(\cdot,t) = \frac{\partial^{\lceil\theta\rceil}}{\partial t^{\lceil\theta\rceil}}(\omega_{\lceil\theta\rceil-\theta} * v)(\cdot,t),$$

where $\lceil\theta\rceil$ is the ceiling function of θ (i.e., the smallest integer is greater than or equal to θ).

Clearly, for $\theta \in (0,1)$ we have

$$\partial_t^\theta v(\cdot,t) = \frac{\partial}{\partial t}(\omega_{1-\theta} * v)(\cdot,t).$$

Therefore, the Caputo fractional derivative of the order $\theta \in (0,1)$ to a function $v(x,t)$ can be given as

$$\mathbf{D}_t^{\theta} v(x,t) = \partial_t^{\theta} v(x,t) - \omega_{1-\theta}(t)v(x,0), \tag{11}$$

if both derivatives exist (see (2.4.8) [2]).

In the first proposition, which subsumes and partially generalizes (in particular, it concerns (iii) in the statement below) Propositions 4.1 and 4.2 from [18], we remind the reader of some useful properties of fractional integrals and derivatives.

Proposition 1. *The following relations hold.*

(i) *Let $\theta, \theta_1 \in (0,1)$, $t \in [0,T]$. Then for any function $w = w(t) \in \mathcal{C}^{\theta_1}([0,T])$, there is*

$$I_t^{\theta} \partial_t^{\theta} w(t) = w(t).$$

If, in addition, $\theta < 2\theta_1$, and $p \geq 2$ is any even integer, it is also true that

$$\partial_t^{\theta} w^p(t) \leq p w^{p-1}(t) \partial_t^{\theta} w(t).$$

(ii) *Let θ be a positive number, $k \in L_1(0,T)$, and let $W(t)$ be a bounded function on $[0,T]$. Then*

$$I_t^{\theta}(k * W)(t) = (k * w)(t) \quad \text{where } w = I_t^{\theta} W(t).$$

(iii) *Let $k(t) \in \mathcal{C}^1([0,T])$, $w_2(t) \in \mathcal{C}^{\theta_2}([0,T])$, $\theta_2 \geq 1$, $\mathbf{D}_t^{\theta} w(t) \in \mathcal{C}([0,T])$. Then the equality holds:*

$$\begin{aligned}(k * w_2 \mathbf{D}_t^{\theta} w)(t) = {} & k(0) w_2(t)(\omega_{1-\theta} * [w - w(0)])(t) + (k' * w_2(\omega_{1-\theta} * [w - w(0)]))(t) \\ & + (k * w_2'(\omega_{1-\theta} * [w - w(0)]))(t), \quad t \in [0,T].\end{aligned}$$

(iv) *For any given positive numbers θ_1 and θ_2, the following equalities are fulfilled:*

$$\omega_{\theta_1} * \omega_{\theta_2} = \omega_{\theta_1+\theta_2}(t), \qquad 1 * \omega_{\theta_1} = \omega_{\theta_1+1}(t), \qquad \omega_{\theta_1}(t) \geq C T^{\theta_1 - 1},$$

$$\omega_{\theta_1} * k^{(l)} \leq C \omega_{\theta_1+1} \leq C \omega_{\theta_1},$$

$l = 0,1$, for any $t \in [0,T]$. The positive constant C depends only on T, θ_1 and either $\|k\|_{\mathcal{C}^1(0,T)}$ if $l = 1$ or $\|k\|_{L_1(0,T)}$ if $l = 0$.

The next result describes the main properties of the function $\frac{d}{dt}(\mathcal{N} * w)(t)$, where a kernel $\mathcal{N} = \mathcal{N}(t)$ is completely monotonic and satisfies the following requirements.

H8. *For any $T > 0$ (including $T = +\infty$) and all $t \in [0,T]$, there holds*

$$\mathcal{N} \in \begin{cases} L_1(0,T), & \text{if } T \text{ is arbitrarily fixed,} \\ L_{1,loc}(\mathbb{R}_+), & \text{otherwise,} \end{cases} \qquad \lim_{t \to 0} \mathcal{N}(t) = +\infty.$$

Moreover, for some $\theta^\star \in (0,1)$ and $t^\star = \min\{1,T\}$, the following inequalities are fulfilled:

$$\begin{aligned} & \int_0^t |\mathcal{N}'(s)| s^{\theta^\star} ds < +\infty, \quad t \in [0,t^\star], \\ & \int_0^{t^\star} |\mathcal{N}'(s)| s^{\theta^\star} ds + \int_{t^\star}^t |\mathcal{N}'(s)| ds < +\infty, \quad t > t^\star, \\ & (-1)^k \frac{d^k \mathcal{N}}{dt^k}(t) \geq 0, \quad k = 0,1,2,\dots. \end{aligned}$$

Clearly, the last inequality in **H8** tells us that the kernel $\mathcal{N}$ is a completely monotonic function.

Proposition 2. *Let assumption* **H8** *hold. Then, for any functions $w_1 = w_1(t)$ and $w_2 = w_2(t)$ satisfying requirements*

$$w_1 \in \mathcal{C}^{\theta_1}([0,T]), \quad w_2 \in \mathcal{C}^{\theta_2}([0,T]), \quad \theta_1 \in [0,1), \quad \theta_2 \in (0,1], \quad \theta_1 + \theta_2 \geq \theta^\star,$$

and

$$\frac{d}{dt}(\mathcal{N} * w_1)(t),\ \frac{d}{dt}(\mathcal{N} * w_2)(t),\ \frac{d}{dt}(\mathcal{N} * w_1 w_2)(t) \in \mathcal{C}([0,T]), \quad \lim_{t\to 0} w_1(t) w_2(t) \mathcal{N}(t) < +\infty,$$

the following relations hold:

(i)

$$\begin{aligned}\frac{d}{dt}(\mathcal{N} * w_1 w_2)(t) = {}& w_1(t)\frac{d}{dt}(\mathcal{N} * w_2)(t) + w_2(t)\frac{d}{dt}(\mathcal{N} * w_1)(t) - w_1(t)w_2(t)\mathcal{N}(t)\\ &+ \int_0^t \mathcal{N}'(t-s)[w_1(t) - w_1(s)][w_2(t) - w_2(s)]ds, \quad t \in [0,T].\end{aligned}$$

(ii) *For any integer even $p \geq 2$, there is*

$$p w_1^{p-1}(t)\frac{d}{dt}(\mathcal{N} * w_1)(t) \geq \frac{d}{dt}(\mathcal{N} * w_1^p)(t).$$

If, in addition, w_1 is non-negative, then this bound holds for integer odd p.

Proof. First, we verify the point (i) of this assertion. It is worth noting that if $\mathcal{N}(t) = \omega_{1-\nu}(t)$ and $\theta^\star \in (\nu, 1)$, this claim is proved in Lemma 1 [11] for any fixed $T > 0$. Here, we extend this result to the case of a more general kind of $\mathcal{N}$.

By the definition of a derivative, we have

$$\frac{d}{dt}(\mathcal{N} * w_1 w_2)(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\int_0^{t+\varepsilon} \mathcal{N}(t+\varepsilon-s)w_1(s)w_2(s)ds - \int_0^t \mathcal{N}(t-s)w_1(s)w_2(s)ds\Big].$$

Then, taking advantage of the easily verified representation

$$w_1(s)w_2(s) = [w_1(s) - w_1(t)][w_2(s) - w_2(t)] - w_1(t)w_2(t) + w_1(t)w_2(s) + w_1(s)w_2(t),$$

we arrive at the equality

$$\begin{aligned}\frac{d}{dt}(\mathcal{N} * w_1 w_2)(t) = {}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\int_0^{t+\varepsilon} \mathcal{N}(t+\varepsilon-s)[w_1(s) - w_1(t)][w_2(s) - w_2(t)]ds\\ &- \int_0^t \mathcal{N}(t-s)[w_1(s) - w_1(t)][w_2(s) - w_2(t)]ds\Big]\\ &+ w_1(t)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\int_0^{t+\varepsilon} \mathcal{N}(t+\varepsilon-s)w_2(s)ds - \int_0^t \mathcal{N}(t-s)w_2(s)ds\Big]\\ &+ w_2(t)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\int_0^{t+\varepsilon} \mathcal{N}(t+\varepsilon-s)w_1(s)ds - \int_0^t \mathcal{N}(t-s)w_1(s)ds\Big]\\ &- w_1(t)w_2(t)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\int_0^{t+\varepsilon} \mathcal{N}(t+\varepsilon-s)ds - \int_0^t \mathcal{N}(t-s)ds\Big].\end{aligned}$$

Finally, keeping in mind **H8** and the smoothness of the functions w_1 and w_2, we end up with the desired equality.

Coming to the proof of point (ii) in this proposition, we first verify the cases of $p = 2$ and $p = 3$. To this end, substituting

$$w_2 = \begin{cases} w_1, & \text{if} \quad p = 2,\\ w_1^2, & \text{if} \quad p = 3,\end{cases}$$

to the equality in (i) of this claim, we deduce the relations

$$\frac{d}{dt}(\mathcal{N} * w_1^2) = 2w_1 \frac{d}{dt}(\mathcal{N} * w_1)(t) - \mathcal{N}(t)w_1^2(t) + \int_0^t \mathcal{N}'(t-s)[w_1(s) - w_1(t)]^2 ds,$$

$$\frac{d}{dt}(\mathcal{N} * w_1^3) = w_1 \Big[\frac{d}{dt}(\mathcal{N} * w_1^2)(t) + w_1(t)\frac{d}{dt}(\mathcal{N} * w_1)(t) - \mathcal{N}(t)w_1^2(t)\Big] + \int_0^t \mathcal{N}'(t-s)[w_1(s) - w_1(t)]^2[w_1(s) + w_1(t)]ds.$$

Appealing to the complete monotonicity of $\mathcal{N}$ and to the non-negativity of the function w_1 (if $p = 3$), we immediately end up with the desired estimates for $p = 2, 3$. Finally, taking advantage of these estimates and exploiting the induction, we complete the proof of (ii) for $p > 3$, and hence, the proof of Proposition 2. □

Introducing the new function

$$\mathcal{N}_\nu(t) = \mathcal{N}(t; \nu_0, \nu_1) = \omega_{1-\nu_0}(t) - \omega_{1-\nu_1}(t) \tag{12}$$

with $\nu_1, \nu_0 \in (0, 1)$, we assert the following claim:

Corollary 1. *Let $0 < \nu_1 < \nu_0 < \nu^* < 1$. Then for any function $w_1 \in \mathcal{C}^{\theta_1}([0, T^*])$, $\theta_1 \in (\nu_0, 1)$, $\lim\limits_{t\to 0} w_1(t)\mathcal{N}_\nu(t) < +\infty$, and for each even integer $p \geq 2$, the inequality*

$$p w_1^{p-1} \frac{d}{dt}(\mathcal{N}_\nu * w_1)(t) \geq \frac{d}{dt}(\mathcal{N}_\nu * w_1^p)(t)$$

holds for all $t \in [0, T^*]$. If additionally w_1 is non-negative, then this bound holds for any integer odd p.

Proof. It is apparent that this statement is a simple consequence of Proposition 2 if $\mathcal{N}_\nu(t)$ meets requirement **H8**. In light of (10) and (12), the kernel $\mathcal{N}_\nu(t)$ satisfies the first four conditions in **H8**. Thus, we are left to check that $\mathcal{N}_\nu(t)$ is completely monotonic for all $t \in [0, T^*]$.

If ν_1 and ν_0 satisfy the assumption of this claim, then definitions of ν^* and T^* provide the positivity of the function $\mathcal{N}_\nu(t)$ for all $t \in [0, T^*]$. Then, straightforward calculations arrive at the equality

$$(-1)^k \frac{d^k \mathcal{N}_\nu}{dt^k}(t) = (-1)^{2k} t^{-k} \mathcal{N}_\nu(t) \prod_{j=0}^{k-1}(\nu_0 + j) + (-1)^{2k} \frac{t^{-\nu_1 - k}}{\Gamma(1-\nu_1)} \Big[\prod_{j=0}^{k-1}(\nu_0 + j) - \prod_{j=0}^{k-1}(\nu_1 + j)\Big], k = 0, 1, 2 \ldots$$

Finally, appealing to the positivity of $\mathcal{N}_\nu(t)$ and bearing in mind the relation $0 < \nu_1 < \nu_0$, we immediately obtain the non-negativity of the function $(-1)^k \frac{d^k \mathcal{N}_\nu}{dt^k}(t)$. This finishes the proof of this corollary. □

The next assertion is related to the fractional differentiation of a product, the so-called one variant of the Leibniz rule in the case of fractional derivatives.

Corollary 2. *Let $w_1 \in \mathcal{C}^1([0, T])$ and $w_2 \in \mathcal{C}([0, T])$. For $\theta \in (0, 1)$, we assume that*

(i) *either $\mathbf{D}_t^\theta w_2 \in \mathcal{C}([0, T])$,*

(ii) *or $\mathbf{D}_t^\theta w_2 \in L_p(0, T)$ with $p \geq 2$.*

Then the equality

$$\mathbf{D}_t^{\theta}(w_1 w_2) = w_1(t)\mathbf{D}_t^{\theta} w_2(t) + w_2(0)\mathbf{D}_t^{\theta} w_1(t) + \frac{\theta}{\Gamma(1-\theta)} \int_0^t \frac{[w_1(t) - w_1(s)][w_2(s) - w_2(0)]}{(t-s)^{1+\theta}} ds \tag{13}$$

holds, and $\mathbf{D}_t^{\theta}(w_1 w_2)$ *has the regularity*

$$\mathbf{D}_t^{\theta}(w_1 w_2) \in \begin{cases} \mathcal{C}([0,T]) & \textit{in the case of } \textbf{(i)}, \\ L_p(0,T) & \textit{in the case of } \textbf{(ii)}. \end{cases}$$

If, in addition, $w_2(0) = 0$*, then for any* $\theta_1 \geq \theta$ *and all* $t \in [0,T]$*, the equality holds*

$$I_t^{\theta_1}(w_1 \partial_t^{\theta} w_2)(t) = I_t^{\theta_1 - \theta}(w_1 w_2)(t) - \theta I_t^{1-\theta+\theta_1}(\mathcal{W}(w_1) w_2)(t) \tag{14}$$

with $\mathcal{W}(w_1) = \int_0^1 \frac{\partial w_1}{\partial z} d\tau$, $z = \tau t + (1-\tau)s$, $0 < s < t$.

Proof. First of all, we remark that under a stronger regularity on the function w_2, representation (13) was proved in Corollary 3.1 [37]. Here, we just extend this result to the case of a weaker assumption on the w_2. Namely, we require that $\mathbf{D}_t^{\theta} w_2$ belongs to either $L_p(0,T)$ or $\mathcal{C}([0,T])$.

Appealing to the definition of the Caputo fractional derivative and taking into account the smoothness of functions w_1 and w_2, we easily conclude that

$$\mathbf{D}_t^{\theta}(w_1 w_2) = w_1(t)\mathbf{D}_t^{\theta} w_2(t) + w_2(0)\mathbf{D}_t^{\theta} w_1(t) + \frac{1}{\Gamma(1-\theta)} w_1'(t) \int_0^t \frac{[w_2(s) - w_2(0)]}{(t-s)^{\theta}} ds$$
$$- \frac{1}{\Gamma(1-\theta)} \int_0^t [w_2(s) - w_2(0)] \frac{\partial}{\partial t} \frac{[w_1(t) - w_1(s)]}{(t-s)^{\theta}} ds.$$

After that, performing differentiation in the last integral arrives at the desired equality. Coming to the smoothness of the function $\mathbf{D}_t^{\theta}(w_1 w_2)$, it is a simple consequence of the obtained representation (13) and the regularity of w_1 and w_2.

Obviously, relation (14) is a simple consequence of (13) and (11). Indeed, in virtue of $w_2(x,0) = 0$, we can rewrite (13) as

$$w_1(t)\partial_t^{\theta} w_2 = \partial_t^{\theta}(w_1 w_2) - \theta I_t^{1-\theta}(\mathcal{W}(w_1) w_2).$$

Finally, computing the fractional integral $I_t^{\theta_1}$ of both sides in this equality and taking into account Proposition 2.2 in [2] and semigroup property to the fractional Riemann-Liouville integral, we end up with (14). This completes the verification of this corollary. □

We now state and prove some inequalities that will be needed to prove estimate (8) in Section 6.2. First, we introduce the function

$$\mathfrak{I}_{\theta}(t) = \mathfrak{I}_{\theta}(t; w_1, w_2) = \int_0^t \frac{[w_1(x,t) - w_1(x,s)][w_2(x,s) - w_2(x,0)]}{(t-s)^{1+\theta}} ds, \tag{15}$$

where $\theta \in (0,1)$, w_1 and w_2 are some given functions whose smoothness provides the boundedness of the singular integral in (15).

Lemma 2. *Let arbitrarily fixed* $T > 0$, $p \geq 2$*, and* $\theta \in (0,1)$*. We assume that* $w_1 \in \mathcal{C}^1(\bar{\Omega}_T)$*,* $w_2 \in L_p(\Omega_T)$ *and* $\mathcal{K} \in \mathcal{C}([0,T])$*. Then, there are the following inequalities:*

(i)

$$\|\mathcal{K}*w_2\|_{L_p(\Omega_T)} \leq C_Y T\|\mathcal{K}\|_{\mathcal{C}([0,T])}\|w_2\|_{L_p(\Omega_T)},$$
$$\|\omega_\theta * w_2\|_{L_p(\Omega_T)} \leq C_Y \frac{T^\theta}{\Gamma(1+\theta)}\|w_2\|_{L_p(\Omega_T)},$$
$$\|\mathcal{K}*\omega_\theta * w_2\|_{L_p(\Omega_T)} \leq C_Y \frac{T^{1+\theta}}{\Gamma(2+\theta)}\|\mathcal{K}\|_{\mathcal{C}([0,T])}\|w_2\|_{L_p(\Omega_T)},$$

where C_Y is the positive constant in the Young inequality for a convolution (see [38]).

(ii)

$$\|\mathfrak{I}_\theta(t)\|_{L_p(\Omega_T)} + \|\mathcal{K}*\mathfrak{I}_\theta(t)\|_{L_p(\Omega_T)}$$
$$\leq C_Y[1+C_Y T\|\mathcal{K}\|_{\mathcal{C}([0,T])}]\frac{T^{1-\theta}}{(1-\theta)}\|w_1\|_{\mathcal{C}^1(\bar{\Omega}_T)}\|W_2\|_{L_p(\Omega_T)},$$

where $W_2(x,t) = w_2(x,t) - w_2(x,0)$.

(iii) *If, for any $\theta_1 \in (0,\theta)$, $\theta_2 \in (\theta_1,\theta]$ and $p > \frac{1}{\theta_2-\theta_1}$, we additionally assume that $w_2 \in \mathfrak{H}_p^{\theta,2}(\Omega_T)$ and $w_2(x,0) = 0$ for all $x \in \bar{\Omega}$. Then, for any small $\varepsilon \in (0,1)$, the estimates hold:*

$$\|\partial_t^{\theta_1} w_2\|_{L_p(\Omega_T)} \leq C_\theta T^{\theta_2-\theta_1}\|w_2\|_{W^{\theta_2,p}((0,T),L_p(\Omega))},$$
$$\|w_2\|_{L_p(\Omega_T)} + \|\mathcal{K}*w_2\|_{L_p(\Omega_T)} \leq C_Y C_\theta[1+C_Y T\|\mathcal{K}\|_{\mathcal{C}([0,T])}]T^\theta\|w_2\|_{W^{\theta,p}((0,T),L_p(\Omega))},$$
$$\|D_x w_2\|_{L_p(\Omega_T)} + \|\mathcal{K}*D_x w_2\|_{L_p(\Omega_T)} \leq \varepsilon[1+C_Y T\|\mathcal{K}\|_{\mathcal{C}([0,T])}]\|w_2\|_{L_p((0,T),W^{2,p}(\Omega))}$$
$$+\mathfrak{C}T^\theta\|w_2\|_{W^{\theta,p}((0,T),L_p(\Omega))}.$$

Here

$$\mathfrak{C} = C_{GN}(1+\varepsilon^{-1})\frac{4^{p-1}C_\theta C_Y}{\Gamma(1+\theta_1)}[1+C_Y T\|\mathcal{K}\|_{\mathcal{C}([0,T])}]$$

and C_{GN} is the constant in the Gagliardo–Nirenberg inequality.

Proof. The inequalities in point (i) are verified with straightforward calculations, where we exploit the Young inequality for a convolution and relations in (iv) of Proposition 1.

Concerning point (ii), this estimate is a simple consequence of the easily verified inequality

$$|\mathfrak{I}_\theta(t)| \leq \|w_1\|_{\mathcal{C}^1(\bar{\Omega}_T)}\int_0^t \frac{|w_2(x,s)-w_2(x,0)|}{(t-s)^\theta}ds,$$

and the Young inequality for a convolution.

As for the verification of the first inequality in (iii), bearing in mind restrictions on p, θ_1, θ_2, and appealing to the embedding Theorem (see (1.4.4.6) in [35]), we conclude that $w_2 \in \mathcal{C}^{\theta_2-\frac{1}{p}}([0,T],L_p(\Omega))$ and

$$\|w_2\|_{\mathcal{C}^{\theta_2-\frac{1}{p}}([0,T],L_p(\Omega))} \leq C\|w_2\|_{W^{\theta_2,p}((0,T),L_p(\Omega))}. \tag{16}$$

Collecting (15), (16) with the homogeneous initial data of w_2 and (15) allows us to apply Theorem 3.1 [37] and deduce the equality

$$\partial_t^{\theta_1} w_2 = \frac{1}{\Gamma(1-\theta_1)}\frac{w_2(x,t)-w_2(x,0)}{t^{\theta_1}} + \frac{\theta_1}{\Gamma(1-\theta_1)}\int_0^t \frac{w_2(x,t)-w_2(x,\tau)}{(t-\tau)^{1+\theta_1}}d\tau.$$

Next, taking advantage of this representation to compute L_p- norm of $\partial_t^{\theta_1} w_2$ and performing standard technical calculations, we have

$$\|\partial_t^{\theta_1} w_2\|_{L_p(\Omega_T)} \leq C\|w_2\|_{\mathcal{C}^{\theta_2-\frac{1}{p}}([0,T],L_p(\Omega))}\Bigg[\bigg(\int_0^T t^{(\theta_2-\theta_1)p-1}dt\bigg)^{\frac{1}{p}} + \bigg(\int_0^T\bigg(\int_0^t (t-\tau)^{\theta_2-\theta_1-\frac{1}{p}-1}d\tau\bigg)^p dt\bigg)^{\frac{1}{p}}\Bigg].$$

Finally, straightforward calculations and inequality (16) arrive at the desired bound.

Consecutive application of formula (3.5.4) in [2] to the difference $|w_2(x,s)-w_2(x,0)|$, Young inequality for a convolution and, finally, the first inequality in (iii) of this claim provides the estimate

$$\begin{aligned}\|w_2\|_{L_p(\Omega_T)} &\leq C_Y\|\partial_t^{\theta_1} w_2\|_{L_p(\Omega_T)}\|\omega_{\theta_1}\|_{L_1(0,T)}\\ &\leq C_\theta C_Y\frac{T^\theta}{\Gamma(1+\theta_1)}\|w_2\|_{W^{\theta,p}((0,T),L_p(\Omega))}\end{aligned} \tag{17}$$

for any $\theta_1\in(0,\theta)$. Finally, using this bound and the Young inequality to manage the term $\|\mathcal{K}*w_2\|_{L_p(\Omega_T)}$, we arrive at the first estimate in (iii).

Coming to the second inequality in (iii), the Gagliardo–Nirenberg and Cauchy inequalities lead to the bound

$$\|D_x w_2\|_{L_p(\Omega)} \leq \varepsilon_1\|D_x^2 w_2\|_{L_p(\Omega)} + C_{GN}[1+\varepsilon_1^{-1}]\|w_2\|_{L_p(\Omega)}$$

for any $\varepsilon_1\in(0,1)$, which together with Jensen's inequality to a sum, in turn, provides

$$\|D_x w_2\|_{L_p(\Omega_T)} \leq 2^{p-1}\varepsilon_1\|D_x^2 w_2\|_{L_p(\Omega_T)} + 2^{p-1}C_{GN}[1+\varepsilon_1^{-1}]\|w_2\|_{L_p(\Omega_T)}.$$

After that, choosing $\varepsilon=\varepsilon_1 2^{p-1}<1$ and applying (17) to control the second term in the right-hand side of the inequality above, we immediately end up with

$$\|D_x w_2\|_{L_p(\Omega_T)} \leq \varepsilon\|w_2\|_{L_p((0,T),W^{2,p}(\Omega))} + 4^{p-1}[1+\varepsilon^{-1}]\frac{C_{GN}C_\theta C_Y T^\theta}{\Gamma(1+\theta_1)}\|w_2\|_{W^{\theta,p}((0,T),L_p(\Omega))}.$$

Finally, collecting this estimate with the first inequality in (i) yields the estimate in (iii). This completes the proof of Lemma 2. □

Remark 8. *It is worth noting that repeating the arguments leading to the first bound in (iii) of Lemma 2 arrives at the following inequalities in the case of $w_2(x,0)\neq 0$:*

$$\begin{aligned}\|\mathbf{D}_t^{\theta_1} w_2\|_{L_p(\Omega_T)} &\leq C_\theta T^{\theta_2-\theta_1}\|w_2\|_{W^{\theta_2,p}((0,T),L_p(\Omega))},\\ \|\partial_t^{\theta_1} W_2\|_{L_p(\Omega_T)} &\leq C_\theta T^{\theta_2-\theta_1}\|W_2\|_{W^{\theta_2,p}((0,T),L_p(\Omega))}\end{aligned}$$

with W_2 being defined in (ii) of Lemma 2.

Finally, for convenience, we remind the reader of the result related to the global classical solvability of the linear problem corresponding to (4)–(6). The result, written as a lemma, is obtained in our previous work [32] (see Theorem 4.1 there) and will be exploited in Section 7 to prove the one-valued solvability of the nonlinear model (4)–(6).

Lemma 3. *Let $T>0$ be any fixed, $\partial\Omega\in\mathcal{C}^{2+\alpha}$, $g(u)\equiv 0$, and let $\nu_0\in(0,1)$, $\nu_1\in(0,\nu_0(2-\alpha)/2)$. We assume that assumptions **H2**–**H5** hold. Then, linear equation (4) with the initial condition (5),* *subject to the Dirichlet boundary condition* (6)*, has a unique classical solution* $u = u(x,t)$ *in* $\bar{\Omega}_T$*, possessing the regularity* $u \in \mathcal{C}^{2+\alpha, \frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)$*. This solution fulfills the estimate*

$$\|u\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)} + \|\mathbf{D}_t^{\nu_1} u\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\nu_0}(\bar{\Omega}_T)} \leq C[\|f\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\nu_0}(\bar{\Omega}_T)} + \|u_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})} + \|\psi\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\partial\Omega_T)}].$$

The generic constant C is independent of the right-hand sides of (4)–(6).

6. Proof of Theorem 1

We start the proof of Theorem 1 with estimating a solution in the space $\mathfrak{H}_p^{\nu_0,2}(\Omega_T)$. On this route, we collect certain results from Section 5.1 in [18] and from Sections 2 and 4 in [23]. The second part of this Section is devoted to obtaining the estimate of u in the Hölder spaces.

6.1. Estimate of $\|u\|_{\mathfrak{H}_p^{\nu_0,2}(\Omega_T)}$

We first study in detail the special case where $\psi \equiv 0$ and $u_0 \equiv 0$, i.e., (5) and (6) are replaced by the simpler conditions

$$u(x,0) = 0 \quad \text{in} \quad \bar{\Omega}, \qquad u(x,t) = 0 \quad \text{on} \quad \partial\Omega_T. \tag{18}$$

In a farther step, we will discuss how to come from the general case to this special one. Here, we will follow the strategy consisting in two main steps. The first one is related to obtaining the estimate in a small time interval $[0, T_1]$. On the second step, we discuss the extension of this estimate to the interval $[T_1, T]$.

Step 1. Let $T_1 \in (0, T]$ be specified below. Keeping in mind the regularity of u, ϱ_0 and ϱ_1 (see assumption **H3**), we apply Corollary 2 to the first two terms in the left-hand side of (4) and rewrite this equation in more suitable form:

$$\varrho_0\mathbf{D}_t^{\nu_0}u - \varrho_1\mathbf{D}t^{\nu_1}u - \mathcal{L}_1 u - \mathcal{K} * \mathcal{L}_2 u + \frac{\nu_0}{\Gamma(1-\nu_0)}\mathfrak{I}_{\nu_0}(t;\varrho_0,u)$$
$$- \frac{\nu_1}{\Gamma(1-\nu_1)}\mathfrak{I}_{\nu_1}(t;\varrho_1,u) = f(x,t).$$

After that, Proposition 4.4 in [18], where we set

$$\begin{aligned}
&\bar{w} = -\mathcal{L}_0 u,\\
&w = f - \varrho_0\mathbf{D}_t^{\nu_0}u + \varrho_1\mathbf{D}_t^{\nu_1}u + (\mathcal{L}_1 - \mathcal{L}_0)u + \mathcal{K} * (\mathcal{L}_2 - \mathcal{L}_0)u\\
&\quad - \frac{\nu_0}{\Gamma(1-\nu_0)}\mathfrak{I}_{\nu_0}(t;\varrho_0,u) + \frac{\nu_1}{\Gamma(1-\nu_1)}\mathfrak{I}_{\nu_1}(t;\varrho_1,u),
\end{aligned}$$

and the point (iii) of Proposition 1 allow us to remove the term $\mathcal{K} * \mathcal{L}_0 u$ from the equation above. Hence, we have

$$\mathbf{D}_t^{\nu_0}u - \frac{1}{\varrho_0}\mathcal{L}_0 u + \sum_{ij=1}^{n}\frac{1}{\varrho_0}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} = \sum_{l=0, l\neq 1}^{8}\frac{1}{\varrho_0}\mathfrak{F}_l(u) + \frac{1}{\varrho_0}\mathfrak{F}_1(f), \tag{19}$$

where denoting the conjugate kernel to $\mathcal{K}$ by $\bar{\mathcal{K}} \in \mathcal{C}^1([0,T])$ (see its properties in Proposition 4.4 [18]), we put

$$\mathfrak{F}_0(u) = \varrho_1\mathbf{D}_t^{\nu_1}u, \qquad \mathfrak{F}_1(f) = f - \bar{\mathcal{K}} \star f, \qquad \mathfrak{F}_2(u) = (\mathcal{L}_1 - \mathcal{L}_0)u + \sum_{ij=1}^{n}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j},$$
$$\mathfrak{F}_3(u) = -\bar{\mathcal{K}} * (\mathcal{L}_1 - \mathcal{L}_0)u, \qquad \mathfrak{F}_4(u) = \bar{\mathcal{K}} * (\mathcal{L}_2 - \mathcal{L}_0)u,$$
$$\mathfrak{F}_5(u) = -\frac{\nu_0}{\Gamma(1-\nu_0)}\mathfrak{I}_{\nu_0}(t;\varrho_0,u) + \frac{\nu_0}{\Gamma(1-\nu_0)}\bar{\mathcal{K}} * \mathfrak{I}_{\nu_0}(t;\varrho_0,u),$$

$$\mathfrak{F}_6(u) = \frac{\nu_1}{\Gamma(1-\nu_1)}\mathfrak{I}_{\nu_1}(t;\varrho_1,u) - \frac{\nu_1}{\Gamma(1-\nu_1)}\bar{\mathcal{K}} * \mathfrak{I}_{\nu_1}(t;\varrho_1,u),$$
$$\mathfrak{F}_7(u) = \bar{\mathcal{K}}(0)\varrho_0(\omega_{1-\nu_0} * u) - \bar{\mathcal{K}} * \frac{\partial\varrho_0}{\partial t}\Big(\omega_{1-\nu_0} * u\Big) + \bar{\mathcal{K}}' * \varrho_0(\omega_{1-\nu_0} * u),$$
$$\mathfrak{F}_8(u) = -\bar{\mathcal{K}}(0)\varrho_1(\omega_{1-\nu_1} * u) + \bar{\mathcal{K}} * \frac{\partial\varrho_1}{\partial t}\Big(\omega_{1-\nu_1} * u\Big) - \bar{\mathcal{K}}' * \varrho_1(\omega_{1-\nu_1} * u).$$

Then, bearing in mind assumptions **H2** and **H3**, we take advantage of Theorem 2.3 in [23] to problem (18), (19) and conclude that

$$\|u\|_{L_p((0,T_1),W^{2,p}(\Omega))} + \|u\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))} \leq C_0 \sum_{l=0}^{8} \|\varrho_0^{-1}\mathfrak{F}_l\|_{L_p(\Omega_{T_1})}, \tag{20}$$

where the positive constant C_0 depends only on ν_0, n, p, the Lebesgue measure of Ω, T and the norms of the coefficients a_{ij} and ϱ_0, ϱ_1.

At this point, setting $\mathfrak{F}_1 = \mathfrak{F}_1(f)$ and $\mathfrak{F}_l = \mathfrak{F}_l(u)$, we examine each term in the right-hand side of (20), separately.

• Keeping in mind assumptions **H2, H3** and the homogeneous initial condition, we can exploit estimates in (iii) of Lemma 2 (with $\theta_1 = \nu_1$, $\theta = \nu_0$, $C_\theta = C_\nu$) and deduce the estimates

$$\begin{aligned}\|\varrho_0^{-1}\mathfrak{F}_0\|_{L_p(\Omega_{T_1})} &\leq \frac{\|\varrho_1\|_{\mathcal{C}(\bar{\Omega}_T)}}{\mu_0}\|\partial_t^{\nu_1}u\|_{L_p(\Omega_{T_1})} \\ &\leq \frac{C_\nu\|\varrho_1\|_{\mathcal{C}(\bar{\Omega}_T)}}{\mu_0}\|u\|_{W^{\nu_1+\nu_2,p}((0,T_1),L_p(\Omega))}\end{aligned}$$

with a some quantity ν_2 satisfying inequalities $0 < \nu_1 + \nu_2 < \nu_0$.

After that, standard calculations and estimate (17) lead to

$$\|u\|_{W^{\nu_1+\nu_2,p}((0,T_1),L_p(\Omega))} \leq \left[\frac{C_\nu C_Y T_1^{\nu_0}}{\Gamma(1+\nu_0)} + T_1^{\nu_0-\nu_1-\nu_2}\right]\|u\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))}.$$

Collecting all these bounds, we end up with

$$\left\|\varrho_0^{-1}\mathfrak{F}_0\right\|_{L_p(\Omega_{T_1})} \leq C_1 T_1^{\nu_3}\|u\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))},$$

where

$$\nu_3 = \nu_0 - \nu_1 - \nu_2, \qquad C_1 = \frac{\|\varrho_1\|_{\mathcal{C}(\bar{\Omega}_T)}}{\mu_0}\left\{\frac{C_\nu C_Y T^{\nu_1+\nu_2}}{\Gamma(1+\nu_0)} + 1\right\}.$$

• Obviously, the Young inequality for a convolution and assumptions **(H2), (H4)** provide

$$\|\varrho_0^{-1}\mathfrak{F}_1\|_{L_p(\Omega_{T_1})} \leq \mu_0^{-1}[1 + T\|\mathcal{K}\|_{\mathcal{C}([0,T])}]\|f\|_{L_p(\Omega_{T_1})}.$$

Here, we appeal to the fact that $\bar{\mathcal{K}}$ is the conjugate kernel to $\mathcal{K}$ (see Proposition 4.4 in [18]), and thus

$$\|\bar{\mathcal{K}}\|_{\mathcal{C}([0,T])} \leq C\|\mathcal{K}\|_{\mathcal{C}([0,T])}. \tag{21}$$

• As for terms $\mathfrak{F}_l$, $l = 2, 3, 4$, they are examined with inequalities in (iii) of Lemma 2 and the bound (21). Hence, we have for any small $\varepsilon \in (0, 1)$ (which will be specified below)

$$\left\|\varrho_0^{-1}\sum_{l=2}^{4}\mathfrak{F}_l\right\|_{L_p(\Omega_{T_1})} \leq \mu_0^{-1}\Bigg[\sum_{i=1}^{n}\left(\|a_i\|_{\mathcal{C}(\bar{\Omega}_T)}+\|b_i\|_{\mathcal{C}(\bar{\Omega}_T)}+\sum_{j=1}^{n}\left\|\frac{\partial a_{ij}}{\partial x_i}\right\|_{\mathcal{C}(\bar{\Omega}_T)}\right)$$
$$+\|a_0\|_{\mathcal{C}(\bar{\Omega}_T)}+\|b_0\|_{\mathcal{C}(\bar{\Omega}_T)}\Bigg]$$
$$\times\left[\sum_{j=1}^{n}\left(\left\|\frac{\partial u}{\partial x_j}\right\|_{L_p(\Omega_{T_1})}+\left\|\bar{K}*\frac{\partial u}{\partial x_j}\right\|_{L_p(\Omega_{T_1})}\right)\right.$$
$$\left.+\|u\|_{L_p(\Omega_{T_1})}+\|\bar{K}*u\|_{L_p(\Omega_{T_1})}\right]$$
$$\leq C_2T_1^{\nu_0}\|u\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))}+C_3\varepsilon\|u\|_{L_p((0,T_1),W^{2,p}(\Omega))},$$

where

$$C_2=\mu_0^{-1}\left[\|a_0\|_{\mathcal{C}(\bar{\Omega}_T)}+\|b_0\|_{\mathcal{C}(\bar{\Omega}_T)}+\sum_{i=1}^{n}\left(\|a_i\|_{\mathcal{C}(\bar{\Omega}_T)}+\|b_i\|_{\mathcal{C}(\bar{\Omega}_T)}+\sum_{j=1}^{n}\left\|\frac{\partial a_{ij}}{\partial x_i}\right\|_{\mathcal{C}(\bar{\Omega}_T)}\right)\right]$$
$$\times\left[\mathfrak{C}n+(1+C_YT\|\mathcal{K}\|_{\mathcal{C}([0,T])})\frac{C_\nu C_Y}{\Gamma(1+\nu_0)}\right],$$
$$C_3=\frac{n}{\mu_0}\left[\|a_0\|_{\mathcal{C}(\bar{\Omega}_T)}+\|b_0\|_{\mathcal{C}(\bar{\Omega}_T)}+\sum_{i=1}^{n}\left(\|a_i\|_{\mathcal{C}(\bar{\Omega}_T)}+\|b_i\|_{\mathcal{C}(\bar{\Omega}_T)}+\sum_{j=1}^{n}\left\|\frac{\partial a_{ij}}{\partial x_i}\right\|_{\mathcal{C}(\bar{\Omega}_T)}\right)\right]$$
$$\times(1+C_YT\|\mathcal{K}\|_{\mathcal{C}([0,T])}).$$

- Concerning the terms $\frac{\mathfrak{F}_5}{\varrho_0}$ and $\frac{\mathfrak{F}_6}{\varrho_0}$, we apply the point (ii) of Lemma 2 and (21) to deduce

$$\left\|\sum_{l=5}^{6}\varrho_0^{-1}\mathfrak{F}_l\right\|_{L_p(\Omega_{T_1})}\leq\frac{C_YT_1^{1-\nu_0}}{\mu_0}\left[\frac{\nu_0\|\varrho_0\|_{\mathcal{C}^1(\bar{\Omega}_T)}}{\Gamma(2-\nu_0)}+\frac{\nu_1\|\varrho_1\|_{\mathcal{C}^1(\bar{\Omega}_T)}}{\Gamma(2-\nu_1)}T^{\nu_0-\nu_1}\right]$$
$$\times[1+C_YT\|\mathcal{K}\|_{\mathcal{C}([0,T])}]\|u\|_{L_p(\Omega_{T_1})}.$$

Then, to estimate $\|u\|_{L_p(\Omega_{T_1})}$, we use the first inequality in (iii) of Lemma 2 and arrive at

$$\left\|\sum_{l=5}^{6}\varrho_0^{-1}\mathfrak{F}_l\right\|_{L_p(\Omega_{T_1})}\leq C_4T_1\|u\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))}$$

with the positive constant

$$C_4=\frac{C_Y^2C_\nu}{\mu_0\Gamma(1+\nu_0)}\left[\frac{\nu_0\|\varrho_0\|_{\mathcal{C}^1(\bar{\Omega}_T)}}{\Gamma(2-\nu_0)}+\frac{\nu_1\|\varrho_1\|_{\mathcal{C}^1(\bar{\Omega}_T)}}{\Gamma(2-\nu_1)}T^{\nu_0-\nu_1}\right][1+C_YT\|\mathcal{K}\|_{\mathcal{C}([0,T])}]^2.$$

- Finally, the term $\frac{\mathfrak{F}_7+\mathfrak{F}_8}{\varrho_0}$ is evaluated via (i) and (iii) of Lemma 2. Thus, we obtain

$$\left\|\sum_{l=7}^{8}\varrho_0^{-1}\mathfrak{F}_l\right\|_{L_p(\Omega_{T_1})}\leq C_5T_1\|u\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))},$$

where

$$C_5=\frac{C_Y^2C_\nu\|\mathcal{K}\|_{\mathcal{C}^1([0,T])}}{\Gamma(1+\nu_0)}\left[1+\frac{\|\varrho_1\|_{\mathcal{C}^1(\bar{\Omega}_T)}+\|\varrho_0\|_{\mathcal{C}^1(\bar{\Omega}_T)}}{\mu_0}\right][1+C_YT\|\mathcal{K}\|_{\mathcal{C}([0,T])}]$$
$$\times\left[\frac{1}{\Gamma(2-\nu_0)}+\frac{T^{\nu_0-\nu_1}}{\Gamma(2-\nu_1)}+\frac{T}{\Gamma(3-\nu_0)}+\frac{T^{1+\nu_0-\nu_1}}{\Gamma(3-\nu_1)}\right].$$

Collecting relation (20) with estimates of $\mathfrak{F}_l$, and choosing ε and T_1 satisfying inequalities

$$\varepsilon = \frac{1}{2C_0C_3} \quad \text{and} \quad C_0(T_1^{\nu_3}C_1 + T_1^{\nu_0}C_2 + C_4T_1 + C_5T_1) < \frac{1}{2},$$

we end up with bounds

$$\begin{aligned} \|u\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))} + \|u\|_{L_p((0,T_1),W^{2,p}(\Omega))} &\leq \frac{2C_0}{\mu_0}[1 + T\|\mathcal{K}\|_{\mathcal{C}([0,T])}]\|f\|_{L_p(\Omega_{T_1})} \\ &\leq C_6\|f\|_{L_p(\Omega_{T_1})} \leq C_6\|f\|_{L_p(\Omega_T)}, \\ \|u\|_{\mathcal{C}(\bar{\Omega}_{T_1})} &\leq C_6\|f\|_{L_p(\Omega_{T_1})} \leq C_6\|f\|_{L_p(\Omega_T)}. \end{aligned} \tag{22}$$

It is worth noting that, in light of the relation between p, ν_0 and n, the second estimate is a simple consequence of the first inequality in (22) and the embedding Theorem (see Theorem 1.4.33 and (1.4.4.6) in [35] and also p. 818 in [23]).

Step 2: Extension of estimate (22) *to whole time interval.* First of all, we discuss the technique which allows us to extend estimate (22) to interval $[0, 3T_1/2]$. After that, we recast this procedure a finite number of times until the entire $[0, T]$ is exhausted. Hence, estimating the first term in the left-hand side of (8) is completed under additional assumption (18). To this end, we introduce a new function

$$\Phi(x,t) = \begin{cases} \varrho_0\mathbf{D}_t^{\nu_0}u - \mathcal{L}_1u, & \text{if} \quad (x,t) \in \Omega_{T_1/2}, \\ [\varrho_0\mathbf{D}_t^{\nu_0}u - \mathcal{L}_1u]|_{t=T_1/2}, & \text{if} \quad x \in \Omega,\, t \geq T_1/2, \end{cases} \tag{23}$$

and define the function $\mathcal{U}(x,t)$ which solves the initial-boundary value problem

$$\begin{cases} \varrho_0\mathbf{D}_t^{\nu_0}\mathcal{U} - \mathcal{L}_1\mathcal{U} = \Phi(x,t) \quad \text{in} \quad \Omega_{3T_1/2}, \\ \mathcal{U}(x,0) = 0 \quad \text{in} \quad \bar{\Omega}, \\ \mathcal{U}(x,t) = 0 \quad \text{on} \quad \partial\Omega_{3T_1/2}. \end{cases}$$

Bearing in mind that the function $u(x,t)$ solves problem (4), (18) and satisfies estimate (22) for $t \in [0, T_1]$, we exploit Theorem 2.3 [23] and Lemma 2, and assert the following result.

Corollary 3. *The following relations hold:*

$$\begin{aligned} \mathcal{U}(x,t) = u(x,t) \quad &\textit{in} \quad \bar{\Omega}_{T_1/2}, \\ \|\Phi\|_{L_p(\Omega_{3T_1/2})} &\leq C_6\|f\|_{L_p(\Omega_T)}, \\ \|\mathcal{U}\|_{W^{\nu_0,p}((0,3T_1/2),L_p(\Omega))} + \|\mathcal{U}\|_{L_p((0,3T_1/2),W^{2,p}(\Omega))} &\leq C_6\|f\|_{L_p(\Omega_T)}, \\ \|\mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{U}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{U}) - \mathcal{L}_1\mathcal{U} - \mathcal{K} * \mathcal{L}_2\mathcal{U}\|_{L_p(\Omega_{3T_1/2})} &\leq C\|f\|_{L_p(\Omega_T)} \end{aligned}$$

with constants C and C_6 being independent of T_1.

Finally, introducing new unknown function

$$\mathfrak{U} = u(x,t) - \mathcal{U}(x,t),$$

and then the new time variable

$$\sigma = t - T_1/2, \qquad \sigma \in [-T_1/2, T_1] \quad \text{for} \quad t \in [0, 3T_1/2],$$

in problem (4), (18), we recast arguments of Section 6.3 in [32] and deduce

$$
\begin{cases} \mathbf{D}_{\sigma}^{\nu_0}(\bar{\varrho}_0\bar{\mathfrak{U}}) - \mathbf{D}_{\sigma}^{\nu_1}(\bar{\varrho}_1\bar{\mathfrak{U}}) - \bar{\mathcal{L}}_1\bar{\mathfrak{U}} - \mathcal{K} * \bar{\mathcal{L}}_2\bar{\mathfrak{U}} = f^{\star}(x,\sigma) \quad \text{in} \quad \Omega_{T_1}, \\ \bar{\mathfrak{U}}(x,0) = 0, \qquad \text{in} \quad \bar{\Omega}, \\ \bar{\mathfrak{U}}(x,\sigma) = 0, \qquad \text{on} \quad \bar{\Omega}_{T_1}, \end{cases} \tag{24}
$$

and, besides, $\bar{\mathfrak{U}}(x,\sigma) = 0$, $\sigma \in [-T_1/2, 0]$.

Here, we put

$$
\begin{aligned}
\bar{\mathfrak{U}}(x,\sigma) &= \mathfrak{U}(x,\sigma+T_1/2), \quad \bar{\varrho}_0(x,\sigma) = \varrho_0(x,\sigma+T_1/2),\\
\bar{\varrho}_1(x,\sigma) &= \varrho_1(x,\sigma+T_1/2), \quad \bar{a}_{ij}(x,\sigma) = a_{ij}(x,\sigma+T_1/2),\\
\bar{a}_i(x,\sigma) &= a_i(x,\sigma+T_1/2), \quad \bar{a}_0(x,\sigma) = a_0(x,\sigma+T_1/2),\\
\bar{b}_i(x,\sigma) &= b_i(x,\sigma+T_1/2), \quad \bar{b}_0(x,\sigma) = b_0(x,\sigma+T_1/2),\\
f^{\star}(x,\sigma) &= f(x,\sigma+T_1/2) - \bar{f}(x,\sigma+T_1/2), \quad \text{where}\\
\bar{f}(x,\sigma+T_1/2) &= \{\mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{U}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{U}) - \mathcal{L}_1\mathcal{U} - \mathcal{K} * \mathcal{L}_2\mathcal{U}\}|_{t=\sigma+T_1/2},
\end{aligned}
$$

and we call $\bar{\mathcal{L}}_i$ the operators $\mathcal{L}_i$ with the bar coefficients. It is easy to verify that the coefficients of the operators $\bar{\mathcal{L}}_i$ and $\bar{\varrho}_0$, $\bar{\varrho}_1$ meet the requirements of Theorem 1.

Then, recasting the arguments leading to (22) (see *Step 1* in this subsection), we deduce

$$
\|\bar{\mathfrak{U}}\|_{L_p((0,T_1),W^{2,p}(\Omega))} + \|\bar{\mathfrak{U}}\|_{W^{\nu_0,p}((0,T_1),L_p(\Omega))} + \|\bar{\mathfrak{U}}\|_{\mathcal{C}(\bar{\Omega}_{T_1})} \leq C_6[\|f\|_{L_p(\Omega_T)} + \|\bar{f}\|_{L_p(\Omega_{T_1})}].
$$

Collecting this estimate with Corollary 3 and the representation of the function u, we arrive at the inequality

$$
\begin{aligned}
&\|u\|_{L_p((0,3T_1/2),W^{2,p}(\Omega))} + \|u\|_{W^{\nu_0,p}((0,3T_1/2),L_p(\Omega))} + \|u\|_{\mathcal{C}(\bar{\Omega}_{3T_1/2})}\\
&\leq C_6[\|f\|_{L_p(\Omega_T)} + \|\bar{f}\|_{L_p(\Omega_{T_1})}] \leq C\|f\|_{L_p(\Omega_T)},
\end{aligned}
$$

which in turn tells us that we extended inequality (22) from $[0, T_1]$ to $[T_1, 3T_1/2]$. It is worth noting that the constant C in this estimate is independent of T_1.

Thus, we finished the evaluation of the first term in the left hand-side of (8) for $t \in [0, T]$ under assumption (18).

Step 3: Removing restriction (18). To this end, we look for a solution of (4)–(6) in the form

$$
u = u_1 + u_2,
$$

where u_1 solves the Cauchy–Dirichlet problem to the homogeneous subdiffusion equation

$$
\begin{cases} \varrho_0\mathbf{D}_t^{\nu_0}u_1 - \mathcal{L}_1u_1 = 0 \qquad \text{in} \qquad \Omega_T, \\ u_1(x,0) = u_0(x) \qquad \text{in} \quad \bar{\Omega}, \\ u_1(x,t) = \psi(x,t) \qquad \text{on} \quad \partial\Omega_T, \end{cases} \tag{25}
$$

while u_2 is a solution of problem (4), (18) with the new right-hand side in the equation

$$
\mathfrak{f} = f + \mathcal{K} * \mathcal{L}_2u_1 + \varrho_1\mathbf{D}_t^{\nu_1}u_1 - \frac{\nu_0}{\Gamma(1-\nu_0)}\mathfrak{J}_{\nu_0}(t;\varrho_0,u_1) + \frac{\nu_1}{\Gamma(1-\nu_1)}\mathfrak{J}_{\nu_1}(t;\varrho_1,u_1).
$$

After that, applying Theorem 2.3 [23] and Lemma 2 to problem (25) and recasting the arguments leading to estimates of $\mathfrak{F}_0(f)$ and $\mathfrak{F}_l(u)$, $l = 5, 6$, (see *Step 1* in this subsection) provide the following relations:

$$
\begin{aligned}
\|u_1\|_{\mathfrak{H}_p^{\nu_0,2}(\Omega_T)} + \|u_1\|_{\mathcal{C}(\bar{\Omega}_T)} &\leq C[\|u_0\|_{W^{2-\frac{2}{p\nu_0},p}(\Omega)} + \|\psi\|_{\mathfrak{H}_p^{\nu_0(1-\frac{1}{2p}),2-\frac{1}{p}}(\partial\Omega_T)}],\\
\|\mathfrak{f}\|_{L_p(\Omega_T)} &\leq C[\|u_0\|_{W^{2-\frac{2}{p\nu_0},p}(\Omega)} + \|f\|_{L_p(\Omega_T)} + \|\psi\|_{\mathfrak{H}_p^{\nu_0(1-\frac{1}{2p}),2-\frac{1}{p}}(\partial\Omega_T)}].
\end{aligned}
$$

Accordingly, we can repeat the whole argument of *Steps 1–2*, so drawing the estimate of the first term in the left hand-side of (8) to the function u_2. As a byproduct, $u = u_1 + u_2$ is the solution satisfying the corresponding estimate in (8) in the general case

$$\|u\|_{\mathfrak{H}_p^{\nu_0,2}(\Omega_T)} + \|u\|_{\mathcal{C}(\bar{\Omega}_T)} \leq C\{\|f\|_{L_p(\Omega_T)} + \|u_0\|_{W^{2-\frac{2}{p\nu_0},p}(\Omega)} + \|\psi\|_{\mathfrak{H}_p^{\nu_0(1-\frac{1}{2p}),2-\frac{1}{p}}(\partial\Omega_T)}\}. \quad (26)$$

We notice that the bound of $\|\mathbf{D}_t^{\nu_1} u\|_{L_p(\Omega_T)}$ is a simple consequence of estimate (21) and Remark 8.

6.2. Conclusion of the Proof of Theorem 1

To complete the proof of this theorem, we are left to obtain the estimate of the corresponding Hölder seminorms to u. We remark that the verification of this estimate follows from Theorem 4.1 in [23] and (26).

For this purpose, it is enough to examine the initial-boundary value problem in the case of homogeneous initial and boundary conditions (18). Namely, in order to convert the general case to this special one, we repeat arguments of *Step 3* in Section 6.1 and take advantage of Theorem 4.1 in [23]. Coming to problem (4), (18), we again rewrite equation (4) in the form of (19) and then apply Theorem 4.1 [23] with $p = q = r$ to problem (19), (18). Thus, we obtain

$$\begin{aligned}\langle u\rangle^{(\alpha)}_{x,\Omega_T} + \langle u\rangle^{(\alpha\nu_0/2)}_{t,\Omega_T} &\leq C\Big[\sum_{l=0}^{8}\|\varrho_0^{-1}\mathfrak{F}_l\|_{L_p(\Omega_T)} + \|u\|_{\mathcal{C}(\bar{\Omega}_T)}\Big]\\ &\leq C\Big[\sum_{l=0}^{8}\|\varrho_0^{-1}\mathfrak{F}_l\|_{L_p(\Omega_T)} + \|f\|_{L_p(\Omega_T)}\Big].\end{aligned} \quad (27)$$

Here, to control the term $\|u\|_{\mathcal{C}(\bar{\Omega}_T)}$, we use (26) with $u_0 = \psi = 0$. Then, to evaluate $\sum_{l=0}^{8}\|\varrho_0^{-1}\mathfrak{F}_l\|_{L_p(\Omega_T)}$, we recast the arguments of *Step 1* of Section 6.1 with $T_1 = T$ and then exploit (26). Thus, we end up with

$$\begin{aligned}\sum_{l=0}^{8}\|\varrho_0^{-1}\mathfrak{F}_l\|_{L_p(\Omega_T)} &\leq C\{\|u\|_{L_p((0,T),W^{2,p}(\Omega))} + \|u\|_{W^{\nu_0,p}((0,T),L_p(\Omega))} + \|f\|_{L_p(\Omega_T)}\}\\ &\leq C\|f\|_{L_p(\Omega_T)}\end{aligned}$$

with the positive C depending only on T, p, ν_0, ν_1, and $|\Omega|$, and the corresponding norms of $\mathcal{K}$ and of the coefficients of the operators $\mathcal{L}_i$.

At last, collecting this estimate with (27) yields

$$\langle u\rangle^{(\alpha)}_{x,\Omega_T} + \langle u\rangle^{(\alpha\nu_0/2)}_{t,\Omega_T} \leq C\|f\|_{L_p(\Omega_T)}.$$

As a result, this inequality and (26) complete the evaluation of the second term in the left-hand side of (8) and, as a consequence, Theorem 1.

7. Proof of Theorem 2

Here, we proceed with a detailed proof of this Theorem in the case of homogeneous initial and boundary conditions, i.e., (18). Indeed, to convert the general case to this special one, we take advantage of Remark 3.1 in [18] and Lemma 3 to the linear model for the unknown function $v = v(x,t) : \Omega_T \to \mathbb{R}$,

$$\begin{cases}\mathbf{D}_t^{\nu_0}(\varrho_0 v) - \mathbf{D}_t^{\nu_1}(\varrho_1 v) - \mathcal{L}_1 v - \mathcal{K} * \mathcal{L}_2 v = f(x,t) + g(u_0) & \text{in} \quad \Omega_T,\\ v(x,t) = \psi(x,t) & \text{on} \quad \partial\Omega_T,\\ v(x,0) = u_0(x) & \text{in} \quad \bar{\Omega},\end{cases}$$

and we obtain the existence of a unique solution $v \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)$ satisfying the bound

$$\begin{aligned}
&\|v\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)} + \|\mathbf{D}_t^{\nu_1} v\|_{\mathcal{C}^{\alpha,\frac{\nu_0\alpha}{2}}(\bar{\Omega}_T)} \\
&\leq C\Big[\|g(u_0)\|_{\mathcal{C}^{\alpha,\frac{\nu_0\alpha}{2}}(\bar{\Omega}_T)} + \|f\|_{\mathcal{C}^{\alpha,\frac{\nu_0\alpha}{2}}(\bar{\Omega}_T)} + \|u_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})} + \|\psi\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\partial\Omega_T)}\Big] \\
&\leq C\Big[1 + \|f\|_{\mathcal{C}^{\alpha,\frac{\nu_0\alpha}{2}}(\bar{\Omega}_T)} + \|u_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})} + \|\psi\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\partial\Omega_T)}\Big] \\
&\equiv C\mathfrak{G}(u_0, f, \psi).
\end{aligned}$$

Here, we used assumption **H6** and Remark 3.1 [18] to handle the term $\|g(u_0)\|_{\mathcal{C}^{\alpha,\frac{\nu_0\alpha}{2}}(\bar{\Omega}_T)}$.

Then, we search a solution of the original problem (4)–(6) in the form

$$u(x,t) = v(x,t) + \mathcal{V}(x,t), \tag{28}$$

where the unknown function $\mathcal{V} = \mathcal{V}(x,t)$ is a solution of the problem

$$\begin{cases} \mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{V}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{V}) - \mathcal{L}_1\mathcal{V} - \mathcal{K} * \mathcal{L}_2\mathcal{V} = F(x,t) + G(\mathcal{V}) & \text{in} \quad \Omega_T, \\ \mathcal{V}(x,t) = 0 & \text{on} \quad \partial\Omega_T, \\ \mathcal{V}(x,0) = 0 & \text{in} \quad \bar{\Omega}. \end{cases} \tag{29}$$

Here, we set

$$F(x,t) = g(v) - g(u_0), \quad G(\mathcal{V}) = g(\mathcal{V} + v) - g(v).$$

Remark 9. *Assumption **(H6)** and the estimates of v readily provide the following inequalities for the functions F and G:*

$$\|F\|_{\mathcal{C}^{\alpha,\frac{\nu_0\alpha}{2}}(\bar{\Omega}_T)} \leq C\mathfrak{G}(u_0, f, \psi),$$

and for all $u_i \in [-\rho, \rho]$ and $u \in \mathbb{R}$,

$$|G(u_1) - G(u_2)| \leq C_\rho|u_1 - u_2|, \qquad |G(u)| \leq L_0(1 + |u|)$$

with

$$L_0 = L[1 + 2\sup_{\bar{\Omega}_T}|v|] \leq L[1 + 2C\mathfrak{G}(u_0, f, \psi)].$$

Moreover, the straightforward calculations and the definition of the function v arrive at the equalities

$$F(x,0) = 0 \quad \textit{for any} \quad x \in \bar{\Omega}, \qquad G(0) = 0 \quad \textit{for each} \quad (x,t) \in \bar{\Omega}_T.$$

Hence, the last relations mean that the compatibility conditions hold in problem (29).

As a result, *Theorem 2 should be proved only in the case of homogeneous initial and boundary conditions, i.e., for problem* (29).

To this end, we exploit the so-called continuation argument, similar to the case of subdiffusion equations with a single-term fractional derivative (i.e., if $\varrho_0 = 1$ and $\varrho_1 = 0$) described in our previous work [18]. This approach is related to the analysis of the family of problems for $\lambda \in [0,1]$:

$$\begin{cases} \mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{V}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{V}) - \mathcal{L}_1\mathcal{V} - \mathcal{K} * \mathcal{L}_2\mathcal{V} = F(x,t) + \lambda G(\mathcal{V}) & \text{in} \quad \Omega_T, \\ \mathcal{V}(x,t) = 0 & \text{on} \quad \partial\Omega_T, \\ \mathcal{V}(x,0) = 0 & \text{in} \quad \bar{\Omega}. \end{cases} \tag{30}$$

Let (30) be solvable on $[0,T]$ for any $\lambda \in \Lambda$. Clearly, for $\lambda = 0$, problems (30) transform to the linear problem studied in [32]. Thus, keeping in mind assumptions **H1–H5** and Remark 9, we can apply Lemma 3 to (30) with $\lambda = 0$ and deduce the global classical solvability in the corresponding classes. Therefore, $0 \in \Lambda$. Then, we have to check that the set Λ is open and closed at the same time. On this step, we use the essential arguments described in Section 5.3 [18] (i.e., in the case of the equation with a single-term fractional derivative in time). Hence, in our consideration here, we restrict ourselves to a detailed description of only the differences in the proof, which emphasize the difficulties involved in the multi-term fractional derivatives (in general with a non-positive kernel $\mathcal{N}_\nu(t)$). We preliminarily observe that these peculiarities are related to producing a priori estimates for the solutions to (30) in $\mathcal{C}(\bar{\Omega}_T)$ and $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)$, uniformly as $\lambda \in [0,1]$, and are stated in the following lemma. The proof of this claim is provided in Section 7.1.

Lemma 4. *Let the assumptions of Theorem 2 hold, and let $\mathcal{V} \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)$ be the solution to problems (30). Then for any $\lambda \in [0,1]$, there are the following estimates:*

$$\begin{aligned} \|\mathcal{V}\|_{\mathcal{C}(\bar{\Omega}_T)} &\leq C[1 + \|F\|_{\mathcal{C}^{\alpha,\alpha\nu_0/2}(\bar{\Omega}_T)}] \\ &\leq C[1 + \mathfrak{G}(u_0, f, \psi)], \\ \|\mathcal{V}\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)} + \|\mathbf{D}_t^{\nu_1}\mathcal{V}\|_{\mathcal{C}^{\alpha,\frac{\nu_0\alpha}{2}}(\bar{\Omega}_T)} &\leq C[1 + \|F\|_{\mathcal{C}^{\alpha,\alpha\nu_0/2}(\bar{\Omega}_T)}] \\ &\leq C[1 + \mathfrak{G}(u_0, f, \psi)]. \end{aligned} \tag{31}$$

The positive constant C is independent of λ and the right-hand sides of (30), and depends only on T and the structural parameters of the problem.

Finally, exploiting Lemma 4 and Theorem 1 (in particular, the estimate $\|u\|_{\mathcal{C}^{\alpha,\alpha\nu_0/2}(\bar{\Omega}_T)}$ in (8)) and recasting step-by-step the arguments of Section 5.2 in [18], we complete the proof of Theorem 2.
Thus, we are left to verify statements in Lemma 4.

7.1. Proof of Lemma 4: Verification of Estimates in (31)

First, we remark that the second estimate in (31) is verified with the standard Schauder technique and by means of Lemma 3, Remark 9 and the estimate of $\|\mathcal{V}\|_{\mathcal{C}(\bar{\Omega}_T)}$ in (31). Hence, to prove Lemma 4, we are left to produce the first inequality in (31). We proceed here with a detailed proof of this estimate in the case of the positive function ϱ_1. This means that the second fractional derivative in time may have a negative kernel. Another case is simpler and is examined either in the similar manner or with arguments from Section 5.1 in [18].

Here, contrary to the case of a single-term fractional derivative in time (see arguments in Lemma 5.2 [18]) we first estimate the maximum of $\mathcal{V}$ in a small time interval. It is worth noting that, in the case of negative ϱ_1, this estimate is obtained straight on the whole time interval $[0,T]$.

Namely, on the first step, exploiting the integral iteration technique adapted to the case of multi-term fractional derivatives, we obtain the bound

$$\|\mathcal{V}\|_{\mathcal{C}(\bar{\Omega}_{T_0})} \leq C\|f\|_{\mathcal{C}(\bar{\Omega}_{T_0})} \tag{32}$$

for each fixed T_0, $0 < T_0 < T^*$.

The second stage deals with the extension of (32) to the whole time interval $[T_0, T]$.
Step 1: Estimates of $\sup_{\bar{\Omega}_{T_0}} |u(x,t)|$. Recasting the arguments of *Step 1* of Section 6.1 leading to representation (19), we rewrite the equation in (30) in the form

$$\mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{V}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{V}) - \mathcal{L}_0\mathcal{V} = \mathfrak{F}_1(F) + \lambda\mathfrak{F}_1(G(\mathcal{V})) + \sum_{l=2}^{6}\mathfrak{F}_l^{\star}(\mathcal{V}), \tag{33}$$

where we set

$$\mathfrak{F}_2^\star = \mathfrak{F}_2(\mathcal{V}) - \sum_{ij=1}^{n} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial \mathcal{V}}{\partial x_j}, \quad \mathfrak{F}_l^\star = \mathfrak{F}_l(\mathcal{V}), \quad l = 3, 4,$$
$$\mathfrak{F}_5^\star = \bar{\mathcal{K}}(0)(\omega_{1-\nu_0} * (\varrho_0 \mathcal{V})) + \bar{\mathcal{K}}' * \omega_{1-\nu_0} * (\varrho_0 \mathcal{V}),$$
$$\mathfrak{F}_6^\star = -\bar{\mathcal{K}}(0)(\omega_{1-\nu_1} * (\varrho_1 \mathcal{V})) - \bar{\mathcal{K}}' * \omega_{1-\nu_1} * (\varrho_1 \mathcal{V}),$$

while $\mathfrak{F}_l$, $l = 1, 2, 3, 4$, are defined in (19). Collecting equalities (11) (where $\mathcal{V}(x, 0) = 0$) and (12) with assumption **H7**, we deduce that

$$\mathbf{D}_t^{\nu_0}(\varrho_0 \mathcal{V}) - \mathbf{D}_t^{\nu_1}(\varrho_1 \mathcal{V}) = \frac{\partial}{\partial t}(\mathcal{N}_\nu * (\varrho_0 \mathcal{V})) + \partial_t^{\nu_1}(\varrho \mathcal{V}).$$

Then, taking into account this equality and multiplying (33) by $p(\varrho_0 \mathcal{V})^{p-1}$ with $p = 2^m$, $m \geq 1$, and then integrating over Ω, we arrive at the inequality (after standard technical calculations with appealing to **H2**)

$$\begin{aligned}
&\int_\Omega p(\varrho_0 \mathcal{V})^{p-1}(x,\tau) \frac{\partial}{\partial \tau}(\mathcal{N}_\nu * (\varrho_0 \mathcal{V}))(x,\tau)dx + \int_\Omega p(\varrho_0 \mathcal{V})^{p-1}(x,\tau)\partial_\tau^{\nu_1}(\varrho \mathcal{V})(x,\tau)dx \\
&+ \frac{p(p-1)\mu_1}{\|\varrho_0\|_{\mathcal{C}(\bar{\Omega}_T)}} \int_\Omega (\varrho_0 \mathcal{V})^{p-2} |\nabla(\varrho_0 \mathcal{V})|^2 dx \\
&\leq \sum_{l=2}^{6} \int_\Omega p(\varrho_0 \mathcal{V})^{p-1} \mathfrak{F}_l^\star(\mathcal{V}) dx + \int_\Omega p(p-1) \sum_{ij=1}^{n} \frac{a_{ij}}{\varrho_0^2} (\varrho_0 \mathcal{V})^{p-1} \frac{\partial(\varrho_0 \mathcal{V})}{\partial x_j} \frac{\partial \varrho_0}{\partial x_i} dx \\
&+ \int_\Omega p(\varrho_0 \mathcal{V})^{p-1} [\mathfrak{F}_1(F) + \lambda \mathfrak{F}_1(G(\mathcal{V}))] dx.
\end{aligned}$$

To handle the first two terms in the left-hand side of this estimate, we use Corollary 1 and statement (i) in Proposition 1, respectively, and we deduce

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial \tau}(\mathcal{N}_\nu * (\varrho_0 \mathcal{V})^p)(x,\tau)dx + \int_\Omega \left(\frac{\varrho_0}{\varrho}\right)^{p-1} \partial_\tau^{\nu_1}(\varrho \mathcal{V})^p(x,\tau)dx \\
&+ \frac{p(p-1)\mu_1}{\|\varrho_0\|_{\mathcal{C}(\bar{\Omega}_T)}} \int_\Omega (\varrho_0 \mathcal{V})^{p-2} |\nabla(\varrho_0 \mathcal{V})|^2 dx \leq \sum_{l=2}^{6} \int_\Omega p(\varrho_0 \mathcal{V})^{p-1} \mathfrak{F}_l^\star(\mathcal{V}) dx \\
&+ \int_\Omega p(p-1)(\varrho_0 \mathcal{V})^{p-1} \sum_{ij=1}^{n} \frac{a_{ij}}{\varrho_0^2} \frac{\partial(\varrho_0 \mathcal{V})}{\partial x_j} \frac{\partial \varrho_0}{\partial x_i} dx \\
&+ \int_\Omega p(\varrho_0 \mathcal{V})^{p-1} [\mathfrak{F}_1(F) + \lambda \mathfrak{F}_1(G(\mathcal{V}))] dx.
\end{aligned}$$

Then, taking into account the definition of $\mathcal{N}_\nu$, Proposition 2.2 in [2] and keeping in mind (14), we compute the fractional integral $I_t^{\nu_0}$ of both sides in this inequality. Hence, we end up with

$$\int_\Omega (\varrho_0 \mathcal{V})^p(x,t)dx + \frac{p(p-1)\mu_1}{\|\varrho_0\|_{\mathcal{C}(\bar{\Omega}_T)}} I_t^{\nu_0} \left(\int_\Omega (\varrho_0 \mathcal{V})^{p-2} |\nabla(\varrho_0 \mathcal{V})|^2 dx \right)(t) \leq \sum_{j=0}^{8} \mathfrak{D}_l(t),$$

where

$$\mathfrak{D}_1(t) = I_t^{\nu_0}\left(\int_\Omega p(\varrho_0\mathcal{V})^{p-1}[\mathfrak{F}_1(F) + \lambda\mathfrak{F}_1(G(\mathcal{V}))]dx\right)(t),$$
$$\mathfrak{D}_2(t) = I_t^{\nu_0}\left(\int_\Omega p(\varrho_0\mathcal{V})^{p-1}\left[\mathfrak{F}_2^\star(\mathcal{V}) + (p-1)\sum_{ij=1}^{n}\frac{a_{ij}}{\varrho_0^2}\frac{\partial(\varrho_0\mathcal{V})}{\partial x_j}\frac{\partial\varrho_0}{\partial x_i}\right]dx\right)(t),$$
$$\mathfrak{D}_l(t) = I_t^{\nu_0}\left(\int_\Omega p(\varrho_0\mathcal{V})^{p-1}\mathfrak{F}_l^\star(\mathcal{V})dx\right)(t), \quad l \in \{3,4,5,6\},$$
$$\mathfrak{D}_7(t) = \nu_1 I_t^{1+\nu_0-\nu_1}\left(\int_\Omega(\varrho\mathcal{V})^p\mathcal{W}\left(\left[\frac{\varrho_0}{\varrho}\right]^{p-1}\right)dx\right)(t),$$
$$\mathfrak{D}_8(t) = I_t^{\nu_0-\nu_1}\left(\int_\Omega(\varrho_0\mathcal{V})^p\frac{\varrho}{\varrho_0}dx\right)(t).$$

We recall that

$$\mathcal{W}\left(\left[\frac{\varrho_0}{\varrho}\right]^{p-1}\right) = \int_0^1(p-1)\left[\frac{\varrho_0}{\varrho}\right]^{p-2}\frac{\partial}{\partial z}\left(\frac{\varrho_0}{\varrho}\right)ds \quad \text{with} \quad z = st + (1-s)\tau,\ \tau \in (0,t).$$

At this point, we evaluate each term $\mathfrak{D}_j$, separately.

• It is worth noting that, the terms $\mathfrak{D}_l$, $l = 1,2,3,4$, are examined with arguments leading to (5.8), (5.10) and (5.11) in [18]. Thus, taking into account Remark 9 and assumptions **H2, H3, H7**, we immediately achieve the estimate

$$\sum_{l=1}^{4}|\mathfrak{D}_l(t)| \le Cp(\|F\|^p_{C(\bar\Omega_T)} + 1) + \frac{\mu_1 p(p-1)}{2\|\varrho_0\|_{C(\bar\Omega_T)}}I_t^{\nu_0}\left(\int_\Omega(\varrho_0\mathcal{V})^{p-2}|\nabla(\varrho_0\mathcal{V})|^2dx\right)(t).$$

• As for $\mathfrak{D}_5(t)$ and $\mathfrak{D}_6(t)$, we pre-observe that the bound of $\mathfrak{D}_6(t)$ is the same as the one of $\mathfrak{D}_5(t)$. Applying the Young inequality to the function $(\varrho_0\mathcal{V})(x,s)(\varrho_0\mathcal{V})^{p-1}(x,\tau)$ and then collecting Proposition 1 with the smoothness of ϱ_0, ϱ and $\bar{\mathcal{K}}$, we end up with

$$|\mathfrak{D}_5(t)| + |\mathfrak{D}_6(t)| \le CpI_t^{\nu_0}\left(\int_\Omega(\varrho_0\mathcal{V})^p dx\right)(t),$$

where the positive constant C depends only on T, ν_0, ν_1, μ_0 and the corresponding norms of $\mathcal{K}$ and ϱ_1.

• By assumption **H7**, we immediately conclude that

$$|\mathfrak{D}_8(t)| \le I_t^{\nu_0-\nu_1}\left(\int_\Omega(\varrho_0\mathcal{V})^p dx\right)(t) \quad \text{and} \quad \mathcal{W}\left(\left[\frac{\varrho_0}{\varrho}\right]^{p-1}\right) \le 0.$$

In particular, the last inequality arrives at the estimate

$$\mathfrak{D}_7(t) \le 0.$$

Now, collecting the estimates of $|\mathfrak{D}_l|$ with the relation

$$|\nabla(\varrho_0\mathcal{V})^{p/2}|^2 \le p(p-1)(\varrho_0\mathcal{V})^{p-2}|\nabla(\varrho_0\mathcal{V})|^2,$$

we conclude that

$$\int_\Omega(\varrho_0\mathcal{V})^p(x,t)dx + I_t^{\nu_0}\left(\int_\Omega|\nabla(\varrho_0\mathcal{V})^{p/2}|^2dx\right)(t)$$
$$\le Cp\left\{\|F\|^p_{C(\bar\Omega_T)} + 1 + (p-1)I_t^{\nu_0}\left(\int_\Omega(\varrho_0\mathcal{V})^p dx\right)(t)\right\} + I_t^{\nu_0-\nu_1}\left(\int_\Omega(\varrho_0\mathcal{V})^p dx\right)(t). \quad (34)$$

In order to evaluate the integral $I_t^{\nu_0}\Big(\int_\Omega(\varrho_0\mathcal{V})^p dx\Big)(t)$, we appeal to the first interpolation inequality in Proposition 4.6 [18] with $\varepsilon=\frac{1}{2Cp(p-1)}$. Hence, we have

$$\begin{aligned}&\int_\Omega(\varrho_0\mathcal{V})^p(x,t)dx+\frac{1}{2}I_t^{\nu_0}\Big(\int_\Omega|\nabla(\varrho_0\mathcal{V})^{p/2}|^2dx\Big)(t)\\&\leq Cp[\|F\|^p_{\mathcal{C}(\bar\Omega_T)}+1]+[2Cp(p-1)]^{\frac{n+2}{2}}\Big\|\int_\Omega(\varrho_0\mathcal{V})^{p/2}dx\Big\|^2_{\mathcal{C}([0,T_0])}\\&+I_t^{\nu_0-\nu_1}\Big(\int_\Omega(\varrho_0\mathcal{V})^p dx\Big)(t).\end{aligned}\tag{35}$$

Exploiting the Gronwall inequality (see Proposition 4.3 [18]) arrives at

$$\int_\Omega(\varrho_0\mathcal{V})^p(x,t)dx\leq AE_{\nu_0-\nu_1}(t^{\nu_0-\nu_1})$$

for any $t\in[0,T_0]$, where we set

$$A=Cp[\|F\|^p_{\mathcal{C}(\bar\Omega_T)}+1]+[2Cp(p-1)]^{\frac{n+2}{2}}\Big\|\int_\Omega(\varrho_0\mathcal{V})^{p/2}dx\Big\|^2_{\mathcal{C}([0,T_0])},$$

while $E_\theta(z)=\sum_{m=0}^{+\infty}\frac{z^m}{\Gamma(m\theta+1)}$ is the classical Mittag–Leffler function of the order θ (see its definition in (2.2.4) [39]).

After that, applying this estimate to handle the last term in the right-hand side of (35) and then taking into account formula (3.7.44) in [39] to compute the fractional integral of the Mittag–Leffler function, we obtain

$$\begin{aligned}\int_\Omega(\varrho_0\mathcal{V})^p(x,t)dx\leq CpE_{\nu_0-\nu_1}(t^{\nu_0-\nu_1})[\|F\|^p_{\mathcal{C}(\bar\Omega_T)}+1\\+2(p-1)(2Cp(p-1))^{\frac{n}{2}}\|(\varrho_0\mathcal{V})^{p/2}\|^p_{\mathcal{C}([0,T_0],L_{p/2}(\Omega))}].\end{aligned}$$

At last, denoting

$$B=4CE_{\nu_0-\nu_1}(T^{\nu_0-\nu_1})\quad\text{and}\quad\mathcal{A}_m=\sup_{t\in[0,T_0]}\Big(\int_\Omega(\varrho_0\mathcal{V})^p dx\Big)^{1/p}$$

with $p=2^m$, $T_0<T^*$, we derive the bound

$$\mathcal{A}_m\leq B^{m2^{-m}}(\|F\|_{\mathcal{C}(\bar\Omega_{T_0})}+1)+B^{nm2^{-m}}\mathcal{A}_{m-1}.\tag{36}$$

At this point, we discuss two possibilities:

(i) either $\max\{\mathcal{A}_{m-1},\|F\|_{\mathcal{C}(\bar\Omega_{T_0})}+1\}=\|F\|_{\mathcal{C}(\bar\Omega_{T_0})}+1$,

(ii) or $\max\{\mathcal{A}_{m-1},\|F\|_{\mathcal{C}(\bar\Omega_{T_0})}+1\}=\mathcal{A}_{m-1}$.

Clearly, in the case (i), passing to the limit as $m\to+\infty$ in (36), we end up with the desired estimate. Conversely, if (ii) holds, then

$$\mathcal{A}_m\leq[B^{m2^{-m}}+B^{nm2^{-m}}]\mathcal{A}_{m-1}<C\prod_{k=1}^m[B+B^n]^{k2^{-k}}\mathcal{A}_1<C\exp\Big\{|\ln[B+B^n]|\sum_{k=1}^{+\infty}\frac{kn}{2^k}\Big\}\mathcal{A}_1,$$

and letting $m\to+\infty$ and having in mind the convergence of the series, we deduce that

$$\sup_{\bar\Omega_{T_0}}|\varrho_0\mathcal{V}|\leq C\mathcal{A}_1.$$

Finally, to control the term $\mathcal{A}_1$, we first put $p = 2$ in (34) and then apply the Gronwall inequality (4.3) in [18] (where we set $k = \omega_{\nu_0}(t) + \omega_{\nu_0-\nu_1}(t)$). As a result, taking into account assumption **H2**, we end up with the desired estimate (32) and, hence, the second inequality in (31) for each $T_0 \in (0, T^*]$.

Step 2: Extension of estimate (32) *to the whole time interval.* To this end, we modify the arguments of *Step 2* in Section 6.1. Indeed, setting there (see (23) and (24))

$$\Phi(x,t) = \begin{cases} \mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{V}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{V}) - \mathcal{L}_1\mathcal{V} - \mathcal{K} * \mathcal{V}, & \text{if} \quad (x,t) \in \bar{\Omega}_{T_0/2}, \\ [\mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{V}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{V}) - \mathcal{L}_1\mathcal{V} - \mathcal{K} * \mathcal{V}]|_{t=T_0/2}, & \text{if} \quad x \in \bar{\Omega}, \quad t \geq T_0/2, \end{cases}$$

we designate $\mathcal{U}(x,t)$ as a solution of the linear problem

$$\begin{cases} \mathbf{D}_t^{\nu_0}(\varrho_0\mathcal{U}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathcal{U}) - \mathcal{L}_1\mathcal{U} - \mathcal{K} * \mathcal{U} = \Phi(x,t) \quad \text{in} \quad \Omega_{3T_0/2}, \\ \mathcal{U}(x,0) = 0 \qquad \text{in} \quad \bar{\Omega}, \\ \mathcal{U}(x,t) = 0 \qquad \text{on} \quad \partial\Omega_{3T_0/2}. \end{cases} \tag{37}$$

Thanks to (32) and (31) with $t \in [0, T_0]$, we have

$$\|\Phi\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_{3T_0/2})} \leq C\|\mathcal{V}\|_{\mathcal{C}^{2+\alpha,\frac{(2+\alpha)\nu_0}{2}}(\bar{\Omega}_{T_0})} \leq C[1 + \mathfrak{G}(u_0, f, \psi)], \quad \Phi(x,0) = 0 \quad \text{if} \quad x \in \partial\Omega \tag{38}$$

with the constant C being independent of T_0 and λ.

Then, appealing to relations (38), we apply Theorem 4.1 in [32] to problem (37) and end up with the one-valued classical solvability of this problem such that

$$\|\mathcal{U}\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_{3T_0/2})} + \|\mathbf{D}_t^{\nu_1}\mathcal{U}\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_{3T_0/2})} \leq C[1 + \mathfrak{G}(u_0, f, \psi)],$$
$$\mathcal{U}(x,t) = \mathcal{V}(x,t) \quad \text{if} \quad (x,t) \in \bar{\Omega}_{T_0/2}. \tag{39}$$

After that, we introduce new unknown function

$$\mathfrak{U}(x,t) = \mathcal{V}(x,t) - \mathcal{U}(x,t)$$

satisfying relations

$$\begin{cases} \mathbf{D}_t^{\nu_0}(\varrho_0\mathfrak{U}) - \mathbf{D}_t^{\nu_1}(\varrho_1\mathfrak{U}) - \mathcal{L}_1\mathfrak{U} - \mathcal{K} * \mathfrak{U} = f^\star + \lambda G^\star(\mathfrak{U}) \quad \text{in} \quad \Omega_{3T_0/2}, \\ \mathfrak{U}(x,0) = 0 \qquad \text{in} \quad \bar{\Omega}, \\ \mathfrak{U}(x,t) = 0 \qquad \text{on} \quad \partial\Omega_{3T_0/2}, \end{cases} \tag{40}$$

where we set

$$f^\star(x,t) = F(x,t) - \Phi(x,t), \quad G^\star(\mathfrak{U}) = G(\mathfrak{U} + \mathcal{U}).$$

By virtue of (38) and (39), we deduce that

$$\mathfrak{U}(x,t) = 0 \quad \text{if} \quad (x,t) \in \bar{\Omega}_{T_0/2}; \quad f^\star(x,0) = 0 \quad \text{and} \quad G^\star(\mathfrak{U})|_{t=0} = 0 \quad \text{if} \quad x \in \partial\Omega.$$

Moreover, the estimate

$$\|f^\star\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar{\Omega}_{3T_0/2})} \leq C[1 + \mathfrak{G}(u_0, f, \psi)]$$

holds, and $G^\star(\mathfrak{U})$ has the properties of $G(\mathcal{V})$ (see Remark 9).

At last, introducing new variable

$$\sigma = t - T_0/2, \quad t \in [0, 3T_0/2],$$

and performing the change of variable in (40) (that is similar to *Step 2* in Section 6.1), and then recasting the arguments of *Step 1* in this subsection, we arrive at estimate (31) for $t \in [0, 3T_0/2]$. Finally, repeating this procedure a finite number of times, we exhaust the entire $[0, T]$, which proves estimate (31) if $t \in [0, T]$ for any fixed $T > 0$.

8. Conclusions

In this art, we discuss the initial-boundary value problem to linear and semilinear multi-term fractional subdiffusion equations with memory terms. We establish sufficient conditions on the order of the fractional derivatives and given parameters in the model, ensuring the well-posedness of these problems in fractional Sobolev and Hölder spaces. The particular case of the studied problems models the oxygen transport through capillaries [4]. Thus, our analytical technique and ideas can be incorporated to study the corresponding inverse problems concerning to identification of the unknown parameters (e.g., the time lag in concentration of oxygen along capillaries, the order of oxygen subdiffusion and so on). On the other hand, our approach can be generalized and employed in order to research linear and nonlinear degenerate subdiffusion equations with multi-term fractional derivatives. These issues will be addressed with possible further research.

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References

1. Hu, Y.; Zhang, X.-Y.; Li, X.-F. Thermoelastic analysis of biological tissue during hyperthermia treatment for moving laser heating using fractional dual-phase-lag bioheat conduction. *Int. J. Thermal Sci.* **2022**, *182*, 107806. [CrossRef]
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
3. Zhang, J.; Liu, F.; Lin, Z.; Anh, V. Analytical and numerical solutions of a multi-term time-factional Burgers fluid model. *Appl. Math. Comput.* **2019**, *356*, 1–22. [CrossRef]
4. Srivastava, V.; Rai, K.N. A multi-term fractional diffusion equation for oxygen delivery through a capillary to tissues. *Math. Comput. Model.* **2010**, *51*, 616–624. [CrossRef]
5. Marales-Delgado, V.F.; Gómez-Aguilar, J.F.; Saad, K.M.; Khan, M.A.; Agarwal, P. Analytical solution for oxygen diffusion from capillary to tissues involving external force effects: A fractional calculus approach. *Phys. A* **2019**, *523*, 48–65. [CrossRef]
6. Metzler, R.; Jeon, J.-H.; Cherstvy, A.G.; Barkai, E. Anomalous diffusion models and their properties: Non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking. *Phys. Chem. Chem. Phys.* **2014**, *16*, 24128–24164. [CrossRef] [PubMed]
7. Sandev, T.; Metzler, R.; Chechkin, A. From continuous time random walks to the generalized diffusion equation. *Fract. Calc. Appl. Anal.* **2018**, *21*, 10–28. [CrossRef]
8. Sandev, T.; Sokolov, I.M.; Metzler, R.; Chechkin, A. Beyond monofractional kinetics. *Chaos Solitons Fractals* **2017**, *102*, 210–217. [CrossRef]
9. Meerschaert, M.M.; Sikorskii, A. *Stochastic Models for Fractional Calculus*; De Gruter Studies in Mathematics, 43; Walter de Gruyter & Co.: Berlin, Germany, 2012.
10. Caputo, M. Models of flux in porous media with memory. *Water Resour. Res.* **2000**, 36, 693–705. [CrossRef]
11. Alsaedi, A.; Ahmad, B. Maximum principle for certain generalized time and space fractional diffusion equations. *Quart. Appl. Math.* **2015**, *73*, 163–175. [CrossRef]
12. Asjad, M.I.; Ullah, N.; Rehman, H.; Baleanu, D. Optical solitons for conformable space-time fractional nonlinear model. *J. Math. Computer Sci.* **2022**, *27*, 28–41. [CrossRef]
13. Cao, Y.; Nikan, O.; Avazzadeh, Z. A localized meshless technique for solving 2*D* nonlinear integro-differential equation with multi-term kernels. *Appl. Numeric. Math.* **2023**, *183*, 140–156. [CrossRef]
14. Janno, J.; Kinash, N. Reconstruction of an order of derivative and a source term in a fractional diffusion equation from final measurements. *Inverse Probl.* **2018**, 34, 02507. [CrossRef]
15. Gal, C.G.; Warma, M. *Fractional-in-Time Semilinear Parabolic Equations and Applications*; Mathematiques at Applications; Springer Nature Switzerland AG: Cham, Switzerland, 2020.
16. Kempainen, J.; Ruotsalainen, K. Boundary integral solution of the time-fractional diffusion equation. *Integr. Equa. Oper. Theory.* **2009**, *64*, 239–249. [CrossRef]

17. Krasnoschok, M.; Pata, V.; Vasylyeva, N. Solvability of linear boundary value problems for subdiffusion equation with memory. *J. Integral Equ. Appl.* **2018**, *30*, 417–445. [CrossRef]
18. Krasnoschok, M.; Pata, V.; Vasylyeva, N. Semilinear subdiffusion with memory in multidimensional domains. *Math. Nachrichten.* **2019**, *292*, 1490–1513. [CrossRef]
19. Krasnoschok, M.; Vasylyeva, N. Linear subdiffusion in weighted fractional Hölder spaces. *Evol. Equ. Control Theory.* **2022**, *11*, 1455–1487. [CrossRef]
20. Lizama, C.; Guérékata, G.M. Bounded mild solutions for semilinear integro-differential equations in Banach spaces. *Integral. Equa. Oper. Theory* **2010**, *68*, 207–227. [CrossRef]
21. Ponce, R. Bounded mild solutions top fractional integro-differential equations in Banach spaces. *Semigroup Forum.* **2015**, *87*, 377–392. [CrossRef]
22. Vergara, V.; Zacher, R. Optimal decay estimates for time fractional and other nonlocal subdiffusion equations via energy methods. *SIAM J. Math. Anal.* **2015**, *47*, 210–239. [CrossRef]
23. Zacher, R. Global strong solvability of a quasilinear subdiffusion problem. *J. Evol. Equ.* **2012**, *12*, 813–831. [CrossRef]
24. Kochubei, A. General fractional calculus, evolution equations and renewal processes. *Integr. Equ. Oper. Theory* **2011**, *71*, 583–600. [CrossRef]
25. Luchko, Y. Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation. *J. Math. Anal. Appl.* **2011**, *374*, 538–548. [CrossRef]
26. Daftardar-Gejji, V.; Bhalekar, S. Boundary value problems for multi-term fractional differential equations. *J. Math. Anal. Appl.* **2008**, *345*, 754–765. [CrossRef]
27. Li, Z.; Liu, Y.; Yamamoto, M. Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients. *Appl. Math. Comput.* **2015**, *257*, 381–397. [CrossRef]
28. Li, Z.; Huang, X.; Yamamoto, M. Initial-boundary value problems for multi-term time-fractional diffusion equations with x-dependent coefficients. *Evol. Equ. Control. Theory* **2020**, *9*, 153–179. [CrossRef]
29. Sin, C.-S. Well-posedness of general Caputo-type fractional differential equations. *Fract. Calc. Appl. Anal.* **2018**, *21*, 819–832. [CrossRef]
30. Bazhlekova, E. Series solution of a nonlocal problem for a time-fractional diffusion wave equation with damping. *Compte Rend. L'Academa Bulg. Sci.* **2013**, *66*, 1091–1096.
31. Soltani Joujehi, A.; Derakhshan, M.H.; Marasi, H.R. An efficient hybrid numerical method for multi-term time fractional partial differential equations in fluid mechanics with convergence and error analysis. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *114*, 106620. [CrossRef]
32. Pata, V.; Siryk, S.V.; Vasylyeva, N. Multi-term fractional linear equation modeling oxygen subdiffusion through capillaries. *arXiv* **2022**, arXiv:2210.05009.
33. Siryk, S.V.; Vasylyeva, N. Initial-boundary value problems to semilinear multi-term fractional differential equations. *arXiv* **2023**, arXiv:2301.07574.
34. Adams, R.A.; Fournier, J.J.F. *Sobolev Spaces*, 2nd ed.; v. 140; Academic Press: New York, NY, USA, 2003.
35. Grisvard, P. *Elliptic Problems in Nonsmooth Domain*; Pitman: Boston, MA, USA, 1985.
36. Ladyzhenskaia, O.A.; Solonnikov, V.A.; Ural'tseva, N.N. *Linear and Quasilinear Parabolic Equations*; Academic Press: New York, NY, USA, 1968.
37. Krasnoschok, M.; Pata, V.; Siryk, S.V.; Vasylyeva, N. Equivalent definitions of Caputo derivatives and applications to subdiffusion equations. *Dyn. Partial. Differ. Equ.* **2020**, *17*, 383–402. [CrossRef]
38. Fournier, J.J.F. Sharpness in Young's inequality for convolution. *Pactice J. Math.* **1977**, *72*, 383–397. [CrossRef]
39. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: Berlin/Heidelberg, Germany, 2014.