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Some Estimates of k-Fractional Integrals for Various Kinds of Exponentially Convex Functions

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Abstract: In this paper, we aim to find unified estimates of fractional integrals involving Mittag–Leffler functions in kernels. The results obtained in terms of fractional integral inequalities are provided for various kinds of convex and related functions. A variant of Hadamard-type inequality is also presented, which shows the upper and lower bounds of fractional integral operators of many kinds. The results of this paper are directly linked with many recently published inequalities.

Keywords: convex function; exponentially convex function; Mittag–Leffler function; fractional integral operators

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1. Introduction

Special functions play very important role in mathematical analysis, complex analysis, geometric function theory, physics, statistics, and many other subjects. Now, they are frequently utilized in fractional calculus. For example, the Mittag–Leffler function is the generalization of exponential, trigonometric, and hyperbolic functions, depending on the well-known gamma function. Likewise, the beta function is also utilized to extend the Mittag–Leffler function.

The Mittag–Leffler function appears in the solutions of fractional differential equations, such as exponential function, which occurs in solving ordinary differential equations. Therefore, the fractional integral operators are defined using a Mittag–Leffler function in their kernels. Integral operators are important tools in many areas, including the theory of integral and differential equations, approximation theory, the theory of Fourier series and Fourier integrals, and summability theory, see [1,2].

Now, integral operators are used routinely in establishing generalized versions of classical inequalities. Among very familiar inequalities, Hadamard-, Ostrowski-, Minkowski-, and Grüss-type inequalities are studied very commonly for fractional integral operators. For instance, the reader can see recently published articles on Hadamard inequality in [3,4], while, for Ostrowski inequality, we refer to [5,6].

Motivated by the recent research on fractional inequalities, the aim of this paper is to estimate the bounds of fractional integral operators in different forms by using a certain type of convexity. A fractional version of Hadamard inequality is provided; a lot of such inequalities are deducible for convex, and exponentially convex functions of almost all kinds that are directly linked with Definition 8.

In [7], the general forms of integral operators of fractional order are defined by using the modified Mittag–Leffler function. Before defining these operators, first we provide the definitions of gamma function, beta function, λ-beta function, and pochhammer symbol, see [8,9].

Definition 1 ([10,11])**.** *The gamma function for $z > 0$ is provided by*

$$\Gamma(z) = \int_0^\infty e^{-w} w^{z-1} dw. \tag{1}$$

The k-analogue of gamma function is provided by

$$\Gamma_k(z) = \int_0^\infty w^{z-1} e^{\frac{-w^k}{k}} dw, \tag{2}$$

where $z \in \mathbb{C}$ with $\Re(z) > 0$ and $k > 0$.

Definition 2 ([10])**.** *The beta function is defined by*

$$\beta(p,q) = \int_0^1 w^{p-1}(1-w)^{q-1} dw,$$

where $\Re(p), \Re(q) > 0$.

Definition 3 ([12])**.** *The definition of $\lambda-$beta function is provided by*

$$\beta_\lambda(p,q) = \int_0^1 w^{p-1}(1-w)^{q-1} e^{-\frac{\lambda}{w(1-w)}} dw,$$

where $\min\{\Re(p), \Re(q)\} > 0$ and $\Re(\lambda) > 0$.

Definition 4 ([10])**.** *The pochhammer symbol for $r \in \mathbb{C}$ is provided by*

$$(r)_{n\mu} = \frac{\Gamma(r+n\mu)}{\Gamma(r)}. \tag{3}$$

Next, we provide the definition of an integral operator that is directly linked with many well-known fractional integral operators.

Definition 5 ([7])**.** *Let $\phi : [a,b] \to \mathbb{R}$ be a differentiable and strictly increasing function, g be a positive and $g \in L_1[a,b]$ where $0 < a < b$. Furthermore, let $u, \rho, z, r, \sigma \in \mathbb{C}$, $\Re(u)$, $\Re(z) > 0$, $\Re(r) > \Re(\rho) > 0$, $\lambda \geq 0$, $\gamma, \epsilon, k > 0$ with $0 < \mu \leq \epsilon + \gamma$. Then, for $t \in [a,b]$ the integral operators ${}^k_\phi\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a^+} g(.,.)$ and ${}^k_\phi\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a^+} g(.,.)$ are defined by;*

$$\left({}^k_\phi\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a^+} g\right)(t;\lambda) = \int_a^t (\phi(t)-\phi(w))^{\frac{\sigma}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(w))^{\frac{\gamma}{k}};\lambda\right) g(w) d(\phi(w)), \tag{4}$$

and

$$\left({}^k_\phi\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,b^-} g\right)(t;\lambda) = \int_t^b (\phi(w)-\phi(t))^{\frac{\sigma}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(w)-\phi(t))^{\frac{\gamma}{k}};\lambda\right) g(w) d(\phi(w)) \tag{5}$$

where the Mittag–Leffler function is provided by

$$E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}(w;\lambda) = \sum_{n=0}^\infty \frac{\beta_\lambda(\rho+n\mu, r-\rho)}{\beta(\rho, r-\rho)} \frac{(r)_{n\mu}}{k\Gamma_k(\gamma n+\sigma)} \frac{w^n}{(u)_{n\epsilon}}. \tag{6}$$

The above Definition 5 in particular cases provides the definitions of the fractional integral operators defined in [13–15].

The goal of this paper is to establish the bounds for the k-fractional integral operators provided in (4) and (5) by using a generalized class of exponentially convex functions. In the following, we provide the definition of convex, exponentially convex, and other important functions that will be helpful to study the linkages of this paper with already published work.

Definition 6 ([16,17])**.** *Let $p, q \in A$ where $A \subseteq \mathbb{R}$ is an interval and $0 \leq w \leq 1$. Then, a real valued function ζ satisfying the inequality*

$$\zeta(wp + (1-w)q) \leq w\zeta(p) + (1-w)\zeta(q), \tag{7}$$

is called convex function on A.

Definition 7. *Let a function $\zeta : A \subseteq \mathbb{R} \to \mathbb{R}$, with A being an interval, satisfy the inequality*

$$\zeta(wp + (1-w)q) \leq w\frac{\zeta(p)}{e^{\xi p}} + (1-w)\frac{\zeta(q)}{e^{\xi q}},$$

$\forall\ p, q \in A$, $0 \leq w \leq 1$, and $\xi \in \mathbb{R}$. Then, ζ is called an exponentially convex function.

The aforementioned definitions motivated the researchers to define many new classes of functions. For example, m-convex [18], exponentially m-convex [19], s-convex [20], exponentially s-convex [21], h-convex [22], exponentially h-convex [23], $(h-m)$-convex [24], exponentially $(h-m)$-convex [25], (s, m)-convex [18], (α, m)-convex [26], and exponentially (α, m)-convex functions [27] are all defined after appearance of convex functions. An exponentially $(\alpha, h-m)$-convex function unifies all the above functions, and is defined as follows:

Definition 8. *Let $h : B \to \mathbb{R}$ be a non-negative function, where B is an interval in $\mathbb{R}$ that contains $(0, 1)$. Then, function $\zeta : [0, b] \to \mathbb{R}$ satisfying the inequality*

$$\zeta(wp + (1-w)q) \leq h(w^{\alpha})\frac{\zeta(p)}{e^{\xi p}} + mh(1-w^{\alpha})\frac{\zeta(q)}{e^{\xi q}},$$

for all $p, q \in [0, b]$, $(\alpha, m) \in [0, 1]^2$, $0 < w < 1$, and $\xi \in \mathbb{R}$; this is called an exponentially $(\alpha, h-m)$-convex function.

The Riemann–Liouville fractional integral operator with respect to an increasing function is defined as follows:

Definition 9 ([28])**.** *Let $g \in Ł_1[a, b]$. Then, the Riemann–Liouville fractional integral operators of order $\sigma \in \mathbb{C}$, $\Re(\sigma) > 0$ are defined as follows:*

$$_{\phi}I^{\sigma}_{a^+}g(t) = \frac{1}{\Gamma(\sigma)}\int_a^t (\phi(t) - \phi(w))^{\sigma-1} g(w)d(\phi(w)),\ t > a, \tag{8}$$

$$_{\phi}I^{\sigma}_{b^-}g(t) = \frac{1}{\Gamma(\sigma)}\int_t^b (\phi(w) - \phi(t))^{\sigma-1} g(w)d(\phi(w)),\ t < b, \tag{9}$$

where ϕ is an increasing function on $[a, b]$.

The classical Riemann–Liouville fractional integral can be obtained by setting $\phi(t) = t$ in the above definition. It can also be noted that $\left({}^k_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,0,a^+}g\right)(t; 0) = {}_{\phi}I^{\sigma}_{a^+}g(t)$ and $\left({}^k_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,0,b^-}g\right)(t; 0) = {}_{\phi}I^{\sigma}_{b^-}g(t)$. From k-fractional integral operators (4) and (5), one can see that:

$$J_{\frac{\sigma}{k},a^+}(t;\lambda) := \left({}^k_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a^+}1\right)(t;\lambda) = k(\phi(t) - \phi(a))^{\frac{\sigma}{k}} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma+k,u,k}\left(z(\phi(t) - \phi(a))^{\frac{\gamma}{k}};\lambda\right), \tag{10}$$

$$J_{\frac{\tau}{k},b^-}(t;\lambda) := \left({}^k_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,b^-}1\right)(t;\lambda) = k(\phi(b) - \phi(t))^{\frac{\tau}{k}} E^{\rho,\epsilon,\mu,r}_{\gamma,\tau+k,u,k}\left(z(\phi(b) - \phi(t))^{\frac{\gamma}{k}};\lambda\right). \tag{11}$$

Remark 1.

(i) The extended Mittag–Leffler function (6) in particular cases produces the related Mittag–Leffler functions defined in [13,14,28–30], *see* [15] *(Remark 1.3).*

(ii) The operators (4) and (5) produce, in particular, several kinds of known fractional integral operators, see [15] *(Remark 1.4).*

The rest of the paper is organized as follows: in the upcoming section in Theorem 1, we establish estimates of fractional integral operators of Definition 5. Then, we outline the consequences of this theorem in the form of corollaries. In Theorem 2, the continuity of the aforementioned operators is proved. In Theorem 3, the estimates of the fractional integrals are provided in the form of a modulus inequality; then, its consequences are discussed. In the last theorem, a Hadamard-like inequality is proven, which generates several important implications.

2. Main Results

Theorem 1. *Let $\psi, \phi : [a_1, a_2] \longrightarrow \mathbb{R}$, be the two functions such that ψ is positive, integrable, and exponentially $(\alpha, h-m)$-convex, $m \in (0,1]$, and ϕ is differentiable and strictly increasing with $\phi' \in L^1[a_1, a_2]$. Then, for $\sigma, \tau \geq k$, $\xi \in \mathbb{R}$, the following fractional integral inequality holds:*

$$\begin{aligned}&\left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(t;\lambda)+\left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(t;\lambda)\leq(t-a_1)J_{\frac{\sigma}{k},a_1^+}(t;\lambda)\Bigg(\frac{\psi(a_1)}{e^{\xi a_1}}\\&\int_0^1 h(\zeta^\alpha)\phi'(t-\zeta(t-a_1))d\zeta+m\frac{\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^\alpha)\phi'(t-\zeta(t-a_1))d\zeta\Bigg)+(a_2-t)\\&J_{\frac{\tau}{k},a_2^-}(t;\lambda)\Bigg(\frac{\psi(a_2)}{e^{\xi a_2}}\int_0^1 h(\zeta^\alpha)\phi'(\zeta(a_2-t)+t)d\zeta+\frac{m\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^\alpha)\phi'(\zeta(a_2-t)+t)d\zeta\Bigg).\end{aligned}\tag{12}$$

Proof. Let $t \in [a_1, a_2]$. Then, for $w \in [a_1, t)$ and $\sigma \geq k$, the following inequality holds:

$$\begin{aligned}&(\phi(t)-\phi(w))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(w))^{\frac{\gamma}{k}};\lambda\right)\phi'(w)\\&\leq(\phi(t)-\phi(a_1))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\phi'(w).\end{aligned}\tag{13}$$

By applying the definition of the exponentially $(\alpha, h-m)$-convex function, for $\xi \in \mathbb{R}$, one can have

$$\psi(w)\leq h\left(\frac{t-w}{t-a_1}\right)^\alpha\frac{\psi(a_1)}{e^{\xi a_1}}+mh\left[1-\left(\frac{t-w}{t-a_1}\right)^\alpha\right]\frac{\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}.\tag{14}$$

The following inequality is yielded after multiplication of the inequalities (13) and (14) and then integrating on the interval $[a_1, t]$:

$$\begin{aligned}&\int_{a_1}^t(\phi(t)-\phi(w))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(w))^{\frac{\gamma}{k}};\lambda\right)\phi'(w)\psi(w)dw\\&\leq\Big(\phi(t)-\phi(a_1)\Big)^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\left[\frac{\psi(a_1)}{e^{\xi a_1}}\int_{a_1}^t h\left(\frac{t-w}{t-a_1}\right)^\alpha\right.\\&\left.\phi'(w)dw+m\frac{\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}\int_{a_1}^t h\left(1-\left(\frac{t-w}{t-a_1}\right)^\alpha\right)\phi'(w)dw\right].\end{aligned}$$

Inequality (15) is obtained by utilizing the definition of left integral operator

$$\begin{aligned}&\left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(t;\lambda)\leq(t-a_1)J_{\frac{\sigma}{k},a_1^+}(t;\lambda)\Bigg(\frac{\psi(a_1)}{e^{\xi a_1}}\int_0^1 h(\zeta^\alpha)\phi'(t-\zeta(t-a_1))d\zeta\\&+m\frac{\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^\alpha)\phi'(t-\zeta(t-a_1))d\zeta\Bigg).\end{aligned}\tag{15}$$

On the other hand, we obtain the inequality (16) for $w \in (t, a_2]$ and $\tau \geq k$:

$$(\phi(w)-\phi(t))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\Big(z(\phi(w)-\phi(t))^{\frac{\gamma}{k}};\lambda\Big)\phi'(w) \\ \leq (\phi(a_2)-\phi(t))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\Big(z(\phi(a_2)-\phi(t))^{\frac{\gamma}{k}};\lambda\Big)\phi'(w). \tag{16}$$

For $\xi \in \mathbb{R}$, the following inequality is acquired by applying the definition of exponentially $(\alpha, h-m)$ convex function ψ:

$$\psi(w) \leq h\left(\frac{w-t}{a_2-t}\right)^{\alpha}\frac{\psi(a_2)}{e^{\xi a_2}} + mh\left[1-\left(\frac{w-t}{a_2-t}\right)^{\alpha}\right]\frac{\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}. \tag{17}$$

The following inequality is yielded after multiplication of inequalities (16) and (17) and then integrating over $[t, a_2]$:

$$\int_t^{a_2}(\phi(w)-\phi(t))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\Big(z(\phi(w)-\phi(t))^{\frac{\gamma}{k}};\lambda\Big)\phi'(w)\psi(w)dw \\ \leq \Big(\phi(a_2)-\phi(t)\Big)^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\Big(z(\phi(a_2)-\phi(t))^{\frac{\gamma}{k}};\lambda\Big)\left[\frac{\psi(a_2)}{e^{\xi a_2}}\int_t^{a_2}h\left(\frac{w-t}{a_2-t}\right)^{\alpha} \right. \\ \left. \phi'(w)dw + m\frac{\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}\int_t^{a_2}h\left(1-\left(\frac{w-t}{a_2-t}\right)^{\alpha}\right)\phi'(w)dw\right].$$

Inequality (18) is obtained by utilizing the definition of the right integral operator

$$\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(t;\lambda) \leq (a_2-t)J_{\frac{\tau}{k},a_2^-}(t;\lambda)\left(\frac{\psi(a_2)}{e^{\xi a_2}}\int_0^1 h(\zeta^{\alpha})\phi'(\zeta(a_2-t)+t)d\zeta \right. \\ \left. +m\frac{\psi(\frac{t}{m})}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^{\alpha})\phi'(\zeta(a_2-t)+t)d\zeta\right). \tag{18}$$

After doing the sum of inequalities (15) and (18), we obtain the inequality (12). □

Corollary 1. *Along with assumptions of Theorem 1, if $\psi \in L_\infty[a_1, a_2]$, then the following inequality is established:*

$$\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(t;\lambda) + \left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(t;\lambda) \leq ||\psi||_\infty\left[(t-a_1)J_{\frac{\sigma}{k},a_1^+}(t;\lambda)\right. \\ \left(\frac{1}{e^{\xi a_1}}\int_0^1 h(\zeta^{\alpha})\phi'(t-\zeta(t-a_1))d\zeta + \frac{m}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^{\alpha})\phi'(t-\zeta(t-a_1))d\zeta\right) + (a_2-t) \\ \left. J_{\frac{\tau}{k},a_2^-}(t;\lambda)\left(\frac{1}{e^{\xi a_2}}\int_0^1 h(\zeta^{\alpha})\phi'(\zeta(a_2-t)+t)d\zeta + \frac{m}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^{\alpha})\phi'(\zeta(a_2-t)+t)d\zeta\right)\right]. \tag{19}$$

Remark 2. *By setting $\alpha = 1$; $m = 1$; $\alpha = m = 1$; $h(t) = t$; $h(t) = t$ and $\alpha = 1$; $h(t) = t$; and $\alpha = m = 1$ in (12), it holds for exponentially $(h-m)$ convex, exponentially (α, h)-convex, exponentially h-convex, exponentially (α, m) convex, exponentially m-convex, and exponentially convex functions, respectively.*

Remark 3.

(i) If we say that $k = 1$, $\phi(t) = t$, $h(t) = t$, and $\xi = 0$ in (12), then we obtain [31] *(Theorem 2.1).*

(ii) If we set $k = 1$, $\phi(t) = t$, $\alpha = 1$, and $\xi = 0$ in (12), we obtain [32] *(Theorem 1).*

Theorem 2. *With the assumptions of Theorem 1 if $\psi \in L_\infty[a_1, a_2]$, the operators defined in (4) and (5) are bounded.*

Proof. Using inequality (15) for $\psi \in L_\infty[a_1, a_2]$, we obtained the following inequality

$$\begin{aligned} &\left| \left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi \right)(t;\lambda) \right| \le ||\psi||_\infty (t-a_1) J_{\frac{\sigma}{k},a_1^+}(t;\lambda) \\ &\int_0^1 \left(\frac{1}{e^{\xi a_1}} h(\zeta^\alpha)\phi'(t-\zeta(t-a_1)) + m\frac{1}{e^{\xi \frac{t}{m}}} h(1-\zeta^\alpha)\phi'(t-\zeta(t-a_1)) \right) d\zeta \\ &\le ||\psi||_\infty (a_2-a_1) J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda) \\ &\int_0^1 \left(\frac{1}{e^{\xi a_1}} h(\zeta^\alpha)\phi'(a_2-\zeta(a_2-a_1)) + m\frac{1}{e^{\xi \frac{q}{m}}} h(1-\zeta^\alpha)\phi'(a_2-\zeta(a_2-a_1)) \right) d\zeta. \end{aligned} \tag{20}$$

Therefore, we obtain

$$\left| \left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi \right)(t;\lambda) \right| \le M||\psi||_\infty, \tag{21}$$

where

$$\begin{aligned} M =& (a_2-a_1) J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda) \int_0^1 \left(\frac{1}{e^{\xi a_1}} h(\zeta^\alpha)\phi'(a_2-\zeta(a_2-a_1)) \right. \\ &\left. + \frac{m}{e^{\xi \frac{q}{m}}} h(1-\zeta^\alpha)\phi'(a_2-\zeta(a_2-a_1)) \right) d\zeta. \end{aligned}$$

One can obtain the inequality (22) by using inequality (18):

$$\left| \left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi \right)(t;\lambda) \right| \le K||\psi||_\infty, \tag{22}$$

where

$$\begin{aligned} K =& (a_2-a_1) J_{\frac{\tau}{k},a_2^-}(p,\lambda) \int_0^1 \left(\frac{1}{e^{\xi a_2}} h(\zeta^\alpha)\phi'(p+\zeta(a_2-a_1)) \right. \\ &\left. + \frac{m}{e^{\xi \frac{a_1}{m}}} h(1-\zeta^\alpha)\phi'(p+\zeta(a_2-a_1)) \right) d\zeta. \end{aligned}$$

Therefore, $\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi \right)(t;\lambda)$ and $\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi \right)(t;\lambda)$ are bounded. □

Theorem 3. *Let $\psi, \phi : [a_1, a_2] \longrightarrow \mathbb{R}$, be functions such that ψ is positive and integrable, that $|\psi'|$ is exponentially $(\alpha, h-m)$-convex, and $m \in (0,1]$ and ϕ are differentiable and strictly increase with $\phi' \in L^1[a_1, a_2]$. Then, for $\sigma, \tau \ge k$, $\xi \in \mathbb{R}$, the following fractional integral inequality for generalized integral operators as (4) and (5) holds:*

$$\begin{aligned} &\left| \left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}(\phi*\psi) \right)(t,w;\lambda) + \left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}(\phi*\psi) \right)(t,w;\lambda) \right| \\ &\le (t-a_1) J_{\frac{\sigma}{k},a_1^+}(t;\lambda) \left(\frac{|\psi'(a_1)|}{e^{\xi a_1}} \int_0^1 h\zeta^\alpha \phi'(t-\zeta(t-a_1))d\zeta + \frac{m|\psi'(\frac{t}{m})|}{e^{\xi\left(\frac{t}{m}\right)}} \right. \\ &\left. \times \int_0^1 h(1-\zeta^\alpha)\phi'(t-\zeta(t-a_1))d\zeta \right) + (a_2-t) J_{\frac{\tau}{k},a_2^-}(t;\lambda) \\ &\left(\frac{|\psi'(a_2)|}{e^{\xi a_2}} \int_0^1 h\zeta^\alpha \phi'(t+\zeta(a_2-t))d\zeta + m\frac{|\psi'(\frac{t}{m})|}{e^{\xi \frac{t}{m}}} \int_0^1 h(1-\zeta^\alpha)\phi'(t+\zeta(a_2-t))d\zeta \right). \end{aligned} \tag{23}$$

where

$$\begin{aligned} &\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}(\phi*\psi) \right)(t,w;\lambda) := \int_{a_1}^t (\phi(t)-\phi(w))^{\frac{\sigma}{k}-1} \\ &E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(w))^{\frac{\gamma}{k}};\lambda \right)\phi'(w)\psi'(w)dw \end{aligned}$$

and

$$\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}(\phi*\psi)\right)(t,w;\lambda) := \int_t^{a_2}(\phi(w)-\phi(t))^{\frac{\tau}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(w)-\phi(t))^{\frac{\gamma}{k}};\lambda\right)\phi'(w)\psi'(w)dw.$$

Proof. Let $t \in [a_1, a_2]$ and $w \in [a_1, t)$. Then, for $\xi \in \mathbb{R}$, applying the definition of exponentially $(\alpha, h-m)$-convexity of $|\psi'|$, the inequality (24) holds:

$$|\psi'(w)| \le h\left(\frac{t-w}{t-a_1}\right)^{\alpha}\frac{|\psi'(a_1)|}{e^{\xi a_1}} + mh\left[1-\left(\frac{t-w}{t-a_1}\right)^{\alpha}\right]\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}. \tag{24}$$

From the above inequality, one can obtain

$$\psi'(w) \le h\left(\frac{t-w}{t-a_1}\right)^{\alpha}\frac{|\psi'(a_1)|}{e^{\xi a_1}} + mh\left[1-\left(\frac{t-w}{t-a_1}\right)^{\alpha}\right]\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}. \tag{25}$$

Now, by multiplying inequalities (13) and (25) we obtain the following inequality:

$$\begin{aligned}&(\phi(t)-\phi(w))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(w))^{\frac{\gamma}{k}};\lambda\right)\psi'(w)\phi'(w)\\ &\le (\phi(t)-\phi(a_1))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\phi'(w)\\ &\left(h\left(\frac{t-w}{t-a_1}\right)^{\alpha}\frac{|\psi'(a_1)|}{e^{\xi a_1}} + mh\left[1-\left(\frac{t-w}{t-a_1}\right)^{\alpha}\right]\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}\right).\end{aligned} \tag{26}$$

Further, the following inequality is yielded after integrating over $[a_1, t]$:

$$\begin{aligned}&\int_{a_1}^t(\phi(t)-\phi(w))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(w))^{\frac{\gamma}{k}};\lambda\right)\phi'(w)\psi'(w)dw\\ &\le (\phi(t)-\phi(a_1))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(t)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\left(\frac{|\psi'(a_1)|}{e^{\xi a_1}}\int_{a_1}^t h\left(\frac{t-w}{t-a_1}\right)^{\alpha}\right.\\ &\left.\phi'(w)dw + m\frac{|\psi'(\frac{t}{m})|}{e^{\xi\frac{t}{m}}}\int_{a_1}^t h\left(1-\left(\frac{t-w}{t-a_1}\right)^{\alpha}\right)\phi'(w)dw\right)\\ &= (t-a_1)J_{\frac{\sigma}{k},a_1^+}(t;\lambda)\left(\frac{|\psi'(a_1)|}{e^{\xi a_1}}\int_0^1 h\zeta^{\alpha}\phi'(t-\zeta(t-a_1))d\zeta + m\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}\right.\\ &\left.\int_0^1 h(1-\zeta^{\alpha})\phi'(t-\zeta(t-a_1))d\zeta\right).\end{aligned} \tag{27}$$

Simplifying the left-hand side of inequality (27), we have

$$\begin{aligned}&\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}(\phi*\psi)\right)(t,w;\lambda) \le (t-a_1)J_{\frac{\sigma}{k},a_1^+}(t;\lambda)\left(\frac{|\psi'(a_1)|}{e^{\xi a_1}}\right.\\ &\left.\int_0^1 h\zeta^{\alpha}\phi'(t-\zeta(t-a_1))d\zeta + m\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^{\alpha})\phi'(t-\zeta(t-a_1))d\zeta\right).\end{aligned} \tag{28}$$

From inequality (24), we have

$$\psi'(w) \ge -\left[h\left(\frac{t-w}{t-a_1}\right)^{\alpha}\frac{|\psi'(a_1)|}{e^{\xi a_1}} + mh\left[1-\left(\frac{t-w}{t-a_1}\right)^{\alpha}\right]\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}\right]. \tag{29}$$

Similarly, one can have the following inequality:

$$\begin{aligned}&\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}(\phi*\psi)\right)(t,w;\lambda) - (t-a_1)J_{\frac{\sigma}{k},a_1^+}(t;\lambda)\left(\frac{|\psi'(a_1)|}{e^{\xi a_1}}\right.\\ &\left.\ge \int_0^1 h\zeta^{\alpha}\phi'(t-\zeta(t-a_1))d\zeta + m\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^{\alpha})\phi'(t-\zeta(t-a_1))d\zeta\right).\end{aligned} \tag{30}$$

The following inequality is obtained from inequalities (28) and (30),

$$\begin{aligned}&\left|\left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}(\phi*\psi)\right)(t,w;\lambda)\right|\\&\leq (t-a_1)J_{\frac{\sigma}{k},a_1^+}(t;\lambda)\left(\frac{|\psi'(a_1)|}{e^{\xi a_1}}\int_0^1 h\zeta^{\alpha}\phi'(t-\zeta(t-a_1))d\zeta+m\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}\right.\\&\left.\int_0^1 h(1-\zeta^{\alpha})\phi'(t-\zeta(t-a_1))d\zeta\right).\end{aligned} \tag{31}$$

Now, for $t\in[a_1,a_2]$ and $w\in(t,a_2]$. Again, by using the exponentially $(\alpha,h-m)$-convexity of $|\psi'|$ for $\xi\in\mathbb{R}$, we have

$$|\psi'(w)|\leq h\left(\frac{w-t}{a_2-t}\right)^{\alpha}\frac{|\psi'(a_2)|}{e^{\xi a_2}}+mh\left[1-\left(\frac{w-t}{a_2-t}\right)^{\alpha}\right]\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}. \tag{32}$$

Proceeding on the same lines as we did to obtain (31), the following inequality holds:

$$\begin{aligned}&\left|\left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}(\phi*\psi)\right)(t,w;\lambda)\right|\leq (a_2-t)J_{\frac{\tau}{k},a_2^-}(t;\lambda)\left(\frac{|\psi'(a_2)|}{e^{\xi a_2}}\right.\\&\left.\int_0^1 h\zeta^{\alpha}\phi'(t+\zeta(a_2-t))d\zeta+m\frac{|\psi'(\frac{t}{m})|}{e^{\xi(\frac{t}{m})}}\int_0^1 h(1-\zeta^{\alpha})\phi'(t+\zeta(a_2-t))d\zeta\right).\end{aligned} \tag{33}$$

From inequalities (31) and (33), using triangular inequality, (23) is established. □

Remark 4. *By setting $\alpha=1$; $m=1$; $\alpha=m=1$; $h(t)=t$; $h(t)=t$ and $\alpha=1$; and $h(t)=t$ and $\alpha=m=1$ in (23), it holds for exponentially $(h-m)$ convex, exponentially (α,h)-convex, exponentially h-convex, exponentially (α,m) convex, exponentially m-convex, and exponentially convex functions, respectively.*

Remark 5.

(i) If we use $k=1$, $\phi(t)=t$, $h(t)=t$, and $\xi=0$ in (23), then we obtain [31] *(Theorem 2.2).*

(ii) If we use $k=1$, $\phi(t)=t$, $\alpha=1$, and $\xi=0$ in (23), then we obtain [32] *(Theorem 2).*

It is easy to prove the next lemma that will be helpful to produce the Hadamard-type estimations for the generalized fractional integral operators.

Lemma 1. *Let $\psi:[a_1,ma_2]\to\mathbb{R}$, $a_1<ma_2$ be exponentially $(\alpha,h-m)$-convex function. If $\frac{\psi\left(\frac{a_1+ma_2-w}{m}\right)}{e^{\xi\left(\frac{a_1+ma_2-w}{m}\right)}}=\frac{\psi(w)}{e^{\xi w}}$ and $m\in(0,1]$, then the following inequality holds:*

$$\psi\left(\frac{a_1+ma_2}{2}\right)\leq\frac{\psi(w)}{e^{\xi w}}\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\right). \tag{34}$$

Proof. Since ψ is an exponentially $(\alpha,h-m)$-convex function, one can write the following inequality:

$$\begin{aligned}&\psi\left(\frac{a_1+ma_2}{2}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)\frac{\psi\left(\frac{w-a_1}{ma_2-a_1}ma_2+\frac{ma_2-w}{ma_2-a_1}a_1\right)}{e^{\xi\left(\frac{w-a_1}{ma_2-a_1}ma_2+\frac{ma_2-w}{ma_2-a_1}a_1\right)}}+mh\left(1-\frac{1}{2^{\alpha}}\right)\\&\frac{\psi\left(\frac{\frac{w-a_1}{ma_2-a_1}a_1+\frac{ma_2-w}{ma_2-a_1}ma_2}{m}\right)}{e^{\xi\left(\frac{\frac{w-a_1}{ma_2-a_1}a_1+\frac{ma_2-w}{ma_2-a_1}ma_2}{m}\right)}}=h\left(\frac{1}{2^{\alpha}}\right)\frac{\psi(w)}{e^{\xi w}}+mh\left(1-\frac{1}{2^{\alpha}}\right)\frac{\psi\left(\frac{a_1+ma_2-w}{m}\right)}{e^{\xi\left(\frac{a_1+ma_2-w}{m}\right)}}.\end{aligned} \tag{35}$$

Hence, by using the condition on ψ, we obtain the required inequality (34). □

Theorem 4. *Let $\psi, \phi : [a_1, a_2] \longrightarrow \mathbb{R}$, $a_1 < ma_2$, be functions such that ψ be positive, integrable, exponentially $(\alpha, h-m)$-convex, and $\frac{\psi\left(\frac{a_1+ma_2-w}{m}\right)}{e^{\xi\left(\frac{a_1+ma_2-w}{m}\right)}} = \frac{\psi(w)}{e^{\xi w}}$, $m \in (0,1]$, and ϕ be differentiable and strictly increasing with $\phi' \in L^1[a_1, a_2]$. Then, for $\sigma, \tau \geq k$, $\xi \in \mathbb{R}$, the following fractional integral inequality holds:*

$$\begin{aligned} &\frac{e^{\xi w}}{h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)}\psi\left(\frac{a_1+ma_2}{2}\right)\left[J_{\frac{\tau}{k},a_2^-}(a_1;\lambda) + J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda)\right] \\ &\leq \left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(a_1;\lambda) + \left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(a_2;\lambda) \\ &\leq \left[J_{\frac{\tau}{k},a_2^-}(a_1;\lambda) + J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda)\right](a_2-a_1)\left(\frac{\psi(a_2)}{e^{\xi a_2}}\int_0^1 h(\zeta^\alpha)\right. \\ &\left.\phi'(a_1+\zeta(a_2-a_1))d\zeta + \frac{m\psi(\frac{a_1}{m})}{e^{\xi\left(\frac{a_1}{m}\right)}}\int_0^1 h(1-\zeta^\alpha)\phi'(a_1+\zeta(a_2-a_1))d\zeta\right). \end{aligned} \tag{36}$$

Proof. For $w \in [a_1, a_2]$ and $\tau > 0$, we have

$$\begin{aligned} &(\phi(w)-\phi(a_1))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(w)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\phi'(w) \\ &\leq (\phi(a_2)-\phi(a_1))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(a_2)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\phi'(w). \end{aligned} \tag{37}$$

For $\xi \in \mathbb{R}$, applying the exponentially $(\alpha, h-m)$ convexity of ψ we have:

$$\psi(w) \leq h\left(\frac{w-a_1}{a_2-a_1}\right)^\alpha \frac{\psi(a_2)}{e^{\xi a_2}} + mh\left[1-\left(\frac{w-a_1}{a_2-a_1}\right)^\alpha\right]\frac{\psi(\frac{a_1}{m}))}{e^{\xi\frac{a_1}{m}}}. \tag{38}$$

The following inequality is yielded after multiplication of inequalities (37) and (38) and then integrating over the interval $[a_1, a_2]$:

$$\begin{aligned} &\int_{a_1}^{a_2}(\phi(w)-\phi(a_1))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(w)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\phi'(w)\psi(w)dw \\ &\leq (\phi(a_2)-\phi(a_1))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(a_2)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right)\left[\frac{\psi(a_2)}{e^{\xi a_2}}\int_{a_1}^{a_2} h\left(\frac{w-a_1}{a_2-a_1}\right)^\alpha\right. \\ &\left.\phi'(w)dw + m\frac{\psi(\frac{a_1}{m})}{e^{\xi\frac{a_1}{m}}}\int_{a_1}^{a_2} h\left(1-\left(\frac{w-a_1}{a_2-a_1}\right)^\alpha\right)\phi'(w)dw\right]. \end{aligned}$$

On simplifying, we obtain the inequalities (39) and (40) which are provided as:

$$\begin{aligned} &\left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(a_1;\lambda) \leq (\phi(a_2)-\phi(a_1))^{\frac{\tau}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(a_2)-\phi(a_1))^{\frac{\gamma}{k}};\lambda\right) \\ &\left(\frac{\psi(a_2)}{e^{\xi a_2}}\int_0^1 h(\zeta^\alpha)\phi'(p+\zeta(a_2-a_1))(a_2-a_1)d\zeta\right. \\ &\left.+m\frac{\psi(\frac{a_1}{m})}{e^{\xi\left(\frac{a_1}{m}\right)}}\int_0^1 h(1-\zeta^\alpha)\phi'(p+\zeta(a_2-a_1))(a_2-a_1)d\zeta\right), \end{aligned} \tag{39}$$

$$\begin{aligned} &\left({}_{\phi}^{k}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(a_1;\lambda) \leq (a_2-a_1)\, J_{\frac{\tau}{k},a_2^-}(a_1;\lambda) \\ &\left(\frac{\psi(a_2)}{e^{\xi a_2}}\int_0^1 h(\zeta^\alpha)\phi'(p+\zeta(a_2-a_1))d\zeta + m\frac{\psi(\frac{a_1}{m})}{e^{\xi\left(\frac{a_1}{m}\right)}}\int_0^1 h(1-\zeta^\alpha)\phi'(p+\zeta(a_2-a_1))d\zeta\right). \end{aligned} \tag{40}$$

Similarly, on the other hand, the inequality (41) holds for $t \in [a_1, a_2]$ and $\sigma > 0$ as follows:

$$\begin{aligned}&(\phi(a_2) - \phi(w))^{\frac{\sigma}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(a_2) - \phi(w))^{\frac{\gamma}{k}}; \lambda\right)\phi'(w) \\ &\leq (\phi(a_2) - \phi(a_1))^{\frac{\sigma}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(a_2) - \phi(a_1))^{\frac{\gamma}{k}}; \lambda\right)\phi'(w).\end{aligned} \tag{41}$$

The following inequality is yielded after multiplication of inequalities (38) and (41), and then integrating over the interval $[a_1, a_2]$:

$$\begin{aligned}&\int_{a_1}^{a_2} (\phi(a_2) - \phi(w))^{\frac{\sigma}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(a_2) - \phi(w))^{\frac{\gamma}{k}}; \lambda\right)\phi'(w)\psi(t)dw \\ &\leq (\phi(a_2) - \phi(a_1))^{\frac{\sigma}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(a_2) - \phi(a_1))^{\frac{\gamma}{k}}; \lambda\right) \left[\frac{\psi(a_2)}{e^{\xi a_2}} \int_{a_1}^{a_2} h\left(\frac{w-a_1}{a_2-a_1}\right)^{\alpha}\right. \\ &\left.\phi'(w)dw + m\frac{\psi(\frac{a_1}{m})}{e^{\xi\frac{a_1}{m}}} \int_{a_1}^{a_2} h\left(1 - \left(\frac{w-a_1}{a_2-a_1}\right)^{\alpha}\right)\phi'(w)dw\right].\end{aligned}$$

Further, we have:

$$\begin{aligned}&\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(a_2;\lambda) \leq (\phi(a_2) - \phi(a_1))^{\frac{\sigma}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(a_2) - \phi(a_1))^{\frac{\gamma}{k}}; \lambda\right) \\ &\left(\frac{\psi(a_2)}{e^{\xi a_2}} \int_0^1 h(\zeta^{\alpha})\phi'(a_1 + \zeta(a_2 - a_1))(a_2 - a_1)d\zeta\right. \\ &\left.+\frac{m\psi(\frac{a_1}{m})}{e^{\xi(\frac{a_1}{m})}} \int_0^1 h(1-\zeta^{\alpha})\phi'(a_1 + \zeta(a_2 - a_1))(a_2 - a_1)d\zeta\right)\end{aligned} \tag{42}$$

$$\begin{aligned}&\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(a_2;\lambda) \leq (a_2 - a_1)\, J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda)\left(\frac{\psi(a_2)}{e^{\xi a_2}}\right. \\ &\left.\int_0^1 h(\zeta^{\alpha})\phi'(a_1 + \zeta(a_2 - a_1))d\zeta + m\frac{\psi(\frac{a_1}{m})}{e^{\xi(\frac{a_1}{m})}} \int_0^1 h(1-\zeta^{\alpha})\phi'(a_1 + \zeta(a_2 - a_1))d\zeta\right).\end{aligned} \tag{43}$$

By summing the inequalities (40) and (43), we have;

$$\begin{aligned}&\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(a_1;\lambda) + \left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(a_2;\lambda) \\ &\leq \left[J_{\frac{\tau}{k},a_2^-}(a_1;\lambda) + J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda)\right](a_2 - a_1)\left(\frac{\psi(a_2)}{e^{\xi a_2}}\right. \\ &\left.\int_0^1 h(\zeta^{\alpha})\phi'(a_1 + \zeta(a_2 - a_1))d\zeta + m\frac{\psi(\frac{a_1}{m})}{e^{\xi(\frac{a_1}{m})}} \int_0^1 h(1-\zeta^{\alpha})\phi'(a_1 + \zeta(a_2 - a_1))d\zeta\right).\end{aligned} \tag{44}$$

Multiplying the inequality (34) with $(\phi(w) - \phi(a_1))^{\frac{\tau}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(w) - \phi(a_1))^{\frac{\gamma}{k}}; \lambda\right)$ $\phi'(w)$ and then integrating it over $[a_1, a_2]$, we obtain

$$\begin{aligned}&\psi\left(\frac{a_1 + ma_2}{2}\right) \int_{a_1}^{a_2} (\phi(w) - \phi(a_1))^{\frac{\tau}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(w) - \phi(a_1))^{\frac{\gamma}{k}}; \lambda\right)\phi'(w)dw \\ &\leq \frac{1}{e^{\xi w}}\left(h\left(\frac{1}{2^{\alpha}}\right) + mh\left(1 - \frac{1}{2^{\alpha}}\right)\right) \\ &\int_{a_1}^{a_2} (\phi(w) - \phi(a_1))^{\frac{\tau}{k}-1} E^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,k}\left(z(\phi(w) - \phi(a_1))^{\frac{\gamma}{k}}; \lambda\right)\phi'(w)\psi(t)dw.\end{aligned} \tag{45}$$

Inequality (46) is obtained by utilizing inequalities (5) and (11)

$$\begin{aligned}&\psi\left(\frac{a_1 + ma_2}{2}\right) J_{\frac{\tau}{k},a_2^-}(a_1;\lambda) \\ &\leq \frac{1}{e^{\xi w}}\left(h\left(\frac{1}{2^{\alpha}}\right) + mh\left(1 - \frac{1}{2^{\alpha}}\right)\right)\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(a_1;\lambda).\end{aligned} \tag{46}$$

Multiplying inequality (34) with $(\phi(a_2)-\phi(w))^{\frac{\sigma}{k}-1}E^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,k}\left(z(\phi(a_2)-\phi(w))^{\frac{\gamma}{k}};\lambda\right)$ $\phi'(w)$ and integrating it over $[a_1,a_2]$, also utilizing (4) and (10), we obtain

$$\begin{aligned}&\psi\left(\frac{a_1+ma_2}{2}\right)J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda)\\&\leq\frac{1}{e^{\xi w}}\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)\right)\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(a_2;\lambda).\end{aligned}\tag{47}$$

Adding the inequalities (46) and (47), we obtain

$$\begin{aligned}&\frac{e^{\xi w}}{h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)}\psi\left(\frac{a_1+ma_2}{2}\right)\left[J_{\frac{\tau}{k},a_2^-}(a_1;\lambda)+J_{\frac{\sigma}{k},a_1^+}(a_2;\lambda)\right]\\&\leq\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\tau,u,z,a_2^-}\psi\right)(a_1;\lambda)+\left({}^{k}_{\phi}\kappa^{\rho,\epsilon,\mu,r}_{\gamma,\sigma,u,z,a_1^+}\psi\right)(a_2;\lambda).\end{aligned}\tag{48}$$

From inequalities (44) and (48), inequality (36) can be obtained. □

Remark 6. *By setting $\alpha=1$; $m=1$; $\alpha=m=1$; $h(t)=t$; $h(t)=t$ and $\alpha=1$; and $h(t)=t$ and $\alpha=m=1$ in (36), it holds for exponentially $(h-m)$ convex, exponentially (α,h)-convex, exponentially h-convex, exponentially (α,m) convex, exponentially m-convex, and exponentially convex functions, respectively.*

3. Conclusions

Using exponential $(\alpha,h-m)$-convexity, we provided the bounds of fractional integral operators incorporating Mittag–Leffler functions. The generalization of the numerous results proved in [31,32] are established. In the form of well-known Hadamard-like inequality, we presented upper as well as lower bounds for operators of various types. The results hold for almost all kinds of convex functions. By utilizing the integral operators studied in this paper, it is possible to generalize Ostrowski-, Gruss-, and Ostrowski-Gruss-type inequalities. The authors are further working on fractional equations for the integral operators utilized in this paper.

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