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A Note on *k*-Bonacci Random Walks

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Abstract: In this work, the probability of return for random walks on $\mathbb{Z}$, whose increment is given by the *k*-bonacci sequence, is determined. Additionally, the Hausdorff, packing and box-counting dimensions of the set of these walks that return an infinite number of times to the origin are given. As an application, we study the return for tribonacci random walks to the first term of the tribonacci sequence.

Keywords: random walks; *k*-bonacci sequence; probability of return; fractal dimension

1. Introduction and Main Results

Random walks are one of the basic objects studied in probability theory. The motivation comes from observations of various random motions in the physical and biological sciences. In 1921, Polya was one of the first to furnish a thorough analysis of Markovian time-discrete random walks on periodic d-dimensional lattices. In these 'Polya walks', the walker is allowed to step with equal probability only to any of its neighbor nodes. He proved for such random walk that the walker is sure to return to its starting node for dimensions d = 1, 2 for the lattice, whereas for dimensions d > 2, a finite escape probability (probability of never returning) exists. This celebrated result has become known as the Polya theorem or recurrence theorem. Since then, it has been studied by several authors, highlighting its importance in several fields [1–7]. In this paper, we consider the *k*-bonacci random walks and we compute the exact probability of return to the origin.

The Fibonacci sequence, commonly denoted by $(f_n)_{n\geq 0}$, is a sequence of integers such that each of them is the sum of the two preceding ones starting from zero and one, where

$$\begin{cases} f_0 = 0, \quad f_1 = 1, \\ f_n = f_{n-1} + f_{n-2}, \qquad for \quad n \geq 2. \end{cases}$$

This sequence was first introduced by Leonardo Fibonacci and is tightly connected to the golden ratio $\varphi = (1+\sqrt{5})/2 = 1.61803398\ldots$. Since then, many researchers have been interested in the study of the properties of this sequence and its applications. For example, one can cite [8], where it was proven that $(f_n)_{n\geq 0}$ increases exponentially with n at a rate given by φ. A more general case was explored in [9], where the author considered the random Fibonacci sequence $(t_n)_{n\geq 0}$ defined by $t_1 = t_2 = 1$ and the following for $n > 2$:

$$t_n = \pm t_{n-1} \pm t_{n-2},$$

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where each $\pm$ sign is independent and either positive or negative with a probability of $1/2$. It is not even obvious that $|t_n|$ should increase with n. However, it was proven that the following is almost certainly true:

$$\lim_{n \to +\infty} \sqrt[n]{|t_n|} = 1.13198824\ldots$$

Later, in [10], the author considered the Fibonacci random walk and determined the probability of its return to the origin. More precisely, he considered the random walk on $\mathbb{Z}$ whose increments are given by $(f_n)_{n \geq 0}$ such that

$$\hat{F}_n = \sum_{i=1}^{n} f_i w_i,$$

where $(w_i)_{i \geq 1}$ is a sequence of independent, identically distributed random variables with $\mathbb{P}(w_i = \pm 1) = 1/2$.

We denote with $\mathbb{N}$ the set of positive integers and consider the space of infinite sequences $\mathcal{A} = \{-1, 1\}^{\mathbb{N}}$. We define for an elementary event $w \in \mathcal{A}$ the set

$$F(w) = \{n \geq 1,\ \hat{F}_n(w) = 0\}.$$

Denoting with "$\sharp B$" the cardinality of a given set B, we set

$$R_i = \{w \in \mathcal{A} \mid \sharp F(w) = i\}, \qquad i \in \mathbb{N}$$

and

$$\mathsf{N} = \Big\{w \in \mathcal{A} \mid \sharp F(w) = \infty\Big\}.$$

It is known from [10] that the probability of R_i is $3/4^{i+1}$. In particular, $\mathbb{P}(\mathsf{N}) = 0$. The idea of studying such types of problems comes from a classical result from Polya [11], who was interested in a simple random walk of a length $n \geq 1$ on the integers, given by

$$S_n = \sum_{j=1}^{n} w_j.$$

This means that S_n is seen as the position after n steps of a walk on the integers of an individual, who is supposed to start its motion at the origin of the lattice (i.e., $S_0 = w_0 = 0$). Pólya [11] showed this walk to be recurrent, which means that it almost surely returns to the origin in a finite number of steps. The reader can also see, for example, refs. [12–14] for more discussions on this problem.

In this paper, we are interested, for a given integer $k \geq 2$, in the k-bonacci random walk given by $\hat{F}_n$, where we take into consideration the k-bonacci sequence $(f_n)_{n \geq 0}$ defined by $f_0 = 0$:

$$f_n = \sum_{j=1}^{k} f_{n-j}, \qquad \text{for} \quad n \geq k+1. \tag{1}$$

as well as the k initializing terms $(f_n)_{1 \leq n \leq k}$ which are supposed to satisfy the following condition:

$$\sum_{j=1}^{n} f_j < f_{n+2} \ \text{and} \ \sum_{j=1}^{n} \pm f_j \neq 0 \qquad \text{for } 1 \leq n \leq k, \tag{2}$$

The condition in Equation (2) is guaranteed, for example, in the following situation:

$$f_n = 1 + \sum_{j=0}^{n-1} f_j, \qquad \text{for} \quad 1 \leq n \leq k.$$

We study the probability of return of the k-bonacci random walk to the origin. For this, we establish a necessary and sufficient condition for the k-bonacci random walk to reach zero at least one time (Proposition 1). Our first main result is the following:

Theorem 1. *For* $i \in \mathbb{N}$, $\quad \mathbb{P}(R_i) = \dfrac{2^k - 1}{2^{k(i+1)}}$.

Next, we are interested in the set N of walks returning infinitely many times to zero. Since $\mathbb{P}(\mathsf{N}) = 0$, it is natural to ask a question about the size of N as a subset of $\mathcal{A}$. Denoting with $\dim_H$, $\dim_P$ and $\dim_B$ the Hausdorff, packing and box-counting dimensions, respectively, we can state our second result as follows:

Theorem 2. $\dim_H(\mathsf{N}) = \dim_P(\mathsf{N}) = \dim_B(\mathsf{N}) = \dfrac{1}{k+1}$.

2. Fractal Dimensions and Preliminary Results

2.1. Fractal Dimensions

For a non-empty subset U of the Euclidian space $\mathbb{R}^n$, the diameter of U is defined as

$$|U| = \sup\{|x - y|, x, y \in U\}.$$

Let I and F be non-empty subsets of $\mathbb{N}$ and $\mathbb{R}^n$, respectively. (I may be either finite or countable.) We say that $(U_i)_{i\in I}$ is a δ covering of F if

$$F \subset \bigcup_{i\in I} U_i \qquad \text{and} \qquad 0 < |U_i| \leq \delta, \quad \forall i \in I.$$

The s-dimensional Hausdorff measure of F is defined as

$$\mathcal{H}^s(F) = \lim_{\delta\to 0^+} \inf\Big\{\sum_{i\in\mathbb{N}} |U_i|^s\Big\},$$

where the infimum is taken over all the countable δ coverings $(U_i)_{i\in\mathbb{N}}$ of F. The Hausdorff dimension of F is defined as

$$\dim_H F = \sup\{s > 0,\ \mathcal{H}^s(F) = \infty\} = \inf\{s > 0,\ \mathcal{H}^s(F) = 0\},$$

with the conventions $\sup\emptyset = 0$ and $\inf\emptyset = \infty$.

The s-dimensional packing measure of F is defined as

$$\mathcal{P}^s(F) = \lim_{\delta\to 0^+} \sup\Big\{\sum_{i\in\mathbb{N}} |B_i|^s\Big\},$$

where the supremum is taken over all the packings $\{B_i\}_{i\in\mathbb{N}}$ of F by balls centered on F and with a diameter smaller than or equal to δ. The packing dimension of F is defined as

$$\dim_P(F) = \sup\{s > 0,\ \mathcal{P}^s(F) = \infty\} = \inf\{s > 0,\ \mathcal{P}^s(F) = 0\}.$$

Let $N_\delta(F)$ be the smallest number of sets with a diameter of at most δ which can cover F. The lower and upper box-counting dimensions of F are respectively defined as follows:

$$\underline{\dim}_B(F) = \liminf_{\delta\to 0} \frac{\log\ N_\delta(F)}{\log(\delta)},$$

and

$$\overline{\dim}_B(F) = \limsup_{\delta\to 0} \frac{\log\ N_\delta(F)}{\log(\delta)}.$$

If $\underline{\dim}_B(F) = \overline{\dim}_B(F)$, then this common value is denoted as $\dim_B(F)$ and referred to as the box-counting dimension (or simply the box dimension) of F such that

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log\ N_\delta(F)}{\log(\delta)}.$$

For more details, the reader can be referred to, for example [15–20].

2.2. Fractal Dimension of the Iterated Function System (IFS)

Let m and p be two positive integers with $p \geq 2$ and X be a non-empty closed set of $\mathbb{R}^m$. A family $\{S_i, i = 1, \ldots, p\}$ of contractive mappings on X is called an iterated function system (IFS) on X [21]. Hutchinson showed in [22] that there is a unique non-empty compact set K of X, called the attractor of $\{S_i, i = 1, \ldots, p\}$, such that

$$K = \bigcup_{i=1}^{p} S_i(K).$$

The local dimension at a point $x \in \mathbb{R}^m$ is defined by

$$d(\mu, x) = \lim_{r \to 0} \frac{\log\ \mu\big(B(x,r)\big)}{\log(r)},$$

where $B(x,r)$ denotes the closed ball of a radius r centered at x. A probability measure μ on $\mathbb{R}^m$ is said to be exactly dimensional if there is a constant C such that

$$d(\mu, x) = C, \qquad \mu - a.e. x \in \mathbb{R}^m.$$

It was proven that the Hausdorff dimension of the measure μ is

$$\underline{\dim}(\mu) = C.$$

This result was first shown by Young [23]. The reader can also be referred to [19,20,24] for more details.

Definition 1. *Let m and $p \geq 2$ be two positive integers, X be a non-empty closed set of $\mathbb{R}^m$ and $\mathcal{S} = \{S_i\}_{1 \leq i \leq p}$ be an IFS on X. Then, $\mathcal{S}$ is said to satisfy the open set condition (OSC) if there is a non-empty, bounded and open set V such that*

1. $\bigcup_{i=1}^{p} S_i(V) \subset V.$
2. $S_i(V) \bigcap S_j(V) = \emptyset, \quad \text{if} \quad i \neq j.$

This definition allows us to recall the following result, which will allow us to calculate the fractal dimension of N:

Theorem 3. [*Theorem 9.3 in* [18]] *Let m and p be two positive integers, with $p \geq 2$, X as a non-empty closed set of $\mathbb{R}^m$ and $\mathcal{S} = \{S_i\}_{1 \leq i \leq p}$ as an IFS on X. Suppose that for $1 \leq i \leq p$, S_i is a similarity with a ratio r_i and attractor F. In addition, suppose that $\mathcal{S}$ satisfies the OSC, and let s be a unique value such that*

$$\sum_{i=1}^{p} r_i^s = 1.$$

Then, we have

$$\dim_H(F) = \dim_B(F) = s.$$

In particular, if $r_1 = \ldots = r_p = r$ for some r, then

$$\dim_H(F) = \dim_B(F) = -\frac{\log\ n}{\log\ r}.$$

The reader can find a proof of the dimension formula for self-similar sets either in [18] or in [25]. It is well known [18] that

$$\dim_H(F) \le \dim_P(F) \le \dim_B(F).$$

As a consequence, Theorem 3 allows us to deduce the following:

Corollary 1. *According to [18], suppose that the conditions of Theorem 3 are satisfied. If we also have $r_1 = \cdots = r_p = r$, then*

$$\dim_H(F) = \dim_P(F) = \dim_B(F) = -\frac{\log\ n}{\log\ r}.$$

2.3. Preliminary Results

We considered the k-bonacci sequence $(f_i)_{i\ge 0}$ defined by Equations (1) and (2). Let $n \ge k+1$. We easily obtain through induction that

$$\sum_{j=1}^{n} f_j < f_{n+2}. \tag{3}$$

Moreover, we have

$$\begin{aligned} f_{n+1} &= \sum_{j=n-k+1}^{n} f_j = f_n + \sum_{j=n-k+1}^{n-1} f_j \\ &= f_n + \sum_{j=n-k}^{n-1} f_j - f_{n-k} \\ &= 2f_n - f_{n-k}. \end{aligned}$$

This means that

$$2f_n \ge 1 + f_{n+1}. \tag{4}$$

Since we have that $f_{n-k} > 1$ for $n \ge k+2$, then the result follows. We give in the following a necessary and sufficient condition to obtain $\sum_{i=1}^{n} w_i f_i = 0$. For this, we consider for any integer $i \ge 2$ the finite sequences

$$v_{i+} = \underbrace{+1,\ +1,\ \ldots,\ +1,}_{i \text{ times}}\ \ -1 \quad \text{and} \quad v_{i-} = \underbrace{-1,\ -1,\ \ldots,\ -1,}_{i \text{ times}}\ \ +1.$$

We also consider, for a given integer $p \ge 1$, the condition $C(k,p)$,

$$(w_p,\ w_{p+1},\ \ldots,\ w_{p+k}) \in \{v_{k+}, v_{k-}\}.$$

It is clear that if the condition $C(k,p)$ holds, then $\sum_{j=p}^{p+k} w_j f_j = 0$.

Proposition 1. *Consider the k-bonacci sequence $(f_i)_{i\ge 0}$ given by Equations (1) and (2), and let $w = (w_i)_{i\ge 0} \in \mathcal{A}$. We have*

$$\hat{F}_n(w) = 0 \tag{5}$$

if and only if $n = (k+1)m$ for some integer $m > 0$ and

$$C\big(k, (k+1)i + 1\big) \quad \textit{holds for all} \quad 0 \le i < m. \tag{6}$$

To prove this proposition, we need to show the following result:

Lemma 1. *Suppose that $C(k,p)$ does not hold for an integer $p \geq 1$. Then, we have*

1. $\Big|\sum_{j=p}^{p+k} w_j f_j\Big| \geq 2 f_p$.
2. $|\hat{F}_{p+k}(w)| > 1$.

Proof.

1. Under Equation (1), we have

$$\Big|\sum_{j=p}^{p+k} w_j f_j\Big| \quad = \Big|\sum_{j=p}^{p+k-1} (w_j + w_{p+k}) f_j\Big|. \tag{7}$$

We supposed that $w_{p+k} = 1$ (the case $w_{p+k} = -1$ is analogous). We considered the set

$$A_{p,k} = \{j, \quad p \leq j \leq p+k-1, \quad w_j + w_{p+k} \neq 0\}.$$

Since $C(k,p)$ does not hold, then $A_{p,k} \neq \emptyset$. Thus, Equation (7) leads to

$$\Big|\sum_{j=p}^{p+k} w_j f_j\Big| = 2 \sum_{j \in A_{p,k}} f_j \geq 2 f_p.$$

2. If $p = 1$, and $C(k,1)$ does not hold, then by using Lemma 1 (1), we obtain

$$|\hat{F}_{k+1}(w)| \geq 2 f_1 > 1.$$

Otherwise, we have

$$|\hat{F}_{p+k}(w)| \geq \Big|\sum_{j=p}^{p+k} w_j f_j\Big| - |\hat{F}_{p-1}(w)|.$$

Since $C(k,p)$ does not hold, then using Lemma 1 (1) again leads to

$$|\hat{F}_{p+k}(w)| \geq 2 f_p - |\hat{F}_{p-1}(w)| \geq 2 f_p - \sum_{j=1}^{p-1} f_j.$$

By applying Equations (3) and (4) successively, we obtain $|\hat{F}_{p+k}(w)| > 2 f_p - f_{p+1} \geq 1$.

□

Proof of Proposition 1. $\Leftarrow$: Obviously, if Equation (6) is realized, then through Equation (1), we obtain Equation (5).

$\Rightarrow$: Conversely, suppose that Equation (5) is insured. Then, using the condition in Equation (2), it becomes $n \geq k+1$. Let m be a unique positive integer such that $n = (k+1)m + t(n)$.

1. If there exits $p \in \{(k+1)j + t(n), 0 \leq j < m\}$ such that $C(k,p)$ is not satisfied, then we set

$$p(n) = \sup\big\{p = (k+1)j + t(n) + 1, \quad 0 \leq j < m, \quad C(k,p) \text{ does not hold}\big\}.$$

Thanks to Lemma 1, we have $\hat{F}_n(w) = \hat{F}_{p(n)+k}(w) \neq 0$.

2. If $t(n) \neq 0$, and $C(k,p)$ is satisfied for all $p \in \{(k+1)j + t(n), 0 \leq j < m\}$, then under the condition in Equation (2), we have

$$\hat{F}_n(w) = \hat{F}_{t(n)}(w) + \sum_{i=0}^{m-1} \Big(\underbrace{\sum_{t=1}^{k+1} w_{(k+1)i+t(n)+t} f_{(k+1)i+t(n)+t}}_{0} \Big) = \hat{F}_{t(n)}(w) \neq 0.$$

Consequently, we must have that $t(n) = 0$ and Equation (6) satisfied. This ends the proof. □

3. Proof of Theorem 1

Let $i \geq 1$. From Proposition 1, we deduce that $\hat{F}_n$ reaches the origin exactly i times if and only if $n \geq (k+1)i$, with

$$\hat{F}_{(k+1)i} = 0 \qquad \text{and} \qquad \hat{F}_{(k+1)(i+1)} \neq 0.$$

Moreover, we have

$$\mathbb{P}\big(\hat{F}_{(k+1)i} = 0\big) = \frac{2^i}{2^{(k+1)i}} = \frac{1}{2^{ki}}$$

and

$$\mathbb{P}\Big(\hat{F}_{(k+1)(i+1)} = 0 \Big/ \hat{F}_{(k+1)i} = 0\Big) = \frac{2}{2^{k+1}} = \frac{1}{2^k}.$$

It follows that

$$\begin{aligned}
\mathbb{P}(R_i) &= \mathbb{P}\Big(\hat{F}_{(k+1)(i+1)} \neq 0 \text{ and } \hat{F}_{(k+1)i} = 0\Big) \\
&= \mathbb{P}\Big(\hat{F}_{(k+1)(i+1)} \neq 0 \Big/ \hat{F}_{(k+1)i} = 0\Big) \times \mathbb{P}\Big(\hat{F}_{(k+1)i} = 0\Big) \\
&= \frac{2^k - 1}{2^{k(i+1)}}.
\end{aligned}$$

4. Proof of Theorem 2

We consider the metric d, defined for any couple $\Big((u_i)_i, (v_i)_i\Big)$ of $\mathcal{A} \times \mathcal{A}$ by

$$d\big((u_i)_i, (v_i)_i\big) = \sum_{i=1}^{\infty} \frac{|u_i - v_i|}{2^i}.$$

Endowed with this metric, $(\mathcal{A}, d)$ becomes a compact metric space. As a direct consequence of Proposition 1, we obtain the following lemma:

Lemma 2. *We have* $\mathsf{N} = \{v_{k+}, v_{k-}\}^{\mathbb{N}}$.

Now, we consider the mappings T_1 and T_2, defined for any $w = (w_i)_i \in \mathcal{A}$ by

$$T_1(w) = (v_{k+},\ w_1,\ w_2,\ \ldots) \qquad \text{and} \qquad T_2(w) = (v_{k-},\ w_1,\ w_2,\ \ldots).$$

For $i \in \{1,2\}$ and for any $(u,v) = \Big((u_j)_j, (v_j)_j\Big) \in \mathcal{A}^2$, we have

$$d\big(T_i(u), T_i(v)\big) = \sum_{j=1}^{\infty} \frac{\big|\big(T_i(u)\big)_j - \big(T_i(v)\big)_j\big|}{2^j} = \sum_{j=k+2}^{\infty} \frac{|u_j - v_j|}{2^j} = \frac{1}{2^{k+1}}\, d(u,v).$$

This means that T_1 and T_2 are contracting similarities on the metric space $(\mathcal{A}, d)$, with contraction rates

$$r_1 = r_2 = \frac{1}{2^{k+1}}.$$

Coming back to [22], one deduces the existence of a unique, compact, self-similar subset F of $\mathcal{A}$ such that $F = T_1(F) \cup T_2(F)$. Lemma 2 implies that $F = N$. Furthermore, we have $T_1(N) \cap T_2(N) = \emptyset$. Hence, N is a self-similar set satisfying the open set condition in the compact metric space $(\mathcal{A}, d)$. Their fractal dimension is then given by Theorem 3:

$$\dim_H(\mathsf{N}) = \dim_P(\mathsf{N}) = \frac{\ln(2)}{\ln(2^{k+1})} = \frac{1}{k+1}.$$

Finally, by taking into account Corollary 1, we obtain the result.

5. Application

We are interested in applying the ideas presented in the previous sections to a class of tribonacci sequences, defined by $f_0 = 0$, $f_1 = 1$, $f_2 = 3$, $f_3 = 6$ and

$$f_i = \sum_{j=1}^{3} f_{i-j}, \quad \text{for} \quad i \geq 4. \tag{8}$$

The return point on which we focus our attention is no longer the origin. Our ideas are still applicable when considering the number of visits of $\hat{F}_n$ to f_1. We considered, for an elementary event $w \in \mathcal{A}$, the set $F_1(w)$, for which $\hat{F}_n(w)$ reaches f_1 after n steps of the walk such that

$$F_1(w) = \{n \geq 1, \ \hat{F}_n(w) = f_1\}.$$

For $i \in \mathbb{N}$, we denote as $R_{1,i}$ the event for which the element $\hat{F}_n$ passes through f_1 exactly i times, where

$$R_{1,i} = \Big\{w \in \mathcal{A} \mid \sharp F_1(w) = i\Big\}.$$

Our first result can be stated as follows:

Theorem 4. *For* $i \in \mathbb{N}$, $\quad \mathbb{P}(R_{1,i}) = \dfrac{7}{2^{3(i+1)+1}}.$

In a similar way, we considered the set N_1 consisting of elements of $\mathcal{A}$, for which $\hat{F}_n$ passes through f_1 an infinite number of times such that

$$\mathsf{N}_1 = \Big\{w \in \mathcal{A} \mid \sharp F_1(w) = \infty\Big\}.$$

Theorem 5. *We have*

$$\dim_H(\mathsf{N}_1) = \dim_P(\mathsf{N}_1) = \dim_B(\mathsf{N}_1) = \frac{1}{4}.$$

In the same spirit of Proposition 1, we have the following:

Proposition 2. *Consider the tribonacci sequence* $(f_i)_{i\geq 0}$ *given by Equation (8), and let* $(w_i)_{i\geq 0} \in \mathcal{A}$ *with* $w_1 = 1$. *We have*

$$\hat{F}_n(w) = f_1 \tag{9}$$

if and only if either $n = 1$ *or* $n = 4m + 1$ *for some integer* $m \geq 1$ *and*

$$\mathsf{C}(3, 4j+2) \quad \textit{holds for all} \quad 0 \leq j < m. \tag{10}$$

Proof. $\Leftarrow$: Obviously, if either $n = 1$ or $n = 4m+1$ for some integer $m \geq 1$, and Equation (10) holds, then thanks to Equation (8), we obtain Equation (9).

$\Rightarrow$: Conversely, supposing that $n \geq 4$, and that $C(3, n-p)$ is not satisfied, we obtain a contradiction through Lemma 1. Otherwise, through arguing by induction, we obtain

$$\hat{F}_n(w) = \hat{F}_{t(n)}(w).$$

If $t(n) \neq 1$, then $\hat{F}_{t(n)}$ is even and positive, and thus $|\hat{F}_{t(n)}| > f_1$. Again, this is a contradiction.

It follows that either $n = 1$ or $t(n) = 1$, and Equation (10) holds. □

5.1. Proof of Theorem 4

We take $i \geq 1$. From Proposition 2, we deduce that $\hat{F}_n$ reaches f_1 exactly i times if and only if $n \geq 4i+1$, with

$$\hat{F}_{4i+1} = 1 \qquad \text{and} \qquad \hat{F}_{4(i+1)+1} \neq 1.$$

Moreover, we have

$$\mathbb{P}(\hat{F}_{4i+1} = 1) = \frac{2^i}{2^{4i+1}} = \frac{1}{2^{3i+1}} \quad \text{and} \quad \mathbb{P}\Big(\hat{F}_{4(i+1)+1} = 1 \Big/ \hat{F}_{4i+1} = 1\Big) = \frac{2}{2^4} = \frac{1}{8}.$$

It follows that

$$\mathbb{P}(R_{1,i}) = \mathbb{P}\Big(\hat{F}_{4(i+1)+1} \neq 1 \Big/ \hat{F}_{4i+1} = 1\Big) \times \mathbb{P}\Big(\hat{F}_{4i+1} = 1\Big) = \frac{7}{2^{3(i+1)+1}}.$$

5.2. Proof of Theorem 5

We have that

$$\mathsf{N}_1 = \{1\} \times \{v_{3+},\, v_{3-}\}^{\mathbb{N}}.$$

We consider the mapping T, defined for any $w = (w_i)_i \in \mathcal{A}$, with

$$T(w) = 1,\ w.$$

For $(u, v) = \Big((u_j)_j, (v_j)_j\Big) \in \mathcal{A}^2$, we have

$$d\big(T(u), T(v)\big) = \sum_{j=1}^{\infty} \frac{|T(u)_j - T(v)_j|}{2^j} = \sum_{j=2}^{\infty} \frac{|u_{j-1} - v_{j-1}|}{2^j} = \frac{1}{2}\, d(u, v).$$

This means that T is a bi-Lipschitz mapping. Since $\mathsf{N}_1 = T(\mathsf{N})$, we have

$$\dim_H(\mathsf{N}_1) = \dim_H(\mathsf{N}) = \frac{1}{4}.$$

Coming back to Theorem 3 and Corollary 1, we deduce that

$$\dim_H(\mathsf{N}_1) = \dim_B(\mathsf{N}_1) = \dim_P(\mathsf{N}_1) = \frac{1}{4}.$$

6. Concluding Remarks and Perspectives

In this paper, we studied the probability of return for random walks on $\mathbb{Z}$, whose increment is given by the k-bonacci sequence. In particular, we generalized the results given in [10] (Theorems 1 and 2). Moreover, we considered the set N of walks returning infinitely many times to zero. Since we had $\mathbb{P}(\mathsf{N}) = 0$ from Theorem 1 , then we were interested in Theorem 2 to describe geometrically the set N by computing its fractal dimension. In Section 5, we considered a particular interesting case—the tribonacci random walks

($k = 3$)—and we studied the return to the first term of this sequence. Finally, we think it is very interesting to make points regarding some remarks and possible extensions of our work:

1. The results given by Theorems 4 and 5 still remain valid if we take $w_1 = -1$. In other words, if we take the tribonacci sequence defined by Equation (8) and consider the set

$$F_{-1}(w) = \{n \geq 1,\ \hat{F}_n(w) = -f_1\}$$

and if we denote for $i \in \mathbb{N}_0$ the event for which elements F_n passes through f_1 exactly i times by $R_{-1,i}$ such that

$$R_{-1,i} = \left\{w \in \mathcal{A} \mid \sharp F_{-1}(w) = i\right\},$$

then we have that $\mathbb{P}(R_{-1,i}) = \dfrac{7}{2^{3(i+1)+1}}$.
Moreover, if the set N_{-1} consists of the elements of $\mathcal{A}$ for which $\hat{F}_n$ passes through $(-f_1)$ an infinite number of times, where

$$\mathsf{N}_{-1} = \left\{w \in \mathcal{A} \mid \sharp F_{-1}(w) = \infty\right\},$$

then N_{-1} is such that $\dim_H(\mathsf{N}_{-1}) = \dim_P(\mathsf{N}_{-1}) = \dim_B(\mathsf{N}_{-1}) = \dfrac{1}{5}$.

2. The results obtained in this work and those given in the previous works concerning the number of returns of $\hat{F}_n$ to the origin or even to $\pm f_1$, as studied in Section 5, depended strongly on the k initializing terms of the k-bonacci sequence (i.e., $(f_i)_{1 \leq i \leq k}$). In particular, thanks to (AS), $\hat{F}_n$ is allowed to visit zero or $\pm f_1$ only one time in its first k steps of the walk, where $(\hat{F}_i)_{1 \leq i \leq k}$. If (AS) is no longer satisfied, then $\hat{F}_n$ can reach the values of zero or $\pm f_1$ more than one time before its $(k+1)$th term $\hat{F}_{k+1}$. Obviously, the equivalences established either in this or in the previous studies are no longer valid. One can think about adapting the techniques used in this work or giving another approach to study such problems.
3. One can ask to think of the possibility of reaching other terms of the k-bonacci sequence and the eventual necessary or sufficient conditions to realize this task with $\hat{F}_n$.

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