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Some New Applications of the Faber Polynomial Expansion Method for Generalized Bi-Subordinate Functions of Complex Order γ Defined by q-Calculus

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Abstract: This work examines a new subclass of generalized bi-subordinate functions of complex order γ connected to the q-difference operator. We obtain the upper bounds ρ_m for generalized bi-subordinate functions of complex order γ using the Faber polynomial expansion technique. Additionally, we find coefficient bounds $|\rho_2|$ and Feke–Sezgo problems $|\rho_3 - \rho_2^2|$ for the functions in the newly defined class, subject to gap series conditions. Using the Faber polynomial expansion method, we show some results that illustrate diverse uses of the Ruschewey q differential operator. The findings in this paper generalize those from previous efforts by a number of prior researchers.

Keywords: quantum (or q-) calculus; analytic functions; univalent functions; q-derivative operator; convex functions; starlike functions; bi-univalent functions; Faber polynomial expansion

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1. Introduction and Definitions

The set of all analytic functions $h(z)$ in the open unit disc $E = \{z : |z| < 1\}$ is denoted by the symbol $\mathfrak{A}$ and every $h \in \mathfrak{A}$ is normalized by

$$h(0) = 0 \text{ and } h'(0) = 1.$$

Thus, every function $h \in \mathfrak{A}$ can be expressed in the following form:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m. \tag{1}$$

Furthermore, $\mathcal{S} \subset \mathfrak{A}$ and every $h \in \mathcal{S}$ is univalent in E. For h_1, $h_2 \in \mathfrak{A}$, and h_1 subordinate to h_2 in E, denoted by

$$h_1(z) \prec h_2(z), \ \ z \in E,$$

if there exists a function w_0, such that $w_0 \in \mathfrak{A}$, with $w_0(0) = 0$, and $|w_0(z)| < 1$, satisfying

$$h_1(z) = h_2(w_0(z)), \ z \in E.$$

Let $\mathcal{S}^*$ represent the class of starlike functions and every $h \in \mathcal{S}^*$, if

$$\text{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0, \ z \in E$$

and C represents the class of convex functions and every $h \in C$, if

$$1 + \mathrm{Re}\left(\frac{zh''(z)}{h'(z)}\right) > 0,\ z \in E.$$

In terms of subordination, these conditions are equivalent as follows:

$$\mathcal{S}^* = \left\{h \in \mathfrak{A} : \frac{zh'(z)}{h(z)} \prec \frac{1+z}{1-z}\right\}$$

and

$$\mathcal{C} = \left\{h \in \mathfrak{A} : 1 + \frac{zh''(z)}{h'(z)} \prec \frac{1+z}{1-z}\right\}.$$

Ma and Minda [1] stated that the aforementioned two classes can be generalized as follows:

$$\mathcal{S}^*(\varphi) = \left\{h \in \mathfrak{A} : \frac{zh'(z)}{h(z)} \prec \varphi(z)\right\}$$

and

$$\mathcal{C}(\varphi) = \left\{h \in \mathfrak{A} : 1 + \frac{zh''(z)}{h'(z)} \prec \varphi(z)\right\}.$$

where $\varphi(z)$ is a positive real part function and is normalized by the condition

$$\varphi(0) = 1,\ \varphi'(0) > 0$$

and φ maps E onto a region that is starlike with respect to 1 and symmetric with respect to the real axis. Ravichandran et al. [2] gave the extension of above two classes in the following way:

$$\mathcal{S}^*(\gamma, \varphi) = \left\{h \in \mathfrak{A} : 1 + \frac{1}{\gamma}\left(\frac{zh'(z)}{h(z)} - 1\right) \prec \varphi(z);\ \ \gamma \in \mathbb{C}\backslash\{0\}\right\}$$

and

$$\mathcal{C}(\gamma, \varphi) = \left\{h \in \mathfrak{A} : 1 + \frac{1}{\gamma}\left(\frac{zh''(z)}{h'(z)}\right) \prec \varphi(z);\ \ \gamma \in \mathbb{C}\backslash\{0\}\right\}.$$

These types of functions are referred to as Ma–Minda starlike and convex functions of γ, $(\gamma \in \mathbb{C}\backslash\{0\})$, respectively.

The Koebe one-quarter theorem (see [3]) states that the image of E under every $h \in \mathcal{S}$ contains a disk of radius one-quarter centered at the origin. Thus, every function $h \in \mathcal{S}$ has an inverse $h^{-1} = g$,

$$g(h(z)) = z,\ \ z \in E$$

and

$$h(g(w)) = w,\ |w| < r_0(h),\ r_0(h) \geq \frac{1}{4}.$$

The series of the inverse function g is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots. \tag{2}$$

A function $h \in \mathfrak{A}$ is called bi-univalent in E if both h and h^{-1} are univalent in E and we denote the class of all bi-univalent functions by Σ.

Lewin [4] developed the idea of class Σ and established that $|a_2| < 1.51$ for every $h \in \Sigma$. Styer and Wright [5] demonstrated the existence of $h \in \Sigma$ for which $|a_2| > \frac{4}{3}$. Since the creation of the class Σ, several researchers have been trying to determine how the geometric properties of the functions in the class and the coefficient bounds are related. Indeed, a strong foundation for the study of bi-univalent functions was laid by authors such as Lewin [4], Brannan and Taha [6], and Srivastava et al. [7]. Only non-sharp estimates of the initial coefficients were produced in these recent works. Coefficient estimates for general subclasses of analytic bi-univalent functions were also obtained in [8]. More recently, in [9], coefficient estimates for general subclasses of analytic bi-univalent functions were also obtained using the integral operator based upon Lucas polynomials, while Oros and Cotirla [10] defined a new subclass of v-fold bi-univalent functions and obtained coefficient estimates and the Fekete–Szego problem. However, the problem of a sharp coefficient bound for $|a_m|$, $(m = 3, 4, 5, \dots)$ is still open.

Recently, Hamidi and Jahangiri [11,12] started to apply the Faber polynomial expansion method to find coefficient bounds $|a_m|$ for $m \geq 3$. The Faber polynomial method was introduced by Faber in [13] and its importance was discussed by Gong [14]. A number of new subclasses of bi-univalent functions have been introduced and studied by considering and involving the Faber polynomial expansion method. In the following article [15] Bult defined some new subclasses of bi-univalent functions and used the Faber polynomial technique to find general coefficient bounds $|a_m|$ for $m \geq 3$, and also discussed the unpredictable behavior of initial coefficient bounds. The general coefficient bounds $|a_m|$ $m \geq 3$ of analytic bi-univalent functions were also obtained recently, by using the subordination properties and Faber polynomial expansion method [16], and also using the same technique that Altinkaya and Yalcin [17] discussed concerning the interesting behavior of coefficient bounds for new subclasses of bi-univalent functions. Furthermore, many authors have applied the technique of Faber polynomials and determined some interesting results for bi-univalent functions.

Jackson [18] presented the idea of the q-calculus operator and defined D_q, while Ismail et al. [19] were the first to use the q-difference operator (D_q) to define a class of q-starlike functions. After that, many researchers introduced several subclasses of analytic functions related to q-calculus, (see, for details, [20–22]). The following articles on differential operators shall be used for the study of the applications of operators: [23–26].

In order to create some new subclasses of analytic and bi-univalent functions, the core definitions and ideas of q-calculus need to be discussed.

Definition 1. *For $\eta, q \in \mathbb{C}$, the q-shifted factorial $(\eta, q)_m$ is defined by*

$$(\eta, q)_m = \begin{cases} 1 & \text{if } m = 0, \\ (1-\eta)(1-\eta q)\dots(1-\eta q^{n-1}) & \text{if } (m \in \mathbb{N}). \end{cases} \tag{3}$$

If $\eta \neq q^{-l}$, $(l \in \mathbb{N}_0)$, then it can be written as:

$$(\eta, q)_\infty = \prod_{m=0}^{\infty} (1 - \eta q^m), \quad (\eta \in \mathbb{C} \text{ and } |q| < 1), \tag{4}$$

when $\eta \neq 0$ and $q \geq 1$, $(\eta, q)_\infty$ diverges. Therefore, whenever we use $(\eta, q)_\infty$ then $|q| < 1$ will be assumed.

Remark 1. *It is noted that when $q \to 1-$ in $(\eta, q)_m$, then (19) reduces to the Pochhammer symbol $(\eta)_m$ defined by*

$$(\eta)_m = \eta(\eta+1)\dots(\eta+m-1) \quad \text{if } m \in \mathbb{N}.$$

If $m = 0$, then $(\eta)_m = 1$.

Definition 2. *The $(\eta, q)_m$ in (19) is precise with respect to the q-Gamma function, which is given below*

$$\digamma_q(\eta) = \frac{(1-q)^{1-\eta}(q, q)_\infty}{(q^\eta, q)_\infty}, \quad (0 < q < 1),$$

or

$$(q^{\eta}, q)_m = \frac{(1-q)^m \digamma_q(\eta+m)}{\digamma_q(\eta)}, \quad (m \in \mathbb{N})$$

and q-factorial $[m]_q!$ *is defined by:*

$$\begin{aligned} [m]_q! &= \prod_{k=1}^{m} [k]_q, \text{ if } (m \in \mathbb{N}) \\ &= 1 \text{ if } m = 0. \end{aligned} \tag{5}$$

It is important to note that ordinary calculus is a limiting case of quantum calculus. It is expected that a study of quantum difference operators will be crucial to the growth of q-function theory, which is essential for combinatory analysis. In addition, the differential and integral operators are widely used in geometric function theory. The most significant feature of our study is that we are investigating the properties of new class of analytic bi-univalent functions under a certain q-derivative operator. Geometric-function-theory-related research on this topic has still not been performed extensively.

In this paper, we first define the q-derivative (q-difference) operator and then consider this operator to define a new class of analytic bi-univalent functions of class Σ.

Definition 3 ([18])**.** *For* $h \in \mathfrak{A}$, *the q-difference operator is defined as:*

$$D_q h(z) = \frac{h(qz) - h(z)}{z(q-1)}, \quad z \in E.$$

Note that, for $m \in \mathbb{N}$ *and* $z \in E$ *and*

$$D_q(z^m) = [m]_q z^{m-1}, \ D_q\left(\sum_{m=1}^{\infty} a_m z^m\right) = \sum_{m=1}^{\infty} [m]_q a_m z^{m-1},$$

where $(0 < q < 1)$, *is defined by*

$$[m]_q = \frac{1-q^m}{1-q}, \text{ and } [0]_q = 0$$

and the q-number shift factorial is given by

$$\begin{aligned} [m]_q! &= [1]_q [2]_q [3]_q \dots [m]_q, \\ [0]_q! &= 1. \end{aligned}$$

The q-generalized Pochhammer symbol is defined by

$$[x]_{q,m} = \frac{\digamma_q(x+m)}{\digamma_q(x)}, \ m \in \mathbb{N}, \ x \in \mathbb{C}. \tag{6}$$

Remark 2. *For* $q \to 1-$, *then* $[x]_{q,m}$ *reduces to* $(x)_m = \frac{\Gamma(x+m)}{\Gamma(x)}$.

Suppose that φ is an analytic function with a positive real part in the unit disk E satisfying

$$\varphi(0) = 1 \text{ and } \varphi'(0) > 0$$

and $\varphi(E)$ is symmetric with respect to the real axis and has the series

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \ \text{ and } \ (B_1 > 0). \tag{7}$$

The q-calculus operator theory is used to solve a wide range of problems in heat transfer and other areas of mathematical physics and engineering that include cylindrical and spherical coordinates. Several remarkable characteristics of new subclasses of analytic functions have been found using q-differential operators, including new subclasses of convex and starlike functions. One of the classic areas of geometric function theory is the study of particular subclasses of starlike functions and its generalization. Therefore, by means of the q-difference operator (D_q) defined in Definition 3 and inspired by the work introduced in [27], a new class of analytic bi-univalent functions of class Σ is introduced. The original results will be proved in the following section using the Faber polynomial approach and two lemmas.

Definition 4. *Let h be the form (1) and $h \in \mathfrak{J}(\lambda, \gamma, q; \varphi)$ if*

$$1 + \frac{1}{\gamma}\left(\frac{zD_q h(z) + \lambda z^2 D_q^2(h(z))}{(1-\lambda)h(z) + \lambda z D_q h(z)} - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wD_q g(w) + \lambda w^2 D_q^2(g(w))}{(1-\lambda)g(w) + \lambda w D_q g(w)} - 1\right) \prec \varphi(w),$$

where, $0 \le \lambda \le 1$, $\gamma \in \mathbb{C}\backslash\{0\}$, $z, w \in E$ and $g = h^{-1}$.

Note: If both h and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(\lambda, \gamma, q; \varphi)$, then h is called a generalized bi-subordinate function of complex order γ.

Remark 3. *For $\lambda = 0$, then we have $\mathfrak{J}(\lambda, \gamma, q; \varphi) = \mathfrak{J}(0, \gamma, q; \varphi)$ and for $\lambda = 1$, then we have $\mathfrak{J}(\lambda, \gamma, q; \varphi) = \mathfrak{J}(1, \gamma, q; \varphi)$.*

Remark 4. *For $q \to 1-$, then $\mathfrak{J}(\lambda, \gamma, q; \varphi) = \mathfrak{J}(\lambda, \gamma; \varphi)$, and introduced by Deniz in [28].*

2. The Faber Polynomial Expansion Method and Its Applications

For the function $h \in \mathfrak{A}$, Airault and Bouali ([29], page 184) used Faber polynomials to show that

$$\frac{zh'(z)}{h(z)} = 1 - \sum_{m=2}^{\infty} [R_{m-1}(a_2, a_3, \ldots a_m)] z^{m-1}, \tag{8}$$

where

$$R_{m-1}(a_2, a_3, \ldots a_m) = \sum_{i_1 + 2i_2 + \ldots (m-1)i_{m-1} = m-1}^{\infty} A(i_1, i_2, i_3, \ldots, i_{m-1}) \left(a_2^{i_1} a_3^{i_2} \ldots a_m^{i_{m-1}}\right)$$

and

$$\begin{aligned} A(i_1, i_2, i_3, \ldots, i_{m-1}) &= (-1)^{(m-1)+2i_1+\cdots+mi_{m-1}} \\ &\quad \times \left(\frac{(i_1 + i_2 + i_3, \cdots + i_{m-1} - 1)!(m-1)}{(i_1!)(i_2!)(i_3!), \ldots (i_{m-1}!)}\right). \end{aligned}$$

The first terms of the Faber polynomial R_{m-1}, $m \ge 2$, are given by (e.g., see ([30], page 52))

$$\begin{aligned} R_1 &= -a_2,\ R_2 = a_2^2 - 2a_3, \\ R_3 &= -a_2^3 + 3a_2a_3 - 3a_4), \\ R_3 &= a_2^4 - 4a_2^2a_3 + 4a_2a_4 + 2a_3^2 - 4a_5). \end{aligned}$$

Using the Faber polynomial technique for the analytic functions h, then the coefficients of its inverse map g can be written as follows (see ([29], page 185)):

$$g(w) = h^{-1}(w) = w + \sum_{m=2}^{\infty} \frac{1}{m} R_{m-1}^{m}(a_2, a_3, \ldots, a_m) w^m,$$

where the coefficients of the m parametric function are

$$\begin{aligned}
R_1^p &= pa_2, \\
R_2^p &= \frac{p(p-1)}{2} a_2^2 + pa_3, \\
R_3^p &= p(p-1)a_2 a_3 + pa_4 + \frac{p(p-1)(p-2)}{3!} a_2^3, \\
R_4^p &= p(p-1)a_2 a_4 + pa_5 + \frac{p(p-1)}{2} a_3^2 + \frac{p(p-1)(p-2)}{2} a_2^2 a_3 + \frac{p!}{(p-4)!4!} a_2^4, \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
R_{m-1}^p &= \frac{p!}{(p-m)!m!} a_2^m + \frac{p!}{(p-m+1)!(m-2)!} a_2^{m-2} a_3 + \frac{p!}{(p-n+2)!(m-3)!} a_2^{m-3} a_4 \\
&+ \frac{p!}{(p-n+3)!(m-4)!} a_2^{m-4} \left(a_5 + \frac{p-n+3}{2} a_3^2 \right) \\
&+ \frac{p!}{(p-n+4)!(m-5)!} a_2^{m-4} (a_6 + (p-m+3) a_3 a_4) + \sum_{\mathrm{i} \geq 6} a_2^{m-\mathrm{i}} Q_{\mathrm{i}},
\end{aligned}$$

and Q_{i} is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_m$, for $6 \leq \mathrm{i} \leq m$; see [31], page 349, and [29], pages 183 and 205. Particularly, the first three terms of R_{m-1}^p are

$$\begin{aligned}
\frac{1}{2} R_1^1 &= -a_2, \ \frac{1}{3} R_2^{-3} = 2a_2^2 - a_3, \\
\frac{1}{4} R_3^3 &= -(5a_2^3 - 5a_2 a_3 + a_4).
\end{aligned}$$

In general, for $r \in \mathbb{Z}$ ($\mathbb{Z} := 0, \pm 1, \pm 2, \ldots$ and $m \geq 2$, there is an expansion of R_m^r of the form:

$$R_m^r = ra_m + \frac{r(r-1)}{2} \mathcal{V}_m^2 + \frac{r!}{(r-3)!3!} \mathcal{V}_m^3 + \cdots + \frac{r!}{(r-m)!(m)!} \mathcal{V}_m^m,$$

where,

$$\mathcal{V}_m^r = \mathcal{V}_m^r(a_2, a_3 \ldots),$$

and by [32], we have

$$\mathcal{V}_m^v(a_2, \ldots, a_m) = \sum_{m=1}^{\infty} \frac{v!(a_2)^{\mu_1} \ldots (a_m)^{\mu_m}}{\mu_1!, \ldots, \mu_m!}, \quad \text{for } a_1 = 1 \text{ and } v \leq m.$$

The sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_m$, which satisfy

$$\begin{aligned}
\mu_1 + \mu_2 + \cdots + \mu_m &= v, \\
\mu_1 + 2\mu_2 + \cdots + m\mu_m &= m.
\end{aligned}$$

Clearly,

$$\mathcal{V}_m^m(a_1, \ldots, a_m) = \mathcal{V}_1^m,$$

and the first and last polynomials are

$$\mathcal{V}_m^m = a_1^m, \text{ and } \mathcal{V}_m^1 = a_m.$$

Geometric function theory has always placed a great deal of importance on establishing bounds for the coefficients. The size of the coefficients can determine a number of properties of analytic functions, including univalency, rate of growth, and distortion. Several scholars have employed a variety of techniques to resolve the aforementioned problems. Similar to univalent functions, the bounds of bi-univalent function coefficients have recently attracted a lot of attention. As a result of the significance of studying the coefficient problems described above, in this section, we consider the q-difference operator and Faber polynomial technique to obtain coefficient estimates $|\rho_m|$ of bi-univalent functions in the family $\mathfrak{J}(\lambda, \gamma, q; \varphi)$ and discuss the unpredictable behavior of initial coefficient bounds $|\rho_2|$ and Feke–Sezgo problems $\left|\rho_3 - \rho_2^2\right|$ in this family, subject to gap series conditions.

Using the Ruscheweyh differential operator, and Ruscheweyh q-differential operator, many scholars have defined new classes of convex and starlike functions. In this study, we also use the Ruscheweyh q-differential operator along with the Faber polynomial method and discuss the applications of our main results. We also investigate the Feketo–Sezego problem and some known consequences of our main results.

Set of Lemmas

The following well-known lemmas are required to prove our main theorems:

Lemma 1 ([3])**.** *Let the function* $p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m$ *and* $Re(p(z)) > 0$ *for* $z \in E$, *then for* $-\infty < \alpha < \infty$

$$\left|p_2 - \alpha p_1^2\right| \le \begin{cases} 2 - \alpha|p_1|^2 & \text{if } \alpha < \frac{1}{2} \\ 2 - (1-\alpha)|p_1|^2 & \text{if } \alpha \ge \frac{1}{2}. \end{cases}$$

Lemma 2 ([28])**.** *Let the function* $\phi(z) = \sum_{m=1}^{\infty} \phi_m z^m$ *so that* $|\phi(z)| < 1$ *for* $z \in E$, *then*

$$\left|\phi_2 + \beta\phi_1^2\right| \le \begin{cases} 1 - (1-\delta)|\beta|^2 & \text{if } \beta > 0 \\ 1 - (1+\delta)|\beta|^2 & \text{if } \beta \le 0. \end{cases} \tag{9}$$

This paper uses the q-difference operator for $\xi \in \mathfrak{A}$, and the new class $\mathfrak{J}(\lambda, \gamma, q; \varphi)$ of generalized bi-subordinate functions of complex order γ is defined. Next, in Theorem 1, upper bounds ρ_m for generalized bi-subordinate functions of complex order γ are proved and in Theorem 2 the initial coefficient bound $|\rho_2|$ and Feke–Sezgo problems $\left|\rho_3 - \rho_2^2\right|$ are investigated by putting the special value of parameters in the class $\mathfrak{J}(\lambda, \gamma, q; \varphi)$, and we obtain some new and known results. In Section 4, we use the Ruscheweyh q-differential operator and investigate some new characteristics of the class of generalized bi-subordinate functions of complex order γ in the form of some new results. In Section 5, we give concluding remarks.

3. Main Results

Theorem 1. *Let* $0 \le \lambda \le 1, \gamma \in \mathbb{C}\backslash\{0\}$. *If both function* $h(z) = z + \sum_{m=2}^{\infty} \rho_m z^m$ *and its inverse map* $g = h^{-1}$ *are in* $\mathfrak{J}(\lambda, \gamma, q; \varphi)$ *and* $\rho_k = 0, 2 \le k \le m-1$, *then*

$$|\rho_m| \le \frac{|\gamma| B_1}{([m]_q - 1)\left(1 + \lambda\left([m]_q - 1\right)\right)}, \quad (B_1 > 0).$$

Proof. If we write

$$\Lambda(h(z)) = (1-\lambda)h(z) + \lambda z D_q h(z),$$

then

$$h \in \mathfrak{J}(\lambda, \gamma, q; \varphi) \Leftrightarrow 1 + \frac{1}{\gamma}\left(\frac{zD_q(\Lambda(h(z)))}{\Lambda(h(z))} - 1\right) \prec \varphi(z)$$

and

$$g = h^{-1} \in \mathfrak{J}(\lambda, \gamma, q; \varphi) \Leftrightarrow 1 + \frac{1}{\gamma}\left(\frac{wD_q(\Lambda(g(w)))}{\Lambda(g(w))} - 1\right) \prec \varphi(w).$$

We notice that

$$a_m = 1 + \lambda\Big([m]_q - 1\Big)\rho_n$$

for

$$\Lambda(h(z)) = z + \sum_{m=2}^{\infty} a_m z^m.$$

Now, using the Faber polynomial expansion (8) for the power series $\mathfrak{J}(\lambda, \gamma, q; \varphi)$ yields:

$$1 + \frac{1}{\gamma}\left(\frac{zD_q(\Lambda(h(z)))}{\Lambda(h(z))} - 1\right) = 1 - \frac{1}{\gamma}\sum_{m=2}^{\infty}[R_{m-1}(a_2, a_3, \ldots a_m)]z^{m-1}, \tag{10}$$

and for the inverse map $g = h^{-1}$, obviously, we have

$$1 + \frac{1}{\gamma}\left(\frac{wD_q(\Lambda(h(w)))}{\Lambda(h(w))} - 1\right) = 1 - \frac{1}{\gamma}\sum_{m=2}^{\infty} R_{m-1}(b_2, b_3, \ldots b_m)w^{m-1} \tag{11}$$

where

$$b_m = 1 + \lambda\Big([m]_q - 1\Big)\tau_n = \frac{1}{m}R^m_{m-1}(a_2, a_3, \ldots, a_m).$$

By the definition of subordination, there exist two Schwarz functions

$$u(z) = \sum_{m=1}^{\infty} c_m z^m$$

and

$$v(w) = \sum_{m=1}^{\infty} d_m w^m,$$

Additionally, we have

$$\varphi(u(z)) = 1 - B_1 \sum_{m=1}^{\infty} R_m^{-1}(c_1, -c_2, \ldots (-1)^{m+1}c_m, B_1, B_2, \ldots B_m)z^m \tag{12}$$

and

$$\varphi(v(w)) = 1 - B_1 \sum_{m=1}^{\infty} R_m^{-1}(d_1, -d_2, \ldots, (-1)^{m+1}d_m, B_1, B_2, \ldots B_m)w^m. \tag{13}$$

In general (e.g., see [28]), the coefficients $R_m^p = R_m^p(k_1, k_2, \ldots, k_m, B_1, B_2, \ldots B_m)$ are given by

$$\begin{aligned} R_m^p \quad = \quad & \frac{p!}{(p-m)!(m)!}k_1^m\frac{B_m}{B_1} + \frac{p!}{(p-m+1)!(m-2)!}k_1^{m-2}k_2\frac{B_{m-1}}{B_1} \\ & + \frac{p!}{(p-m+2)!(m-4)!}k_1^{m-3}k_3\left(\frac{B_{m-2}}{B_1}\right) \\ & + \frac{p!}{(p-m+3)!(m-4)!}k_1^{m-4}\left[k_4\left(\frac{B_{m-3}}{B_1}\right) + \left(\frac{p-m+3}{2}\right)k_2^2\left(\frac{B_{m-2}}{B_1}\right)\right] \\ & + \frac{p!}{(p-m+4)!(m-5)!}k_1^{m-5}\left[k_5\left(\frac{B_{m-4}}{B_1}\right) + (p-m+4)k_2k_3\left(\frac{B_{m-3}}{B_1}\right)\right] \\ & + \sum_{j\geq 6} k_1^{m-j}Q_j, \end{aligned}$$

where Q_j in the variables $k_2, k_3, \ldots k_m$ is a homogeneous polynomial of degree j.

Evaluating the coefficients of Equations of (10) and (12) yields

$$\frac{1}{\gamma}R_{m-1}(a_2, a_3, \ldots a_m) = B_1 R_m^{-1}(c_1, -c_2, \ldots (-1)^m c_m, B_1, B_2, \ldots B_m). \tag{14}$$

However, using the facts $|c_m| \leq 1$ and $|d_m| \leq 1$ (e.g., see [3]), and under the assumption $2 \leq k \leq m-1$ and $a_k = 0$, respectively, we have

$$\frac{1}{\gamma}\left([m]_q - 1\right)a_m = \frac{1}{\gamma}\left([m]_q - 1\right)\left(1 + \lambda\left([m]_q - 1\right)\right)\rho_m = -B_1 c_{m-1}. \tag{15}$$

Evaluating the coefficients of Equations (11) and (13) yields

$$\frac{1}{\gamma}R_{m-1}(b_2, b_3, \ldots b_m) = B_1 R_m^{-1}(d_1, -d_2, \ldots (-1)^m d_m, B_1, B_2, \ldots B_m), \tag{16}$$

which by the hypothesis, we obtain

$$-\frac{1}{\gamma}\left([m]_q - 1\right)b_m = -B_1 d_{m-1}.$$

Note that, for, $2 \leq k \leq m-1$, $b_m = -a_m$ and $a_k = 0$; therefore

$$\frac{1}{\gamma}\left([m]_q - 1\right)a_m = \frac{1}{\gamma}\left([m]_q - 1\right)\left(1 + \lambda\left([m]_q - 1\right)\right)\rho_m = -B_1 d_{m-1}. \tag{17}$$

Taking the absolute values of either of Equations (15) or (17) we obtain the required bound.

This completes Theorem 1. □

For $\lambda = 0$, in Theorem 1, we obtain a new corollary, which is given below.

Corollary 1. *Let $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z)$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(0, \gamma, q; \varphi)$ and $\rho_k = 0, 2 \leq k \leq m-1$, then*

$$|\rho_m| \leq \frac{|\gamma| B_1}{[m]_q - 1}, \quad B_1 > 0.$$

For $\lambda = 1$, in Theorem 1, we obtain a new corollary, which is given below.

Corollary 2. *Let $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z)$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(1,\gamma,q;\varphi)$ and $\rho_k = 0, 2 \le k \le m-1$, then*

$$|\rho_m| \le \frac{|\gamma|B_1}{[m]_q\left([m]_q - 1\right)}, \quad B_1 > 0.$$

For $q \to 1-$ in Theorem 1, we obtain a known corollary that was proven in [28].

Corollary 3 ([28])**.** *Let $0 \le \lambda \le 1$, $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z)$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(\lambda,\gamma;\varphi)$ and $\rho_k = 0, 2 \le k \le m-1$, then*

$$|\rho_m| \le \frac{|\gamma|B_1}{(m-1)(1+\lambda(m-1))}, \quad B_1 > 0.$$

For $\lambda = 0$, and $q \to 1-$ in Theorem 1, we obtain a known corollary that was proven in [28].

Corollary 4 ([28])**.** *Let $0 \le \lambda \le 1$, $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z)$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(\gamma;\varphi)$ and $\rho_k = 0, 2 \le k \le m-1$, then*

$$|\rho_m| \le \frac{|\gamma|B_1}{m-1}, \quad B_1 > 0.$$

Theorem 2. *Let $0 \le \lambda \le 1$, $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z) = z + \sum_{m=2}^{\infty} \rho_m z^m$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(\lambda,\gamma,q;\varphi)$, then*

$$|\rho_2| \le \left\{ \begin{array}{ll} \sqrt{\frac{|\gamma|B_1}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}} & \text{if } B_1 \ge |B_2| \\ \sqrt{\frac{|\gamma|B_2}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}} & \text{if } B_1 < |B_2| \end{array} \right\}$$

and

$$\left|\rho_3 - \rho_2^2\right| \le \left\{ \begin{array}{ll} \frac{|\gamma||B_1|}{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)} & \text{if } B_1 \ge |B_2| \\ \frac{|\gamma||B_2|}{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)} & \text{if } B_1 < |B_2| \end{array} \right\}.$$

Proof. For $m = 2$, Equations (14) and (16), respectively, yield

$$\rho_2 = \frac{\gamma B_1 c_1}{\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)} \text{ and } \rho_2 = \frac{-\gamma B_1 d_1.}{\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)}. \tag{18}$$

If we take the absolute values of any of these two equations, and apply $|c_m| \le 1$ and $|d_m| \le 1$ (e.g., see Duren [3]), we obtain

$$|\rho_2| \le \frac{|\gamma B_1|}{\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)}.$$

For $m = 3$, Equations (14) and (16), respectively, yield

$$\frac{1}{\gamma}\left(\begin{array}{c} \left([3]_q - 1\right)\left(1 + \lambda\left([3]_q - 1\right)\right)\rho_3 \\ -\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)^2\rho_2^2 \end{array} \right) = B_1 c_2 + B_2 c_1^2 \tag{19}$$

and

$$\frac{1}{\gamma}\left(\begin{array}{c} -\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)\rho_3 \\ +\left\{\begin{array}{c} 2\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right) \\ -\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2 \end{array}\right\}\rho_2^2 \end{array}\right)=B_1d_2+B_2d_1^2. \tag{20}$$

By combining the two equations mentioned above and finding $|\rho_2|$, we arrive at

$$\rho_2^2=\frac{\gamma\left(B_1c_2+B_2c_1^2+B_1d_2+B_2d_1^2\right)}{2\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}.$$

Or

$$|\rho_2|^2\leq\frac{|\gamma|B_1\left(\left|c_2+\frac{B_2}{B_1}c_1^2\right|+\left|d_2+\frac{B_2}{B_1}d_1^2\right|\right)}{2\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}. \tag{21}$$

If $B_2\leq 0$, and $\delta=\frac{B_2}{B_1}$, then by using Lemma 2 for (21), we obtain

$$|\rho_2|^2\leq\frac{|\gamma|B_1\left[1-\left(\frac{B_1+B_2}{B_1}\right)|c_1|^2\right]+\left[1-\left(\frac{B_1+B_2}{B_1}\right)|d_1|^2\right]}{2\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}. \tag{22}$$

If $B_1+B_2>0$, then (22) yields

$$|\rho_2|\leq\sqrt{\frac{|\gamma|B_1}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}}.$$

If $B_1+B_2<0$, then for the maximum values of $|c_1|=|d_1|$

$$\begin{aligned}|\rho_2|^2 &\leq \frac{2|\gamma|B_1\left[1-\left(\frac{B_1+B_2}{B_1}\right)\right]}{2\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}} \\ &= \frac{-|\gamma||B_2|}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}.\end{aligned}$$

If $B_2>0$, and $\delta=\frac{B_2}{B_1}$, then by using Lemma 2 on (21), we obtain

$$|\rho_2|^2\leq\frac{|\gamma|B_1\left\{\left[1-\left(\frac{B_1-B_2}{B_1}\right)|c_1|^2\right]+\left[1-\left(\frac{B_1-B_2}{B_1}\right)|d_1|^2\right]\right\}}{2\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}. \tag{23}$$

If $B_1-B_2>0$, then (23) yields

$$|\rho_2|\leq\sqrt{\frac{|\gamma|B_1}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}}.$$

If $B_1 - B_2 < 0$, then for the maximum values of $|c_1| = |d_1|$, we have

$$\begin{aligned} |\rho_2|^2 &\leq \frac{2|\gamma|B_1\left[1-\left(\frac{B_1+B_2}{B_1}\right)\right]}{2\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}} \\ &= \frac{|\gamma||B_2|}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}. \end{aligned}$$

Therefore

$$|\rho_2| \leq \sqrt{\frac{|\gamma||B_2|}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\right\}}}.$$

Now we subtract (19) and (20), and $B_1 > 0$, we have

$$\rho_3 - \rho_2^2 = \frac{\gamma B_1}{2\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)}\left\{\left(c_2+\frac{B_2}{B_1}c_1^2\right)-\left(d_2+\frac{B_2}{B_1}d_1^2\right)\right\}. \tag{24}$$

If we take the absolute values of the two sides of (24), we obtain

$$\left|\rho_3-\rho_2^2\right| \leq \frac{|\gamma|B_1\left\{\left|c_2+\frac{B_2}{B_1}c_1^2\right|+\left|d_2+\frac{B_2}{B_1}d_1^2\right|\right\}}{2\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)}. \tag{25}$$

If $B_2 \leq 0$, and $\delta = \frac{B_2}{B_1}$, then by using Lemma 2 on (25), we obtain

$$\left|\rho_3-\rho_2^2\right| \leq \frac{|\gamma|B_1\left\{\left[1-\left(\frac{B_1+B_2}{B_1}\right)|c_1|^2\right]+\left[1-\left(\frac{B_1+B_2}{B_1}\right)|d_1|^2\right]\right\}}{2\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)}. \tag{26}$$

If $B_1 + B_2 > 0$, then (26) yields

$$\left|\rho_3-\rho_2^2\right| \leq \frac{|\gamma|B_1}{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)}.$$

If $B_1 + B_2 < 0$, then for the maximum values of $|c_1| = |d_1|$, inequality (26) yields

$$\begin{aligned} \left|\rho_3-\rho_2^2\right| &\leq \frac{2|\gamma|B_1\left[1-\left(\frac{B_1+B_2}{B_1}\right)\right]}{2\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)} \\ &= \frac{-|\gamma||B_2|}{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)}. \end{aligned}$$

If $B_2 > 0$, and $\delta = \frac{B_2}{B_1}$, then by using Lemma 2 for (25) we obtain

$$\left|\rho_3-\rho_2^2\right| \leq \frac{|\gamma|B_1\left\{\left[1-\left(\frac{B_1-B_2}{B_1}\right)|c_1|^2\right]+\left[1-\left(\frac{B_1-B_2}{B_1}\right)|d_1|^2\right]\right\}}{2\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)}. \tag{27}$$

If $B_1 - B_2 > 0$, then (27) yields

$$\left|\rho_3 - \rho_2^2\right| \leq \frac{|\gamma|B_1}{\left([3]_q - 1\right)\left(1 + \lambda\left([3]_q - 1\right)\right)}.$$

If $B_1 - B_2 < 0$, then for the maximum values of $|c_1| = |d_1|$, the inequality (27) yields

$$\begin{aligned} \left|\rho_3 - \rho_2^2\right| &\leq \frac{2|\gamma|B_1\left[1 - \left(\frac{B_1+B_2}{B_1}\right)\right]}{2\left([3]_q - 1\right)\left(1 + \lambda\left([3]_q - 1\right)\right)} \\ &= \frac{|\gamma||B_2|}{\left([3]_q - 1\right)\left(1 + \lambda\left([3]_q - 1\right)\right)}. \end{aligned}$$

This concludes the proof of Theorem 2. □

Taking $\lambda = 0$ in Theorem 2, we obtain a new corollary.

Corollary 5. *Let $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z) = z + \sum_{m=2}^{\infty} \rho_m z^m$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(\gamma, q; \varphi)$, then*

$$|\rho_2| \leq \left\{ \begin{array}{ll} \sqrt{\frac{|\gamma||B_1|}{[3]_q - [2]_q}} & \text{if } B_1 \geq |B_2| \\ \sqrt{\frac{|\gamma||B_2|}{[3]_q - [2]_q}} & \text{if } B_1 < |B_2| \end{array} \right\}$$

and

$$\left|\rho_3 - \rho_2^2\right| \leq \left\{ \begin{array}{ll} \frac{|\gamma||B_1|}{[3]_q - 1} & \text{if } B_1 \geq |B_2| \\ \frac{|\gamma||B_2|}{[3]_q - 1} & \text{if } B_1 < |B_2| \end{array} \right\}.$$

Taking $\lambda = 1$ in Theorem 2, we obtain the following new corollary.

Corollary 6. *Let $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z) = z + \sum_{m=2}^{\infty} \rho_m z^m$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(1, \gamma, q; \varphi)$, then*

$$|\rho_2| \leq \left\{ \begin{array}{ll} \sqrt{\frac{|\gamma|B_1}{\left\{\left([3]_q - 1\right)[3]_q - \left([2]_q - 1\right)[2]_q^2\right\}}} & \text{if } B_1 \geq |B_2| \\ \sqrt{\frac{|\gamma||B_2|}{\left\{[3]_q\left([3]_q - 1\right) - \left([2]_q - 1\right)[2]_q^2\right\}}} & \text{if } B_1 < |B_2| \end{array} \right\}$$

and

$$\left|\rho_3 - \rho_2^2\right| \leq \left\{ \begin{array}{ll} \frac{|\gamma|B_1}{[3]_q\left([3]_q - 1\right)} & \text{if } B_1 \geq |B_2| \\ \frac{|\gamma||B_2|}{[3]_q\left([3]_q - 1\right)} & \text{if } B_1 < |B_2| \end{array} \right\}.$$

Taking $q \to 1-$ in Theorem 2, we obtain the known corollary proved in [28].

Corollary 7. *Let $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$. If both function $h(z) = z + \sum_{m=2}^{\infty} \rho_m z^m$ and its inverse map $g = h^{-1}$ are in $\mathfrak{J}(\lambda, \gamma; \varphi)$, then*

$$|\rho_3| \leq \left\{ \begin{array}{ll} \sqrt{\frac{|\gamma|B_1}{2(1+2\lambda)-(1+\lambda)^2}} & if\ B_1 \geq |B_2| \\ \sqrt{\frac{|\gamma||B_2|}{2(1+2\lambda)-(1+\lambda)^2}} & if\ B_1 < |B_2| \end{array} \right\}$$

and

$$\left|\rho_3 - \rho_2^2\right| \leq \left\{ \begin{array}{ll} \frac{|\gamma|B_1}{2(1+2\lambda)} & if\ B_1 \geq |B_2| \\ \frac{|\gamma||B_2|}{2(1+2\lambda)} & if\ B_1 < |B_2| \end{array} \right\}.$$

4. Applications

Kanas and Raducanu [21] defined the Ruscheweyh q-differential operator as follows: For $f \in \mathcal{A}$,

$$\mathcal{R}_q^{\delta} h(z) = h(z) * F_{q,\delta+1}(z) \qquad (\delta > -1,\ z \in E) \tag{28}$$

where

$$F_{q,\delta+1}(z) = z + \sum_{m=2}^{\infty} \frac{\digamma_q(m+\delta)}{[m-1]!\,\Gamma_q(1+\delta)} z^m = z + \sum_{m=2}^{\infty} \frac{[\delta+1]_{m-1}}{[m-1]!} z^m. \tag{29}$$

We note that

$$\lim_{q\to 1-} F_{q,\delta+1}(z) = \frac{z}{(1-z)^{\delta+1}}, \qquad \lim_{q\to 1-} \mathcal{R}_q^{\delta} h(z) = h(z) * \frac{z}{(1-z)^{\delta+1}}.$$

Making use of (28) and (29), we have

$$\mathcal{R}_q^{\delta} h(z) = z + \sum_{m=2}^{\infty} \frac{\digamma_q(m+\delta)}{[m-1]!\,\Gamma_q(1+\delta)} a_m z^m = z + \sum_{m=2}^{\infty} \psi_m a_m z^m \qquad (z \in E), \tag{30}$$

where $\digamma_q$ is the q-generalized Pochhammer symbol defined in (6) and

$$\psi_m = \frac{\digamma_q(m+\delta)}{[m-1]_q!\,\Gamma_q(1+\delta)}. \tag{31}$$

From (30), we note that

$$\begin{array}{rcl} \mathcal{R}_q^{0} h(z) & = & h(z), \\ \mathcal{R}_q^{1} h(z) & = & zD_q h(z), \\ \mathcal{R}_q^{\delta} h(z) & = & \dfrac{zD_q^{\delta}(z^{\delta-1}h(z))}{[\delta]_q!} \quad (\delta \in \mathbb{N}). \end{array}$$

We also have

$$D_q(\mathcal{R}_q^{\delta} h(z)) = 1 + \sum_{m=2}^{\infty} [m]_q \psi_m a_m z^{m-1}. \tag{32}$$

Remark 5. *When $q \to 1-$, then the Ruscheweyh q-differential operator reduces to the differential operator defined by Ruscheweyh [33].*

Definition 5. *Let h be of the form (1) and $h \in \mathfrak{J}(\lambda, \gamma, \psi, q; \varphi)$ if*

$$1 + \frac{1}{\gamma}\left(\frac{zD_q\left(\mathcal{R}_q^{\delta} h(z)\right) + \lambda z^2 D_q^2\left(\mathcal{R}_q^{\delta} h(z)\right)}{(1-\lambda)\mathcal{R}_q^{\delta} h(z) + \lambda z D_q\left(\mathcal{R}_q^{\delta} h(z)\right)} - 1\right) \prec \varphi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{zD_q\left(\mathcal{R}_q^{\delta}g(z)\right)+\lambda w^2 D_q^2\left(\mathcal{R}_q^{\delta}g(w)\right)}{(1-\lambda)\left(\mathcal{R}_q^{\delta}g(w)\right)+\lambda wD_q\left(\mathcal{R}_q^{\delta}g(w)\right)}-1\right)\prec\varphi(w),$$

where $0\le\lambda\le1$, $\gamma\in\mathbb{C}\backslash\{0\}$, $z,w\in E$ *and* $g=h^{-1}$.

Theorem 3. *Let* $0\le\lambda\le1$, $\gamma\in\mathbb{C}\backslash\{0\}$. *If both function* $h(z)=z+\sum_{m=2}^{\infty}\rho_m z^m$ *and its inverse map* $g=h^{-1}$ *are in* $\mathfrak{J}(\lambda,\gamma,\psi,q;\varphi)$ *and* $\rho_k=0$, $2\le k\le m-1$, *then*

$$|\rho_m|\le\frac{|\gamma|B_1}{\psi_m\left([m]_q-1\right)\left(1+\lambda\left([m]_q-1\right)\right)}.$$

Proof. If we write

$$\Lambda(h(z))=(1-\lambda)\mathcal{R}_q^{\delta}h(z)+\lambda zD_q\left(\mathcal{R}_q^{\delta}h(z)\right),$$

then

$$h\in\mathfrak{J}(\lambda,\gamma,\psi,q;\varphi)\Leftrightarrow1+\frac{1}{\gamma}\left(\frac{zD_q\left(\Lambda\left(\mathcal{R}_q^{\delta}h(z)\right)\right)}{\Lambda\left(\mathcal{R}_q^{\delta}h(z)\right)}-1\right)\prec\varphi(z)$$

and

$$g=h^{-1}\in\mathfrak{J}(\lambda,\gamma,\psi,q;\varphi)\Leftrightarrow1+\frac{1}{\gamma}\left(\frac{wD_q\left(\Lambda\mathcal{R}_q^{\delta}g(w)\right)}{\Lambda(g(w))}-1\right)\prec\varphi(w).$$

We see that

$$a_m=\psi_m\left(1+\lambda\left([m]_q-1\right)\right)$$

for

$$\Lambda(h(z))=z+\sum_{m=2}^{\infty}a_m z^m.$$

Now, an application of Faber polynomial expansion to the power series $\mathfrak{J}(\lambda,\gamma,\psi,q;\varphi)$ yields:

$$1+\frac{1}{\gamma}\left(\frac{zD_q(\Lambda(h(z)))}{\Lambda(h(z))}-1\right)=1-\frac{1}{\gamma}\sum_{m=2}^{\infty}[R_{m-1}(a_2,a_3,\dots a_m)]z^{m-1},$$

After that, by using the similar method of Theorem 1, we can obtain Theorem 3. □

Theorem 4. *Let* $0\le\lambda\le1$, $\gamma\in\mathbb{C}\backslash\{0\}$. *If both function* $h(z)=z+\sum_{m=2}^{\infty}\rho_m z^m$ *and its inverse map* $g=h^{-1}$ *are in* $\mathfrak{J}(\lambda,\gamma,\psi,q;\varphi)$, *then*

$$|\rho_2|\le\begin{cases}\sqrt{\frac{|\gamma|B_1}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)\psi_3-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\psi_2^2\right\}}} & \text{if } B_1\ge|B_2|\\ \sqrt{\frac{|\gamma|B_2}{\left\{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)\psi_3-\left([2]_q-1\right)\left(1+\lambda\left([2]_q-1\right)\right)^2\psi_2^2\right\}}} & \text{if } B_1<|B_2|\end{cases}$$

and

$$\left|\rho_3-\rho_2^2\right|\le\begin{cases}\frac{|\gamma|B_1}{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)\psi_3} & \text{if } B_1\ge|B_2|\\ \frac{|\gamma|B_2}{\left([3]_q-1\right)\left(1+\lambda\left([3]_q-1\right)\right)v} & \text{if } B_1<|B_2|\end{cases}.$$

Proof. For $m = 2$, Equations (14) and (16), respectively, yield

$$\rho_2 = \frac{\gamma B_1 c_1}{\psi_2\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)} \text{ and } \rho_2 = \frac{-B_1 d_1.}{\psi_2\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)}.$$

If we take the absolute values of any of these two equations, and apply $|c_m| \leq 1$ and $|d_m| \leq 1$ (e.g., see Duren [3]), we obtain

$$|\rho_2| \leq \frac{|\gamma B_1|}{\psi_2\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)}.$$

For $m = 3$, Equations (14) and (16), respectively, yield

$$\frac{1}{\gamma}\begin{pmatrix} \left([3]_q - 1\right)\left(1 + \lambda\left([3]_q - 1\right)\right)\psi_3\rho_3 \\ -\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)^2\psi_2^2\rho_2^2 \end{pmatrix} = B_1 c_2 + B_2 c_1^2$$

and

$$\frac{1}{\gamma}\left(\begin{array}{c} -\left([3]_q - 1\right)\left(1 + \lambda\left([3]_q - 1\right)\right)\psi_3\rho_3 \\ +\left\{\begin{array}{c} 2\left([3]_q - 1\right)\left(1 + \lambda\left([3]_q - 1\right)\right)\psi_3 \\ -\left([2]_q - 1\right)\left(1 + \lambda\left([2]_q - 1\right)\right)^2\psi_2^2 \end{array}\right\}\rho_2^2 \end{array}\right) = B_1 d_2 + B_2 d_1^2.$$

By using the similar method of Theorem 2, we can obtain the required result of Theorem 4. □

5. Conclusions

In order to introduce a new class of generalized bi-subordinate functions of complex order γ in the open unit disk E, we used the idea of convolution and q-calculus in the current work. We produced estimates for the general coefficients in their Taylor–Maclaurin series expansions in the open unit disk E for functions that belong to the class of analytic and bi-univalent functions. Our approach is mostly based on the Faber polynomial expansion technique. In addition, we listed some corollaries and applications of our primary findings.

The application of the idea of subordination and the Faber polynomial technique for producing findings involving the newly defined operators can be identified when additional research proposals are produced. Additionally, the method that has been presented in this paper might also apply to define a number of new subclasses of meromorphic, multivalent, and harmonic functions and can be investigated for a number of new properties of these classes. The only innovation in the types of studies that can be conducted in these classes will come from the researchers themselves and how the findings presented here motivate them.

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