fractal and fractional

Article

Mathematical Model of Heat Conduction for a Semi-Infinite Body, Taking into Account Memory Effects and Spatial Correlations

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Abstract: One of the promising approaches to the description of many physical processes is the use of the fractional derivative mathematical apparatus. Fractional dimensions very often arise when modeling various processes in fractal (multi-scale and self-similar) environments. In a fractal medium, in contrast to an ordinary continuous medium, a randomly wandering particle moves away from the reference point more slowly since not all directions of motion become available to it. The slowdown of the diffusion process in fractal media is so significant that physical quantities begin to change more slowly than in ordinary media.This effect can only be taken into account with the help of integral and differential equations containing a fractional derivative with respect to time. Here, the problem of heat and mass transfer in media with a fractal structure was posed and analytically solved when a heat flux was specified on one of the boundaries. The second initial boundary value problem for the heat equation with a fractional Caputo derivative with respect to time and the Riesz derivative with respect to the spatial variable was studied. A theorem on the semigroup property of the fractional Riesz derivative was proved. To find a solution, the problem was reduced to a boundary value problem with boundary conditions of the first kind. The solution to the problem was found by applying the Fourier transform in the spatial variable and the Laplace transform in time. A computational experiment was carried out to analyze the obtained solutions. Graphs of the temperature distribution dependent on the coordinate and time were constructed.

Keywords: heat transfer; fractional derivative; integral transformations; memory effects; spatial correlations

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1. Introduction

In a fractal medium, in contrast to an ordinary continuous medium, a randomly wandering particle moves away from the reference point more slowly since not all directions of motion become available to it.

The transition from the application of the approach to the description of heat transfer processes to the fractal description can be carried out using the apparatus of fractional calculus [1–7]. As indicated in [8], to consider nonlocalities in space, fractional derivatives with respect to spatial variables are used, and to take into account memory effects, fractional derivatives with respect to time are used. At present, mathematical models described by fractional differential equations affect the study of various physical processes. These include processes such as filtration processes in complex, inhomogeneous porous media [9], the transformation of temperature and humidity fields in low layers of the atmosphere [3,10,11], the kinetics of dispersive charge-carrier transfer in semiconductor structures [12], anomalous diffusion and diffusion particles in inhomogeneous media [13–15], and thermal conductivity [6,16].

This paper is devoted to the study of thermal conductivity for a semi-bounded body with a fractal structure when a heat flux is specified on one of the boundaries. A boundary value problem for the heat equation with boundary conditions of the second kind, fractional derivatives of Caputo with respect to time, and fractional derivatives of Riesz with respect to the spatial variable is studied.

2. Mathematical Statement of the Problem

The classical theory of heat conduction is based on the local Fourier law, which relates the heat flux vector q to the temperature gradient:

$$q = -\lambda \cdot \mathrm{grad} T,$$

where λ is the thermal conductivity of the solid.

Combined with the law of conservation of energy:

$$\rho c \frac{\partial T}{\partial n} = -\mathrm{div} q,$$

where ρ is the mass density and c is the heat capacity, the Fourier law is reduced to the parabolic heat conduction equation:

$$\frac{\partial T}{\partial t} = a \Delta T, \tag{1}$$

where $a = \lambda/(\rho c)$ is the coefficient of thermal diffusivity.

Deng and Ge [14] studied heat transfer in a fractal medium using the fractional Helmholtz equation of the form:

$$\frac{\partial^{2\alpha} T(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\beta} T(x,y)}{\partial y^{2\beta}} + k^2 T(x,y) = f(x,y),$$

where $0 < \alpha \le 1, \frac{1}{2} < \beta \le 1$.

He and Liu [16] used a fractional version of the Fourier law:

$$\lambda^{2\beta} \frac{\partial^{\beta} T}{\partial x^{\beta}} = q. \tag{2}$$

Mathematically, the transition from the deterministic representation of the heat transfer model to its fractal description can be carried out using the apparatus of fractional differentiation and integration [2,3,13]. In particular, for the mathematical formalization of the characteristics of fractal media, fractional derivatives with respect to spatial coordinates are used, and for the representation of the memory effect, a fractional derivative with respect to time is used.

Consider a generalization of Equation (1) to fractional order derivatives:

$$\partial_{0t}^{\alpha} T(\xi,\tau) = \bar{a}\, {}^{R}D_{0x}^{2\beta} T(\xi,\tau), \tau > 0, 0 < \xi < \infty, \tag{3}$$

where

$$\partial_{0t}^{\alpha} T(\xi,\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\tau} \frac{T_t^{/}(\xi,s)}{(\tau - s)^{\alpha}} ds$$

is the Caputo partial fractional derivative,

$${}^{R}D_{0x}^{2\beta} T(\xi,\tau) = \frac{1}{2 \cdot \Gamma(2-2\beta) \cos\left(\frac{\pi}{2}(2-2\beta)\right)} \frac{d^2}{d\xi^2} \int_0^{\infty} \frac{T(z,\tau)}{|\xi - z|^{2\beta - 1}} dz$$

is the Riesz partial fractional derivative on the semiaxis [2], $0 < \alpha < 1$, $\frac{1}{2} < \beta < 1$, $T(\xi,\tau)$ is the temperature, $\tau = t/t_0, \xi = x/x_0$ are the dimensionless time and coordinate, respectively, t_0, x_0 are the characteristic time and coordinate, respectively, and $\overline{a} = a \cdot t_0/x_0^2$ is the dimensionless thermal diffusivity.

Let us study the case in which a heat flux is specified at one end of a region, that is, consider the boundary condition of the second kind:

$$\xi = 0 : \lambda^{2\beta R} D_{0\xi}^{\beta} T(\xi,\tau) + q_c = 0, \tag{4}$$

where $q_c = -\frac{Q}{\lambda}$, $Q = \frac{P}{S}$ is the specific power of surface heat release, P is the power of the heat source, and S is the area of heating the edge of the region.

We will assume that the other boundary of the region is significantly removed from the gradient zone, and that a temperature equal to the ambient temperature is established at this boundary:

$$T(\infty,\tau) = T_0, \left.\frac{\partial T(\xi,\tau)}{\partial \xi}\right|_{\xi=\infty} = 0. \tag{5}$$

We supplement the problem with the initial conditions $T(\xi,\tau_0) = T_0, 0 \le \xi < +\infty$.

3. Semigroup Property of the Fractional Riesz Derivative

Let us formulate and prove two lemmas.

Lemma 1. *Let $f(x) \in C(\Omega)$, where $\Omega = (-\infty,+\infty)$. Then*

$$\int_{-\infty}^{\infty} \frac{dx}{|\xi - x|^{\beta}} \int_{-\infty}^{\infty} \frac{f(s)ds}{|x-s|^{\beta}} = \widetilde{B}(\beta,\beta) \int_{-\infty}^{\infty} \frac{f(s)ds}{|\xi - s|^{2\beta-1}} \tag{6}$$

where $\widetilde{B}(\beta,\beta) = B(1-\beta,1-\beta) + B(1-\beta,2\beta-1) + B(1-\beta,2\beta-1)$ *and* $\frac{1}{2} < \beta < 1$.

Proof of Lemma 1. Let us transform the integral on the left side of equality (6).

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dx}{|\xi-x|^{\beta}} \int_{-\infty}^{\infty} \frac{f(s)ds}{|x-s|^{\beta}} \\
&= \left\{ \int_{-\infty}^{\xi} \frac{dx}{(\xi-x)^{\beta}} + \int_{\xi}^{\infty} \frac{dx}{(x-\xi)^{\beta}} \right\} \cdot \left\{ \int_{-\infty}^{x} \frac{f(s)ds}{(x-s)^{\beta}} + \int_{x}^{\infty} \frac{f(s)ds}{(s-x)^{\beta}} \right\} \\
&= \int_{-\infty}^{\xi} \frac{dx}{(\xi-x)^{\beta}} \int_{-\infty}^{x} \frac{f(s)ds}{(x-s)^{\beta}} + \int_{-\infty}^{\xi} \frac{dx}{(\xi-x)^{\beta}} \int_{x}^{\infty} \frac{f(s)ds}{(s-x)^{\beta}} \\
&+ \int_{\xi}^{\infty} \frac{dx}{(x-\xi)^{\beta}} \int_{-\infty}^{x} \frac{f(s)ds}{(x-s)^{\beta}} + \int_{\xi}^{\infty} \frac{dx}{(x-\xi)^{\beta}} \int_{x}^{\infty} \frac{f(s)ds}{(s-x)^{\beta}}.
\end{aligned} \tag{7}$$

Let us represent the integrals that make up the second and third terms of the right side of equality (7) in the form:

$$\int_{-\infty}^{\xi} \frac{dx}{(\xi-x)^{\beta}} \int_{x}^{\infty} \frac{f(s)ds}{(s-x)^{\beta}} = \int_{-\infty}^{\xi} \frac{dx}{(\xi-x)^{\beta}} \int_{x}^{\xi} \frac{f(s)ds}{(s-x)^{\beta}} + \int_{-\infty}^{\xi} \frac{dx}{(\xi-x)^{\beta}} \int_{\xi}^{\infty} \frac{f(s)ds}{(s-x)^{\beta}} \tag{8}$$

and

$$\int_{\xi}^{\infty} \frac{dx}{(x-\xi)^{\beta}} \int_{-\infty}^{x} \frac{f(s)ds}{(x-s)^{\beta}} = \int_{\xi}^{\infty} \frac{dx}{(x-\xi)^{\beta}} \int_{-\infty}^{\xi} \frac{f(s)ds}{(x-s)^{\beta}} + \int_{\xi}^{\infty} \frac{dx}{(x-\xi)^{\beta}} \int_{\xi}^{x} \frac{f(s)ds}{(x-s)^{\beta}}. \tag{9}$$

Substituting (8) and (9) into (7), we obtain

$$\begin{aligned}&\int_{-\infty}^{\xi}\frac{dx}{|\xi-x|^{\beta}}\int_{x}^{\infty}\frac{f(s)ds}{|x-s|^{\beta}}\\&=\int_{-\infty}^{\xi}\frac{dx}{(\xi-x)^{\beta}}\int_{-\infty}^{x}\frac{f(s)ds}{(x-s)^{\beta}}+\int_{-\infty}^{\xi}\frac{dx}{(\xi-x)^{\beta}}\int_{x}^{\xi}\frac{f(s)ds}{(s-x)^{\beta}}+\int_{-\infty}^{\xi}\frac{dx}{(\xi-x)^{\beta}}\int_{\xi}^{\infty}\frac{f(s)ds}{(s-x)^{\beta}}\\&+\int_{\xi}^{\infty}\frac{dx}{(x-\xi)^{\beta}}\int_{-\infty}^{\xi}\frac{f(s)ds}{(x-s)^{\beta}}+\int_{\xi}^{\infty}\frac{dx}{(x-\xi)^{\beta}}\int_{\xi}^{x}\frac{f(s)ds}{(x-s)^{\beta}}+\int_{\xi}^{\infty}\frac{dx}{(x-\xi)^{\beta}}\int_{x}^{\infty}\frac{f(s)ds}{(s-x)^{\beta}}.\end{aligned}\tag{10}$$

Replacing the order of integration in the first term of equality (10), we obtain

$$\begin{aligned}\int_{-\infty}^{\xi}\frac{dx}{(\xi-x)^{\beta}}\int_{-\infty}^{x}\frac{f(s)ds}{(x-s)^{\beta}}&=\int_{-\infty}^{\xi}f(s)ds\int_{s}^{\xi}\frac{dx}{(\xi-x)^{\beta}(x-s)^{\beta}}=\langle t=x-s\rangle\\&=\int_{-\infty}^{\xi}f(s)ds\int_{0}^{\xi-s}\frac{dt}{(\xi-s-t)^{\beta}t^{\beta}}.\end{aligned}$$

Further, in the resulting integral, making the change $t=(\xi-s)z$, we obtain

$$\begin{aligned}\int_{-\infty}^{\xi}f(s)ds\int_{0}^{\xi-s}\frac{dt}{(\xi-s-t)^{\beta}t^{\beta}}&=\int_{-\infty}^{\xi}\frac{f(s)ds}{(\xi-s)^{2\beta-1}}\int_{0}^{1}(1-z)^{1-\beta-1}z^{1-\beta-1}dz\\&=B(1-\beta,1-\beta)\int_{-\infty}^{\xi}\frac{f(s)ds}{(\xi-s)^{2\beta-1}}.\end{aligned}$$

The remaining terms in equality (10) can be calculated similarly. Finally, we obtain

$$\int_{-\infty}^{\infty}\frac{dx}{|\xi-x|^{\beta}}\int_{-\infty}^{\infty}\frac{f(s)ds}{|x-s|^{\beta}}=\widetilde{B}(\beta,\beta)\int_{-\infty}^{\infty}\frac{f(s)ds}{|\xi-s|^{2\beta-1}}.$$

□

Lemma 2. *Let* $f(x)\in C(\Omega)$ *where* $\Omega=(-\infty,+\infty)$*. Then*

$$\int_{-\infty}^{\infty}\frac{ds}{|x-s|^{\beta}}\frac{d}{ds}\int_{-\infty}^{\infty}\frac{f(z)dz}{|s-z|^{\beta}}=\beta\cdot\widetilde{\widetilde{B}}(\beta,\beta)\int_{-\infty}^{\infty}\frac{sign(x-s)f(s)ds}{|x-s|^{2\beta}},\tag{11}$$

where $\widetilde{\widetilde{B}}(\beta,\beta)=B(-\beta,2\beta)-B(1-\beta,2\beta)-B(-\beta,1-\beta),\frac{1}{2}<\beta<1$, *and* $\mathrm{sign}(x)$ *is a sign function.*

Proof of Lemma 2. The integral on the left side of equality (11) can be represented as:

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{ds}{|x-s|^{\beta}}\frac{d}{ds}\int_{-\infty}^{\infty}\frac{f(z)dz}{|s-z|^{\beta}}\\&=\int_{-\infty}^{\infty}ds\left\{\frac{d}{ds}\left[\frac{1}{|x-s|^{\beta}}\int_{-\infty}^{\infty}\frac{f(z)dz}{|s-z|}\right]-\left[\frac{d}{ds}\frac{1}{|x-s|^{\beta}}\right]\int_{-\infty}^{\infty}\frac{f(z)dz}{|s-z|^{\beta}}\right\}.\end{aligned}\tag{12}$$

The integral of the first term in equality (12), which is equal to the integrand, vanishes. In the second integral, we use the following relation:

$$\frac{d}{ds}\frac{1}{|x-s|^{\beta}}=-\beta\cdot\frac{\mathrm{sign}(x-s)}{|x-s|^{\beta+1}}.$$

Thus, we obtain

$$\int_{-\infty}^{\infty} \frac{ds}{|x-s|^\beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-x|^\beta} = -\beta \int_{-\infty}^{\infty} \frac{\text{sign}(x-s)ds}{|x-s|^{\beta+1}} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-z|^\beta}. \tag{13}$$

The integral on the right side of equality (13) is calculated similarly to integral (6). Then, we obtain

$$\int_{-\infty}^{\infty} \frac{ds}{|x-s|^\beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-z|^\beta} = \beta \cdot \widetilde{\widetilde{B}}(\beta,\beta) \int_{-\infty}^{\infty} \frac{\text{sign}(x-s)f(s)ds}{|x-s|^{2\beta}}. \tag{14}$$

□

Let us prove a theorem on the semigroup property of the fractional Riesz derivative.

Theorem 1. *Let* $f(x) \in C(\Omega)$ *where* $\Omega = (-\infty, +\infty)$. *Then, there is the equality*

$$^R D^\beta \Big(D^\beta \big(f(x) \big) \Big) = {}^R D^\gamma f(x), \tag{15}$$

where $\gamma = 2\beta$.

Proof of Theorem 1. To prove equality (15), we represent it in the form:

$$^R D^\beta \Big({}^R D^\beta \big(f(x) \big) \Big) = {}^R D^{1+\lambda} f(x), \tag{16}$$

where $\gamma = 1 + \lambda$, and λ is determined from the condition of fulfillment of equality (16). We have

$$\frac{d}{dx} \int_{-\infty}^{\infty} \frac{ds}{|x-s|^\beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-z|^\beta} = \frac{A^2}{B} \frac{d^2}{dx^2} \int_{-\infty}^{\infty} \frac{f(s)ds}{|x-s|^\lambda} \tag{17}$$

where $A = 2\Gamma(1-\beta)\cos\left(\frac{\pi}{2}(1-\beta)\right), B = 2\Gamma(2-\gamma)\cos\left(\frac{\pi}{2}(2-\gamma)\right)$.

Consider the relation

$$\int_{-\infty}^{\infty} \frac{ds}{|x-s|^\beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-z|^\beta} = \frac{A^2}{B} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{f(s)ds}{|x-s|^\lambda}, \tag{18}$$

from which (17) is implied. We multiply expression (18) by the factor $|\xi - x|^\alpha$ and integrate over x. Therefore, we have

$$\int_{-\infty}^{\infty} \frac{dx}{|\xi-x|^\beta} \int_{-\infty}^{\infty} \frac{ds}{|x-s|^\beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-z|^\beta} = \frac{A^2}{B} \int_{-\infty}^{\infty} \frac{dx}{|\xi-x|^\beta} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{f(s)ds}{|x-s|^\lambda}.$$

Let us introduce the following notation:

$$I_1 = \int_{-\infty}^{\infty} \frac{dx}{|\xi-x|^\beta} \int_{-\infty}^{\infty} \frac{f(s)ds}{|x-s|^\beta},\ I_2 = \int_{-\infty}^{\infty} \frac{ds}{|x-s|^\beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-z|^\beta}.$$

According to Lemma 1:

$$I_1 = \widetilde{B}(\beta,\beta) \int_{-\infty}^{\infty} \frac{f(s)ds}{|\xi-s|^{2\beta-1}}, \tag{19}$$

and according to Lemma 2:

$$I_2 = \int_{-\infty}^{\infty} \frac{ds}{|x-s|^\beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{f(z)dz}{|s-z|^\beta} = \beta \cdot \tilde{\tilde{B}}(\beta, \beta) \int_{-\infty}^{\infty} \frac{\text{sign}(x-s) f(s) ds}{|x-s|^{2\beta}}. \tag{20}$$

Substituting relations (19) and (20) into (17), we finally obtain the following expression:

$$\begin{aligned} &\tilde{B}(\beta,\beta)\cdot(2\beta-1)\cdot\tilde{\tilde{B}}(2\beta-1,\beta)\cdot \int_{-\infty}^{\infty} \tfrac{\text{sign}(\xi-x)f(x)dx}{|\xi-x|^{3\beta-1}} \\ &= \tfrac{A^2}{B}\cdot\beta\cdot\tilde{\tilde{B}}(\beta,\lambda)\cdot \int_{-\infty}^{\infty} \tfrac{\text{sign}(\xi-x)f(x)dx}{|\xi-x|^{\beta+\lambda}}. \end{aligned} \tag{21}$$

Requiring the equality of the integrals in (21), we obtain $3\beta - 1 = \beta + \lambda$. Hence, we have $\lambda = 2\beta - 1$. Taking this into account, equality (21) will take the form:

$$\tilde{B}(\beta,\beta)\cdot(2\beta-1)\cdot\tilde{\tilde{B}}(2\beta-1,\beta) = \frac{A^2}{B}\cdot\beta\cdot\tilde{\tilde{B}}(\beta,2\beta-1). \tag{22}$$

Let us show that relation (22) is satisfied identically. Indeed, it is easy to show the equality:

$$(2\beta-1)\cdot\tilde{\tilde{B}}(2\beta-1,\beta) = \beta\cdot\tilde{\tilde{B}}(\beta,2\beta-1). \tag{23}$$

Actually,

$$\begin{aligned} &\beta\cdot\tilde{\tilde{B}}(\beta,\beta) = \beta(B(-\beta,2\beta) - B(1-\beta,2\beta) - B(-\beta,1-\beta)) = \beta\tfrac{\Gamma(-\beta)\Gamma(2\beta)}{\Gamma(\beta)} \\ &-\beta\tfrac{\Gamma(1-\beta)\Gamma(2\beta)}{\Gamma(1+\beta)} - \beta\tfrac{\Gamma(-\beta)\Gamma(1-\beta)}{\Gamma(1-2\beta)} = -\tfrac{\Gamma(1-\beta)\Gamma(2\beta)}{\Gamma(\beta)} - \tfrac{\Gamma(1-\beta)\Gamma(2\beta)}{\Gamma(\beta)} \\ &+\tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(1-2\beta)} = -\tfrac{\Gamma(2\beta)}{\Gamma(\beta)\Gamma(\beta)}[\Gamma(\beta)\Gamma(1-\beta)+\Gamma(\beta)\Gamma(1-\beta)] + \tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(1-2\beta)}. \end{aligned}$$

From the resulting expression, it follows that:

$$\begin{aligned} \beta\cdot\tilde{\tilde{B}}(\beta,2\beta-1) = \tfrac{\Gamma(3\beta-1)}{\Gamma(\beta)\Gamma(2\beta-1)}[\Gamma(\beta)\Gamma(1-\beta) + \Gamma(2\beta-1)\Gamma(2-2\beta)] \\ +\tfrac{\Gamma(1-\beta)\Gamma(2-2\beta)}{\Gamma(2-3\beta)}, \end{aligned} \tag{24}$$

$$\begin{aligned} (2\beta-1)\cdot\tilde{\tilde{B}}(2\beta-1,\beta) = -\tfrac{\Gamma(3\beta-1)}{\Gamma(\beta)\Gamma(2\beta-1)}[\Gamma(2\beta-1)\Gamma(2-2\beta)+\Gamma(\beta)\Gamma(1-\beta)] \\ +\tfrac{\Gamma(2-2\beta)\Gamma(1-\beta)}{\Gamma(2-3\beta)}. \end{aligned} \tag{25}$$

From these equalities, it follows that

$$(2\beta-1)\cdot\tilde{\tilde{B}}(2\beta-1,\beta) = \beta\cdot\tilde{\tilde{B}}(\beta,2\beta-1)$$

Equality (23) follows from the obvious equality of the right-hand sides of equalities (24) and (25). Then, (22) takes the form $\tilde{B}(\beta,\beta) = \frac{A^2}{B}$.

Further, for $\tilde{B}(\beta,\beta)$, we have

$$\begin{aligned} \tilde{B}(\beta,\beta) &= B(1-\beta,1-\beta) + 2B(1-\beta,2\beta-1) = \tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(2-2\beta)} \\ &+2\tfrac{\Gamma(1-\beta)\Gamma(2\beta-1)}{\Gamma(\beta)} = 2\tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(2-2\beta)}\left(\tfrac{1}{2} + \tfrac{\Gamma(2\beta-1)\Gamma(2-2\beta}{\Gamma(\beta)\Gamma(1-\beta)}\right). \end{aligned}$$

Using the relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we obtain

$$\begin{aligned}\widetilde{B}(\beta,\beta) &= 2\tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(2-2\beta)}\left(\tfrac{1}{2}+\tfrac{\sin(\pi\beta)}{\sin(\pi(2\beta-1))}\right) &&= 2\tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(2-2\beta)}\left(\tfrac{1}{2}-\tfrac{\sin(\pi\beta)}{\sin(2\pi\beta)}\right)\\ &= 2\tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(2-2\beta)}\left(\tfrac{1}{2}-\tfrac{1}{2\cos(\pi\beta)}\right) &&= 2\tfrac{\Gamma(1-\beta)\Gamma(1-\beta)}{\Gamma(2-2\beta)}\tfrac{\sin^2(\pi\beta/2)}{(-1)\cos(\pi\beta)}\\ & &&= \tfrac{(2\Gamma(1-\beta)\sin(\pi(1-\beta)/2))^2}{2\Gamma(2-2\beta)\cos(\pi(2-2\beta)/2)}.\end{aligned}$$

Given the definitions of A and B, we obtain

$$\widetilde{B}(\beta,\beta) = \frac{A^2}{B}.$$

Thus, for $\lambda = 2\beta - 1$, the relation (22) turns into the identity

$$\frac{A^2}{B} = \frac{A^2}{B}.$$

Substituting the value in (14), we obtain the required equality (17).□

4. Solution of the Problem

Problems (3)–(5) will be solved by reducing this problem to a problem with boundary conditions of the first kind. According to equality (2), we have

$$q(\xi,\tau) = -\lambda^{2\beta}\cdot {}^{R}D^{\beta}_{0\xi}T(\xi,\tau). \tag{26}$$

According to Theorem 1, the fractional Riesz derivative satisfies the equality (15), i.e.,

$${}^{R}D^{\beta}_{0\xi}\left({}^{R}D^{\beta}_{0\xi}(T(\xi,\tau)\right) = {}^{R}D^{2\beta}_{0\xi}T(\xi,\tau). \tag{27}$$

Let us differentiate the left and right parts of Equation (3). Then the equation will take the form:

$$\partial^{\alpha}_{0\tau}\left({}^{R}D^{\beta}_{0\xi}T(\xi,\tau)\right) = \overline{a}^{R}D^{\beta}_{0\xi}\left(D^{2\beta}_{0\xi}T(\xi,\tau)\right). \tag{28}$$

Using equalities (26) and (28), we rewrite problems (3)–(5) in the form:

$$\partial^{\alpha}_{0\tau}q(\xi,\tau) = \overline{a}^{R}D^{2\beta}_{0\tau}q(\xi,\tau), \tag{29}$$

where

$$\begin{gathered} q(\xi,0) = 0,\\ q(0,\tau) = q_c, q(\infty,\tau) = 0. \end{gathered} \tag{30}$$

Let the function $q(\xi,\tau)$ be continuous in the domain $D = (0,+\infty)\times[0,T]$, and $q^{/}_{\tau}(\xi,\tau) \in L[0,T]$. Then, for $0 < \alpha < 1$ the derivative $D^{\alpha}_{0\tau}q(\xi,\tau)$ exists, and almost everywhere on $[0,T]$ there is the representation:

$${}^{RL}D^{\alpha}_{0\tau}q(\xi,\tau) = \frac{q(\xi,0)}{\Gamma(1-\alpha)}\cdot\tau^{-\alpha} + \frac{1}{\Gamma(`1-\alpha)}\int_0^{\tau}\frac{q^{/}_{t}(\xi,s)}{(\tau-s)^{\alpha}}ds, \tag{31}$$

where

$${}^{RL}D^{\alpha}_{0\tau}q(\xi,\tau) = \frac{1}{\Gamma(`1-\alpha)}\frac{\partial}{\partial\tau}\int_0^{\tau}\frac{q(x,s)}{(\tau-s)^{\alpha}}ds$$

is the fractional Riemann–Liouville derivative.

Since $q(\xi, 0) = 0$, equality (31)then takes the form:

$$D_{0t}^{\alpha} q(\xi,\tau) = \frac{0}{\Gamma(1-\alpha)} \cdot \tau^{-\alpha} + \frac{1}{\Gamma('1-\alpha)} \int\limits_0^t \frac{q_t^{/}(\xi,s)}{(\tau-s)^{\alpha}} ds = \partial_{0t}^{\alpha} q(\xi,\tau). \tag{32}$$

Taking into account (32), problems (29) and (30) can be rewritten in the form:

$$^{RL}D_{0t}^{\alpha} q(\xi,\tau) = \bar{a}^{R} D_{0x}^{2\beta} q(\xi,\tau). \tag{33}$$

The initial and boundary conditions are determined by (10).

The solution of (33) can be found using the Fourier and Laplace transforms. Performing cosine Fourier transforms in the spatial variable and Laplace transforms in time, we obtain the following expression for the image:

$$p^{\alpha} \bar{q}(k,p) = \frac{q_c \cdot a^2}{k^{1-2\beta} \cdot p} - \bar{a} \cdot k^{2\beta} \bar{q}(k,p), \tag{34}$$

i.e.,

$$\bar{q}(k,p) = \frac{\bar{a} \cdot q_c \cdot k^{2\beta-1}}{p \cdot (p^{\alpha} + \bar{a} \cdot k^{2\beta})}, \tag{35}$$

where $E_{\alpha,1}(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ is the Mittag-Leffler function.

We have [7]

$$\frac{1}{p(p^{\alpha} + \bar{a} \cdot k^{2\beta})} = \frac{1}{\bar{a} \cdot k^{\beta}} \int\limits_0^{\infty} e^{-p\tau} (1 - E_{\alpha,1}(-\bar{a} \cdot k^{2\beta} \tau^{\alpha})) d\tau. \tag{36}$$

Taking into account (36), equality (35) can be written as:

$$\bar{q}(k,p) = q_c \cdot k^{-1} \int\limits_0^{\infty} e^{-p\tau} (1 - E_{\alpha,1}(-\bar{a} \cdot k^{2\beta} \tau^{\alpha})) d\tau. \tag{37}$$

Applying the inverse cosine Fourier transform, we obtain the expression for the original function:

$$q(\xi,\tau) = \frac{2 \cdot q_c}{\pi} \int\limits_0^{+\infty} \frac{\cos(k\xi)}{k} (1 - E_{\alpha,1}(-\bar{a} \cdot k^{2\beta} \tau^{\alpha})) dk. \tag{38}$$

Let us study the question of the uniqueness of solution (18). Let $q_1(\xi,\tau)$ and $q_2(\xi,\tau)$ be the solutions of problem (33), and satisfy the initial and boundary conditions of (30). Let also $q_1(\xi,\tau), q_2(\xi,\tau) \in C[\overline{D}]$, $q_1(\xi,\tau), q_2(\xi,\tau) \in C^{2,1}[D]$.

Let us denote $q(\xi,\tau) = q_1(\xi,\tau) - q_2(\xi,\tau)$. Then, according to the maximum principle, we have

$$\max_{\overline{D}} q(\xi,\tau) = \max_{\Gamma} q(\xi,\tau) = 0, \tag{39}$$

$$\min_{\overline{D}} q(\xi,\tau) = \min_{\Gamma} q(\xi,\tau) = 0. \tag{40}$$

From equalities (39) and (40), it follows that $q(\xi,\tau) = 0$ in the area $\overline{D}$; that is, $q_1(\xi,\tau) = q_2(\xi,\tau)$.

To find the solution $T(\xi,\tau)$, we substitute the corresponding expression from (38) into expression (6) instead of $q(x,t)$ and apply the Riesz fractional integration operator to both parts:

$$q(\xi,\tau) = -\lambda^{2\beta} \cdot {}^{R}D_{\xi}^{\beta} T(\xi,\tau).$$

Therefore:

$$T(\xi,\tau) = T_0 \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} + \frac{2 \cdot q_c}{\lambda \cdot \pi} R_\beta q(\xi,\tau), \tag{41}$$

i.e.,

$$R_\beta q(\xi,\tau) = \frac{1}{2\Gamma(\beta)\cos(\pi\alpha/2)} \int_{-\infty}^{+\infty} \frac{q(z,\tau)}{|\xi - z|^{1-\beta}} dz.$$

5. Results and Discussion

Figures 1 and 2 show graphs of solution (41) for various values of the fractional derivative parameters α and β.

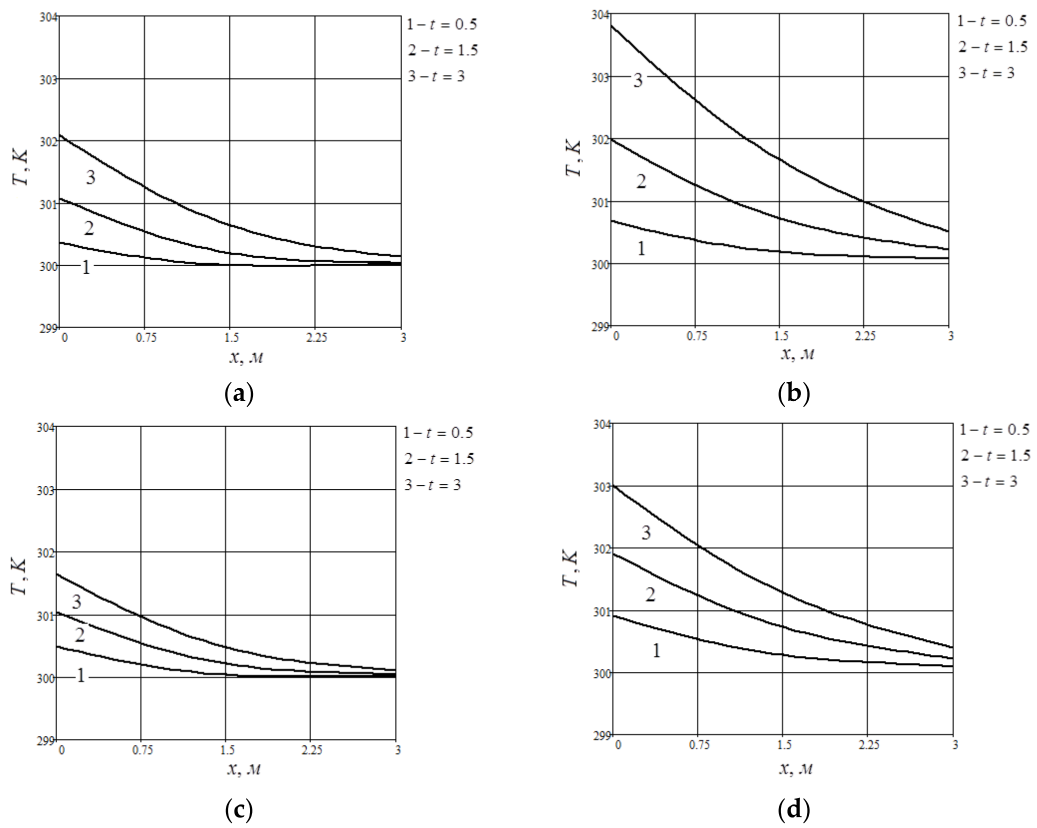
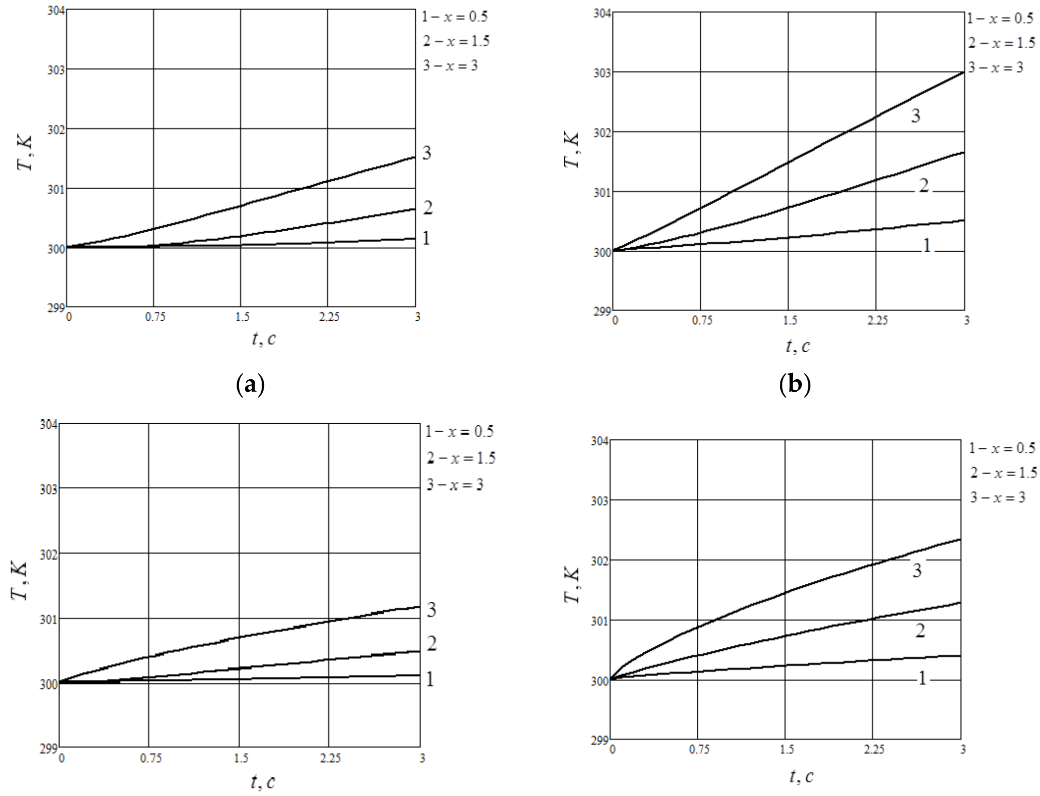


Figure 1. Graphs of temperature distribution in a semi-infinite body at different moments of time at: (**a**) $\alpha = 1,\ \beta = 2$; (**b**) $\alpha = 1,\ \beta = 1.7$; (**c**) $\alpha = 0.7,\ \beta = 2$; (**d**) $\alpha = 0.7,\ \beta = 1.7$. For all cases: $q_c = 1.8 \frac{\mathrm{W}}{\mathrm{m}^2}$, $\lambda = 2.5 \frac{\mathrm{W}}{\mathrm{m \cdot K}}$, $T_0 = 300$ K.

As can be seen from Figures 1 and 2, spatial correlations and memory effects have different influences on the final decision. With a decrease in the index of the spatial derivative (β), an acceleration of the thermal conductivity processes is observed without a significant effect on the nature of the spatial and temporal dependences, while a decrease in the time derivative index (α) leads to a significant slowdown of the processes while changing the nature of the nonlinearity of the time dependences.



(**a**) (**b**) (**c**) (**d**)

Figure 2. Temperature dependences on time in a semi-infinite body for: (**a**) $\alpha = 1,\ \beta = 2$; (**b**) $\alpha = 1, \beta = 1.7$; (**c**) $\alpha = 0.7,\ \beta = 2$; (**d**) $\alpha = 0.7,\ \beta = 1.7$. For all cases: $q_c = 1.8\frac{\text{W}}{\text{m}^2}$, $\lambda = 2.5\frac{\text{W}}{\text{m}\cdot\text{K}}$, $T_0 = 300$ K.

6. Conclusions

We present a mathematical model of the thermal conductivity of a semi-infinite body that takes into account memory effects and spatial correlations. Graphs of the dependence of temperature on the spatial coordinate and time are constructed. When switching to a fractional time derivative, the heat transfer process slows down with a change in the nature of the time dependence. Thus, the transition to fractional derivatives makes it possible to study ultraslow heat transfer processes, which are typical for media with a fractal structure.

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