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Fekete–Szegö Problem and Second Hankel Determinant for a Class of Bi-Univalent Functions Involving Euler Polynomials

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Abstract: Some well-known authors have extensively used orthogonal polynomials in the framework of geometric function theory. We are motivated by the previous research that has been conducted and, in this study, we solve the Fekete–Szegö problem as well as give bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant for functions in the class $\mathcal{G}_{\Sigma}(v,\sigma)$ of analytical and bi-univalent functions, implicating the Euler polynomials.

Keywords: analytic function; bi-univalent function; Fekete–Szegö problem; second Hankel determinant; Euler polynomials

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Citation: Riaz, S.; Shaba, T.G.; Xin, Q.; Tchier, F.; Khan, B.; Malik, S.N. Fekete–Szegö Problem and Second Hankel Determinant for a Class of Bi-Univalent Functions Involving Euler Polynomials. *Fractal Fract.* **2023**, 7, 295. https://doi.org/10.3390/fractalfract7040295

Academic Editors: Carlo Cattani, Gheorghe Oros and Georgia Irina Oros

Received: 24 January 2023
Revised: 23 March 2023
Accepted: 28 March 2023
Published: 29 March 2023

1. Introduction

Let the collection of all functions f be expressed by $\mathcal{A}$ and has the following form of series.

$$f(\xi) = \xi + \sum_{l=2}^{\infty} s_l \xi^l = \xi + s_2\xi^2 + s_3\xi^3 + \cdots + s_l\xi^l + \cdots, \quad s_l \in \mathbb{C}, \tag{1}$$

which are holomorphic in $\mathcal{U}$ where

$$\mathcal{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$$

in the complex plane. If a function never yields the same value twice, it is said to be univalent in $\mathcal{U}$. Mathematically

$$\xi_1 \neq \xi_2 \text{ for all points } \xi_1 \text{ and } \xi_2 \text{ in } \mathcal{U} \text{ implies } f(\xi_1) \neq f(\xi_2).$$

Let $\mathcal{S}$ represent the family of all univalent functions in $\mathcal{A}$ as well. As the families of starlike and convex functions of order ϕ, respectively, the sets $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ are some of the significant and well-researched subclasses of $\mathcal{S}$, therefore, have been added here as follows (see [1,2]).

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{S} : \Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \phi, \;\; \phi \in [0,1), \;\; \xi \in \mathcal{U} \right\}$$

and

$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) > \phi, \ \ \phi \in [0,1), \ \ \xi \in \mathcal{U} \right\}.$$

Remark 1. *It is easy to seen that*

$$\mathcal{S}^*(0) = \mathcal{S}^* \quad \textit{and} \quad \mathcal{C}(0) = \mathcal{C},$$

where $\mathcal{S}^*$ and $\mathcal{C}$ are the well-known function classes of starlike and convex functions, respectively.

Suppose g and f be analytical functions in $\mathcal{U}$. For an analytic function w with

$$|\omega(\xi)| < 1 \text{ and } \omega(0) = 0 \ \ (\xi \in \mathcal{U}),$$

The function f is considered to be subordinate to g if the relation below holds, that is

$$g(\omega(\xi)) = f(\xi).$$

In addition to that, if the function $g \in \mathcal{S}$, then the following equivalency exists:

$$f(\xi) \prec g(\xi) \ \text{ if } \ g(0) = f(0)$$

and

$$f(\mathcal{U}) \subset g(\mathcal{U}).$$

For details, see [1]. The inverse function for every $f \in \mathcal{S}$, is defined by

$$\mathcal{F}(f(\xi)) = \xi, f(\mathcal{F}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right) \text{ and } (\xi, w \in \mathcal{U}),$$

where

$$\mathcal{F}(w) = w - s_2 w^2 + (2s_2^2 - s_3)w^3 + (-5s_2^3 + 5s_2 s_3 - s_4)w^4 + \cdots . \tag{2}$$

A function f which is analytic is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. The classes of all such function is denoted by Σ.

The housebreaking research of Srivastava et al. [3] in fact, in the past decades, revitalized the examination of bi-univalent functions. Following the study of Srivastava et al. [3], numerous unique subclasses of the class Σ were presented and similarly explored by numerous authors. The function classes $H_\Sigma(\gamma, \varepsilon, \mu.\varsigma; \alpha)$ and $H_\Sigma(\gamma, \varepsilon, \mu.\varsigma; \beta)$ as an illustration, were defined and Srivastava et al. [4] produced estimates for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Many authors were motivated by the work of Srivastava and have defined a number of other subclasses of analytic and bi-univalent functions, and for their defined functions classes different types of results were obtained. In this paper, motivated by the work of Srivastava, we define certain new classes of bi-univalent functions and obtain some remarkable results for our defined function's classes, including, for example, the initial bonds for the coefficients, the Fekete–Szegö problem and the second Hankel determinant.

The theory of special functions, originating from their numerous applications, is a very old branch of analysis. The long existing interest in them has recently grown due to their new applications and further generalizations. The contemporary intensive development of this theory touches various unexpected areas of applications and is based on the tools of numerical analysis and computer algebra system, used for analytical evolutions and graphical representations of special functions. Additionally, in Computer Science, special functions are used as activation functions, which play a significant role in this area. Particularly, orthogonal polynomials are an important and intriguing class of special functions. Many branches of the natural sciences contain them, including discrete mathematics, theta functions, continuous fractions, Eulerian series, elliptic functions, etc.; see [5,6], also [7–9].

In pure mathematics, the functions mentioned above have numerous uses. A lot of researchers have started working in a variety of fields as a result of the widespread use of these functionalities. Modern geometric function theory research focuses on the geometric features of special functions, including hypergeometric functions, Bessel functions, and certain other related functions. We refer to [10,11] and any relevant references in relation to some of the geometric characteristics of these functions. In this paper, we develop a new class of bi-univalent functions and use a particular special function, the Euler polynomial.

Using the generating function, the Eulers polynomials $\mathcal{E}_m(v)$ are frequently defined (see, e.g., [12,13]):

$$L(v,t) = \frac{2e^{tv}}{e^t+1} = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{t^m}{m!}, \quad |t| < \pi \tag{3}$$

An explicit formula for $\mathcal{E}_m(v)$ is given by

$$\mathcal{E}_n(v) = \sum_{m=0}^{n} \frac{1}{2^m} \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} (v+k)^n$$

Now $\mathcal{E}_m(v)$ in terms of $\mathcal{E}_k$ can be obtained from the equation above as:

$$\mathcal{E}_m(v) = \sum_{k=0}^{m} \binom{m}{k} \frac{\mathcal{E}_k}{2^k} \left(v - \frac{1}{2}\right)^{m-k}. \tag{4}$$

The initial Euler polynomials are:

$$\begin{aligned}
\mathcal{E}_0(v) &= 1 \\
\mathcal{E}_1(v) &= \frac{2v-1}{2} \\
\mathcal{E}_2(v) &= v^2 - v \\
\mathcal{E}_3(v) &= \frac{4v^3 - 6v^2 + 1}{4} \\
\mathcal{E}_4(v) &= v^4 - 2v^3 + v.
\end{aligned} \tag{5}$$

Geometric function theory continues to struggle with the subject of determining bounds on the coefficients. The size of their coefficients can have an impact on a variety of aspects of analytic functions, including univalency, rate of growth, and distortion. The Fekete–Szegö problem, Hankel determinants, and many other formulations of efficient problems include an estimate of general or l^{th} coefficient bounds. The coefficient concerns discussed above were addressed by several researchers using various approaches. Here, the functional of Fekete–Szegö for a function $f(\zeta) \in \mathcal{S}$ is quite significant, and is denoted by $\mathcal{L}_\beta(f) = |s_3 - \beta s_2^2|$. By giving this functional, Fekete and Szegö [14] invalidated the Littlewood and Parley's claim that the modulus of coefficients of odd functions $f \in \mathcal{S}$ are less than or equal to 1. Much attention has been paid to the functional, especially in several subfamilies of univalent functions (see [15,16]).

Pommerenke [17] investigated and defined below the l^{th}-Hankel determinant, denoted by $H_s(l)(s, l \in \mathcal{N} = \{1,2,3,\cdots\})$, for any function $f \in \mathcal{S}$ in geometric function theory:

$$H_s(l) = \begin{vmatrix} j_l & j_{l+1} & \cdots & j_{l+s-1} \\ j_{l+1} & j_{l+2} & \cdots & j_{l+s} \\ j_{l+2} & j_{l+3} & \cdots & j_{l+s+1} \\ \vdots & \vdots & \ldots & \vdots \\ j_{l+s-1} & j_{l+s} & \cdots & j_{l+2(s-1)} \end{vmatrix}$$

For certain s and l values,

$$H_2(1) = \begin{vmatrix} j_1 & j_2 \\ j_2 & j_3 \end{vmatrix} = |j_3 - j_2^2| \text{ and } H_2(2) = \begin{vmatrix} j_2 & j_3 \\ j_3 & j_4 \end{vmatrix} = |j_2 j_4 - j_3^2|. \tag{6}$$

We see that the determinant $|H_2(1)|$ corresponds with the $\mathcal{L}_1(f)$, implying that $\mathcal{L}_\beta(f)$ is a generalization of $|H_2(1)|$. Following that, many additional subclasses of univalent functions paid close attention to the problem of determining bounds on coefficients. Recent research in this area includes the papers in [18,19].

In this study, we define the new subclass introduced and studied in the present paper, denoted by $\mathcal{G}_\Sigma(v,\sigma)$, consisting of bi-univalent functions satisfying a certain subordination involving Eulers polynomials. We solve the Fekete–Szegö problem for functions in the class $\mathcal{G}_\Sigma(v,\sigma)$ and in the special instances, as well as provide bound estimates for the coefficients.

Definition 1. *For $f \in \mathcal{G}_\Sigma(v,\sigma)$, suppose the following subordination is true:*

$$(1-\sigma)\frac{\xi f'(\xi)}{f(\xi)} + \sigma\left(\frac{f'(\xi)+\xi f''(\xi)}{f'(\xi)}\right) \prec L(v,\xi) = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{\xi^m}{m!} \tag{7}$$

and

$$(1-\sigma)\frac{w\mathcal{F}'(w)}{\mathcal{F}(w)} + \sigma\left(\frac{\mathcal{F}'(w)+w\mathcal{F}''(w)}{\mathcal{F}'(w)}\right) \prec L(v,w) = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{w^m}{m!}, \tag{8}$$

where $\sigma \geq 0$, $v \in (\frac{1}{2},1]$, $\xi, w \in \mathcal{U}$, $L(v,w)$ is given by (3), and $\mathcal{F} = f^{-1}$ is given by (2). It could be seen that both the functions f and and its inverse $\mathcal{F} = f^{-1}$ are univalent in $\mathcal{U}$, so we can conclude that the function f is bi-univalent belonging to the function class $\mathcal{G}_\Sigma(v,\sigma)$.

Remark 2. *Setting $\sigma = 0$ in Definition 1, we have bi-starlike function class $f \in \mathcal{S}^*_\Sigma(v)$, which fulfilled the following conditions:*

$$\frac{\xi f'(\xi)}{f(\xi)} \prec L(v,\xi) = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{\xi^m}{m!} \tag{9}$$

and

$$\frac{w\mathcal{F}'(w)}{\mathcal{F}(w)} \prec L(v,w) = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{w^m}{m!}, \tag{10}$$

where $\xi, w \in \mathcal{U}$, $L(v,w)$ is given by (3), and $\mathcal{F} = f^{-1}$ is given by (2).

Remark 3. *Setting $\sigma = 1$ in Definition 1, we have bi-convex function class $f \in \mathcal{C}_\Sigma(v)$, which fulfilled the following conditions:*

$$\frac{f'(\xi)+\xi f''(\xi)}{f'(\xi)} \prec L(v,\xi) = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{\xi^m}{m!} \tag{11}$$

and

$$\frac{\mathcal{F}'(w)+w\mathcal{F}''(w)}{\mathcal{F}'(w)} \prec L(v,w) = \sum_{m=0}^{\infty} \mathcal{E}_m(v)\frac{w^m}{m!}, \tag{12}$$

where $L(v,w)$ is given by (3), and $\mathcal{F} = f^{-1}$ is given by (2).

Next, let $\mathcal{P}$ represent the class including those functions, analytic in $\mathcal{U}$, and having series form given below as:

$$\alpha(\xi) = 1 + \sum_{l=1}^{\infty} \alpha_l \xi^l, \tag{13}$$

such that

$$\Re\{\alpha(\xi)\} > 0 \qquad (\forall\, \xi \in \mathcal{U}).$$

Lemma 1. *[1] Let $\alpha \in \mathcal{P}$ be given by*

$$\alpha(\xi) = 1 + \alpha_1\xi + \alpha_2\xi^2 + \cdots \quad (\xi \in \mathcal{U}) \tag{14}$$

then

$$|\alpha_l| \leq 2 \quad (l \in \{1, 2, 3, \cdots\}). \tag{15}$$

Lemma 2. *[20] Let $\alpha \in \mathcal{P}$ be given by (14), then*

$$2\alpha_2 = \alpha_1^2 + x(4 - \alpha_1^2) \tag{16}$$

and

$$4\alpha_3 = \alpha_1^3 + 2\alpha_1(4 - \alpha_1^2)x - \alpha_1(4 - \alpha_1^2)x^2 + 2(4 - \alpha_1^2)(1 - |x|^2)\xi \tag{17}$$

for some x, ξ, $|x| \leq 1$, and $|\xi| \leq 1$.

2. Coefficients Bounds for the Functions of Class $\mathcal{G}_\Sigma(v, \sigma)$

Theorem 1. *Let $f \in \mathcal{G}_\Sigma(v, \sigma)$. Then:*

$$|s_2| \leq \sqrt{\Omega_1(\sigma, v)},$$

$$|s_3| \leq \frac{(2v-1)^2}{4(1+\sigma)^2} + \frac{2v-1}{4(1+2\sigma)}$$

and

$$|s_4| \leq \frac{(1+4\sigma)(2v-1)^3}{12(1+2\sigma)(1+\sigma)^3} + \frac{(15+45\sigma)(2v-1)^2}{48(1+\sigma)(1+2\sigma)^2} + \frac{4v^3 - 6v^2 + 1}{72(1+2\sigma)}$$

where

$$\Omega_1(\sigma, v) = \frac{(2v-1)^3}{|2(\sigma+1)(2\sigma + 2(\sigma-1)v^2 - 2(3\sigma+1)v + 1)|}. \tag{18}$$

Proof. Let $f \in \Sigma$ given by (1) be in the class $\mathcal{G}_\Sigma(v, \sigma)$. Then

$$(1-\sigma)\frac{\xi f'(\xi)}{f(\xi)} + \sigma\left(\frac{f'(\xi) + \xi f''(\xi)}{f'(\xi)}\right) = L(v, a(\xi)) \tag{19}$$

and

$$(1-\sigma)\frac{w\mathcal{F}'(w)}{\mathcal{F}(w)} + \sigma\left(\frac{\mathcal{F}'(\xi) + w\mathcal{F}''(w)}{\mathcal{F}'(w)}\right) = L(v, b(w)) \tag{20}$$

We define $\alpha, \delta \in \mathcal{P}$ as follows:

$$\alpha(\xi) = \frac{1 + a(\xi)}{1 - a(\xi)} = 1 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \cdots$$

$$\Rightarrow \quad a(\xi) = \frac{\alpha(\xi) - 1}{\alpha(\xi) + 1} \quad (\xi \in \mathcal{U}) \tag{21}$$

and

$$\delta(w) = \frac{1 + b(w)}{1 - b(w)} = 1 + \delta_1 w + \delta_2 w^2 + \delta^3 w^3 + \cdots$$

$$\Rightarrow \quad b(w) = \frac{\delta(w) - 1}{\delta(w) + 1} \quad (w \in \mathcal{U}). \tag{22}$$

From (21) and (22), we obtain

$$a(\xi) = \frac{\alpha_1}{2}\xi + \left(\frac{\alpha_2}{2} - \frac{\alpha_1^2}{4}\right)\xi^2 + \left(\frac{\alpha_3}{2} - \frac{\alpha_1\alpha_2}{2} + \frac{\alpha_1^3}{8}\right)\xi^3 + \cdots \tag{23}$$

and

$$b(w) = \frac{\delta_1}{2}w + \left(\frac{\delta_2}{2} - \frac{\delta_1^2}{4}\right)w^2 + \left(\frac{\delta_3}{2} - \frac{\delta_1\delta_2}{2} + \frac{\delta_1^3}{8}\right)w^3 + \cdots . \tag{24}$$

Taking it from (23) and (24), we have:

$$\begin{aligned} L(v, a(\xi)) = \mathcal{E}_0(v) + \frac{\mathcal{E}_1(v)}{2}\alpha_1\xi + \left[\frac{\mathcal{E}_1(v)}{2}\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\alpha_1^2\right]\xi^2 \\ + \left[\frac{\mathcal{E}_1(v)}{2}\left(\alpha_3 - \alpha_1\alpha_2 + \frac{\alpha_1^3}{4}\right) + \frac{\mathcal{E}_2(v)}{4}\alpha_1\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\alpha_1^3\right]\xi^3 + \cdots \end{aligned} \tag{25}$$

and

$$\begin{aligned} L(v, b(w)) = \mathcal{E}_0(v) + \frac{\mathcal{E}_1(v)}{2}\delta_1 w + \left[\frac{\mathcal{E}_1(v)}{2}\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\delta_1^2\right]w^2 \\ + \left[\frac{\mathcal{E}_1(v)}{2}\left(\delta_3 - \delta_1\delta_2 + \frac{\delta_1^3}{4}\right) + \frac{\mathcal{E}_2(v)}{4}\delta_1\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\delta_1^3\right]w^3 + \cdots . \end{aligned} \tag{26}$$

It follows from (19), (20), (25) and (26) that we have:

$$(1+\sigma)s_2 = \frac{\mathcal{E}_1(v)}{2}\alpha_1 \tag{27}$$

$$-(1+3\sigma)s_2^2 + 2(1+2\sigma)s_3 = \frac{\mathcal{E}_1(v)}{2}\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\alpha_1^2 \tag{28}$$

$$\begin{aligned} (1+7\sigma)s_2^3 - 3(1+5\sigma)s_2s_3 + 3(1+3\sigma)s_4 = \frac{\mathcal{E}_1(v)}{2}\left(\alpha_3 - \alpha_1\alpha_2 + \frac{\alpha_1^3}{4}\right) \\ + \frac{\mathcal{E}_2(v)}{4}\alpha_1\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\alpha_1^3 \end{aligned} \tag{29}$$

$$-(1+\sigma)s_2 = \frac{\mathcal{E}_1(v)}{2}\delta_1 \tag{30}$$

$$(3+5\sigma)s_2^2 - 2(1+2\sigma)s_3 = \frac{\mathcal{E}_1(v)}{2}\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_2(v)}{8}\delta_1^2 \tag{31}$$

$$\begin{aligned} -3(1+3\sigma)s_4 + (12+30\sigma)s_2s_3 - (10+22\sigma)s_2^3 = \frac{\mathcal{E}_1(v)}{2}\left(\delta_3 - \delta_1\delta_2 + \frac{\delta_1^3}{4}\right) \\ + \frac{\mathcal{E}_2(v)}{4}\delta_1\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{\mathcal{E}_3(v)}{48}\delta_1^3. \end{aligned} \tag{32}$$

Adding (27) and (30) and further simplification, we have

$$\alpha_1 = -\delta_1, \quad \alpha_1^2 = \delta_1^2 \text{ and } \alpha_1^3 = -\delta_1^3. \tag{33}$$

When (27) and (30) are squared and added, the following result is obtained:

$$2(1+\sigma)^2 s_2^2 = \frac{\mathcal{E}_1^2(v)(\alpha_1^2+\delta_1^2)}{4} \tag{34}$$

$$\Rightarrow s_2^2 = \frac{\mathcal{E}_1^2(v)(\alpha_1^2+\delta_1^2)}{8(1+\sigma)^2}. \tag{35}$$

Additionally, adding (28) and (31) gives

$$2(1+\sigma)s_2^2 = \frac{2\mathcal{E}_1(v)(\alpha_2+\delta_2)+\alpha_1^2(\mathcal{E}_2(v)-2\mathcal{E}_1(v))}{4}$$

$$8(1+\sigma)s_2^2 = 2\mathcal{E}_1(v)(\alpha_2+\delta_2)+\alpha_1^2(\mathcal{E}_2(v)-2\mathcal{E}_1(v)). \tag{36}$$

Applying (33) in (34)

$$\alpha_1^2 = \frac{4(1+\sigma)^2}{\mathcal{E}_1^2(v)} s_2^2. \tag{37}$$

In (36), replacing α_1^2 with the following results:

$$|s_2|^2 \le \frac{2\mathcal{E}_1^3(v)(|\alpha_2|+|\delta_2|)}{2|2(1+\sigma)\mathcal{E}_1^2(v)-(1+\sigma)^2[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]|}. \tag{38}$$

Applying Lemma 1 and (5), we obtain:

$$|s_2| \le \sqrt{\Omega_1(\sigma,v)}$$

where $\Omega_1(\sigma,v)$ is given by (18).

Subtracting (31) and (28) and with some computation, we have

$$s_3 = s_2^2 + \frac{\mathcal{E}_1(v)(\alpha_2-\delta_2)}{8(1+2\sigma)} \tag{39}$$

$$s_3 = \frac{\mathcal{E}_1^2(v)\alpha_1^2}{4(1+\sigma)^2} + \frac{\mathcal{E}_1(v)(\alpha_2-\delta_2)}{8(1+2\sigma)} \tag{40}$$

Applying Lemma 1 and (5), we obtain:

$$|s_3| \le \frac{(2v-1)^2}{4(1+\sigma)^2} + \frac{2v-1}{4(1+2\sigma)} \tag{41}$$

By removing (32) from (29), we arrive at:

$$\begin{aligned} s_4 = {} & \frac{(1+4\sigma)\mathcal{E}_1^3(v)}{12(1+3\sigma)(1+\sigma)^3}\alpha_1^3 + \frac{(15+45\sigma)\mathcal{E}_1^2(v)(\alpha_2-\delta_2)}{96(1+\sigma)(1+2\sigma)(1+3\sigma)}\alpha_1 + \frac{\mathcal{E}_1(v)(\alpha_3-\delta_3)}{12(1+3\sigma)} \\ & + \frac{[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](\alpha_2+\delta_2)}{24(1+3\sigma)}\alpha_1 + \frac{[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{144(1+3\sigma)}\alpha_1^3. \end{aligned} \tag{42}$$

Applying Lemma 1 and (5), we obtain:

$$|s_4| \le \frac{(1+4\sigma)(2v-1)^3}{12(1+3\sigma)(1+\sigma)^3} + \frac{(15+45\sigma)(2v-1)^2}{48(1+\sigma)(1+2\sigma)(1+3\sigma)} + \frac{4v^3-6v^2+1}{72(1+3\sigma)}.$$

□

If we put $\sigma = 0$ in Theorem 1, then we have the next corollary.

Corollary 1. *Let $f \in \mathcal{S}^*_\Sigma(v)$. Then:*

$$|s_2| \leq \sqrt{\frac{(2v-1)^3}{|2(2v^2+2v-1)|}},$$

$$|s_3| \leq \frac{v(2v-1)}{2}$$

and

$$|s_4| \leq \frac{(2v-1)^3}{12} + \frac{15(2v-1)^2}{48} + \frac{4v^3-6v^2+1}{72}$$

For $\sigma = 1$, we arrive at the next corollary of Theorem 1.

Corollary 2. *Let $f \in \mathcal{C}_\Sigma(v)$. Then:*

$$|s_2| \leq \sqrt{\frac{(2v-1)^3}{|4(3-8v)|}},$$

$$|s_3| \leq \frac{(2v-1)(6v+13)}{192},$$

and

$$|s_4| \leq \frac{5(2v-1)^3}{384} + \frac{5(2v-1)^2}{96} + \frac{4v^3-6v^2+1}{288}.$$

3. Fekete–Szegö Inequalities for the Functions of Class $\mathcal{G}_\Sigma(v,\sigma)$

Theorem 2. *Let $f \in \mathcal{G}_\Sigma(v,\sigma)$. Then, for some $\mu \in \mathbb{R}$,*

$$\left|s_3 - \mu s_2^2\right| \leq \begin{cases} 2|1-\mu|\Omega_1(\sigma,v) & \left(|1-\mu|\Omega_1(\sigma,v) \geq \frac{2v-1}{4(1+2\sigma)}\right) \\ \frac{2v-1}{2(1+2\sigma)} & \left(|1-\mu|\Omega_1(\sigma,v) < \frac{2v-1}{4(1+2\sigma)}\right), \end{cases}$$

where $\Omega_1(\sigma,v)$ is given by (18).

Proof. From (39), we obtain:

$$s_3 - \mu s_2^2 = s_2^2 + \frac{\mathcal{E}_1(v)(\alpha_2-\delta_2)}{8(1+2\sigma)} - \mu s_2^2$$

Applying the popular triangular inequality, we obtain:

$$|s_3 - \mu s_2^2| \leq \frac{2v-1}{4(1+2\sigma)} + |1-\mu|\Omega_1(\sigma,v)$$

If:

$$|1-\mu|\Omega_1(\sigma,v) \geq \frac{2v-1}{4(1+2\sigma)}$$

Furthermore, we obtain

$$|s_3 - \mu s_2^2| \leq 2|1-\mu|\Omega_1(\sigma,v)$$

where

$$|1-\mu| \geq \frac{2v-1}{4(1+2\sigma)\Omega_1(\sigma,v)}$$

and if:

$$|1-\mu|\Omega_1(\sigma,v) \leq \frac{2v-1}{4(1+2\sigma)}$$

then, we obtain:

$$|s_3-\mu s_2^2| \leq \frac{2v-1}{2(1+2\sigma)}$$

where

$$|1-\mu| \leq \frac{2v-1}{4(1+2\sigma)\Omega_1(\sigma,v)}$$

and $\Omega_1(\sigma,v)$ is given in (18). □

By putting $\sigma=0$ in the above Theorem 2, we obtain the following result.

Corollary 3. *Let $f \in \mathcal{S}^*_\Sigma(v)$. Then, for some $\mu \in \mathbb{R}$,*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 2|1-\mu|\Omega_1(\sigma,v) & \left(|1-\mu|\Omega_1(\sigma,v) \geq \frac{2v-1}{4}\right) \\ \frac{2v-1}{2} & \left(|1-\mu|\Omega_1(\sigma,v) \leq \frac{2v-1}{4}\right), \end{cases}$$

where

$$\Omega_1(v) = \frac{(2v-1)^3}{|2(2v^2+2v-1)|}. \tag{43}$$

Letting $\sigma=1$ in Theorem 2, we can obtain the next result.

Corollary 4. *Let $f \in \mathcal{C}_\Sigma(v)$. Then, for some $\mu \in \mathbb{R}$,*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 2|1-\mu|\Omega_1(v) & \left(|1-\mu|\Omega_1(v) \geq \frac{2v-1}{12}\right) \\ \frac{2v-1}{6} & \left(|1-\mu|\Omega_1(v) \leq \frac{2v-1}{12}\right), \end{cases}$$

where

$$\Omega_1(v) = \frac{(2v-1)^3}{|4(3-8v)|}. \tag{44}$$

4. Second Hankel Determinant for the Class $\mathcal{G}_\Sigma(v,\sigma)$

Theorem 3. *Let the function $f(\xi)$ be in the class $\mathcal{G}_\Sigma(v,\sigma)$. Then:*

$$H_2(2) = \left|s_2s_4-s_3^2\right| \leq \begin{cases} T(2,v) & (B_1 \geq 0 \;\; and \;\; B_2 \geq 0) \\ \max\left\{\left(\frac{2v-1}{4(1+2\sigma)}\right)^2, T(2,v)\right\} & (B_1 > 0 \;\; and \;\; B_2 < 0) \\ \left(\frac{2v-1}{4(1+2\sigma)}\right)^2 & (B_1 \leq 0 \;\; and \;\; B_2 \leq 0) \\ \max\{T(g_0,v), T(2,v)\} & (B_1 < 0 \;\; and \;\; B_2 > 0). \end{cases}$$

where

$$T(2,v) = \frac{2(1+4\sigma)\mathcal{E}_1^4(v)}{3(1+3\sigma)(1+\sigma)^4} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{18(1+\sigma)(1+3\sigma)} + \frac{\mathcal{E}_1^4(v)}{(1+\sigma)^4}$$

$$T(g_0,t) = \frac{\mathcal{E}_1^2(v)}{4(1+2\sigma)^2} + \frac{9B_2^4(1+\sigma)^4}{4(1+2\sigma)^2(1+3\sigma)B_1^3} + \frac{3B_2^3(1+\sigma)^2}{4(1+2\sigma)^2(1+3\sigma)B_1^2}.$$

$$B_1 = \mathcal{E}_1(v)\bigg[24\mathcal{E}_1^3(v)(1+4\sigma)(1+2\sigma)^2 + 2(6\mathcal{E}_1(v) - 6\mathcal{E}_2(v) + \mathcal{E}_3(v))(1+\sigma)^3(1+2\sigma)^2$$
$$+ 36\mathcal{E}_1^3(v)(1+3\sigma)(1+2\sigma)^2 - 24\mathcal{E}_1(v)(1+\sigma)^3(1+2\sigma)^2 + 9\mathcal{E}_1(v)(1+\sigma)^4(1+3\sigma) - 9\mathcal{E}_1^2(v)$$
$$(1+\sigma)^2(1+3\sigma)(1+2\sigma)\bigg]r^4$$

$$B_2 = \mathcal{E}_1(v)\bigg[3(1+2\sigma)(1+3\sigma)\mathcal{E}_1^2(v) + 4\mathcal{E}_1(v)(1+\sigma)(1+2\sigma)^2 + 4(\mathcal{E}_2(v) - 2\mathcal{E}_1(v))$$
$$(1+\sigma)(1+2\sigma)^2 + 8\mathcal{E}_1(v)(1+\sigma)(1+2\sigma)^2 - 6\mathcal{E}_1(v)(1+\sigma)^2(1+3\sigma)\bigg]r^2.$$

Proof. From (27) and (42), we have

$$s_2 s_4 = \frac{(1+4\sigma)\mathcal{E}_1^4(v)}{24(1+3\sigma)(1+\sigma)^4}\alpha_1^4 + \frac{(15+45\sigma)\mathcal{E}_1^3(v)(\alpha_2-\delta_2)}{192(1+\sigma)^2(1+2\sigma)(1+3\sigma)}\alpha_1^2 + \frac{\mathcal{E}_1^2(v)(\alpha_3-\delta_3)}{24(1+\sigma)(1+3\sigma)}\alpha_1$$
$$+ \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v) - 2\mathcal{E}_1(v)](\alpha_2+\delta_2)}{48(1+\sigma)(1+3\sigma)}\alpha_1^2 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v) - 6\mathcal{E}_2(v) + \mathcal{E}_3(v)]}{288(1+\sigma)(1+3\sigma)}\alpha_1^4$$

With some calculations, we have

$$s_2 s_4 - s_3^2 = \frac{(1+4\sigma)\mathcal{E}_1^4(v)}{24(1+3\sigma)(1+\sigma)^4}\alpha_1^4 + \frac{\mathcal{E}_1^3(v)(\alpha_2-\delta_2)}{64(1+\sigma)^2(1+2\sigma)}\alpha_1^2 + \frac{\mathcal{E}_1^2(v)(\alpha_3-\delta_3)}{24(1+\sigma)(1+3\sigma)}\alpha_1$$
$$+ \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v) - 2\mathcal{E}_1(v)](\alpha_2+\delta_2)}{48(1+\sigma)(1+3\sigma)}\alpha_1^2 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v) - 6\mathcal{E}_2(v) + \mathcal{E}_3(v)]}{288(1+\sigma)(1+3\sigma)}\alpha_1^4$$
$$- \frac{\mathcal{E}_1^4(v)}{16(1+\sigma)^4}\alpha_1^4 - \frac{\mathcal{E}_1^2(v)(\alpha_2-\delta_2)^2}{64(1+2\sigma)^2}$$

By using Lemma 2,

$$\alpha_2 - \delta_2 = \frac{(4-\alpha_1^2)(x-u)}{2} \tag{45}$$

$$\alpha_2 + \delta_2 = \alpha_1^2 + \frac{(4-\alpha_1^2)(x+u)}{2} \tag{46}$$

and

$$\alpha_3 - \delta_3 = \frac{\alpha_1^3}{2} + \frac{4-\alpha_1^2}{2}\alpha_1(x+u) - \frac{4-\alpha_1^2}{4}\alpha_1(x^2+u^2)$$
$$+ \frac{4-\alpha_1^2}{2}\Big[(1-|x|^2\xi) - (1-|u|^2)w\Big] \tag{47}$$

for some x, u, ξ, w with $|x| \leq 1,\ \ |u| \leq 1,\ \ |\xi| \leq 1,\ \ |w| \leq 1, |\alpha_1| \in [0,2]$ and substituting $(\alpha_2+\delta_2),\ \ (\alpha_2-\delta_2)$ and $(\alpha_3-\delta_3)$, and after some straightforward simplifications, we have

$$
\begin{aligned}
s_2s_4 - s_3^2 = & \frac{(1+4\sigma)\mathcal{E}_1^4(v)}{24(1+3\sigma)(1+\sigma)^4}\alpha_1^4 + \frac{\mathcal{E}_1^3(v)(4-\alpha_1^2)(x-u)}{128(1+\sigma)^2(1+2\sigma)}\alpha_1^2 + \frac{\mathcal{E}_1^2(v)}{48(1+\sigma)(1+3\sigma)}\alpha_1^4 \\
& + \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)(x+u)}{48(1+\sigma)(1+3\sigma)}\alpha_1^2 - \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)(x^2+u^2)}{96(1+\sigma)(1+3\sigma)}\alpha_1^2 \\
& + \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)[(1-|x|^2\xi)-(1-|y|^2)w]}{48(1+\sigma)(1+3\sigma)} + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]}{48(1+\sigma)(1+3\sigma)}\alpha_1^4 \\
& + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](4-\alpha_1^2)(x+u)}{96(1+\sigma)(1+3\sigma)}\alpha_1^2 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{288(1+\sigma)(1+3\sigma)}\alpha_1^4 \\
& - \frac{\mathcal{E}_1^4(v)}{16(1+\sigma)^4}\alpha_1^4 - \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)^2(x-u)^2}{256(1+2\sigma)^2}
\end{aligned}
$$

Let $r=\alpha_1$, assume without any restriction that $r\in[0,2]$, $\eta_1=|x|\le 1$, $\eta_2=|u|\le 1$ and applying triangular inequality, we have

$$
\begin{aligned}
|s_2s_4 - s_3^2| \le & \left\{ \frac{(1+4\sigma)\mathcal{E}_1^4(v)}{24(1+3\sigma)(1+\sigma)^4}r^4 + \frac{\mathcal{E}_1^2(v)}{48(1+\sigma)(1+3\sigma)}r^4 + \frac{\mathcal{E}_1^2(v)(4-r^2)}{24(1+\sigma)(1+3\sigma)}r \right. \\
& \left. + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]}{48(1+\sigma)(1+3\sigma)}r^4 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{288(1+\sigma)(1+3\sigma)}r^4 + \frac{\mathcal{E}_1^4(v)}{16(1+\sigma)^4}r^4 \right\} \\
& + \left\{ \frac{\mathcal{E}_1^3(v)(4-r^2)}{128(1+\sigma)^2(1+2\sigma)}r^2 + \frac{\mathcal{E}_1^2(v)(4-r^2)}{48(1+\sigma)(1+3\sigma)}r^2 \right. \\
& \left. + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](4-r^2)}{96(1+\sigma)(1+3\sigma)}r^2 \right\}(\eta_1+\eta_2) + \left\{ \frac{\mathcal{E}_1^2(v)(4-r^2)}{96(1+\sigma)(1+3\sigma)}r^2 \right. \\
& \left. - \frac{\mathcal{E}_1^2(v)(4-r^2)}{48(1+\sigma)(1+3\sigma)}r \right\}(\eta_1^2+\eta_2^2) + \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)^2}{256(1+2\sigma)^2}(\eta_1+\eta_2)^2
\end{aligned}
$$

and equivalently, we have

$$
\begin{aligned}
|s_2s_4 - s_3^2| &\le \Upsilon_1(v,r) + \Upsilon_2(v,r)(\eta_1+\eta_2) + \Upsilon_3(v,r)(\eta_1^2+\eta_2^2) + \Upsilon_4(v,r)(\eta_1+\eta_2)^2 \\
&= J(\eta_1,\eta_2)
\end{aligned} \tag{48}
$$

where

$$
\begin{aligned}
\Upsilon_1(v,r) = & \left\{ \frac{(1+4\sigma)\mathcal{E}_1^4(v)}{24(1+3\sigma)(1+\sigma)^4}r^4 + \frac{\mathcal{E}_1^2(v)}{48(1+\sigma)(1+3\sigma)}r^4 + \frac{\mathcal{E}_1^2(v)(4-r^2)}{24(1+\sigma)(1+3\sigma)}r \right. \\
& + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)]}{48(1+\sigma)(1+3\sigma)}r^4 + \frac{\mathcal{E}_1(v)[6\mathcal{E}_1(v)-6\mathcal{E}_2(v)+\mathcal{E}_3(v)]}{288(1+\sigma)(1+3\sigma)}r^4 \\
& \left. + \frac{\mathcal{E}_1^4(v)}{16(1+\sigma)^4}r^4 \right\} \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_2(v,r) = & \left\{ \frac{\mathcal{E}_1^3(v)(4-r^2)}{128(1+\sigma)^2(1+2\sigma)}r^2 + \frac{\mathcal{E}_1^2(v)(4-r^2)}{48(1+\sigma)(1+3\sigma)}r^2 \right. \\
& \left. + \frac{\mathcal{E}_1(v)[\mathcal{E}_2(v)-2\mathcal{E}_1(v)](4-r^2)}{96(1+\sigma)(1+3\sigma)}r^2 \right\} \ge 0
\end{aligned}
$$

$$
\Upsilon_3(v,r) = \left\{ \frac{\mathcal{E}_1^2(v)(4-r^2)}{96(1+\sigma)(1+3\sigma)}r^2 - \frac{\mathcal{E}_1^2(v)(4-r^2)}{48(1+\sigma)(1+3\sigma)}r \right\} \le 0
$$

$$\Upsilon_4(v,r) = \frac{\mathcal{E}_1^2(v)(4-\alpha_1^2)^2}{256(1+2\sigma)^2} \geq 0$$

where $0 \leq r \leq 2$. We now maximize the function $J(\eta_1, \eta_2)$ in the closed square

$$\Psi = \{(\eta_1, \eta_2) : \eta_1 \in [0,1], \eta_2 \in [0,1]\} \quad for \quad r \in [0,2].$$

The maximum of $J(\eta_1, \eta_2)$ with reference to r must be explored, taking into consideration the cases where $r = 0, r = 2$, and $r \in (0,2)$. Given a fixed value of r, the coefficients of the function $J(\eta_1, \eta_2)$ in (48) are dependent on m.

The First Case

When $r = 0$,

$$J(\eta_1, \eta_2) = \Upsilon_4(v,0) = \frac{\mathcal{E}_1^2(v)}{16(1+2\sigma)^2}(\eta_1 + \eta_2)^2.$$

Clearly the function $J(\eta_1, \eta_2)$ attains its maximum at (η_1, η_2) and

$$max\{J(\eta_1, \eta_2) : \eta_1, \eta_2 \in [0,1]\} = J(1,1) = \frac{\mathcal{E}_1^2(v)}{4(1+2\sigma)^2}. \quad (49)$$

The Second Case

In the case of $r = 2$, $J(\eta_1, \eta_2)$ is represented as a constant function with regard to m, giving us

$$J(\eta_1, \eta_2) = \Upsilon_1(v,2) = \left\{ \frac{2(1+4\sigma)\mathcal{E}_1^4(v)}{3(1+2\sigma)(1+\sigma)^4} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{18(1+\sigma)(1+2\sigma)} + \frac{\mathcal{E}_1^4(v)}{(1+\sigma)^4} \right\}.$$

The Third Case

When $r \in (0,2)$, let $\eta_1 + \eta_2 = d$ and $\eta_1 \cdot \eta_2 = Y$ in this case, then (48) can be of the form

$$J(\eta_1, \eta_2) = \Upsilon_1(v,r) + \Upsilon_2(v,r)d + (\Upsilon_3(v,r) + \Upsilon_4(v,r))d^2 - 2\Upsilon_3(v,r)l = Y(d,q) \quad (50)$$

where, $d \in [0,2]$ ald $q \in [0,1]$. Now, we need to investigate the maximum of

$$Y(d,q) \in \Theta = \{(d,q) : d \in [0,2], q \in [0,1]\}. \quad (51)$$

By differentiating $Y(d,q)$ partially, we have

$$\frac{\partial Y}{\partial c} = \Upsilon_2(v,r) + 2(\Upsilon_3(v,r) + \Upsilon_4(v,r))d = 0$$

$$\frac{\partial Y}{\partial l} = -2\Upsilon_3(v,r) = 0.$$

These findings demonstrate that $Y(d,r)$ has no critical point in the square Ψ, and, consequently, $J(\eta_1, \eta_2)$has no critical point in the same region.
Because of this, the function $J(\eta_1, \eta_2)$ is unable to reach its maximum value inside of Ψ.
The maximum of $J(\eta_1, \eta_2)$ on the square's Ψ boundary will then be examined.
For $\eta_1 = 0$, $\eta_2 \in [0,1]$ (also, for $\eta_2 = 0, \eta_1 \in [0,1]$) and

$$J(0, \eta_2) = \Upsilon_1(v,r) + \Upsilon_2\eta_2 + (\Upsilon_3(v,r) + \Upsilon_4(v,r))\eta_2^2 = D(\eta_2). \quad (52)$$

Now, since $\Upsilon_3(v,r) + \Upsilon_4(v,r) \geq 0$, then we have

$$D'(\eta_2) = \Upsilon_2(v,r) + 2[\Upsilon_3(v,r) + \Upsilon_4(v,r)]\eta_2 > 0$$

which implies that $D(\eta_2)$ is an increasing function. Therefore, for a fixed $r \in [0,2)$ and $v \in (1/2,1]$, the maximum occurs at $\eta_2 = 1$. Thus, from (52),

$$\max\{r(0,\eta_2):\eta_2\in[0,1]\}=J(0,1)$$
$$=Y_1(v,r)+Y_2(v,r)+Y_3(v,r)+Y_4(v,r). \tag{53}$$

For $\eta_1=1,\eta_2\in[0,1]$ (also, for $\eta_2=1$, $\eta_1\in[0,1]$) and

$$J(1,\eta_2)=Y_1(v,r)+Y_2(v,r)+Y_3(v,r)+Y_4(v,r)+[Y_2(v,r)$$
$$+2Y_4(v,r)]\eta_2+[Y_3(v,r)+Y_4(v,r)]\eta_2^2=N(\eta_2) \tag{54}$$

$$N'(\eta_2)=[Y_2(v)+2Y_4(v)]+2[Y_3(v)+Y_4(v)]\eta_2. \tag{55}$$

We know that $Y_3(v)+Y_4(v)\geq 0$, then

$$N'(\eta_2)=[Y_2(v)+2Y_4(v)]+2[Y_3(v)+Y_4(v)]\eta_2>0.$$

Therefore, the function $N(\eta_2)$ is an increasing function and the maximum occurs at $\eta_2=1$. From (54), we have

$$\max\{J(1,\eta_2):\eta_2\in[0,1]\}=J(1,1)$$
$$=Y_1(v,r)+2[Y_2(v,r)+Y_3(v,r)]+4Y_4(v,r). \tag{56}$$

Hence, for every $r\in(0,2)$, taking it from (53) and (56), we have

$$Y_1(v,r)+2[Y_2(v,r)+Y_3(v,r)]+4Y_4(v,r)$$
$$>Y_1(v,r)+Y_2(v,r)+Y_3(v,r)+Y_4(v,r).$$

Therefore,

$$\begin{aligned}&\max\{J(\eta_1,\eta_2):\eta_1\in[0,1],\eta_2\in[0,1]\}\\=\;&Y_1(v,r)+2[Y_2(v,r)+Y_3(v,r)]+4Y_4(v,r).\end{aligned}$$

Since,

$$D(1)\leq N(1)\quad for\quad r\in[0,2]\quad and\quad v\in[1,1],$$

then

$$\max\{J(\eta_1,\eta_2)\}=J(1,1)$$

occurs on the boundary of square Ψ.
Let $T:(0,2)\to\mathbb{R}$ defined by

$$T(v,r)=\max\{J(\eta_1,\eta_2)\}=J(1,1)=Y_1(v,r)+2Y_2(v,r)+2Y_3(v,r)+4Y_4(v,r). \tag{57}$$

Now, inserting the values of $Y_1(v,r),Y_2(v,r),Y_3(v,r)$ and $Y_4(v,r)$ into (57) and with some calculations, we have

$$T(v,r)=\frac{\mathcal{E}_1^2(v)}{4(1+2\sigma)^2}+\frac{B_1}{576(1+\sigma)^4(1+2\sigma)^2(1+3\sigma)}r^4+\frac{B_2}{48(1+\sigma)^2(1+2\sigma)^2(1+3\sigma)}r^2,$$

where

$$B_1 = \mathcal{E}_1(v)\Bigg[24\mathcal{E}_1^3(v)(1+4\sigma)(1+2\sigma)^2 + 2(6\mathcal{E}_1(v) - 6\mathcal{E}_2(v) + \mathcal{E}_3(v))(1+\sigma)^3(1+2\sigma)^2$$
$$+ 36\mathcal{E}_1^3(v)(1+3\sigma)(1+2\sigma)^2 - 24\mathcal{E}_1(v)(1+\sigma)^3(1+2\sigma)^2 + 9\mathcal{E}_1(v)(1+\sigma)^4(1+3\sigma) - 9\mathcal{E}_1^2(v)$$
$$(1+\sigma)^2(1+3\sigma)(1+2\sigma)\Bigg]r^4$$

$$B_2 = \mathcal{E}_1(v)\Bigg[3(1+2\sigma)(1+3\sigma)\mathcal{E}_1^2(v) + 4\mathcal{E}_1(v)(1+\sigma)(1+2\sigma)^2 + 4(\mathcal{E}_2(v) - 2\mathcal{E}_1(v))$$
$$(1+\sigma)(1+2\sigma)^2 + 8\mathcal{E}_1(v)(1+\sigma)(1+2\sigma)^2 - 6\mathcal{E}_1(v)(1+\sigma)^2(1+3\sigma)\Bigg]r^2.$$

If $T(v,r)$ achieves a maximum value inside of $r \in [0,2]$ and by using some basic mathematics, we have

$$T'(v,r) = \frac{B_1}{144(1+\sigma)^4(1+2\sigma)^2(1+3\sigma)}r^3 + \frac{B_2}{24(1+\sigma)^2(1+2\sigma)^2(1+3\sigma)}r.$$

In virtue of the signs of B_1 and B_2, we must now investigate the sign of the function $T'(v,r)$.
1st result:
Suppose $B_1 \geq 0$ and $B_2 \geq 0$ then,

$T'(v,r) \geq 0$. This shows that $T(v,r)$ is an increasing function on the boundary of $r \in [0,2]$ that is $r = 2$. Therefore,

$$\max\{T(v,r) : r \in (0,2)\} = \frac{2(1+4\sigma)\mathcal{E}_1^4(v)}{3(1+3\sigma)(1+\sigma)^4} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{18(1+\sigma)(1+3\sigma)} + \frac{\mathcal{E}_1^4(v)}{(1+\sigma)^4}$$

2nd result:
If $B_1 > 0$ and $B_2 < 0$ then,

$$T'(v,r) = \frac{B_1 r^3 + 6B_2 r(1+\sigma)^2}{144(1+\sigma)^4(1+2\sigma)^2(1+3\sigma)} = 0 \tag{58}$$

at critical point

$$r_0 = \sqrt{\frac{-6B_2(1+\sigma)^2}{B_1}} \tag{59}$$

is a critical point of the function $T(v,r)$. Now,

$$T''(r_0) = \frac{-B_2}{8(1+\sigma)^2(1+2\sigma)^2(1+3\sigma)} + \frac{B_2}{24(1+\sigma)^2(1+2\sigma)^2(1+3\sigma)} > 0.$$

Therefore, r_0 is the minimum point of the function $T(v,r)$. Hence, $T(v,r)$ can not have a maximum.
3rd result:
If $B_1 \leq 0$ and $B_2 \leq 0$ then,

$$T'(v,r) \leq 0.$$

Therefore, $T(v,r)$ is a decreasing function on the interval $(0,2)$. Consequently,

$$\max\{T(v,r) : r \in (0,2)\} = T(0) = \frac{\mathcal{E}_1^2(v)}{4(1+2\sigma)^2}. \tag{60}$$

4th result:
If $B_1 < 0$ and $B_2 > 0$

$$T''(v_0, r) = \frac{-B_2}{12(1+\sigma)^2(1+2\sigma)^2(1+3\sigma)} < 0.$$

Therefore, $T''(v, r) < 0$. Hence, g_0 is the maximum point of the function $T(v, r)$ and $r = g_0$ is the maximum value. Likewise

$$max\{T(v, r) : r \in (0, 2)\} = T(g_0, s)$$

$$T(g_0, t) = \frac{\mathcal{E}_1^2(v)}{4(1+2\sigma)^2} + \frac{9B_2^4(1+\sigma)^4}{4(1+2\sigma)^2(1+3\sigma)B_1^3} + \frac{3B_2^3(1+\sigma)^2}{4(1+2\sigma)^2(1+3\sigma)B_1^2}.$$

□

Taking $\sigma = 0$ in Theorem 3, we have the next corollary.

Corollary 5. *Let the function $f(\xi)$ given by (1) be in the class $\mathcal{S}^*_\Sigma(v)$. Then:*

$$H_2(2) = \left|a_2a_4 - a_3^2\right| \leq \begin{cases} T(2, v) & (B_1 \geq 0 \quad and \quad B_2 \geq 0) \\ \max\left\{\frac{(2v-1)^2}{16}, T(2, v)\right\} & (B_1 > 0 \quad and \quad B_2 < 0) \\ \frac{(2v-1)^2}{16} & (B_1 \leq 0 \quad and \quad B_2 \leq 0) \\ \max\{T(g_0, v), T(2, v)\} & (B_1 < 0 \quad and \quad B_2 > 0). \end{cases}$$

where

$$T(2, v) = \frac{5\mathcal{E}_1^4(v)}{3} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{18}$$
$$T(g_0, v) = \frac{\mathcal{E}_1^2(v)}{4} + \frac{3B_2^4(3B_2 + B_1)}{4B_1^3}.$$

$$B_1 = \mathcal{E}_1(v)[60\mathcal{E}_1^3(v) + 2(\mathcal{E}_3(v) - 6\mathcal{E}_2(v)) - 3\mathcal{E}_1(v) - 9\mathcal{E}_1^2(v)]r^4$$
$$B_2 = \mathcal{E}_1(v)[3\mathcal{E}_1^2(v) - 2(2\mathcal{E}_2(v) - \mathcal{E}_1(v))]r^2.$$

Taking $\sigma = 1$ in Theorem 3, we have the next corollary.

Corollary 6. *Let the function $f(\xi)$ given by (1) be in the class $\mathcal{C}_\Sigma(v)$. Then:*

$$H_2(2) = \left|a_2a_4 - a_3^2\right| \leq \begin{cases} T(2, v) & (B_1 \geq 0 \quad and \quad B_2 \geq 0) \\ \max\left\{\frac{(2v-1)^2}{144}, T(2, v)\right\} & (B_1 > 0 \quad and \quad B_2 < 0) \\ \frac{(2v-1)^2}{144} & (B_1 \leq 0 \quad and \quad B_2 \leq 0) \\ \max\{T(g_0, v), T(2, v)\} & (B_1 < 0 \quad and \quad B_2 > 0). \end{cases}$$

where

$$T(2, v) = \frac{11\mathcal{E}_1^4(v)}{96} + \frac{\mathcal{E}_1(v)\mathcal{E}_3(v)}{144}$$
$$T(g_0, v) = \frac{\mathcal{E}_1^2(v)}{36} + \frac{B_2^4}{B_1^3} + \frac{B_2^2}{12B_1^2}.$$

$$B_1 = \mathcal{E}_1(v)[2376\mathcal{E}_1^3(v) + 144(\mathcal{E}_3(v) - 6\mathcal{E}_2(v)) - 288\mathcal{E}_1(v) - 432\mathcal{E}_1^2(v)]r^4$$
$$B_2 = \mathcal{E}_1(v)[36\mathcal{E}_1^2(v) - 24(\mathcal{E}_1(v) - 3\mathcal{E}_2(v))]r^2.$$

5. Conclusions

The many well-known mathematicians have been studied the special functions, as well as polynomials in the recent years, due to the fact that they are used in a wide variety of mathematical and other scientific fields as indicated in the introduction section. The subject of this paper is a novel subclass of analytical and univalent functions which have been defined by using Euler polynomial. We solved the Fekete–Szegö problem, as well as provided bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant for functions in the class $\mathcal{G}_\Sigma(v, \sigma)$. One can extend the above results for a class of certain q-Starlike functions, as mentioned in [21–27].

Author Contributions: Conceptualization, Q.X. and B.K.; Methodology, Q.X. and B.K.; Software, T.G.S. and F.T.; Validation, S.R. and S.N.M.; Formal analysis, S.R. and S.N.M.; Investigation, Q.X. and B.K.; Resources, T.G.S. and F.T.; Writing—original draft, B.K.; Writing—review & editing, S.N.M.; Visualization, T.G.S.; Supervision, S.R. and B.K.; Project administration, F.T.; Funding acquisition, F.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: This research was supported by the researchers Supporting Project Number (RSP2023R401), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: No data is used in this work.

Conflicts of Interest: The authors declare no conflict of interest.

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