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An Analysis on the Optimal Control for Fractional Stochastic Delay Integrodifferential Systems of Order $1 < \gamma < 2$

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Abstract: The purpose of this paper is to investigate the optimal control for fractional stochastic integrodifferential systems of order $1 < \gamma < 2$. To ensure the existence and uniqueness of mild solutions, we first gather a novel list of requirements. Further, the existence of optimal control for the stated issue is given by applying Balder's theorem. Additionally, we extend our existence outcomes with infinite delay. The outcomes are obtained via fractional calculus, Hölder's inequality, the cosine family, stochastic analysis techniques, and the fixed point approach. The theory is shown by an illustration, as well.

Keywords: stochastic differential equation; optimal controls; integrodifferential systems; fractional calculus; fixed point theorem

MSC: 26A33; 34K50; 47D09; 47H10; 49J15

1. Introduction

Fractional differential equations have gained popularity over the past three decades due to their effectiveness in mathematical modeling. In contrast to integer-order models, noninteger-order models encourage the discovery of more precise solutions. It turns out that the fractional derivatives more accurately yield the heritable features of several physical phenomena. As a result of its uses in several fields of science, engineering, economics, and optimal problem solving, fractional calculus has attracted many scientists. By utilizing the concepts of fractional calculus, the cosine family, and the fixed point approach, the researchers in [1,2] established the existence of mild solutions and controllability outcomes for fractional systems with or without a nonlocal condition. The approximate controllability outcomes of the Hilfer fractional differential system have been recently examined in [3]. Refer to the textbooks [4–6] for further details on fractional differential systems and their uses.

On the other hand, the concept of stochastic differential systems and their uses have received a great deal of consideration (see [7–10] and the references therein). In [11], the researchers examined the controllability outcomes of neutral stochastic fractional integrodifferential system involving infinite delay with the help of fractional calculus and the fixed point approach, along with stochastic analysis theory. Using fractional calculus and Bohnenblust-Karlin's fixed point strategy, the authors of [12] explored the approximate controllability of fractional neutral stochastic functional integrodifferential systems. Recently, in [13], the authors discussed the asymptotic stability and mean square stability of stochastic differential systems of order $\gamma \in (1, 2]$.

Moreover, in the design and study of control systems, the optimal and approximate controllability problems are significant (see [14–23] and the references therein). By applying the fractional derivatives of Reimann-Liouville, the author of [24] developed a fractional optimal control issue and provided a numerical approach for solving it. In [25], the authors investigated the optimal controls and approximate controllability for fractional differential

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equations of order $\gamma \in (1,2)$ via a fixed point approaches. The author of [26] formulated the requirements for fractional optimal control of equations involving fixed delay. Using the Banach fixed point strategy, in [27], the authors established requirements for the existence of optimal control for the fractional control equations of order $\gamma \in (1,2]$. Furthermore, the authors of [28] investigated the optimal control of fractional evolution systems in the α-norm of order $\gamma \in (1,2)$ involving nonlocal circumstances.

The authors of [29] investigated the optimal controls of system governed by impulsive Hilfer fractional evolution system involving delay and Clarke subdifferential. In [30], by using the compactness of the fractional resolvent operator family, the authors examined the existence of mild solutions and optimal controls for stochastic evolution systems involving the Sobolev type. Moreover, the authors of [31] established the optimal control for stochastic fractional differential systems that were non-instantaneous and impulsive. Integrodifferential systems, which are used in a diversity of scientific fields where an aftereffect or delay must be considered, such as biology, control theory, and medicine, have increasingly attracted attention. In [32], the authors obtained the solvability and optimal controls for fractional stochastic integrodifferential systems with impulsive via Leray-Schauder fixed point approach. The authors of [33] studied the optimal control issue for few integrodifferential systems. In addition, controllability and optimal control results for fractional stochastic integrodifferential equations involving Poisson jumps have been obtained [34].

Recently, in [35], the researchers employed the cosine family, fractional calculus, and the Banach fixed point theorem to develop the optimal control issue and approximate the controllability outcomes for fractional integrodifferential systems involving infinite delay of order $\gamma \in (1,2)$. Using Schauder's fixed point approach, in [36], the existence and optimal control outcomes for fractional mixed Volterra-Fredholm type integrodifferential equations of order $1 < \gamma < 2$ involving sectorial operators have been investigated.

Based on the information provided above, the main aim of the present manuscript is to study the existence and uniqueness of mild solutions to the fractional stochastic integrodifferential systems of order $1 < \gamma < 2$ through stochastic analysis theory, fractional calculus, the cosine family, and the Banach fixed point approach. Additionally, we discover a formulation for fractional optimal control governed by the fractional stochastic integrodifferential systems. Further, with the help of Balder's theorem, the existence of optimal control for the given problem is studied. In addition, the optimal control for fractional stochastic integrodifferential systems of order $1 < \gamma < 2$ involving infinite delay is examined. The obtained outcomes are novel and are thought to contribute to the theory of stochastic fractional optimal control.

Consider the following fractional stochastic integrodifferential systems involving the control term, where the fractional derivative is described in the sense of the Caputo:

$$\begin{cases} {}^{C}D_t^{\gamma} y(t) = Ay(t) + \mathbb{B}(t)\varkappa(t) + h\Big(t, y(t), \int_0^t g(t,s,y(s))ds\Big)\frac{dW(t)}{dt}, \ t \in \mathcal{I}, \\ y(0) = y_0, \ y'(0) = y_1, \end{cases} \tag{1}$$

where ${}^{C}D_t^{\gamma}$ is the fractional derivative, whose order is $\gamma \in (1,2)$ on a separable Hilbert space $\mathscr{Y}$; $\mathcal{I} = [0,\vartheta]$; $A : \mathcal{D}(A) \subset \mathscr{Y} \to \mathscr{Y}$ stand for the infinitesimal generator of a strongly continuous cosine family $\{\mathscr{V}(t),\ t \geq 0\}$; the control function $\varkappa$ takes values from another separable reflexive Hilbert space $\mathscr{X}$; $\mathbb{B} : \mathscr{X} \to \mathscr{Y}$ is a linear operator; h and g are the appropriate functions to be defined further; y_0 and y_1 are $\mathfrak{J}_0$-measurable $\mathscr{Y}$-valued random variables; and W indicates a Wiener process.

The manuscript has been split into several segments: Required preparations are given in Section 2. The existence and uniqueness outcomes of the mild solutions for (1) are verified in Section 3. The existence of optimal control is discussed in Section 4. Then, by utilizing the fixed point approach, the existence and uniqueness of mild solutions for (22) are shown in Section 5. In Section 6, we explored the existence of optimal control for the stated issue. Finally, an illustration is created to validate the theoretical outcomes.

2. Preliminaries

Consider the following: $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}}, \langle\cdot,\cdot\rangle_{\mathscr{Y}})$ and $(\mathscr{X}, \|\cdot\|_{\mathscr{X}}, \langle\cdot,\cdot\rangle_{\mathscr{X}})$ represent two real separable Hilbert spaces, including their vector norms and inner products. $L(\mathscr{X},\mathscr{Y})$ stand for the space of bounded and linear operators from $\mathscr{X}$ into $\mathscr{Y}$. Take into consideration a complete probability space $(\Omega, \mathfrak{J}, \wp)$, including a normal filtration $\{\mathfrak{J}_t\}_{t\in[0,\vartheta]}$. $W(t) = \{W(t), t \geq 0\}$ stand for a $\mathcal{Q}$-Wiener process determined on $(\Omega, \mathfrak{F}, \wp)$, including the covariance operator $\mathcal{Q}$ such that $Tr\mathcal{Q} < \infty$. Assume that there exists a complete orthonormal system $\{\breve{\xi}_{\mathfrak{m}}\}$ in $\mathscr{X}$, a bounded sequence of nonnegative real numbers $\{\lambda_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathbb{N}}$ such that

$$\mathcal{Q}\breve{\xi}_{\mathfrak{m}} = \lambda_{\mathfrak{m}}\breve{\xi}_{\mathfrak{m}}, \quad \breve{\xi}_{\mathfrak{m}} \geq 0,\ \mathfrak{m} \in \mathbb{N},$$

and a sequence of independent real-valued Brownian motion $\{\beta_{\mathfrak{m}}\}_{\mathfrak{m}\geq 1}$ such that

$$\langle W(t), \breve{\xi}\rangle_{\mathscr{X}} = \sum_{\mathfrak{m}=1}^{\infty} \sqrt{\lambda_{\mathfrak{m}}}\langle \breve{\xi}_{\mathfrak{m}}, \breve{\xi}\rangle \beta_{\mathfrak{m}}(t),\ \breve{\xi} \in \mathscr{X}, t \in [0,\vartheta],$$

and $\mathfrak{J}_t = \mathfrak{J}_t^W$ is the σ-algebra induced by $\{W(s) : 0 < s \leq t\}$.

The space of all Hilbert–Schmidt operators from $\mathcal{Q}^{1/2}\mathscr{X}$ into $\mathscr{Y}$ is described as L_2^0, which is $L_2(\mathcal{Q}^{1/2}\mathscr{X}, \mathscr{Y})$, including the inner product $\langle \bar{\phi}, \bar{\varphi}\rangle_{L_2^0} = Tr[\bar{\phi}\mathcal{Q}\bar{\varphi}^*]$. The Banach space of all the continuous functions from $[0,\vartheta]$ into $L_2(\Omega, \mathscr{Y})$ fulfilling the requirements $\sup_{t\in[0,\vartheta]} E\|y(t)\|^2 < \infty$ is determined as $\mathscr{C}([0,\vartheta], L_2(\Omega,\mathscr{Y}))$. It is a Banach space with the norm $\|y\|_{\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y}))} = \left(\sup_{t\in[0,\vartheta]} E\|y(t)\|^2\right)^{\frac{1}{2}}$.

Definition 1 ([4]). *The fractional integral of order for $g : [0,\infty) \to \Re$ with the lower limit zero is represented as*

$$I^{\gamma}g(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{g(s)}{(t-s)^{1-\gamma}}ds,\ t > 0,\ \gamma \in \Re^+,$$

provided the right hand be pointwise defined on $[0,\infty)$.

Definition 2 ([4]). *The Riemann-Liouville derivative of order for $g : [0,\infty) \to \Re$ with the lower limit zero is represented as*

$${}^{L}D^{\gamma}g(t) = \frac{1}{\Gamma(\mathfrak{n}-\gamma)}\frac{d^{\mathfrak{n}}}{dt^{\mathfrak{n}}}\int_0^t \frac{g(s)}{(t-s)^{\gamma+1-\mathfrak{n}}}ds,\ t > 0,\ \mathfrak{n}-1 < \gamma < \mathfrak{n}.$$

Definition 3 ([4]). *The Caputo derivative of order for $g : [0,\infty) \to \Re$ with the lower limit zero is represented as*

$${}^{C}D^{\gamma}g(t) = {}^{L}D^{\gamma}\left(g(t) - \sum_{i=0}^{\mathfrak{n}-1}\frac{t^i}{i!}g^{(i)}(0)\right),\ t > 0,\ \mathfrak{n}-1 < \gamma < \mathfrak{n}.$$

Remark 1. (**1**) *Provided that $g(t) \in \mathscr{C}^{\mathfrak{n}}[0,\infty)$, next*

$${}^{C}D^{\gamma}g(t) = \frac{1}{\Gamma(\mathfrak{n}-\gamma)}\int_0^t \frac{g^{(\mathfrak{n})}(s)}{(t-s)^{\gamma+1-\mathfrak{n}}}ds = I^{\mathfrak{n}-\gamma}g^{(\mathfrak{n})}(t),\ t > 0,\ \mathfrak{n}-1 < \gamma < \mathfrak{n}.$$

(**2**) *Given that g is an abstract function with values in $\mathscr{Y}$, then the integrals that appear in Definitions 1 and 2 are taken in Bochner's sense.*

(**3**) ${}^{C}D^{\gamma}(Constant) = 0.$

Definition 4 ([37]). *$\{\mathscr{V}(t)\}_{t\in\Re} : \mathscr{Y} \to \mathscr{Y}$ is said to be a strongly continuous cosine family if*

(ı) $\mathscr{V}(s+t) + \mathscr{V}(s-t) = 2\mathscr{V}(s)\mathscr{V}(t), \forall\, s,t$ *belongs to* $\Re$;

(ıı) $\forall$ *y belongs to* $\mathscr{Y}$, $\mathscr{V}(t)y$ *is continuous in t on* $\Re$;
(ııı) $\mathscr{V}(0) = I$.

Consider $\{\mathscr{W}(t)\}_{t\in\Re}$ the sine family related with $\{\mathscr{V}(t)\}_{t\in\Re}$ is represented as

$$\mathscr{W}(t)y = \int_0^t \mathscr{V}(s)yds, \quad y \in \mathscr{Y}, \quad t \in \Re. \tag{2}$$

Further, provided that

$$Ay = \frac{d^2}{dt^2}\mathscr{V}(0)y, \ \forall\, y \in \mathcal{D}(A), \tag{3}$$

In the above $\mathcal{D}(A) = \{y \in \mathscr{Y} : \mathscr{V}(t)y \in \mathscr{C}^2(\Re, \mathscr{Y})\}$, where A is a closed, densely-defined operator belonging to $\mathscr{Y}$. Define a family $\mathbb{V} = \{y \in \mathscr{Y} : \mathscr{V}(t)y \in \mathscr{C}^1(\Re, \mathscr{Y})\}$.

Lemma 1 ([37]). *Consider* $\{\mathscr{W}(t)\}_{t\in\Re}$ *is a strongly continuous cosine family in* $\mathscr{Y}$. *The subsequent are hold*

(ı) $\exists\, \mathcal{K} \geq 1$ *and* $\omega \geq 0 \in \|\mathscr{V}(t)\|_{L_\vartheta(\mathscr{Y})} \leq \mathcal{K}e^{\omega|t|}$ $\forall$ *t belongs to* $\Re$;
(ıı) $\|\mathscr{W}(t_2) - \mathscr{W}(t_1)\|_{L_\vartheta} \leq \mathcal{K}\left|\int_{t_1}^{t_2} e^{\omega|s|}ds\right|$ $\forall\, t_1, t_2$ *belongs to* $\Re$;
(ııı) *Provided that y belongs to* $\mathbb{V}$, *next* $\mathscr{V}(t)y \in \mathcal{D}(A)$ *and* $\frac{d}{dt}\mathscr{V}(t)y = A\mathscr{W}(t)y$.

Lemma 2. *Consider* $\{\mathscr{V}(t)\}_{t\in\Re} \in \mathscr{Y}$. *Next,*

$$\lim_{t\to 0} t^{-1}\mathscr{W}(t)y = y, \ \forall\, y \text{ belongs to } \mathscr{Y}.$$

Lemma 3 ([37]). *Consider* $\{\mathscr{V}(t)\}_{t\in\Re}$ *in* $\mathscr{Y}$ *fulfilling* $\|\mathscr{V}(t)\|_{L_\vartheta} \leq \mathcal{K}e^{\omega|t|}, t \in \Re$, *and A is the infinitesimal generator of* $\{\mathscr{V}(t)\}_{t\in\Re}$. *Next,* $\forall$ *Re* $\Lambda > \omega$, Λ^2 *belongs to* $\rho(A)$ *and*

$$\Lambda R(\Lambda^2; A)y = \int_0^\infty e^{-\Lambda t}\mathscr{V}(t)ydt, \quad R(\Lambda^2; t)y = \int_0^\infty e^{-\Lambda t}\mathscr{W}(t)ydt, \quad \forall\, y \text{ in } \mathscr{Y}.$$

In $\mathscr{Y}$, A is a closed dense operator, $\exists\, \mathcal{K} \geq 1$ such that $\|\mathscr{V}(t)\|_{L_\vartheta} \leq \mathcal{K}$ for $t \geq 0$. Fix $\eta = \frac{\gamma}{2}$ $\forall\, \gamma \in (1,2)$, as described in [1,5].

Definition 5 ([1]). *A stochastic process* $y(t)$ *belonging to* $\mathscr{C}([0,\vartheta], L_2(\Omega, \mathscr{Y}))$ *is called a mild solution of* (1), *provided for every* $\varkappa(\cdot) \in \mathscr{U}_{ad}$ *there exists a* $\vartheta = \vartheta(\varkappa) > 0$, $y(t)$ *is measurable and the subsequent integral system is fulfilled:*

$$\begin{aligned} y(t) = {} & \mathscr{V}_\eta(t)y_0 + \mathcal{N}_\eta(t)y_1 + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)h\Big(s, y(s), \int_0^s g(s, \iota, y(\iota))d\iota\Big)dW(s) \\ & + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds, \ t \in \mathcal{I}. \end{aligned} \tag{4}$$

In the above

$$\mathscr{V}_\eta(t) = \int_0^\infty \mathcal{P}_\eta(\theta)\mathscr{V}(t^\eta\theta)d\theta, \qquad \mathcal{N}_\eta(t) = \int_0^t \mathscr{V}_\eta(s)ds,$$
$$\mathcal{M}_\eta(t) = \int_0^\infty \eta\theta\mathcal{P}_\eta(\theta)\mathscr{W}(t^\eta\theta)d\theta, \qquad \mathcal{P}_\eta(\theta) = \frac{1}{\eta}\theta^{-1-\frac{1}{\eta}}\mu_\eta(\theta^{-\frac{1}{\eta}}) \geq 0,$$
$$\mu_\eta(\theta) = \frac{1}{\pi}\sum_{\mathfrak{n}=1}^{\infty}(-1)^{\mathfrak{n}-1}\theta^{-\mathfrak{n}\eta-1}\frac{\Gamma(\mathfrak{n}\eta+1)}{\mathfrak{n}!}\sin(\mathfrak{n}\pi\eta), \ \theta \in (0,\infty),$$

and $\mathcal{P}_\eta(\cdot)$ is the Wright-type function of Mainardi defined on $(0,\infty)$ such that

$$\mathcal{P}_\eta(\theta) \geq 0 \text{ for } \theta \in (0,\infty) \text{ and } \int_0^\infty \mathcal{P}_\eta(\theta)d\theta = 1.$$

Lemma 4 ([1])**.** *The subsequent characteristics hold to $\mathscr{V}_\eta(t)$, $\mathcal{N}_\eta(t)$ and $\mathcal{M}_\eta(t)$:*

($\imath$) $\{\mathscr{V}_\eta(t), t \geq 0\}$ *is strongly continuous, that is, $\forall\, y \in \mathscr{Y}$ and $\forall\, t',\ t'' \geq 0$, one can obtain*

$$\|\mathscr{V}_\eta(t'')y - \mathscr{V}_\eta(t')y\|_{L_\vartheta} \to 0, \text{ when } t'' \to t'.$$

($\imath\imath$) $\{\mathcal{N}_\eta(t)\}$ *and* $\{\mathcal{M}_\eta(t)\}$ *are uniformly continuous, that is, $\forall\, t',\ t'' \geq 0$, and one can obtain*

$$\|\mathcal{N}_\eta(t'')y - \mathcal{N}_\eta(t')y\|_{L_\vartheta} \to 0,\ \|\mathcal{M}_\eta(t'')y - \mathcal{M}_\eta(t')y\|_{L_\vartheta} \to 0, \text{ when } t'' \to t'.$$

($\imath\imath\imath$) $\forall\, t \geq 0$, *the operators $\mathscr{V}_\eta(t)$, $\mathcal{N}_\eta(t)$ and $\mathcal{M}_\eta(t)$ are linear and bounded operators, that is, $\forall$ $y \in \mathscr{Y}$, the subsequent:*

$$\|\mathscr{V}_\eta(t)y\| \leq \mathcal{K}\|y\|, \quad \|\mathcal{N}_\eta(t)y\| \leq \mathcal{K}\|y\|t, \quad \|\mathcal{M}_\eta(t)y\| \leq \frac{\mathcal{K}}{\Gamma(2\eta)}\|y\|t^\eta.$$

Remark 2. *Nothing that, from Lemma 4 ($\imath\imath$) and ($\imath\imath\imath$), $\forall\, t,s \geq 0$, $y \in \mathscr{Y}$*

$$\lim_{t\to 0} t^{\eta-1}\mathcal{M}_\eta(t)y = 0,\ \|t^{\eta-1}\mathcal{M}_\eta(t) - s^{\eta-1}\mathcal{M}_\eta(s)\|_{L_\vartheta} \to 0,\ t \to s.$$

Now consider the Henry-Gronwall inequality [38].

Lemma 5. *Consider $x, \hbar : [0,\vartheta] \to [0,\infty)$ are continuous functions. Provided that $\hbar$ is nondecreasing, $\alpha \geq 0$ and $\gamma > 0$ such that*

$$x(t) \leq \hbar(t) + \alpha\int_0^t (t-s)^{\gamma-1}x(s)ds,\ t \in [0,\vartheta], \tag{5}$$

next

$$x(t) \leq \hbar(t) + \int_0^t \left[\sum_{\mathfrak{n}=1}^\infty \frac{(\alpha\Gamma(\gamma))^\mathfrak{n}}{\Gamma(\mathfrak{n}\gamma)}(t-s)^{\mathfrak{n}\alpha-1}\hbar(s)\right]ds,\ t \in [0,\vartheta]. \tag{6}$$

Provided that $\hbar(t) = c$, constant on $[0,\vartheta]$, next, the above inequality is reduced to

$$x(t) \leq cE_\gamma\big(\alpha\Gamma(\gamma)t^\gamma\big),\ t \in [0,\vartheta].$$

In the above the Mittag-Leffler function E_γ [39] is determined as

$$E_\gamma(z) = \sum_{\mathfrak{n}=0}^\infty \frac{z^\mathfrak{n}}{\Gamma(\mathfrak{n}\gamma+1)},\ z \in \mathbb{C},\ Re(\gamma) > 0.$$

We recommend that readers see [40] for further information on generalized Henry-Gronwall inequalities.

Lemma 6. *Provided that $\|f\|$ is the Lebesgue integrable, next, a measurable function $f : [0,\vartheta] \to \mathscr{Y}$ is Bochner integrable.*

3. Existence and Uniqueness of Mild Solution

To prove the existence and uniqueness of mild solutions for (1), we list the subsequent basic conditions:

(**H1**). *A is the infinitesimal generator of a strongly continuous cosine family $\{\mathscr{V}(t)\}_{t>0}$ on $\mathscr{Y}$.*

(H2). *$h : \mathcal{I} \times \boldsymbol{\mathscr{Y}} \times \boldsymbol{\mathscr{Y}} \to L_2^0$ fulfills:*

- $(\imath)$ *$\forall\,(\delta, y) \in \boldsymbol{\mathscr{Y}} \times \boldsymbol{\mathscr{Y}}$, $h(\cdot, \delta, y) : \mathcal{I} \to L_2^0$ is measurable.*
- $(\imath\imath)$ *Arbitrary $\delta_1, \delta_2, y_1, y_2 \in \boldsymbol{\mathscr{Y}}$ fulfilling $E\|\delta_1\|^2, E\|\delta_2\|^2, E\|y_1\|^2, E\|y_2\|^2 \le \mathfrak{q}$, $\exists\, L_h(\mathfrak{q}) > 0$ such that*

$$E\|h(t, \delta_1, y_1) - h(t, \delta_2, y_2)\|^2 \le L_h(\mathfrak{q})(E\|\delta_1 - \delta_2\|^2 + E\|\xi_1 - \xi_2\|^2),\ \textit{for all } t \in [0, \vartheta].$$

- $(\imath\imath\imath)$ *$\exists\, d_h > 0$ such that*

$$E\|h(t, \delta, y)\|^2 \le d_h(1 + E\|\delta\|^2 + E\|y\|^2),\ \textit{for all } \delta, y \in \boldsymbol{\mathscr{Y}},\ t \in [0, \vartheta].$$

(H3). *$g : \Delta \times \boldsymbol{\mathscr{Y}} \to \boldsymbol{\mathscr{Y}}$ fulfills:*

- $(\imath)$ *$g(t, s, \cdot) : \boldsymbol{\mathscr{Y}} \to \boldsymbol{\mathscr{Y}}$ is continuous $\forall\,(t, s) \in \Delta = \{(t, s) \in \mathcal{I} \times \mathcal{I} \mid s \in [0, t]\}$.*
- $(\imath\imath)$ *For arbitrary $(t, s) \in \Delta$ and $\delta_1, \delta_2 \in \boldsymbol{\mathscr{Y}}$ fulfilling $E\|\delta_1\|^2, E\|\delta_2\|^2 \le \mathfrak{q}$, $\exists\, L_g(\mathfrak{q}) > 0$ such that*

$$E\|g(t, s, \delta_1) - g(t, s, \delta_2)\|^2 \le L_g(\mathfrak{q})(E\|\delta_1 - \delta_2\|^2).$$

- $(\imath\imath\imath)$ *$\exists\, d_g > 0$ such that*

$$E\|g(t, s, \delta)\|^2 \le d_g(1 + E\|\delta\|^2),\ \forall\, \delta \in \boldsymbol{\mathscr{Y}}.$$

(H4). *Let $\varkappa \in \mathscr{X}$ be the control function and the operator $\mathbb{B}$ in $L_\infty([0, \vartheta], L(\mathscr{X}, \boldsymbol{\mathscr{Y}}))$, $\|\mathbb{B}\|_\infty$ denote the norm of operator $\mathbb{B}$.*

(H5). *Multivalued maps $\boldsymbol{\mathscr{U}}(\cdot) : [0, \vartheta] \to \mathcal{V}(\mathscr{X})$ (where $\mathcal{V}(\mathscr{X})$ is a class of nonempty closed, convex subsets of $\mathscr{X}$) are measurable and $\boldsymbol{\mathscr{U}}(\cdot) \subseteq \eth$, where $\eth$ is a bounded set of $\mathscr{X}$.*

Fix the admissible set,

$$\mathscr{U}_{ad} = \Big\{\kappa(\cdot) : [0, \vartheta] \times \Omega \ \to \mathscr{X} \text{ such that } \kappa \text{ is } \mathfrak{J}_t\text{-adapted stochastic process} \\ \text{and } E\int_0^\vartheta \|\kappa(t)\|^r dt < \infty\Big\}.$$

Clearly, $\mathscr{U}_{ad} \neq \emptyset$ by [41] and $\mathscr{U}_{ad} \subset L^r([0, \vartheta], \mathscr{X})$ $(1 < r < +\infty)$ that is bounded, closed, and convex. It is evident that $\mathbb{B}\varkappa \in L^r([0, \vartheta], \mathscr{X})\ \forall\, \varkappa \in \mathscr{U}_{ad}$.

To present the solvability of (1), we need the subsequent significant a priori estimation.

Lemma 7. *Suppose that (4) is the mild solution of (1) on $[0, \vartheta]$ related to $\varkappa$ in $\mathscr{U}_{ad}$. Next, there exists a constant $\nu > 0$ such that*

$$E\|y(t)\|^2 \le \nu,\ \forall\, t \in [0, \vartheta].$$

Proof. Provided that y is the mild solution of system (1), next, (4) is fulfilled. From requirements $(\mathbf{H2})(\imath\imath\imath)$, $(\mathbf{H3})(\imath\imath\imath)$ and Hölder's inequality, one can obtain

$$\begin{aligned}
&E\|y(t)\|^2 \\
&\le 4E\|\mathscr{V}_\eta(t)y_0\|^2 + 4E\|\mathcal{N}_\eta(t)y_1\|^2 + 4E\|\int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds\|^2 \\
&\quad + 4E\|\int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)h\Big(s, y(s), \int_0^s g(s, \iota, y(\iota))d\iota\Big)dW(s)\|^2 \\
&\le 4\mathcal{K}^2 E\|y_0\|^2 + 4\mathcal{K}^2\vartheta^2 E\|y_1\|^2 \\
&\quad + 4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 \|\mathbb{B}\|_\infty^2 E\left[\int_0^t (t-s)^{2\eta-1}\|\varkappa(s)\|ds\right]^2 \\
&\quad + 4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{4\eta-2}E\|h\Big(s, y(s), \int_0^s g(s, \iota, y(\iota))d\iota\Big)\|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&\le 4\mathcal{K}^2E\|y_0\|^2+4\mathcal{K}^2\vartheta^2E\|y_1\|^2\\
&\quad+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|_\infty^2\left[\left(\int_0^t(t-s)^{\frac{r(2\eta-1)}{r-1}}ds\right)^{\frac{r-1}{r}}\left(E\int_0^t\|\varkappa(s)\|^r ds\right)^{\frac{1}{r}}\right]^2\\
&\quad+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2Tr(\mathcal{Q})\int_0^t(t-s)^{4\eta-2}d_h\Big(1+E\|y(s)\|_{\mathscr{Y}}^2+d_g\vartheta\int_0^s(1+E\|y(\iota)\|_{\mathscr{Y}}^2)d\iota\Big)ds\\
&\le 4\mathcal{K}^2E\|y_0\|^2+4\mathcal{K}^2\vartheta^2E\|y_1\|^2+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|_\infty^2\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}}\vartheta^{4\eta-\frac{2}{r}}\|\varkappa\|_{L^r([0,\vartheta],\mathscr{X})}^2\\
&\quad+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2Tr(\mathcal{Q})\frac{\vartheta^{4\eta-1}}{4\eta-1}d_h\Big(1+d_g\vartheta^2\Big)\\
&\quad+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2Tr(\mathcal{Q})d_h\Big(1+d_g\vartheta^2\Big)\int_0^t(t-s)^{4\eta-2}E\|y(s)\|_{\mathscr{Y}}^2ds. \qquad (7)
\end{aligned}
$$

In $\mathscr{Y}$, we use Gronwall's inequality to determine the boundedness of $y(\cdot)$, that is, $E\|y(t)\|^2\le\nu,\ \forall\, t\in[0,\vartheta]$. □

Theorem 1. *Under the requirements* (**H1**)–(**H5**)*, the system* (1) *has a unique mild solution on* $\mathcal{I}$ *for* $\varkappa(\cdot)\in\mathscr{U}_{ad}$ *and for some* r*, such that* $2\eta r>1$*.*

Proof. Deteremine an operator $\Phi:\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y}))\to\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y}))$ as

$$
\begin{aligned}
(\Phi y)(t)&=\mathscr{V}_\eta(t)y_0+\mathcal{N}_\eta(t)y_1+\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)h\Big(s,y(s),\int_0^s g(s,\iota,y(\iota))d\iota\Big)dW(s)\\
&\quad+\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds. \qquad (8)
\end{aligned}
$$

Showing that Φ has a fixed point in $\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y}))$ is sufficient to verify the existence of the mild solution to (1). Consider

$$
\mathcal{Q}_{\mathfrak{q}}=\big\{y\in\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y})):\|y\|^2_{\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y}))}\le\mathfrak{q},\ t\in[0,\vartheta]\big\},
$$

where $\mathfrak{q}$ is a positive constant. Clearly, $\Phi(\mathcal{Q}_{\mathfrak{q}})$ is a bounded and closed subset of $\mathscr{C}([0,\vartheta],$ $L_2(\Omega,\mathscr{Y}))$. For any $y(\cdot)\in\mathcal{Q}_{\mathfrak{q}}$, and one can obtain

$$
\begin{aligned}
&E\|(\Phi y)(t)\|^2\\
&=E\|\mathscr{V}_\eta(t)y_0+\mathcal{N}_\eta(t)y_1+\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)h\Big(s,y(s),\int_0^s g(s,\iota,y(\iota))d\iota\Big)dW(s)\\
&\quad+\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds\|^2\\
&\le 4E\|\mathscr{V}_\eta(t)y_0\|^2+4E\|\mathcal{N}_\eta(t)y_1\|^2+4E\|\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds\|^2\\
&\quad+4E\|\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)h\Big(s,y(s),\int_0^s g(s,\iota,y(\iota))d\iota\Big)dW(s)\|^2\\
&\le 4\|\mathscr{V}_\eta(t)\|^2E\|y_0\|^2+4\|\mathcal{N}_\eta(t)\|^2E\|y_1\|^2\\
&\quad+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|_\infty^2E\left[\int_0^t(t-s)^{2\eta-1}\|\varkappa(s)\|ds\right]^2\\
&\quad+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2Tr(\mathcal{Q})\int_0^t(t-s)^{4\eta-2}E\|h\Big(s,y(s),\int_0^s g(s,\iota,y(\iota))d\iota\Big)\|^2ds\\
&\le 4\mathcal{K}^2E\|y_0\|^2+4\mathcal{K}^2\vartheta^2E\|y_1\|^2\\
&\quad+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|_\infty^2\left[\left(\int_0^t(t-s)^{\frac{r(2\eta-1)}{r-1}}ds\right)^{\frac{r-1}{r}}\left(E\int_0^t\|\varkappa(s)\|^r ds\right)^{\frac{1}{r}}\right]^2
\end{aligned}
$$

$$+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{4\eta-2} d_h\Big(1+E\|y(s)\|^2_{\mathscr{Y}}+E\|\int_0^s g(s,\iota,y(\iota))d\iota\|^2\Big)ds$$
$$\le 4\mathcal{K}^2E\|y_0\|^2+4\mathcal{K}^2\vartheta^2E\|y_1\|^2$$
$$+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|^2_\infty\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}}\vartheta^{4\eta-\frac{2}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{X})}$$
$$+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{4\eta-2} d_h\Big(1+E\|y(s)\|^2_{\mathscr{Y}}+d_g\vartheta\int_0^s (1+E\|y(\iota)\|^2_{\mathscr{Y}})d\iota\Big)ds$$
$$\le 4\mathcal{K}^2E\|y_0\|^2+4\mathcal{K}^2\vartheta^2E\|y_1\|^2$$
$$+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|^2_\infty\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}}\vartheta^{4\eta-\frac{2}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{Y})}$$
$$+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\frac{\vartheta^{4\eta-1}}{4\eta-1}d_h\Big(1+\mathfrak{q}+d_g(1+\mathfrak{q})\vartheta^2\Big). \tag{9}$$

Now, consider

$$4\mathcal{K}^2E\|y_0\|^2+4\mathcal{K}^2\vartheta^2E\|y_1\|^2+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|^2_\infty\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}}\vartheta^{4\eta-\frac{2}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{X})}$$
$$+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\frac{\vartheta^{4\eta-1}}{4\eta-1}d_h\Big(1+\mathfrak{q}+d_g(1+\mathfrak{q})\vartheta^2\Big)<\mathfrak{q}. \tag{10}$$

Next,

$$4\mathcal{K}^2E\|y_0\|^2+4\mathcal{K}^2\vartheta^2E\|y_1\|^2+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|^2_\infty\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}}$$
$$\times\vartheta^{4\eta-\frac{2}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{X})}+4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\frac{\vartheta^{4\eta-1}}{4\eta-1}d_h\Big(1+d_g\vartheta^2\Big)$$
$$<\mathfrak{q}\left(1-4\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\frac{\vartheta^{4\eta-1}}{4\eta-1}d_h\Big(1+d_g\vartheta^2\Big)\right). \tag{11}$$

The right hand side will become positive, provided that

$$\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\frac{\vartheta^{4\eta-1}}{4\eta-1}d_h\Big(1+d_g\vartheta^2\Big)<1. \tag{12}$$

This suggests that, when ϑ is fulfilled (12), we conclude that $\Phi:\mathscr{Q}_{\mathfrak{q}}\to\mathscr{Q}_{\mathfrak{q}}$.

Next, we will demonstrate that Φ is a contraction map. For every $y,\widetilde{y}\in\mathscr{Q}_{\mathfrak{q}}$, provided that $t\in[0,\vartheta]$, next one can obtain

$$E\|(\Phi y)(t)-(\Phi\widetilde{y})(t)\|^2=E\|\int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\Big[h\Big(s,y(s),\int_0^s g(s,\iota,y(\iota))d\iota\Big)$$
$$-h\Big(s,\widetilde{y}(s),\int_0^s g(s,\iota,\widetilde{y}(\iota))d\iota\Big)\Big]dW(s)\|^2$$
$$\le\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{4\eta-2}E\|h\Big(s,y(s),\int_0^s g(s,\iota,y(\iota))d\iota\Big)$$
$$-h\Big(s,\widetilde{y}(s),\int_0^s g(s,\iota,\widetilde{y}(\iota))d\iota\Big)\|^2ds$$

$$\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{4\eta-2} L_h(\mathfrak{q}) \Big(E\|y(s)-\widetilde{y}(s)\|^2_{\mathscr{Y}}$$
$$+ E\| \int_0^s [g(s,\iota,y(\iota)) - g(s,\iota,\widetilde{y}(\iota))] d\iota \|^2 \Big) ds$$
$$\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{4\eta-2} L_h(\mathfrak{q}) \Big(E\|y(s)-\widetilde{y}(s)\|^2_{\mathscr{Y}}$$
$$+ \vartheta \int_0^s L_g(\mathfrak{q}) E\|y(\iota)-\widetilde{y}(\iota)\|^2_{\mathscr{Y}} d\iota \Big) ds$$
$$\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{4\eta-2} L_h(\mathfrak{q}) \Big(\sup_{s\in[0,\vartheta]} E\|y(s)-\widetilde{y}(s)\|^2_{\mathscr{Y}}$$
$$+ \vartheta^2 L_g(\mathfrak{q}) \sup_{s\in[0,\vartheta]} E\|y(\iota)-\widetilde{y}(\iota)\|^2_{\mathscr{Y}} \Big) ds,$$

which implies that

$$\sup_{s\in[0,\vartheta]} E\|(\Phi y)(t) - (\Phi\widetilde{y})(t)\|^2_{\mathscr{Y}} \leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q}) \frac{\vartheta^{4\eta-1}}{4\eta-1} L_h(\mathfrak{q}) \Big(1+\vartheta^2 L_g(\mathfrak{q})\Big)$$
$$\times \sup_{s\in[0,\vartheta]} E\|y(s)-\widetilde{y}(s)\|^2_{\mathscr{Y}}. \tag{13}$$

Therefore,

$$\|(\Phi y) - (\Phi\widetilde{y})\|^2_{\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y}))} \leq L_0 \|y-\widetilde{y}\|^2_{\mathscr{C}([0,\vartheta],L_2(\Omega,\mathscr{Y}))}. \tag{14}$$

where

$$L_0 = \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q}) \frac{\vartheta^{4\eta-1}}{4\eta-1} L_h(\mathfrak{q}) \Big(1+\vartheta^2 L_g(\mathfrak{q})\Big) < 1. \tag{15}$$

Thus Φ is a contraction mapping on $\mathscr{Q}_{\mathfrak{q}}$. It follows from the contraction mapping principle, i.e., that Φ has a unique fixed point y in $\mathscr{Q}_{\mathfrak{q}}$, which is the mild solution of (1). By taking into account the equation on intervals $[0,\tilde{\vartheta}]$, $[\tilde{\vartheta}, 2\tilde{\vartheta}] \cdots$, including $\tilde{\vartheta}$ fulfilling (15), the additional condition on may be eliminated with ease. □

4. Optimal Control Outcomes

Take into account the subsequent Lagrange problem (LP):
Find $(y^0, \varkappa^0) \in \mathscr{C}([0,\vartheta], L_2(\Omega,\mathscr{Y})) \times \mathscr{U}_{ad}$ such that

$$\mathcal{J}(y^0,\varkappa^0) \leq \mathcal{J}(y^{\varkappa},\varkappa), \text{ for all } \varkappa \in \mathscr{U}_{ad}.$$

In the above

$$\mathcal{J}(y^{\varkappa},\varkappa) = E\left\{ \int_0^{\vartheta} \Im(t, y^{\varkappa}(t), \varkappa(t)) dt \right\}, \tag{16}$$

and the mild solution of (1) is described as $y^{\varkappa}$ in relation to $\varkappa$ that $\in\mathscr{U}_{ad}$. To prove the existence of solution for (LP), we impose the subsequent requirements:

(**H6**) $(\imath)$ On $\mathscr{X}$ $\forall\, y \in\mathscr{Y}$ and almost $t \in[0,\vartheta]$, $\Im(t,y,\cdot)$ is convex.
$(\imath\imath)$ The $\mathfrak{J}_t$-measurable functional $\Im : [0,\vartheta] \times \mathscr{Y} \times \mathscr{X} \to \Re \cup \{\infty\}$.
$(\imath\imath\imath)$ On $\mathscr{Y} \times \mathscr{X}$ for a.e. $t \in [0,\vartheta]$, $\Im(t,\cdot,\cdot)$ is sequentially lower semicontinuous.
$(\imath v)$ There exist $d_1 \geq 0$, $e > 0$, ℓ_1 is nonnegative and $\ell_1 \in L^1([0,\vartheta],\Re)$ such that

$$\ell_1(t) + d_1 E\|y\|^2_{\mathscr{Y}} + e\|\varkappa\|^r_{\mathscr{X}} \leq \Im(t,y,\varkappa).$$

Theorem 2. *Provided that the requirements of Theorem 1 and* (**H6**) *are true, assume that* $\mathbb{B}$ *is a strongly continuous operator, next the Lagrange problem has at least one optimal pair, i.e., there exists an admissible control pair* $(y^0, \varkappa^0) \in \mathscr{C}([0, \vartheta], L_2(\Omega, \mathscr{Y})) \times \mathscr{U}_{ad}$ *such that*

$$\mathcal{J}(y^0, \varkappa^0) = E\left\{ \int_0^{\vartheta} \Im(t, y^0(t), \varkappa^0(t))dt \right\} \leq \mathcal{J}(y^{\varkappa}, \varkappa), \tag{17}$$

for all $(y^{\varkappa}, \varkappa) \in \mathscr{C}([0, \vartheta], L_2(\Omega, \mathscr{Y})) \times \mathscr{U}_{ad}$.

Proof. Provided that $\inf\{\mathcal{J}(y^{\varkappa}, \varkappa) \,|\, \varkappa \in \mathscr{U}_{ad}\} = +\infty$, next we clearly get the outcome. Consider

$$\inf\{\mathcal{J}(y^{\varkappa}, \varkappa) \,|\, \varkappa \in \mathscr{U}_{ad}\} = v < +\infty.$$

From (**H6**), we get $v > -\infty$. $\exists$ minimizing sequence feasible pair by the definition of infimum

$$\{(y^{\mathfrak{m}}, \varkappa^{\mathfrak{m}})\} \subset \mathscr{P}_{ad} \equiv \{(y, \varkappa) \,|\, y \text{ is a mild solution of system (1) corresponding to } \varkappa \in \mathscr{U}_{ad}\}$$

such that $\mathcal{J}(y^{\mathfrak{m}}, \varkappa^{\mathfrak{m}}) \to v$ when $\mathfrak{m} \to +\infty$. For $\mathfrak{m} = 1, 2, \ldots$, $\{\varkappa^{\mathfrak{m}}\} \subseteq L^r([0, \vartheta], \mathscr{X})$ which is also bounded. Next, $\exists$ $\{\varkappa^{\mathfrak{m}}\}$, and $\varkappa^0 \in L^r([0, \vartheta], \mathscr{X})$ such that $\varkappa^{\mathfrak{m}}$ weakly convergent to $\varkappa^0$ in $L^r([0, \vartheta], \mathscr{X})$. By means of Marzur lemma $\mathscr{U}_{ad}$ is closed convex, $\varkappa^0 \in \mathscr{U}_{ad}$.

Consider $\{y^{\mathfrak{m}}\}$ is the sequence of solutions of (1) corresponding to $\{\varkappa^{\mathfrak{m}}\}$, y^0 is the mild solution of (1) corresponding to the control $\varkappa^0$. $y^{\mathfrak{m}}$ and y^0 fulfill the subsequent integral systems,

$$\begin{aligned} y^{\mathfrak{m}}(t) = {} & \mathscr{V}_\eta(t)y_0 + \mathcal{N}_\eta(t)y_1 + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)h\Big(s, y^{\mathfrak{m}}(s), \int_0^s g(s, \iota, y^{\mathfrak{m}}(\iota))d\iota\Big)dW(s) \\ & + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa^{\mathfrak{m}}(s)ds, \end{aligned} \tag{18}$$

and

$$\begin{aligned} y^0(t) = {} & \mathscr{V}_\eta(t)y_0 + \mathcal{N}_\eta(t)y_1 + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)h\Big(s, y^0(s), \int_0^s g(s, \iota, y^0(\iota))d\iota\Big)dW(s) \\ & + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa^0(s)ds. \end{aligned} \tag{19}$$

By Lemma 7 and boundedness of $\{\varkappa^{\mathfrak{m}}\}$, $\{\varkappa^0\}$, make it simple to demonstrate that $\exists\, \mathfrak{q} > 0$ such that $E\|y^{\mathfrak{m}}\|^2 \leq \mathfrak{q}$, $E\|y^0\|^2 \leq \mathfrak{q}$. For $t \in [0, \vartheta]$, one can get

$$\begin{aligned} & E\|y^{\mathfrak{m}}(t) - y^0(t)\|^2 \\ & \leq 2E\Big\|\int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\Big[h\Big(s, y^{\mathfrak{m}}(s), \int_0^s g(s, \iota, y^{\mathfrak{m}}(\iota))d\iota\Big) \\ & \quad - h\Big(s, y^0(s) + \int_0^s g(s, \iota, y^0(\iota))d\iota\Big)\Big]dW(s)\Big\|^2 \\ & \quad + 2E\Big\|\int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)[\mathbb{B}(s)\varkappa^{\mathfrak{m}}(s) - \mathbb{B}(s)\varkappa^0(s)]ds\Big\|^2 \\ & \leq \frac{2\mathcal{K}^2 Tr(\mathcal{Q})L_h(\mathfrak{q})(1 + L_g(\mathfrak{q})\vartheta^2)}{(\Gamma(2\eta))^2}\int_0^t (t-s)^{4\eta-2}E\|y^{\mathfrak{m}}(s) - y^0(s)\|^2 ds \\ & \quad + 2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\left(\frac{r-1}{2\eta r - 1}\right)^{\frac{2(r-1)}{r}}\eta^{4\eta - \frac{2}{r}}\left[\left(E\int_0^t \|\mathbb{B}(s)\varkappa^{\mathfrak{m}}(s) - \mathbb{B}(s)\varkappa^0(s)\|^r ds\right)^{\frac{1}{r}}\right]^2, \end{aligned}$$

it suggest that there exists a constant $\mathbb{K}^* > 0$ such that

$$E\|y^{\mathfrak{m}}(t) - y^0(t)\|^2 \leq \mathbb{K}^*\|\mathbb{B}\varkappa^{\mathfrak{m}} - \mathbb{B}\varkappa^0\|^2_{L^r([0,\vartheta],\mathscr{X})}, \quad t \in [0, \vartheta]. \tag{20}$$

Hence, $\mathbb{B}$ is strongly continuous, one can obtain

$$\|\mathbb{B}\varkappa^{\mathfrak{m}} - \mathbb{B}\varkappa^0\|^2_{L^r([0,\vartheta],\mathscr{X})} \xrightarrow{s} 0 \text{ as } \mathfrak{m} \to \infty. \tag{21}$$

Next, we get

$$E\|y^{\mathfrak{m}}(t) - y^0(t)\|^2 \xrightarrow{s} 0 \text{ as } \mathfrak{m} \to \infty,$$

which is equivalent to

$$\|y^{\mathfrak{m}} - y^0\|^2_{\mathscr{C}} \xrightarrow{s} 0 \text{ as } \mathfrak{m} \to \infty.$$

Therefore

$$y^{\mathfrak{m}} \xrightarrow{s} y^0 \text{ in } \mathscr{C} \text{ when } \mathfrak{m} \to \infty.$$

Hence, from [42], we can conclude that

$$(y, \varkappa) \to E\left\{ \int_0^{\vartheta} \Im(t, y(t), \varkappa(t))dt \right\}$$

is sequentially lower semicontinuous in the strong topology of $L^1([0,\vartheta],\mathscr{Y})$ and weak topology of $L^r([0,\vartheta],\mathscr{X}) \subset L^1([0,\vartheta],\mathscr{X})$.

Therefore, $\mathcal{J}$ is weakly lower semicontinuous on $L^r([0,\vartheta],\mathscr{X})$, and from $(\mathbf{H6})(\imath v)$, $\mathcal{J} > -\infty$, $\mathcal{J}$ succeeds its minimum at $\varkappa^0 \in \mathscr{U}_{ad}$, i.e.,

$$v = \lim_{\mathfrak{m}\to\infty} E\left\{ \int_0^{\vartheta} \Im(t, y^{\mathfrak{m}}(t), \varkappa^{\mathfrak{m}}(t))dt \right\} \geq E\left\{ \int_0^{\vartheta} \Im(t, y^0(t), \varkappa^0(t))dt \right\} = \mathcal{J}(y^0, \varkappa^0) \geq v.$$

This completes the proof. □

5. Integrodifferential Systems with Delay

Recently, there has been an increase in interest in studying systems with memory or aftereffects, that is, the impact of infinite delay on state equations, in various fields of science and engineering, (see [8,9,11] and reference therein). Thus, stochastic integrodifferential systems involving infinite delay must be discussed. The solvability and optimal controls of fractional integrodifferential evolution equations involving infinite delay have been studied in [43]. The authors of [44] examined the optimal control issues for a semilinear evolution system involving infinite delay. Moreover, in [45], the authors discussed the problems of optimal control and time-optimal control for a neutral integrodifferential evolution system with infinite delay.

Here, we discuss the optimal control for fractional stochastic integrodifferential systems with infinite delay as follows:

$$\begin{cases} {}^C D_t^{\gamma} y(t) = Ay(t) + \mathbb{B}(t)\varkappa(t) + \widetilde{h}\left(t, y_t, \int_0^t \widetilde{g}(t,s,y_s)ds\right)\frac{dW(t)}{dt}, \ t \in \mathcal{I}, \\ y(t) = \jmath(t) \in L^2(\Omega, \mathscr{G}), \ t \in (-\infty, 0], \ y'(0) = y_1 \in \mathscr{Y}. \end{cases} \tag{22}$$

In the above A and $\mathbb{B}$ are defined as in the previous segment. The histories $y_t : (-\infty, 0] \to \mathscr{Y}$ is determined as $y_t(s) = y(t+s)$, $s \leq 0$ and $\in$the phase space $\mathscr{G}$. $\jmath = \{\jmath(t) : -\infty < t \leq 0\}$ is an $\mathfrak{J}_0$-measurable, $\mathscr{G}$-valued random variable independent of the Wiener process W with finite second moments. Consider $L_2^0(\Omega, \mathscr{Y})$ is the family of all $\mathfrak{J}_0$-measurable $\mathscr{Y}$-valued random variables $y(0)$.

By [46], the phase space $\mathscr{G}$ is introduced now. Consider $(\mathscr{G}, \|\cdot\|)$ is a linear space of functions mapping $(-\infty, 0]$ into $\mathscr{Y}$ and fulfill the subsequent axioms:

(**A1**) Provided that $y : (-\infty, \vartheta] \rightarrow \mathscr{Y}$, is such that $y_0 \in \mathscr{G}$, next $\forall\, t \in [0, \vartheta]$, the subsequent characteristics are true:

(a) y_t is in $\mathscr{G}$,

(b) $\|y(t)\| \leq \mathcal{H}\|y_t\|_{\mathscr{G}}$,

(c) $\|y_t\|_{\mathscr{G}} \leq \overline{M}(t) \sup\{\|y(s)\| : 0 \leq s \leq t\} + \overline{N}(t)\|y_0\|_{\mathscr{G}}$.

In the above $\mathcal{H} \geq 0$, the continuous function $\overline{M} : [0, \vartheta] \rightarrow [0, +\infty)$, $\overline{N} : [0, +\infty) \rightarrow [0, +\infty)$ is locally bounded operator and $\mathcal{H}$, $\overline{M}$, $\overline{N}$ are not dependent of $y(\cdot)$.

(A2) y_t is a $\mathscr{G}$-valued function in $\mathcal{I}$, where $y(\cdot)$ from (**A1**).

(A3) The space $\mathscr{G}$ is complete.

Further, $\widetilde{h} : \mathcal{I} \times \mathscr{G} \times \mathscr{Y} \rightarrow L_2^0$, $\widetilde{g} : \mathcal{I} \times \mathcal{I} \times \mathscr{G} \rightarrow \mathscr{Y}$ are appropriate functions fulfilling the subsequent requirements:

(**H7**).

($\imath$) $\forall\, (\delta, \xi) \in \mathscr{Y} \times \mathscr{Y}$, *the function* $\widetilde{h}(\cdot, \delta, \xi) : \mathcal{I} \rightarrow L_2^0$ *is measurable.*

($\imath\imath$) *For arbitrary* $\delta_1, \delta_2 \in \mathscr{G}$, $\xi_1, \xi_2 \in \mathscr{Y}$ *fulfilling* $\|\delta_1\|^2_{\mathscr{G}}$, $\|\delta_2\|^2_{\mathscr{G}}$, $E\|\xi_1\|^2$, $E\|\xi_2\|^2 \leq \tau$, *there exists a* $M_{\widetilde{h}}(\tau) > 0$ *such that*

$$E\|\widetilde{h}(t, \delta_1, \xi_1) - \widetilde{h}(t, \delta_2, \xi_2)\|^2 \leq M_{\widetilde{h}}(\tau)(\|\delta_1 - \delta_2\|^2_{\mathscr{G}} + E\|\xi_1 - \xi_2\|^2), \quad \text{for all } t \in \mathcal{I}.$$

($\imath\imath\imath$) *There exists a* $c_{\widetilde{h}} > 0$ *such that*

$$E\|\widetilde{h}(t, \delta, \xi)\|^2 \leq c_{\widetilde{h}}(1 + \|\delta\|^2_{\mathscr{G}} + E\|\xi\|^2), \text{ for all } \delta \in \mathscr{G},\ \xi \text{ belongs to } \mathscr{Y} \text{ and } t \in \mathcal{I}.$$

(**H8**).

($\imath$) $\forall\, (t, s) \in \mathcal{I} \times \mathcal{I}$, *the function* $\widetilde{g}(t, s, \cdot) : \mathscr{G} \rightarrow \mathscr{Y}$ *is continuous.*

($\imath\imath$) *For arbitrary* $(t, s) \in \mathcal{I} \times \mathcal{I}$ *and* $\delta_1, \delta_2 \in \mathscr{G}$ *fulfilling* $\|\delta_1\|^2_{\mathscr{G}}$, $\|\delta_2\|^2_{\mathscr{G}} \leq \tau$, *there exists a* $M_{\widetilde{g}}(\tau) > 0$ *such that*

$$E\|\widetilde{g}(t, s, \delta_1) - \widetilde{g}(t, s, \delta_2)\|^2 \leq M_{\widetilde{g}}(\tau)(\|\delta_1 - \delta_2\|^2_{\mathscr{G}}).$$

($\imath\imath\imath$) *There exists a* $c_{\widetilde{g}} > 0$ *such that*

$$E\|\widetilde{g}(t, s, \delta)\|^2 \leq c_{\widetilde{g}}(1 + \|\delta\|^2_{\mathscr{G}}), \text{ for all } \delta \in \mathscr{G}.$$

Definition 6 ([1])**.** *A stochastic process* $y : (-\infty, \vartheta] \rightarrow \mathscr{Y}$ *is called a mild solution of* (22) *provided that* $y_0 = \jmath \in L^2(\Omega, \mathscr{G})$ *on* $(-\infty, 0]$ *fulfilling* $y_0 \in L^2_0(\Omega, \mathscr{Y})$, $\forall\, \varkappa(\cdot) \in \mathscr{U}_{ad}$ *there exists a* $\vartheta = \vartheta(\varkappa) > 0$, *and the subsequent integral system is fulfilled:*

$$y(t) = \begin{cases} \mathscr{V}_\eta(t)\phi(0) + \mathcal{N}_\eta(t)y_1 \\ + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\Big(s, y_s, \int_0^s \widetilde{g}(s, \iota, y_\iota)d\iota\Big)dW(s) \\ + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds,\ 0 \leq t \leq \vartheta, \\ \phi(t),\ -\infty < t \leq 0. \end{cases} \tag{23}$$

In the above

$$\mathscr{V}_\eta(t) = \int_0^\infty \mathcal{P}_\eta(\theta)\mathscr{V}(t^\eta\theta)d\theta, \qquad \mathcal{N}_\eta(t) = \int_0^t \mathscr{V}_\eta(s)ds,$$

$$\mathcal{M}_\eta(t) = \int_0^\infty \eta\theta\mathcal{P}_\eta(\theta)\mathscr{W}(t^\eta\theta)d\theta, \qquad \mathcal{P}_\eta(\theta) = \frac{1}{\eta}\theta^{-1-\frac{1}{\eta}}\mu_\eta(\theta^{-\frac{1}{\eta}}) \geq 0,$$

$$\mu_\eta(\theta) = \frac{1}{\pi}\sum_{\mathfrak{n}=1}^{\infty}(-1)^{\mathfrak{n}-1}\theta^{-\mathfrak{n}\eta-1}\frac{\Gamma(\mathfrak{n}\eta+1)}{\mathfrak{n}!}\sin(\mathfrak{n}\pi\eta),\ \theta\in(0,\infty),$$

and the Mainardi's Wright-type function $\mathcal{P}_\eta(\cdot)$ is deteremined on $(0,\infty)$ such that

$$\mathcal{P}_\eta(\theta)\geq 0,\ \forall\,\theta\in(0,\infty)\ \textit{and}\ \int_0^\infty \mathcal{P}_\eta(\theta)d\theta = 1.$$

Determine $\mathscr{G}\mathscr{C} = \{y:(-\infty,0]\to\mathscr{Y},\ y|_{(-\infty,0]}\in\mathscr{G}$ and $y|_{\mathcal{I}}\in\mathscr{C}(\mathcal{I},\mathscr{Y})\}$, and assume that $\|\cdot\|_{\mathscr{G}\mathscr{C}}$ is the seminorm in $\mathscr{G}\mathscr{C}$ represented as

$$\|y\|_{\mathscr{G}\mathscr{C}} = \|y_0\|_{\mathscr{G}} + \sup_{s\in\mathcal{I}}\left(E\|y(s)\|^2\right)^{1/2}.$$

It is clear that $(\mathscr{G}\mathscr{C},\|\cdot\|_{\mathscr{G}\mathscr{C}})$ is a Banach space.

Additionally, we fix $\mathscr{G}\mathscr{C}^0 = \{z\in\mathscr{G}\mathscr{C} : z_0 = 0\in\mathscr{G}\}$ and consider $\|\cdot\|_{\mathscr{G}\mathscr{C}^0}$ is the seminorm in $\mathscr{G}\mathscr{C}^0$, represented as

$$\|z\|_{\mathscr{G}\mathscr{C}^0} = \|z_0\|_{\mathscr{G}} + \sup_{s\in\mathcal{I}}\left(E\|z(s)\|^2\right)^{1/2} = \sup_{s\in\mathcal{I}}\left(E\|z(s)\|^2\right)^{1/2}.$$

It is clear that $(\mathscr{G}\mathscr{C}^0,\|\cdot\|_{\mathscr{G}\mathscr{C}^0})$ is a Banach space.

Lemma 8. *Assume that $\jmath(0)$ in $\mathscr{Y}$, $(\mathbf{H7})(\imath\imath)$, and $(\mathbf{H8})(\imath\imath)$ are fulfilled. Suppose (22) is mildly solvable on $(-\infty,\vartheta]$ related to $\varkappa\in\mathscr{U}_{ad}$. Next, there exists a constant $\tau>0$ such that*

$$E\|y(t)\|^2\leq\tau,\ \forall\,t\in\mathcal{I}. \tag{24}$$

Proof. Given that the system (22) is mildly solvable on $(-\infty,\vartheta]$ with regard to $\varkappa\in\mathscr{U}_{ad}$, by Definition 6, consider y is a mild solution of (22), with respect to $\varkappa$ on $(-\infty,\vartheta]$, next y fulfills (23). Take a look at the $y(t) = z(t)+\widetilde{\jmath}(t)$, where $\widetilde{\jmath}:(-\infty,\vartheta]\to\mathscr{Y}$, described as

$$\widetilde{\jmath}(t) = \begin{cases}\jmath(t),\ t\in(-\infty,0],\\ \mathscr{V}_\eta(t)\jmath(0)+\mathcal{N}_\eta(t)y_1,\ t\in[0,\vartheta].\end{cases} \tag{25}$$

Obviously, y fulfills the system (23) if

$$\begin{cases}z_0 = 0,\ t\in(-\infty,0],\\ z(t) = \int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\left(s,z_s+\widetilde{\jmath}_s,\int_0^s\widetilde{g}(s,\iota,z_\iota+\widetilde{\jmath}_\iota)d\iota\right)dW(s)\\ \qquad\quad+\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds,\ t\in\mathcal{I}.\end{cases} \tag{26}$$

For $t\in\mathcal{I}$, one can obtain

$$\begin{aligned}E\|z(t)\|^2 &\leq 2E\|\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\left(s,z_s+\widetilde{\jmath}_s,\int_0^s\widetilde{g}(s,\iota,z_\iota+\widetilde{\jmath}_\iota)d\iota\right)dW(s)\|^2\\ &\quad+2E\|\int_0^t(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds\|^2\\ &\leq 2\left(\frac{\mathcal{K}}{\Gamma(2\vartheta)}\right)^2 Tr(\mathcal{Q})\int_0^t(t-s)^{4\eta-2}E\|\widetilde{h}\left(s,\varkappa_t+\widetilde{\jmath}_t,\int_0^t\widetilde{g}(s,\iota,\varkappa_\iota+\widetilde{\jmath}_\iota)d\iota\right)\|^2ds\\ &\quad+2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|_\infty^2 E\left[\int_0^t(t-s)^{2\eta-1}\|\varkappa(s)\|ds\right]^2\\ &\leq 2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})\int_0^t(t-s)^{4\eta-2}c_{\widetilde{h}}\left(1+\|z_s+\widetilde{\jmath}_s\|_{\mathscr{G}}^2+\vartheta^2 c_{\widetilde{g}}\left(1+\|z_\iota+\widetilde{\jmath}_\iota\|_{\mathscr{G}}^2\right)\right)ds\end{aligned}$$

$$+2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\|\mathbb{B}\|_\infty^2\left[\left(\int_0^t (t-s)^{\frac{r(2\eta-1)}{r-1}}ds\right)^{\frac{r-1}{r}}\left(E\int_0^t \|\varkappa(s)\|^r ds\right)^{\frac{1}{r}}\right]^2$$
$$\leq c+2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q})c_{\widetilde{h}}(1+\vartheta^2 c_{\widetilde{g}})\int_0^t (t-s)^{4\eta-2}\|z_s+\widetilde{\jmath}_s\|_{\mathscr{G}}^2 ds. \tag{27}$$

In the above inequality,

$$c=\frac{2\mathcal{K}^2 c_{\widetilde{h}}(1+\vartheta^2 c_{\widetilde{g}})\vartheta^{4\eta-1}Tr(\mathcal{Q})}{(\Gamma(2\eta))^2(4\eta-1)}+\frac{2\mathcal{K}^2\|\mathbb{B}\|_\infty^2}{(\Gamma(2\eta))^2}\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}}\vartheta^{4\eta-\frac{2}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{X})}.$$

Consider $M_\vartheta=\sup\{\overline{M}(t):t\in\mathcal{I}\}$ and $N_\vartheta=\sup\{\overline{N}(t):t\in\mathcal{I}\}$. Next,

$$\begin{aligned}\|z_s+\widetilde{\jmath}_s\|_{\mathscr{G}}^2 &\leq 2(\|z_s\|_{\mathscr{G}}^2+\|\widetilde{\jmath}_s\|_{\mathscr{G}}^2)\\
&\leq 4(\overline{M}(t))^2\sup\{E\|z(s)\|^2:0\leq s\leq t\}+4(\overline{N}(t))^2\|z_0\|_{\mathscr{G}}^2\\
&\quad+4(\overline{M}(t))^2\sup\{E\|\widetilde{\jmath}(s)\|^2:0\leq s\leq t\}+4(\overline{N}(t))^2\|\widetilde{\jmath}_0\|_{\mathscr{G}}^2\\
&\leq 4M_\vartheta^2\sup\{E\|z(s)\|^2:0\leq s\leq t\}+4M_\vartheta^2(2\mathcal{K}^2E\|\phi(0)\|^2+2\mathcal{K}^2\vartheta^2E\|y_1\|^2)\\
&\quad+4N_\vartheta^2\|\jmath\|_{\mathscr{G}}^2.\end{aligned} \tag{28}$$

Fix

$$\begin{aligned}x(t)=&\,4M_\vartheta^2\sup\{E\|z(s)\|^2:0\leq s\leq t\}+4M_\vartheta^2(2\mathcal{K}^2E\|\jmath(0)\|^2+2\mathcal{K}^2\vartheta^2E\|y_1\|^2)\\
&+4N_\vartheta^2\|\jmath\|_{\mathscr{G}}^2.\end{aligned} \tag{29}$$

Next

$$\|z_s+\widetilde{\jmath}_s\|_{\mathscr{G}}^2\leq x(t),$$

It suggests that (27) may be expressed as

$$E\|z(t)\|^2\leq c+\frac{2\mathcal{K}^2Tr(\mathcal{Q})c_{\widetilde{h}}(1+\vartheta^2c_{\widetilde{g}})}{(\Gamma(2\eta))^2}\int_0^t(t-s)^{4\eta-2}x(s)ds. \tag{30}$$

When applying Equation (30) in the definition of x, one can obtain

$$\begin{aligned}x(t)\leq&\,4M_\vartheta^2(2\mathcal{K}^2E\|\phi(0)\|^2+2\mathcal{K}^2\vartheta^2E\|y_1\|^2)+4N_\vartheta^2\|\phi\|_{\mathscr{G}}^2+4M_\vartheta^2c\\
&+\frac{8M_\vartheta^2\mathcal{K}^2Tr(\mathcal{Q})c_{\widetilde{h}}(1+\vartheta^2c_{\widetilde{g}})}{(\Gamma(2\eta))^2}\int_0^t(t-s)^{4\eta-2}x(s)ds.\end{aligned} \tag{31}$$

Using Lemma 8, there exists $\mathscr{N}>0$ such that

$$x(t)\leq\mathscr{N}(4M_\vartheta^2(2\mathcal{K}^2E\|\phi(0)\|^2+2\mathcal{K}^2\vartheta^2E\|y_1\|^2)+4N_\vartheta^2\|\phi\|_{\mathscr{G}}^2+4M_\vartheta^2c):=\mathscr{M},\ t\in\mathcal{I}.$$

Next

$$\begin{aligned}E\|z(t)\|^2&\leq c+\frac{2\mathcal{K}^2Tr(\mathcal{Q})c_{\widetilde{h}}(1+\vartheta^2c_{\widetilde{g}})}{(\Gamma(2\eta))^2}\int_0^t(t-s)^{4\eta-2}\mathscr{M}ds\\
&\leq c+\frac{2\mathcal{K}^2Tr(\mathcal{Q})c_{\widetilde{h}}(1+\vartheta^2c_{\widetilde{g}})\vartheta^{4\eta-1}}{(\Gamma(2\eta))^2(4\eta-1)}\mathscr{M}:=\mathscr{M}^*.\end{aligned} \tag{32}$$

As a result,

$$E\|y(t)\|^2\leq 2E\|z(t)\|^2+4(E\|\mathscr{V}_\eta(t)\jmath(0)\|^2+E\|\mathcal{N}_\eta(t)y_1\|^2)$$

$$\leq 2\mathcal{K}^* + 4(\mathcal{K}^2 E\|\jmath(0)\|^2 + \mathcal{K}^2\vartheta^2 E\|y_1\|^2) := \tau, \ \forall\, t \in \mathcal{I}. \tag{33}$$

The evidence is finished. □

Remark 3. *It is not difficult to see* $\|y\|_{\mathscr{G}\mathscr{C}} \leq \|\jmath\|_{\mathscr{G}} + \tau := \tau^*$*, according to the seminorm, as expressed in* $\mathscr{G}\mathscr{C}$*.*

Theorem 3. *Provided that the assumptions* $(\mathbf{H1}), (\mathbf{H4}), (\mathbf{H5}), (\mathbf{H7})$*, and* $(\mathbf{H8})$ *are fulfilled,* $\jmath(0) \in \mathscr{Y}$*. Next, for each* $\varkappa \in \mathscr{U}_{ad}$ *and* $1 < r < \infty$ *such that* $2\eta r > 1$*, and Equation (22) is mildly solvable on* $(-\infty, \vartheta]$ *with respect to* $\varkappa$*, and the mild solution is unique.*

Proof. Consider

$$\mathscr{G}\mathscr{C}|_{\vartheta_1} = \{y : (-\infty, 0] \to \mathscr{Y},\ y|_{(-\infty,0]} \text{ belongs to } \mathscr{G} \text{ and } y|_{[0,\vartheta_1]} \in \mathscr{C}([0, \eta_1], \mathscr{Y})\}$$

and

$$\mathbb{S}(1, \vartheta_1) := \{y \in \mathscr{G}\mathscr{C}|_{\vartheta_1} \sup_{s\in[0,\vartheta_1]} E\|y(s) - \phi(0)\|^2 \leq 1,\ y(s) = \jmath(s) \text{ for } s \in (-\infty, 0]\}.$$

Next, $\mathbb{S}(1, \vartheta_1) \subseteq \mathscr{G}\mathscr{C}|_{\vartheta_1}$ is a closed convex collection of $\mathscr{G}\mathscr{C}|_{\vartheta_1}$. We can easily obtain the function $\widetilde{h}(s, y_s, \int_0^s \widetilde{g}(s, \iota, y_\iota)d\iota)$ is a measurable function $\forall\, s \in [0, t]$, $t \in [0, \vartheta_1]$ because it is mentioned in $(\mathbf{H7})(\imath)$ and $(\mathbf{H8})(\imath)$. Consider $y \in \mathbb{S}(1, \vartheta_1)$, $\exists\, \tau^* = 8E\|\jmath(0)\|^2 + 8\vartheta_1^2 E\|y_1\|^2 + 4 + 2\|\jmath\|^2_{\mathscr{G}} > 0$ such that

$$\|y\|^2_{\mathscr{G}\mathscr{C}|_{\vartheta_1}} \leq \tau^*. \tag{34}$$

Using $(\mathbf{H7})(\imath\imath)$ and $(\mathbf{H8})(\imath\imath)$, one can obtain

$$\begin{aligned} E\left\|\widetilde{h}\left(s, y_s, \int_0^s \widetilde{g}(s, \iota, y_\iota)d\iota\right)\right\|^2 &\leq c_{\widetilde{h}}\left(1 + \|y_s\|^2_{\mathscr{G}} + E\left\|\int_0^s \widetilde{g}(s, \iota, y_\iota)ds\right\|^2\right) \\ &\leq c_{\widetilde{h}}\left(1 + \tau^* + \vartheta^2 c_{\widetilde{g}}(1 + \tau^*)\right) \\ &= \mathbb{K}^*,\ \forall\, t \in [0, \vartheta_1]. \end{aligned} \tag{35}$$

By Lemma 4($\imath$) and (35), one can obtain

$$\begin{aligned} &E\left\|\int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\left(s, y_s, \int_0^s \widetilde{g}(s, \iota, y_\iota)d\zeta\right)dW(s)\right\|^2 \\ &\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{4\eta-2} E\left\|\widetilde{h}\left(s, y_s, \int_0^s \widetilde{g}(s, \iota, y_\iota)d\iota\right)\right\|^2 ds \\ &\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{4\eta-2}\mathbb{K}^* ds \\ &\leq \left(\frac{\mathcal{M}}{\Gamma(2\eta)}\right)^2 \frac{Tr(\mathcal{Q})\mathbb{K}^*\eta^{4\eta-1}}{4\eta-1}. \end{aligned} \tag{36}$$

In this case, $(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\left(s, y_s, \int_0^s \widetilde{g}(s, \iota, y_\iota)d\iota\right)$ is treated as Bochner integrable, with regard to s in $[0, t]$ $\forall\, t$ in $[0, \vartheta_1]$, due to Lemma 6.

Furthermore, by Lemma 4($\imath$), $(\mathbf{H7})$, $(\mathbf{H8})$, and $2\eta r > 1$, one can obtain

$$E\left\|\int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds\right\|^2$$

$$\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 \|\mathbb{B}\|_\infty^2 E\left[\int_0^t (t-s)^{2\eta-1}\|\varkappa(s)\|ds\right]^2$$
$$\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 \|\mathbb{B}\|_\infty^2 \left[\left(\int_0^t (t-s)^{\frac{r(2\eta-1)}{r-1}}ds\right)^{\frac{r-1}{r}}\left(E\int_0^t \|\varkappa(s)\|^r ds\right)^{\frac{1}{r}}\right]^2$$
$$\leq \left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 \|\mathbb{B}\|_\infty^2 \left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}} \vartheta^{4\eta-\frac{2}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{X})}. \tag{37}$$

In this case, $(t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)$ is also treated as a Bochner integrable, with regard to $s \in [0,t]$ $\forall\, t \in [0,\vartheta_1]$, due to Lemma 6 once more.

Let $\Upsilon : \mathbb{S}(1,\vartheta_1) \to \mathscr{GC}|_{\vartheta_1}$ be determined by

$$(\Upsilon y)(t) = \begin{cases} \jmath(t),\ t \in (-\infty,0], \\ \mathscr{V}_\eta(t)\phi(0) + \mathcal{N}_\eta(t)y_1 \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\Big(s,y_s,\int_0^s \widetilde{g}(s,\iota,y_\iota)d\iota\Big)dW(s) \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds,\ 0 \leq t \leq \vartheta_1. \end{cases} \tag{38}$$

Consider $y(t) = z(t) + \widetilde{\jmath}(t)$, where the function $\widetilde{\jmath} : (-\infty,\vartheta] \to \mathscr{Y}$ is expressed by (25). Next, y fulfills (23) iff $z_0 = 0$ and

$$z(t) = \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\big(s,z_s+\widetilde{\jmath}_s,\int_0^s \widetilde{g}(s,\iota,z_\iota+\widetilde{\jmath}_\iota)d\iota\big)dW(s)$$
$$+ \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds,\ t \in \mathcal{I}. \tag{39}$$

Determine

$$\mathscr{GC}|^0_{\vartheta_1} = \{z \in \mathscr{GC}|_{\vartheta_1} : z_0 = 0 \in \mathscr{G}\}$$

and let $\|\cdot\|_{\mathscr{GC}|^0_{\vartheta_1}}$ be the seminorm in $\mathscr{GC}|^0_{\vartheta_1}$ determined by

$$\|z\|_{\mathscr{GC}|^0_{\vartheta_1}} = \|z_0\|_{\mathscr{G}} + \sup_{s\in[0,\vartheta_1]}\left(E\|z(s)\|^2\right)^{1/2} = \sup_{s\in[0,\vartheta_1]}\left(E\|z(s)\|^2\right)^{1/2}.$$

$(\mathscr{GC}|^0_{\vartheta_1}, \|\cdot\|_{\mathscr{GC}|^0_{\vartheta_1}})$ is a Banach space.

Set

$$\mathbb{S}^0(1,\vartheta_1) := \left\{z \in \mathscr{GC}|^0_{\vartheta_1} \,|\, \sup_{s\in[0,\vartheta_1]} E\|z(s)\|^2 \leq 1,\ z(s) = 0,\ \text{for all}\ s \in (-\infty,0]\right\}.$$

Because $\mathbb{S}^0(1,\vartheta_1) \subseteq \mathscr{GC}|^0_{\vartheta_1}$ is a closed convex subset of $\mathscr{GC}|^0_{\vartheta_1}$.

Determine $\Upsilon^0 : \mathbb{S}^0(1,\vartheta_1) \to \mathscr{GC}|^0_{\vartheta_1}$ as

$$(\Upsilon^0 z)(t) = \begin{cases} 0,\ t \in (-\infty,0], \\ \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\Big(s,z_s+\widetilde{\jmath}_s,\int_0^s \widetilde{g}(s,\iota,z_\iota+\widetilde{\jmath}_\iota)d\iota\Big)ds \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds,\ 0 \leq t \leq \vartheta_1. \end{cases} \tag{40}$$

Next, we verify that Υ^0 is a contraction mapping on $\mathbb{S}^0(1,\vartheta_1)$ with $\vartheta_1 > 0$. For $t \in [0,\vartheta_1]$, we obtain that

$$E\|(\Upsilon^0 z)(t)\|^2$$

$$\leq 2E\| \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\left(s, z_s+\widetilde{\jmath}_s, \int_0^s \widetilde{g}(s,\iota,z_\iota+\widetilde{\jmath}_\iota)d\iota\right)dW(s)\|^2$$
$$+ 2E\| \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds\|^2$$
$$\leq 2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\left[\frac{Tr(\mathcal{Q})\mathbb{K}^* t^{4\eta-1}}{4\eta-1} + \|\mathbb{B}\|_\infty^2\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2(r-1)}{r}} t^{4\eta-\frac{2}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{X})}\right]. \quad (41)$$

Consider

$$\vartheta_{11} = \left[\frac{\big(\Gamma(2\eta)\big)^2}{2\mathcal{K}^2\left[\frac{Tr(\mathcal{Q})\mathbb{K}^*\vartheta^{\frac{2}{r}-1}}{4\eta-1} + \|\mathbb{B}\|_\infty^2\left(\frac{r-1}{2\eta r-1}\right)^{\frac{2r-1)}{r}}\|\varkappa\|^2_{L^r([0,\vartheta],\mathscr{X})}\right]}\right]^{\frac{r}{4r\eta-2}},$$

Next, $\forall\, t \leq \vartheta_{11}$, and it comes from (41), in which

$$E\|(\Upsilon^0 z)(t)\|^2 \leq 1. \quad (42)$$

Moreover, for $-\infty < t \leq 0$, $(\Upsilon^0 z)(t) = 0$. Hence, $\Upsilon^0(\mathbb{S}^0(1,\vartheta_1)) \subseteq \mathbb{S}^0(1,\vartheta_1)$.

For each $t \in [0,\vartheta_1]$, $z, \widetilde{z} \in \mathbb{S}^0(1,\vartheta_1)$ and $\|z\|^2_{\mathscr{G}\mathscr{C}|^0_\vartheta}, \|\widetilde{z}\|^2_{\mathscr{G}\mathscr{C}|^0_\vartheta} \leq \tau^*$. For $t \in [0,\vartheta]$, using Lemma 4 $(\imath)$, $(\mathbf{H7})(\imath\imath)$, $(\mathbf{H8})(\imath\imath)$, we obtain

$$E\|(\Upsilon^0 z)(t) - (\Upsilon^0\widetilde{z})(t)\|^2$$
$$\leq E\| \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\left[\widetilde{h}\left(s, z_s+\widetilde{\jmath}_s, \int_0^s \widetilde{g}(s,\iota,z_\iota+\widetilde{\jmath}_\iota)d\iota\right)\right.$$
$$\left. - \widetilde{h}\left(s, \widetilde{z}_s+\widetilde{\jmath}_s, \int_0^s \widetilde{g}(s,\iota,\widetilde{z}_\iota+\widetilde{\jmath}_\iota)d\iota\right)\right]dW(s)\|^2$$
$$\leq \frac{\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau^*)}{\big(\Gamma(2\eta)\big)^2}\int_0^t (t-s)^{4\eta-2}\|z_s-\widetilde{z}_s\|^2_{\mathscr{G}}ds$$
$$+ \frac{\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau^*)M_{\widetilde{g}}(\tau^*)\vartheta^2}{\big(\Gamma(2\eta)\big)^2}\int_0^t (t-s)^{4\eta-2}\|z_s-\widetilde{z}_s\|^2_{\mathscr{G}}ds$$
$$\leq \frac{\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau^*)\big(1+M_{\widetilde{g}}(\tau^*)\vartheta^2\big)M_\vartheta^2}{\big(\Gamma(2\eta)\big)^2}\int_0^t (t-s)^{4\eta-2}\sup_{s\in\mathcal{I}} E\|z(s)-\widetilde{z}(s)\|^2 ds, \quad (43)$$

It suggests that

$$\sup_{t\in\mathcal{I}} E\|(\Upsilon^0 z)(t) - (\Upsilon^0\widetilde{z})(t)\|^2$$
$$\leq \frac{\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau^*)(1+M_{\widetilde{g}}(\tau^*)\vartheta^2))M_\vartheta^2}{\big(\Gamma(2\eta)\big)^2}\frac{t^{4\eta-1}}{4\eta-1}\sup_{s\in\mathcal{I}} E\|z(s)-\widetilde{z}(s)\|^2. \quad (44)$$

Therefore,

$$\|\Upsilon^0 z - \Upsilon^0\widetilde{z}\|^2_{\mathscr{G}\mathscr{C}|^0_{\vartheta_1}} \leq \frac{\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau^*)(1+M_{\widetilde{g}}(\tau^*)\vartheta^2)M_\vartheta^2}{\big(\Gamma(2\eta)\big)^2}\frac{t^{4\eta-1}}{4\eta-1}\|z-\widetilde{z}\|^2_{\mathscr{G}\mathscr{C}|^0_{\vartheta_1}}.$$

Let

$$\vartheta_{12} = \frac{1}{2}\left[\frac{(\Gamma(2\eta))^2(4\eta-1)}{\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau^*)(1+M_{\widetilde{g}}(\tau^*)\vartheta^2))M_\vartheta^2}\right]^{\frac{1}{4\eta-1}}, \quad (45)$$

$\vartheta_1 = \min\{\vartheta_{11}, \vartheta_{12}\}$. On $\mathbb{S}^0(1, \vartheta_1)$, Υ^0 is a contraction mapping. Because of the contraction mapping concept, Υ^0 has a unique fixed point z that $\in \mathbb{S}^0(1, \vartheta_1)$. Therefore, $y(t) = z(t) + \tilde{j}(t)$ is unique mild solution of (22), with regard to $\varkappa$ on $(-\infty, \vartheta_1]$.

Consider $\vartheta_{21} = \vartheta_1 + \vartheta_{11}$, $\vartheta_{22} = \vartheta_1 + \vartheta_{12}$, $\Delta\vartheta \min\{\vartheta_{21} - \vartheta_1, \vartheta_{12}\} > 0$. Similarly, we say that Equation (22) has unique mild solutions over $(-\infty, \Delta\vartheta]$. By performing the preceding operations in every interval $[\Delta\vartheta, 2\Delta\vartheta], [2\Delta\vartheta, 3\Delta\vartheta], \cdots$, and employing the methods of stages, we may immediately reach the global existence of mild solutions for (22). □

6. Optimal Control Outcomes with Infinite Delay

Let the Lagrange problem (LP):

Find $\varkappa \in \mathscr{U}_{ad}$ such that

$$\mathscr{J}(y^0, \varkappa^0) \leq \mathscr{J}(y^\varkappa, \varkappa), \ \forall \ \varkappa \text{ belongs to } \mathscr{U}_{ad}.$$

In the above

$$\mathscr{J}(y^\varkappa, \varkappa) = E\left\{ \int_0^\vartheta \Psi(t, y_t^\varkappa, y^\varkappa(t), \varkappa(t))dt \right\}, \tag{46}$$

and the mild solution of (22) is described as $y^\varkappa$ corresponding to $\varkappa \in \mathscr{U}_{ad}$. To prove the existence of a solution for (LP), we offer the subsequent conditions:

(**H9**).

- ($\imath$) *On $\mathscr{X}$ $\forall\ y \in \mathscr{G}$, $z \in \mathscr{Y}$ and almost all $t \in \mathcal{I}$, $\Psi(t, y, z, \cdot)$ and is convex.*
- ($\imath\imath$) *The $\mathfrak{J}_t$-measurable functional $\Psi : \mathcal{I} \times \mathscr{G} \times \mathscr{Y} \times \mathscr{X} \to \Re \cup \{\infty\}$.*
- ($\imath\imath\imath$) *On $\mathscr{G} \times \mathscr{Y} \times \mathscr{X}$ for a.e. $t \in \mathcal{I}$, $\Psi(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous.*
- ($\imath v$) *There exist constants $d_2 > 0$, $e \geq 0$, $\mathfrak{a} > 0$, ℓ_2 is nonnegative and $\ell_2 \in L^1(\mathcal{I}, \Re)$ such that*

$$\ell_2(t) + d_2\|y\|^2_{\mathscr{G}} + eE\|z\|^2 + \mathfrak{a}\|\varkappa\|^r_{\mathscr{X}} \leq \Psi(t, y, z, \varkappa).$$

Theorem 4. *Provided that the condtion* (**H9**) *and requirements of Theorem 3 are fulfilled, $\mathbb{B}$ is a strongly continuous operator. Next the Lagrange problem has at least one optimal pair, i.e., there exists an admissible control $\varkappa^0 \in \mathscr{U}_{ad}$ such that*

$$\mathscr{J}(y^0, \varkappa^0) = E\left\{ \int_0^\vartheta \Psi(t, y_t^0, y^0(t), \varkappa^0(t))dt \right\} \leq \mathscr{J}(y^\varkappa, \varkappa), \ \forall \ \varkappa \in \mathscr{U}_{ad}. \tag{47}$$

Proof. Assume $\inf\{\mathscr{J}(y^\varkappa, \varkappa) |\ \varkappa \in \mathscr{U}_{ad}\} = +\infty$, then we clearly obtain the outcome. Let

$$\inf\{\mathscr{J}(y^\varkappa, \varkappa) \mid \varkappa \in \mathscr{U}_{ad}\} = \varepsilon < +\infty.$$

From (**H9**), we obtain $\varepsilon > -\infty$. There exists a minimizing sequence feasible pair by definition of infimum

$$\{(y^\mathfrak{m}, \varkappa^\mathfrak{m})\} \subset \widetilde{\mathscr{P}}_{ad} \equiv \{(y, \varkappa) \mid y \text{ is a mild solution of system (22) corresponding to } \varkappa \in \mathscr{U}_{ad}\}$$

such that $\mathscr{J}(y^\mathfrak{m}, \varkappa^\mathfrak{m}) \to \varepsilon$ when $\mathfrak{m} \to +\infty$. Since $\{\varkappa^\mathfrak{m}\} \subseteq \mathscr{U}_{ad}$, $\mathfrak{m} = 1, 2, \cdots$, $\{\varkappa^\mathfrak{m}\}$ is a bounded subset of $L^r([0, b], \mathscr{X})$, there exists a subsequence, relabeled as $\{\varkappa^\mathfrak{m}\}$ and $\varkappa^0 \in L^r([0, \vartheta], \mathscr{X})$ such that $\varkappa^\mathfrak{m}$ weakly convergent to $\varkappa^0$ in $L^r([0, \vartheta], \mathscr{X})$. By Marzur lemma $\mathscr{U}_{ad}$ is closed, $\varkappa^0 \in \mathscr{U}_{ad}$.

Consider $\{y^{\mathfrak{m}}\} \subset \mathscr{GC}$ is the corresponding sequence of solutions of the integral system

$$y^{\mathfrak{m}}(t) = \begin{cases} \jmath(t),\ t \text{ in } (-\infty, 0], \\ \mathscr{V}_\eta(t)\jmath(0) + \mathcal{M}_\eta(t)y_1 \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\widetilde{h}\Big(s, z_s + \widetilde{\jmath}_s, \int_0^s \widetilde{g}(s,\iota,z_\iota + \widetilde{\jmath}_\iota)d\iota\Big)dW(s) \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\mathbb{B}(s)\varkappa(s)ds,\ t \in \mathcal{I}. \end{cases} \tag{48}$$

From Lemma 8 and Remark 3, $\exists\ \tau > 0$ such that

$$\|y^{\mathfrak{m}}\|^2_{\mathscr{GC}} \le \tau,\ \mathfrak{m} = 0, 1, 2, \dots. \tag{49}$$

Consider $y^{\mathfrak{m}}(t) = z^{\mathfrak{m}}(t) + \widetilde{\jmath}(t)$, here $z^{\mathfrak{m}} \in \mathscr{GC}^0$ and $\widetilde{\jmath} : (-\infty, \vartheta] \to \mathscr{Y}$. For $t \in \mathcal{I}$, one can obtain

$$\begin{aligned} &E\|z^{\mathfrak{m}}(t) - z^0(t)\|^2 \\ &\le 2E\Big\| \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)\Big[\widetilde{h}\Big(s, z_s^{\mathfrak{m}} + \widetilde{\jmath}_s, \int_0^s \widetilde{g}(s,\iota,z_\iota^{\mathfrak{m}} + \widetilde{\jmath}_\iota)d\iota\Big) \\ &\quad - \widetilde{h}\Big(s, z_s^0 + \widetilde{\jmath}_s, \int_0^s \widetilde{g}(s,\iota,z_\iota^0 + \widetilde{\jmath}_\iota)d\iota\Big)\Big]dW(s)\Big\|^2 \\ &\quad + 2E\Big\| \int_0^t (t-s)^{\eta-1}\mathcal{M}_\eta(t-s)[\mathbb{B}(s)\varkappa^{\mathfrak{m}}(s) - \mathbb{B}(s)\varkappa^0(s)]ds\Big\|^2 \\ &\le \frac{2\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau)\big(1 + M_{\widetilde{g}}(\tau)\vartheta^2\big)}{(\Gamma(2\eta))^2} \int_0^t (t-s)^{4\eta-2}\|z_s^{\mathfrak{m}} - z_s^0\|^2_{\mathscr{G}}ds \\ &\quad + 2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2 E\left[\left(\int_0^t (t-s)^{\frac{r(2\eta-1)}{r-1}}ds\right)^{\frac{r-1}{r}}\left(\int_0^t \|\mathbb{B}(s)\varkappa^{\mathfrak{m}}(s) - \mathbb{B}(s)\varkappa^0(s)\|^r ds\right)^{\frac{1}{r}}\right]^2 \\ &\le \frac{2\mathcal{K}^2 Tr(\mathcal{Q})M_{\widetilde{h}}(\tau)[1 + M_{\widetilde{g}}(\tau)\vartheta^2]M_\vartheta^2}{(\Gamma(2\eta))^2} \int_0^t (t-s)^{4\eta-2}\sup_{s\in\mathcal{I}} E\|z^{\mathfrak{m}}(s) - z^0(s)\|^2 ds \\ &\quad + 2\left(\frac{\mathcal{K}}{\Gamma(2\eta)}\right)^2\left(\frac{r-1}{2\eta r - 1}\right)^{\frac{2(r-1)}{r}}\vartheta^{4\eta-\frac{2}{r}}\left[\left(E\int_0^t \|\mathbb{B}(s)\varkappa^{\mathfrak{m}}(s) - \mathbb{B}(s)\varkappa^0(s)\|^r ds\right)^{\frac{1}{r}}\right]^2, \end{aligned}$$

which deduces that there exists a constant $\mathbb{M} > 0$ such that

$$\sup_{t\in\mathcal{I}} E\|z^{\mathfrak{m}}(t) - z^0(t)\|^2 \le \mathbb{M}\|\mathbb{B}\varkappa^{\mathfrak{m}} - \mathbb{B}\varkappa^0\|^2_{L^r([0,\vartheta],\mathscr{X})},\ t \in \mathcal{I}. \tag{50}$$

Hence, $\mathbb{B}$ is strongly continuous, we get

$$\|\mathbb{B}\varkappa^{\mathfrak{m}} - \mathbb{B}\varkappa^0\|^2_{L^r([0,\vartheta],\mathscr{X})} \xrightarrow{s} 0 \text{ as } \mathfrak{m} \to \infty. \tag{51}$$

Next,

$$\|z^{\mathfrak{m}} - z^0\|^2_{\mathscr{GC}^0} \xrightarrow{s} 0 \text{ as } \mathfrak{m} \to \infty,$$

which is equivalent to

$$\|y^{\mathfrak{m}} - y^0\|^2_{\mathscr{GC}} \xrightarrow{s} 0 \text{ as } \mathfrak{m} \to \infty.$$

This yields that

$$y^{\mathfrak{m}} \xrightarrow{s} y^0 \in \mathscr{GC} \text{ as } \mathfrak{m} \to \infty.$$

From [42], we can conclude that

$$(y_t \times y, \varkappa) \to E\left\{ \int_0^{\vartheta} \Psi(t, y_t, y(t), \varkappa(t))dt \right\}$$

is sequentially lower semicontinuous in the strong topology of $L^1([0,\vartheta], \mathscr{G} \times \mathscr{Y})$ and weak topology of $L^r([0,\vartheta], \mathscr{X}) \subset L^1([0,\vartheta], \mathscr{X})$.

Hence, $\mathscr{J}$ is weakly lower semicontinuous on $L^r([0,\vartheta], \mathscr{X})$, and from $(\mathbf{H9})(\imath v)$, $\mathscr{J} > -\infty$, $\mathscr{J}$ reaches its minimum at $\varkappa^0$ and $\in \mathscr{U}_{ad}$, i.e.,

$$\begin{aligned} \varepsilon &= \lim_{\mathfrak{m}\to\infty} E\left\{ \int_0^{\vartheta} \Psi(t, y_t^{\mathfrak{m}}, y^{\mathfrak{m}}(t), \varkappa^{\mathfrak{m}}(t))dt \right\} \\ &\geq E\left\{ \int_0^{\vartheta} \Psi(t, y_t^0, y^0(t), \varkappa^0(t))dt \right\} = \mathscr{J}(y^0, \varkappa^0) \geq \varepsilon. \end{aligned}$$

This completes the proof. □

7. Example

Assume that $\mho \subset \Re^N$ is a bounded domain. Consider the following optimal control issue for the fractional differential equation with infinite delay

$$\begin{cases} {}^C D_t^{\gamma} y(t,\mathfrak{z}) - \Delta y(t,\mathfrak{z}) \\ \quad = \varrho\Big(t, \int_{-\infty}^{t} \rho_1(s-t)y(s,\mathfrak{z})ds, \int_0^t \int_{-\infty}^0 \rho_2(s,\mathfrak{z},\xi - s)y(\xi,\mathfrak{z})d\xi ds\Big) \frac{dW(t)}{dt} \\ \qquad + \int_{\mho} \breve{\wp}_1(\mathfrak{z},s)\varkappa(s,t)ds, \; t \in \mathcal{I}, \; \mathfrak{z} \in \mho, \\ y(t,\mathfrak{z}) = 0, \; t \in \mathcal{I}, \; \mathfrak{z} \in \partial\mho, \\ y(t,\mathfrak{z}) = \jmath(t,\mathfrak{z}), \; y'(0,\mathfrak{z}) = y_1(\mathfrak{z}), \; t \in (-\infty, 0], \; \mathfrak{z} \in \mho. \end{cases} \tag{52}$$

In the above ${}^C D_t^{\gamma}$ denotes the Caputo fractional partial derivative, whose order is γ in $\left(\frac{3}{2}, 2\right)$; the Wiener process is represented by $W(t)$; $\breve{\wp}_1$ is a continuous function that maps from $\mho \times \mho$ into $\Re$; $\varkappa \in L^2(\mathcal{I} \times \mho)$; and $\breve{\wp}_1 : \mho \times \mho \to \mathbb{R}$ is continuous.

We also take into account the subsequent requirements:

$(\mathcal{S}_1)$ $\rho_1(s) \geq 0$ is continuous in $(-\infty, 0]$ and $\int_{-\infty}^0 \rho_1^2(s)ds < \infty$.

$(\mathcal{S}_2)$ ϱ is continuous in $\mathcal{I} \times \mho \times \mho$ fulfills:

- $(\imath)$ $\exists\, \mathcal{K}_{\varrho} > 0$ such that

$$E\|\varrho(t,\delta_1,y_1) - \varrho(t,\delta_2,y_2)\|^2 \leq \mathcal{K}_{\varrho}(\hat{\mu}\|\delta_1 - \delta_2\|^2 + E\|y_1 - y_2\|^2), \text{ for all } t \in \mathcal{I}.$$

  In the above $\hat{\mu} = (-\frac{1}{2\hbar} \int_{-\infty}^0 \rho_1^2(s)ds)^{\frac{1}{2}}$.

- $(\imath\imath)$ $\exists\, c_{\varrho} > 0$ such that

$$E\|\varrho(t,\delta,y)\| \leq c_{\varrho}(1 + \hat{\mu}\|\delta\|^2 + E\|y\|^2), \text{ for all } t \in \mathcal{I}.$$

$(\mathcal{S}_3)$ $\rho_2(t,\mathfrak{z},s)$ is continuous in $\mathcal{I} \times \mho \times (-\infty, 0]$ and $\int_{-\infty}^0 \rho_2(t,\mathfrak{z},s)ds = \alpha(t,\mathfrak{z}) < \infty$ and $c_j = \sup\{\alpha(t,\mathfrak{z}) : t \in \mathcal{I}, \; \mathfrak{z} \in \mho\}$.

Consider $\mathscr{Y} = \mathscr{X} = L^2(\mho)$ and A is the Laplace operator, with Dirichlet boundary conditions that are represented by $A = \Delta$ and

$$\mathcal{D}(A) = \{\widetilde{h} \in \mathcal{W}_0^1(\mho), \; A\widetilde{h} \in L^2(\mho)\}.$$

Consider $\mathcal{D}(A) = \mathcal{W}_0^1(\mho) \cap \mathcal{W}^2(\mho)$. According to [47], A can generates the uniformly bounded strongly continuous cosine family $\mathscr{V}(t)$ for $t \geq 0$. Let $\lambda_{\mathfrak{m}} = \mathfrak{m}^2\pi^2$ and $\breve{\xi}_{\mathfrak{m}}(\mathfrak{z}) =$

$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin(\mathfrak{m}\pi\mathfrak{z})$, $\mathfrak{m}=1,2,\ldots$. It is obvious that $\{-\lambda_{\mathfrak{m}},\breve{\xi}_{\mathfrak{m}}\}_{\mathfrak{m}=1}^{\infty}$ is the eigensystem of the operator A, then $0<\lambda_1\leq\lambda_2\leq\cdots$, $\lambda_{\mathfrak{m}}\to\infty$ as $\mathfrak{m}\to\infty$, and $\{\breve{\xi}_{\mathfrak{m}}\}_{\mathfrak{m}=1}^{\infty}$ forms an orthonormal basis of $\mathscr{Y}$. Furthermore,

$$Ay=-\sum_{\mathfrak{m}=1}^{\infty}\lambda_{\mathfrak{m}}\langle y,\breve{\xi}_{\mathfrak{m}}\rangle\breve{\xi}_{\mathfrak{m}},\ y\in\mathcal{D}(A), \tag{53}$$

where $\langle\cdot,\cdot\rangle$ represents inner product in $\mathscr{Y}$. Therefore, the cosine function is described as

$$\mathscr{V}(t)y=\sum_{\mathfrak{m}=1}^{\infty}\cos\left(\lambda_{\mathfrak{m}}^{\frac{1}{2}}t\right)\langle y,\breve{\xi}_{\mathfrak{m}}\rangle\breve{\xi}_{\mathfrak{m}},\ y\in\mathscr{Y}, \tag{54}$$

and the sine function is associated with cosine function given by

$$\mathscr{W}(t)y=\sum_{\mathfrak{m}=1}^{\infty}\frac{1}{\lambda_{\mathfrak{m}}^{\frac{1}{2}}}\sin\left(\lambda_{\mathfrak{m}}^{\frac{1}{2}}t\right)\langle y,\breve{\xi}_{\mathfrak{m}}\rangle\breve{\xi}_{\mathfrak{m}},\ y\in\mathscr{Y}, \tag{55}$$

and $\|\mathscr{V}(t)\|_{L_\vartheta}\leq 1,\forall\, t\geq 0$. Therefore, (**H1**) is true. The controls are functions $\varkappa:\mathscr{V}_{\mathfrak{z}}(\mho)\to\Re$, such that $\varkappa\in L^2(\mathscr{V}_{\mathfrak{z}}(\mho))$. It is mentioned that $t\to\varkappa(\cdot,t)$ is measurable. The family

$$\mathscr{U}(t)=\{\varkappa\in\mathscr{X}:\|\varkappa\|_{\mathscr{X}}^2\leq\kappa\}.$$

In the above $\kappa\in L^2(\mathcal{I},\Re^+)$. We allocate the admissible controls $\mathscr{U}_{ad}$ to all $\varkappa\in L^2(\mathscr{V}_{\mathfrak{z}}(\mho))$ such that $\|\varkappa(\cdot,t)\|_{L^2(\mho)}\leq\kappa(t)$, a.e. $t\in\mathcal{I}$.

Suppose that the phase space

$$\mathscr{G}=\left\{\mathscr{H}\in\mathscr{C}((-\infty,0],\mathscr{Y}):\lim_{s\to-\infty}e^{\hbar s}\mathscr{H}(s)\text{ exists in }\mathscr{Y}\right\} \tag{56}$$

is determined as $\hbar<0$ and assume that

$$\|\mathscr{H}\|_{\mathscr{G}}=\sup_{-\infty<s\leq 0}\left\{e^{\hbar s}\left(E\|\mathscr{H}(s)\|^2\right)^{1/2}\right\}. \tag{57}$$

$(\mathscr{G},\|\cdot\|_{\mathscr{G}})$ is a Banach space that fulfills $(\mathcal{S}_1)-(\mathcal{S}_3)$, with $\mathcal{H}=1$, $\overline{M}(t)=\sup\{1,e^{-\hbar t}\}$, $\overline{N}(t)=e^{-\hbar t}$.

In order to $(t,\mathscr{H})\in\mathcal{I}\times\mathscr{G}$, in which $\mathscr{H}(s)(\mathfrak{z})=\jmath(s,\mathfrak{z})$, $(s,\mathfrak{z})\in(-\infty,0]\times\mho$, consider

$$\begin{aligned}
&y(t)(\mathfrak{z})=y(t,\mathfrak{z}),\\
&\widetilde{g}(t,\mathscr{H})(\mathfrak{z})=\int_{-\infty}^{0}\rho_2(t,\mathfrak{z},s)\mathscr{H}(s)(\mathfrak{z})ds,\\
&\widetilde{h}\left(t,\mathscr{H},\int_0^t\widetilde{g}(s,\mathscr{H})ds\right)(\mathfrak{z})=\varrho\left(\int_{-\infty}^{0}\rho_1(s)\mathscr{H}(s)(\mathfrak{z})ds,\int_0^t\rho_2(s,\mathscr{H})(\mathfrak{z})ds\right),\\
&\mathbb{B}(t)\varkappa(t)(\mathfrak{z})=\int_{\mho}\breve{\wp}_1(\mathfrak{z},s)\varkappa(s,t)ds.
\end{aligned}$$

Because Equation (52) has been transformed (22). Now, we consider the subsequent cost function:

$$\mathscr{J}(y^{\varkappa},\varkappa)=E\left\{\int_0^{\vartheta}\Psi(t,y_t^{\varkappa},y^{\varkappa}(t),\varkappa(t))dt\right\}. \tag{58}$$

In the above $\Psi : \mathcal{I} \times \mathscr{C}^{1,0}((-\infty,0] \times \mho) \times L^2(\mathcal{I} \times \mho) \to \Re \cup \{+\infty\}$ for $y \in \mathscr{C}^{1,0}((-\infty,\vartheta] \times \mho)$ and $\varkappa \in L^2(\mho \times \mathcal{I})$,

$$\Psi(t, y_t^{\varkappa}, y^{\varkappa}(t), \varkappa(t))(\mathfrak{z}) \\ = \int_{\mho} \int_{-\infty}^{0} |y^{\varkappa}(t+s,\mathfrak{z})|^2 ds d\mathfrak{z} + \int_{\mho} |y^{\varkappa}(t,\mathfrak{z})|^2 d\mathfrak{z} + \int_{\mho} |\varkappa(\mathfrak{z},t)|^2 d\mathfrak{z}. \quad (59)$$

Theorem 4's properties are fulfilled. Hence, (52) has at least one optimal pair.

8. Conclusions

In this paper, by utilizing fractional calculus, Hölder's inequality, stochastic analysis techniques, and the fixed point theorem, we established the existence and uniqueness of mild solutions for the fractional stochastic integrodifferential systems of order $1 < \gamma < 2$. Additionally, we discussed the existence of optimal control for the proposed problem through Balder's theorem. Now, there has been a growing interest in many areas of science and engineering in the study of systems containing memory or aftereffects, i.e., the effect of infinite delay on state equations. So, we extended the given fractional system to infinite delay. The existence and uniqueness of mild solutions for the fractional stochastic integrodifferential systems of order $1 < \gamma < 2$ involving infinite delay have been examined by using the Banach fixed point approach. Further, we verified the existence of optimal control for the stated problem. An application has been offered at the end to support the validity of the study. In the future, we will investigate the optimal control issue for the fractional stochastic integrodifferential systems of order $1 < \gamma < 2$ via sectorial operator. Further, we will extend the system with noninstantaneous impulses.

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