fractal and fractional

Article

Fear Effect on a Predator–Prey Model with Non-Differential Fractional Functional Response

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Abstract: In this paper, we study the factor of the fear effect in a predator–prey model with prey refuge and a non-differentiable fractional functional response due to the group defense. Since the functional response is non-differentiable, the dynamics of this system are considerably different from the dynamics of a classical predator–prey system. The persistence, the stability and the existence of the steady states are investigated. We examine the Hopf bifurcation at the unique positive equilibrium. Direct Hopf bifurcation is studied via the central manifold theorem. When the value of the fear factor decreases and is less than a threshold κ_H, the limit cycle appears, and it disappears through a loop of heteroclinic orbits when the value of the fear factor is equal to a value κ_{het}.

Keywords: predator–prey model; fear effect; group defense; type IV functional response; Hopf bifurcation

MSC: 34D23; 37C60

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1. Introduction

In ecological webs, there are four main types of interactions between species: commensalism, mutualism, predation, and competition [1]. Numerous differential-equation-based systems have been developed to describe the dynamics of these interactions. Among all these four types, predation has received the most attention from academics and has been widely investigated in a variety of scenarios due to the significance and prevalence of predation in the real world. Suppose that the predator and prey densities change continuously with time. The following differential equations represent a generalized predator–prey model containing logistic growth:

$$\begin{aligned} \frac{dz}{dt} &= \varrho z - \zeta z - \xi z^2 - \phi(z)s, \\ \frac{ds}{dt} &= \beta\phi(z)s - \delta s. \end{aligned} \tag{1}$$

The equations in system (1) depict the dynamics of prey and predator, respectively. The interpretations of z, s, $\phi(z)$, ϱ, ξ, β, ζ, and δ are summarized in Table 1. The functional response $\phi(z)$ is a major feature in any predator–prey model and it takes different forms depending on the scenario (for example, see [2–12]). In Table 2, we summarize a number of traditional forms of the functional response.

Table 1. A summary of the model parameters and their interpretation.

Symbol	Interpretation	Assumptions
z	Prey density	
s	Predator density	

Table 1. *Cont.*

Symbol	Interpretation	Assumptions
$\phi(z)$	Number of prey successfully attacked per predator	$\phi(0) = 0$, $\phi'(z) > 1$
ϱ	Prey population birth rate	
ζ	Rate of natural mortality in prey populations	
ξ	Mortality rate as a result of interspecies competition	
κ	Level of fear	
μ	Capacity of a refuge at t	$\mu \in (0, 1)$
γ	Attack rate per predator and prey	
β	Prey conversion to the predator	
δ	Per capita death rate of the predator	
α	Efficiency of aggregation for prey	$\alpha \in (0, 1)$
σ	Handling time per prey	
z^*	Incipient limiting level	

Table 2. Several forms of traditional functional responses.

Functional Response Type	$\phi(z)$	Reference
type I	$\phi(z) = \begin{cases} \gamma z & \text{if } z \leq z^*, \\ \frac{1}{c} & \text{if } z > z^*. \end{cases}$	[13]
type II	$\phi(z) = \frac{\gamma z}{1 + \gamma\sigma z}.$	[14]
type III	$\phi(z) = \frac{\gamma z^\theta}{1 + \gamma\sigma z^\theta}, \quad \theta > 1.$	[15]
type IV	$\phi(z) = \frac{\gamma g(z)}{1 + \gamma\sigma g(z)},$ different expressions.	[16]

There is a growing belief that the sheer existence of a predator may change the behavior and physiology of prey to the point that it might have an impact on prey populations that is even stronger than direct predation [17–19]. According to Cresswell, all animals exhibit a range of anti-predator responses in response to perceived predation danger, including changes in habitat use, foraging behaviors, alertness, and physiological changes [20]. According to Zanette et al. [21], the ability of parents of song sparrows to produce offspring was reduced by 40% merely due to their fear of predators. Field studies demonstrate that the fear effect would lower productivity. Therefore, this factor has drawn the attention of numerous academics [22–30]. Thus, we amend system (1) by multiplying the production term by a factor $\psi(\kappa, s)$ that takes into account the cost of anti-predator defense brought on by fear, resulting in

$$\begin{aligned} \frac{dz}{dt} &= \varrho\psi(\kappa, s)z - \zeta z - \xi z^2 - \phi(z)s, \\ \frac{ds}{dt} &= \beta\phi(z)s - \delta s. \end{aligned} \tag{2}$$

According to [31], $\psi(\kappa, s)$ meets the following conditions:

$$\begin{aligned} &\psi(0, s) = 1, \quad \lim_{\kappa\to\infty} \psi(\kappa, s) = 0, \quad \frac{\partial\psi(\kappa, s)}{\partial\kappa} = 0, \\ &\psi(\kappa, 0) = 1, \quad \lim_{s\to\infty} \psi(\kappa, s) = 0, \quad \frac{\partial\psi(\kappa, s)}{\partial s} = 0. \end{aligned} \tag{3}$$

Several functions fulfill the conditions in (3), for example

(i) $\psi_1(k, s) = \frac{1}{1 + \kappa s}$

(ii) $\psi_1(k, s) = e^{-\kappa s}$

(iii) $\psi_1(k,s) = \dfrac{e^{-\kappa s}}{1+\omega \sin \kappa s}$, where $\omega \in (0,1)$.

In this paper, we consider $\psi(\kappa, s) = \dfrac{1}{1+\kappa s}$. On the other hand, refuge can be defined to include any technique employed by prey that minimizes the predation risk. Most researchers have demonstrated that refugia have a stabilizing impact on the prey–predator model. Assume that the capacity of the refugia is $\hat{\mu}$. There are two different perspectives on this quantity:

(i) $\hat{\mu} = \mu z$, the refuge capacity is proportional to the density of prey;
(ii) $\hat{\mu} = \mu$, the refuge capacity is constant.

We modify the functional response to incorporate prey refuges to be a function with respect to $(z - \hat{\mu})$, where $\hat{\mu} = \mu z$, and system (1) becomes as follows:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\varrho z}{1+\kappa s} - \zeta z - \xi z^2 - \phi(z - \mu z)s, \\ \frac{ds}{dt} &= \beta\phi(z-\mu z)s - \delta s. \end{aligned} \tag{4}$$

In addition, cooperative behavior is widespread among organisms [32], such as safety in numbers (group defense), pack hunting, parental care, animal migration, and clumping. Some animals find safety in numbers by existing in large groups: buffalo live in herds [33], numerous fish species (including tuna) congregate in large schools [34], and geese gather in flocks as they move [35]. Living in a group allows animals to protect themselves. For example, white rhinos and gnus create defensive circles [36]. Ajraldi et al. investigated the group defense technique using $\gamma\sqrt{z}$ as a functional response [37]. After this, a more general functional response γz^{α} to describe the group defense was developed by Venturino and Petrovskii [38], where the "α" interpretations are as in Table 1. Depending on Venturino's functional response, many authors have investigated various scenarios for predator–prey models containing group defense [39–41]. For example, the existence and uniqueness of limit cycles and nonexistence of periodic orbits was examined in [42], and a bifurcation analysis of a predator–prey model with cooperative predator hunting and a non-differentiable functional response was investigated by Y. Du et al. [43]. Other researchers took various factors into consideration, such as cannibalism [44], multiplicative noise [45], Leslie–Gower terms [46], the Allee effect [47], prey harvesting [48], and predator harvesting [49]. No author has considered how refuge or fear may affect systems that include Venturino's group defense. The following system of nonlinear ordinary differential equations provides a model for the interaction between the predator and prey populations with group defense in prey, the fear effect, and prey refuge:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\varrho z}{1+\kappa s} - \zeta z - \xi z^2 - \frac{\gamma(z-\mu z)^{\alpha} s}{1+\sigma\gamma(z-\mu z)^{\alpha}}, \quad z \geq 0, \\ \frac{ds}{dt} &= \frac{\beta\gamma(z-\mu z)^{\alpha} s}{1+\sigma\gamma(z-\mu z)^{\alpha}} - \delta s. \end{aligned} \tag{5}$$

For more details, see Figure 1. In addition, Figure 2 shows a graphical representation of the fractional functional response $\phi(z) = \dfrac{\gamma(z-\mu z)^{\alpha}}{1+\sigma\gamma(z-\mu z)^{\alpha}}$ with $\alpha < 1$.

In this paper, we pay particular attention to answering the following question: how do group defense, the fear factor, and the refuge affect the qualitative dynamics of the model? We summarize our findings and the contributions of the paper as follows:

1. We consider the type IV functional response, and it is nondifferentiable on the s-axis. The functional response ultimately increases (when $z \geq \frac{1}{1-\mu}$) as the efficiency of aggregation for prey increases and it decreases as the refuge capacity increases.
2. The predator population falls into decay if the per capita death rate of the predator is greater than a constant $\theta = \beta\gamma(1-\mu)^{\alpha}(\frac{\varrho}{\xi})^{\alpha}$ that depends on several parame-

ters. Note that this θ decreases as the capacity of a refuge at t increases, and θ increases (decreases) as the value of the efficiency of aggregation for prey increases if $\frac{(1-\mu)\varrho}{\xi} > 1$ ($\frac{(1-\mu)\varrho}{\xi} < 1$).

3. Because of the z^{α} term, the Jacobian matrix is indeterminate at the origin. Therefore, it is impossible to carry out a stability analysis by merely looking at its eigenvalues. We use the definition of stability to prove that if $\varrho < \zeta$, then $E_0(0,0)$ is stable, and if $\varrho > \zeta$, then $E_0(0,0)$ is unstable.
4. Under some conditions, the coexistence state of system (5) is stable and the alteration in the fear factor's value has no bearing on this stability.
5. We examine the Hopf bifurcation at the unique positive equilibrium. When the fear factor's value decreases, the limit cycle appears when the fear factor's value is less than κ_H, and it disappears when the fear factor's value is equal to κ_{het} through a loop of heteroclinic orbits.

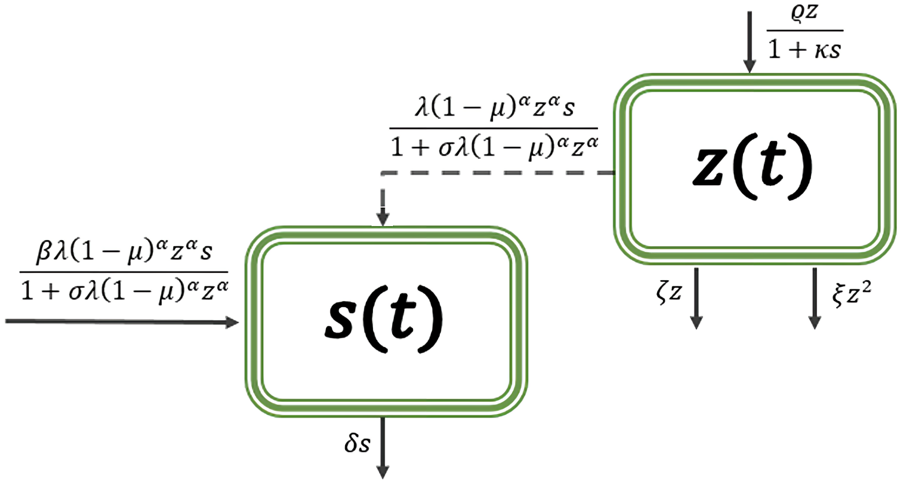
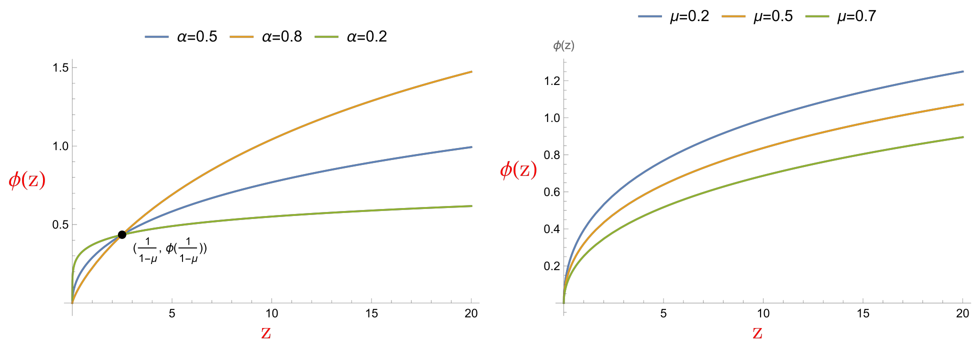


Figure 1. Food chain diagram of system (5).



(**a**) (**b**)

Figure 2. The graphical representation of the functional response. (**a**) The functional response $\phi(z) = \frac{\gamma(z-\mu z)^{\alpha}}{1+\sigma\gamma(z-\mu z)^{\alpha}}$ has been plotted for different values of α when $\gamma = 0.5$, $\mu = 0.6$, and $\sigma = 0.3$. This figure shows that the value of the functional response ultimately increases (when $z \geq \frac{1}{1-\mu}$) as the efficiency of aggregation for prey increases. (**b**) The functional response $\phi(z) = \frac{\gamma(z-\mu z)^{\alpha}}{1+\sigma\gamma(z-\mu z)^{\alpha}}$ has been plotted for different values of μ when $\gamma = 0.5, \alpha = 0.5$, and $\sigma = 0.3$. This figure shows that the value of the functional response decreases as the refuge capacity increases.

2. Boundedness and Positivity

Lemma 1. *For system (5), the first quadrant $\mathbb{R}^2_+$ is a positive invariant set.*

Proof of Lemma 1. For system (5), it is not difficult to show that the set $\{(z,s), s = 0\}$ is an invariant set. This means that any orbit of system (5) that touches the z-axis stays forever on it. On the other hand, since z^α is a nondifferentiable function over $z = 0$, the solution of system (5) that belongs in $\{(z,s), z = 0\}$ is not unique. We can reduce system (5) to

$$\begin{aligned} \frac{dz}{dt} &= 0, \\ \frac{ds}{dt} &= -\delta s. \end{aligned} \tag{6}$$

Along the s-axes, the solution of system (6) moves closer to the origin. This means that any orbit of system (5) that touches the s-axis stays forever on it. □

Lemma 2. *All solutions of system (5) with an initial value in $\mathbb{R}^2_+$ are bounded.*

Proof of Lemma 2. Let $(z(t), s(t))$ be any solution of system (5) with $(z_0, s_0) \in \mathbb{R}^2_+$. If $z_0 > \frac{\varrho}{\xi}$, $\frac{dz}{dt} = \frac{\varrho z}{1+\kappa s} - \zeta z - \xi z^2 - \frac{\gamma(z-\mu z)^\alpha s}{1+\sigma\gamma(z-\mu z)^\alpha} < \varrho z(1 - \frac{z}{\frac{\varrho}{\xi}}) < 0$ if $z > \frac{\varrho}{\xi}$, this means that $z(t)$ decreases when $z > \frac{\varrho}{\xi}$. When $z = \frac{\varrho}{\xi}$, $\frac{dz}{dt} < \varrho z(1 - \frac{z}{\frac{\varrho}{\xi}}) - \frac{\gamma(z-\mu z)^\alpha s}{1+\sigma\gamma(z-\mu z)^\alpha} = -\frac{\gamma(z-\mu z)^\alpha s}{1+\sigma\gamma(z-\mu z)^\alpha} < 0$. Therefore, $z(t) < \lambda_1 = \max\{z_0, \frac{\varrho}{\xi}\}$. Let $x = \beta z + s$, hence

$$\frac{dx}{dt} = \beta(\frac{\varrho z}{1+\kappa s} - \zeta z - \xi z^2) - \delta s < \beta(\varrho + \delta)z - \delta x.$$

Then,

$$\frac{dx}{dt} + \delta x < \lambda_2, \quad \text{where } \lambda_2 = \beta(\varrho + \delta)\lambda_1 > 0.$$

According to Lemma (1.1, [50]), we obtain

$$x < \frac{\lambda_2}{\delta}, \quad t \geq t_0.$$

Then, we have

$$\beta z + s < \frac{\beta\varrho + \delta}{\delta}\lambda_1, \quad t \geq t_0.$$

In other words, $s(t)$ is bounded. □

Remark 1. *The region $\{(z,s) : z > \frac{\varrho}{\xi}, s \geq 0\}$ has no equilibrium.*

3. Non-Persistence

Theorem 1. *For the initial value (z_0, s_0) in $\mathbb{R}^2_+$, if*

$$z^{1-\alpha}(0) < \frac{(1-\alpha)\gamma(1-\mu)^\alpha s(0)}{((1-\alpha)\varrho + 1)(1 + \sigma\gamma(1-\mu)^\alpha\lambda_1^\alpha)} \tag{7}$$

where $\lambda_1 = \max\{z_0, \frac{\varrho}{\xi}\}$, then the prey population falls into decay.

Proof of Theorem 1. We can easly show that $s(t) \geq s_0 e^{-t}, \forall t$. From the proof of Lemma 2, recalling $z(t) \leq \lambda_1 = \max\{z_0, \frac{\varrho}{\xi}\}$, from the first equation of system (5),

$$\frac{dz}{dt} \leq \varrho z - \frac{\gamma(z-\mu z)^\alpha s}{1+\sigma\gamma(z-\mu z)^\alpha} \leq \varrho z - \frac{\gamma(1-\mu)^\alpha z^\alpha s_0 e^{-t}}{1+\sigma\gamma(1-\mu)^\alpha\lambda_1^\alpha}.$$

Suppose that

$$\frac{d\hat{z}}{dt} = \varrho\hat{z} - \frac{\gamma(1-\mu)^{\alpha}\hat{z}^{\alpha}s_0e^{-t}}{1+\sigma\gamma(1-\mu)^{\alpha}\lambda_1^{\alpha}} \tag{8}$$

with $\hat{z}(0) = z(0) = z_0$. By using the comparison theorem of ODE, we obtain $z(t) \leq \hat{z}(t), \forall t$. To solve Equation (8), suppose $\hat{z}(t) = x(t)e^{\varrho t}$; then,

$$\frac{dx}{dt} = -\frac{s_0\gamma(1-\mu)^{\alpha}x^{\alpha}e^{-((1-\alpha)\varrho+1)t}}{1+\sigma\gamma(1-\mu)^{\alpha}\lambda_1^{\alpha}}. \tag{9}$$

By direct calculation, we have

$$x^{1-\alpha}(t) = x^{1-\alpha}(0) - \frac{(1-\alpha)\gamma(1-\mu)^{\alpha}s_0}{((1-\alpha)\varrho+1)(1+\sigma\gamma(1-\mu)^{\alpha}\lambda_1^{\alpha})}[1-e^{-((1-\alpha)\varrho+1)t}]. \tag{10}$$

By the definition of $x(t)$, it is clear that $x^{1-\alpha}(0) \geq 0$ and $x(t)$ is a decreasing function. Thus, $x(\tau) = 0$ for a certain τ if and only if

$$\hat{z}^{1-\alpha}(0) = x^{1-\alpha}(0) < \frac{(1-\alpha)\gamma(1-\mu)^{\alpha}s_0}{((1-\alpha)\varrho+1)(1+\sigma\gamma(1-\mu)^{\alpha}\lambda_1^{\alpha})}. \tag{11}$$

It is no secret that $x(\tau) = 0$ means $\hat{z}(\tau) = 0$. Recalling $z(t) \leq \hat{z}(t), \forall t$. Hence, $z(\tau) \leq 0$ when $\hat{z}(\tau) = 0$, since R_+^2 is an invariant set; then, $\hat{z}(t) = 0,\ t \geq \tau$. □

Theorem 2. *If $\delta > \beta\gamma(1-\mu)^{\alpha}(\frac{\varrho}{\xi})^{\alpha}$, then the predator population falls into decay.*

Proof of Theorem 2. For any solution $(z(t), s(t))$ of system (5), it is easy to prove that there is $\tau \geq 0$ such that $z(t) \leq \frac{\varrho}{\xi}$ for $t \geq \tau$. From the second equation of system (5),

$$\begin{aligned} \frac{ds}{dt} &\leq \beta\gamma(1-\mu)^{\alpha}z^{\alpha}s - \delta s \\ &\leq (\beta\gamma(1-\mu)^{\alpha}(\frac{\varrho}{\xi})^{\alpha} - \delta)s, \quad t \geq \tau. \end{aligned} \tag{12}$$

This implies $s(t) \leq s_0e^{-(\delta-\beta\gamma(1-\mu)^{\alpha}\lambda_1^{\alpha})},\ t \geq \tau$. Therefore, for $\delta > \beta\gamma(1-\mu)^{\alpha}(\frac{\varrho}{\xi})^{\alpha}$, we have $\lim\limits_{t\to\infty} s(t) = 0$. □

4. Steady States and Their Stability

From system (5), z-zero-growth isocline is determined by

$$\frac{\varrho z}{1+\kappa s} - \zeta z - \xi z^2 = \frac{\gamma(z-\mu z)^{\alpha}s}{1+\sigma\gamma(1-\mu)^{\alpha}z^{\alpha}},$$

and s-zero-growth isoclines are $s = 0$ and $\frac{\beta\gamma(z-\mu z)^{\alpha}}{1+\sigma\gamma(z-\mu z)^{\alpha}} = \delta$. We know that the intersection of z-zero-growth isocline and s-zero-growth isocline yields the equilibrium points. For any equilibrium point $E_*(z_*, s_*)$, the Jacobian matrix of the system (5) around E_* is given by

$$J(E_*) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{13}$$

where

$$a_{11} = -\zeta + \frac{\alpha\gamma^2\sigma s_*(1-\mu)^{2\alpha} z_*^{2\alpha-1}}{(\gamma\sigma(1-\mu)^\alpha z_*^\alpha + 1)^2} - \frac{\alpha\gamma s_*(1-\mu)^\alpha z_*^{\alpha-1}}{\gamma\sigma(1-\mu)^\alpha z_*^\alpha + 1} - 2\xi z_* + \frac{\varrho}{\kappa s_* + 1}$$
$$a_{12} = -\frac{\gamma(1-\mu)^\alpha z_*^\alpha}{\gamma\sigma(1-\mu)^\alpha z_*^\alpha + 1} - \frac{\kappa\varrho z_*}{(\kappa s_* + 1)^2}$$
$$a_{21} = \frac{\alpha\beta\gamma s_*(1-\mu)^\alpha z_*^{\alpha-1}}{\gamma\sigma(1-\mu)^\alpha z_*^\alpha + 1} - \frac{\alpha\beta\gamma^2\sigma s_*(1-\mu)^{2\alpha} z_*^{2\alpha-1}}{(\gamma\sigma(1-\mu)^\alpha z_*^\alpha + 1)^2}$$
$$a_{22} = \frac{\beta\gamma(1-\mu)^\alpha z_*^\alpha}{\gamma\sigma(1-\mu)^\alpha z_*^\alpha + 1} - \delta$$

4.1. The Trivial Steady State

The trivial steady state $E_0(0,0)$ always exists. In this equilibrium point, both populations fall into decay. Because of the z^α term, system (5) is not linearizable and the Jacobian matrix becomes indeterminate. In other words, (13) cannot be calculated for $z = 0$ and $s = 0$ to determine the stability of origin. In the next theorems, we will discuss the stability of E_0.

Theorem 3. *If $\varrho < \zeta$, then E_0 is stable.*

Proof of Theorem 3. Let $(z_*(t), s_*(t))$ be any solution of system (5). From the first equation of system (5),

$$\frac{dz_*}{dt} \le (\varrho - \zeta) z_*.$$

Since $\varrho < \zeta$, $\lim_{t\to\infty} z_*(t) = 0$. This means that the prey population falls into decay, and we can reduce system (5) to

$$\begin{aligned} \frac{dz_*}{dt} &= 0, \\ \frac{ds_*}{dt} &= -\delta s_*. \end{aligned} \tag{14}$$

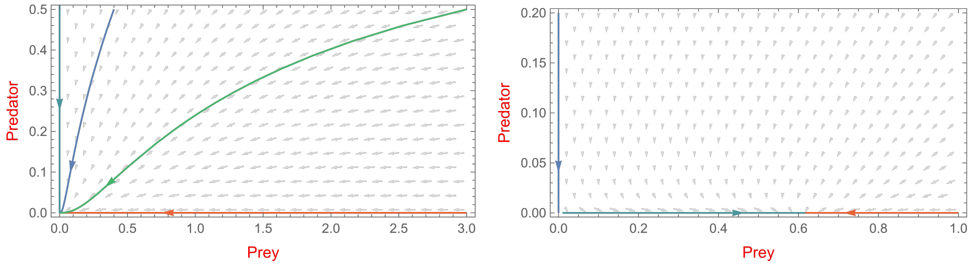
It is clear that $\lim_{t\to\infty} s_*(t) = 0$. The proof is completed. □

Theorem 4. *If $\varrho > \zeta$, then E_0 is an unstable point.*

Proof of Theorem 4. Let $\Omega = \{(z,0) \in \mathbb{R}_+^2 : 0 < z < \frac{\varrho-\zeta}{\xi}\}$, then $\lim_{t\to\infty} z_*(t) = \frac{\varrho-\zeta}{\xi}$ for any solution $(z_*, 0)$ with initial values in Ω. For $\lambda > 0$, let $\Omega_\lambda = \{(z,s) \in \mathbb{R}_+^2 : |(z,s)| < \lambda\}$. It is clear that $\Omega \cap \Omega_\lambda \neq \phi$, for all $\lambda > 0$. Let $0 < \epsilon < \frac{0.1(\varrho-\zeta)}{\xi}$. Then, for all $\lambda > 0$, there is $(z_\lambda, 0) \in \Omega \cap \Omega_\lambda$ such that $\lim_{t\to\infty} z_*(t) = \frac{\varrho-\zeta}{\xi} > \epsilon$. Here, $(z_*(t), 0)$ is the solution of system (5) with initial value $(z_\lambda, 0)$. Therefore, by the definition of stability, E_0 is unstable. □

Example 1. *For $\kappa = 0.25855$, $\zeta = 0.57511$, $\xi = 0.11722$, $\gamma = 0.31937$, $\mu = 0.29078$, $\alpha = 0.48276$, $\sigma = 0.4976$, $\beta = 0.19344$, and $\delta = 0.34488$.*

1. *Take $\varrho = 0.44741$, then $\varrho < \zeta$ and E_0 is stable (see Figure 3a);*
2. *Take $\varrho = 0.64741$, then $\varrho > \zeta$ and E_0 is unstable (see Figure 3b).*



(a) (b)

Figure 3. Phase plane analysis of system (5). (**a**) In Example 1, E_0 is a stable point when $\varrho < \zeta$. (**b**) In Example 1, E_0 is an unstable point when $\varrho > \zeta$.

4.2. The Predator-Free Steady State

If $\varrho > \zeta$ the predator-free equilibrium point $E_1(\frac{\varrho-\zeta}{\xi}, 0)$ exists, which means that the predator becomes extinct and the prey survives. At this equilibrium $E_1(\frac{\varrho-\zeta}{\xi}, 0)$, the Jacobian matrix J_{E_1} is given by

$$J_{E_1} = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}, \tag{15}$$

where b_{11}, b_{12}, and b_{22} are given by

$$b_{11} = -2(\varrho-\zeta) - \zeta + \varrho$$
$$b_{12} = -\frac{\gamma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha}{\gamma\sigma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha + 1} - \frac{\kappa\varrho(\varrho-\zeta)}{\xi}$$
$$b_{22} = \frac{\beta\gamma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha}{\gamma\sigma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha + 1} - \delta$$

Next, we present a theorem on the stability of E_1.

Theorem 5.

(i) Assuming that $\varrho > \zeta$ and $\theta_1 < \theta_2$ hold, then $E_1(\frac{\varrho-\zeta}{\xi}, 0)$ is a stable node.

(ii) Assuming that $\varrho > \zeta$ and $\theta_1 > \theta_2$ hold, then $E_1(\frac{\varrho-\zeta}{\xi}, 0)$ is an unstable saddle point.

Here,

$$\theta_1 = \beta\gamma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha$$
$$\theta_2 = \gamma\delta\sigma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha + \delta$$

Proof of Theorem 5. The eigenvalues of J_{E_1} are $\lambda_1 = \zeta - \varrho < 0$ and

$$\lambda_2 = \frac{\beta\gamma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha - \gamma\delta\sigma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha - \delta}{\gamma\sigma(1-\mu)^\alpha \left(\frac{\varrho-\zeta}{\xi}\right)^\alpha + 1}.$$

If $\theta_1 < \theta_2$, then $\lambda_2 < 0$, and hence E_1 is a stable node. If $\theta_1 > \theta_2$, then $\lambda_2 > 0$, and hence E_1 is an unstable saddle point. □

Example 2. For $\varrho = 0.58201$, $\kappa = 0.30764$, $\zeta = 0.11826$, $\xi = 0.25073$, $\gamma = 0.44606$, $\mu = 0.14733$, $\alpha = 0.26169$, $\sigma = 0.39219$, $\beta = 0.21466$, and $\delta = 0.31582$.

1. $\varrho > \zeta$, system (2) has a non-trivial boundary equilibrium $E_1(1.8496, 0)$;
2. $\theta_1 = 0.1079 < 0.3238 = \theta_2$ and E_1 is a stable node (see Figure 4a).

Example 3. For $\varrho = 0.48636$, $\kappa = 0.38314$, $\zeta = 0.12118$, $\xi = 0.18622$, $\gamma = 0.58601$, $\mu = 0.53635$, $\alpha = 0.39764$, $\sigma = 0.34461$, $\beta =, 0.59255$, and $\delta = 0.25618$.

1. $\varrho > \zeta$, system (3) has a non-trivial boundary equilibrium $E_1(1.9610, 0)$;
2. $\theta_1 = 0.3343 > 0.3060 = \theta_2$ and E_1 is an unstable saddle point (see Figure 4b).

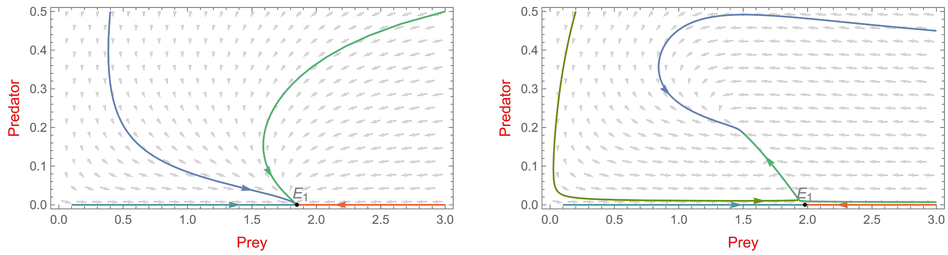


Figure 4. Phase plane analysis of system (5). (**a**) In Example 2, E_1 exists and it is a stable node when $\varrho > \zeta$ and $\theta_1 < \theta_2$. (**b**) In Example 3, E_1 exists and it is an unstable saddle point when $\varrho > \zeta$ and $\theta_1 > \theta_2$.

4.3. The Steady State of Coexistence

Theorem 6.

1. *If $\beta < \sigma\delta$, then system (5) has no positive equilibrium.*
2. *If $\varrho < \zeta + \xi z_1$, then system (5) has no positive equilibrium.*
3. *If $\beta > \sigma\delta$ and $\varrho > \zeta + \xi z_1$, then system (5) has a unique positive equilibrium $E_2(z_1, s_1)$. Moreover, s_1 is strictly decreasing with respect to κ*

where

$$\begin{aligned} z_1 &= z = \left(\frac{\delta}{\gamma(1-\mu)^\alpha(\beta-\sigma\delta)}\right)^{\frac{1}{\alpha}}, \\ s_1 &= \frac{-(\vartheta_1\kappa+\vartheta_2)+\sqrt{(\vartheta_1\kappa+\vartheta_2)^2+4\kappa\vartheta_2(\varrho z_1-\vartheta_1)}}{2\kappa\vartheta_2}, \\ \vartheta_1 &= \zeta z_1 + \xi z_1^2, \\ \vartheta_2 &= \frac{\delta}{\beta}. \end{aligned}$$

Proof of Theorem 6. From the s-zero-growth isocline, $\dfrac{\beta\gamma(z-\mu z)^\alpha}{1+\sigma\gamma(z-\mu z)^\alpha} - \delta = 0$, therefore

$$z = \left(\frac{\delta}{\gamma(1-\mu)^\alpha(\beta-\sigma\delta)}\right)^{\frac{1}{\alpha}}.$$

It is clear that if $\beta < \sigma\delta$, then $z \notin \mathbb{R}^{++}$, which means that there is no positive equilibrium. Now, suppose $\beta > \sigma\delta$, then $z_1 = \left(\dfrac{\delta}{\gamma(1-\mu)^\alpha(\beta-\sigma\delta)}\right)^{\frac{1}{\alpha}} > 0$. It is not difficult to show that $\dfrac{\gamma(1-\mu)^\alpha z_1^\alpha}{1+\sigma\gamma(1-\mu)^\alpha z_1^\alpha} = \dfrac{\delta}{\beta}$. From the z-zero-growth isocline, $\dfrac{\varrho z_1}{1+\kappa s} - \vartheta_1 - \vartheta_2 s = 0$, thus

$$\vartheta_2\kappa s^2 + (\vartheta_1\kappa + \vartheta_2)s + (\vartheta_1 - \varrho z_1) = 0. \tag{16}$$

It is clear that if $(\vartheta_1 - \varrho z_1) > 0$ $(\varrho < \zeta + \xi z_1)$, there is no positive root of (16). If $(\vartheta_1 - \varrho z_1) < 0$ $(\varrho > \zeta + \xi z_1)$, then there exists a unique positive root s_1 of (16), where

$$s_1 = \frac{-(\vartheta_1\kappa + \vartheta_2) + \sqrt{(\vartheta_1\kappa + \vartheta_2)^2 + 4\kappa\vartheta_2(\varrho z_1 - \vartheta_1)}}{2\kappa\vartheta_2}$$

Therefore, there is a unique positive equilibrium $E_2(z_1, s_1)$ (see Figure 5). □

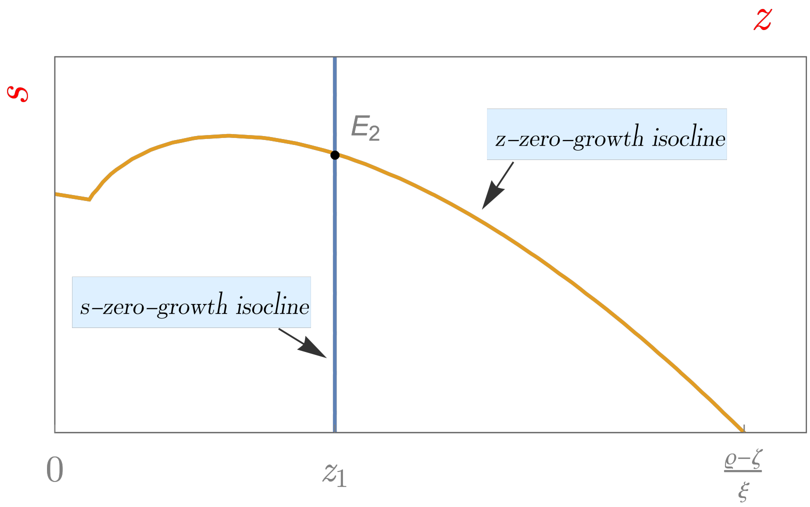


Figure 5. The intersection of z-zero-growth isocline and s-zero-growth isocline yields the equilibrium points. When $z_1 < \frac{\varrho-\zeta}{\xi}$, there is a unique positive equilibrium.

The Jacobian matrix of the system (5) around E_2 is given by

$$J_{E_2} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{bmatrix}, \tag{17}$$

where c_{11}, c_{12}, and c_{21} are given by

$$\begin{aligned} c_{11} &= -\zeta + \frac{\varrho}{\kappa s_1 + 1} + \frac{\alpha\gamma^2\sigma s_1(1-\mu)^{2\alpha} z_1^{2\alpha-1}}{(\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1)^2} - \frac{\alpha\gamma s_1(1-\mu)^\alpha z_1^{\alpha-1}}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1} - 2\xi z_1 \\ c_{12} &= -\frac{\kappa\varrho z_1}{(\kappa s_1+1)^2} - \frac{\gamma(1-\mu)^\alpha z_1^\alpha}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1} \\ c_{21} &= \frac{\alpha\beta\gamma s_1(1-\mu)^\alpha z_1^{\alpha-1}}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1} - \frac{\alpha\beta\gamma^2\sigma s_1(1-\mu)^{2\alpha} z_1^{2\alpha-1}}{(\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1)^2} \end{aligned} \tag{18}$$

and therefore

$$\begin{aligned} \operatorname{tr} J_{E_2} &= \frac{\varrho}{\kappa s_1 + 1} - \zeta - 2\xi z_1 + \frac{\alpha\gamma^2\sigma s_1(1-\mu)^{2\alpha} z_1^{2\alpha-1}}{(1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha)^2} - \frac{\alpha\gamma s_1(1-\mu)^\alpha z_1^{\alpha-1}}{1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha} \\ \det J_{E_2} &= \Big(\frac{\kappa\varrho z_1}{(\kappa s_1+1)^2} + \frac{\gamma(1-\mu)^\alpha z_1^\alpha}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1}\Big)\Big(\frac{\alpha\beta\gamma s_1(1-\mu)^\alpha z_1^{\alpha-1}}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1} - \frac{\alpha\beta\gamma^2\sigma s_1(1-\mu)^{2\alpha} z_1^{2\alpha-1}}{(\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1)^2}\Big) \end{aligned} \tag{19}$$

since $\frac{\gamma\sigma(1-\mu)^\alpha z_1^\alpha}{1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha} < 1$, thus $\det J_{E_2} > 0$.

5. The Effect of Fear

In this part, we will examine the effect of fear on the dynamics of system (5) by performing bifurcation analysis, taking the level of fear κ as a bifurcation parameter. Let us look at the tr J_{E_2} sign. Recalling that the s-zero-growth isocline is $\frac{\gamma(z-\mu z)^\alpha s}{1+\sigma\gamma(z-\mu z)^\alpha} = \frac{\delta}{\beta}s$, therefore, by substituting it into the z-zero-growth isocline, we obtain $s = \frac{\beta}{\delta}(\frac{\varrho z}{1+\kappa s} - \zeta z - \xi z^2)$. Hence,

$$\begin{aligned}
\operatorname{tr} J_{E_2} &= \frac{\varrho}{\kappa s_1+1} - \zeta - 2\xi z_1 + \frac{\alpha\gamma^2\sigma s_1(1-\mu)^{2\alpha} z_1^{2\alpha-1}}{(1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha)^2} - \frac{\alpha\gamma s_1(1-\mu)^\alpha z_1^{\alpha-1}}{1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha} \\
&= \frac{\varrho}{\kappa s_1+1} - \zeta - 2\xi z_1 + \Big[\frac{\alpha\gamma(1-\mu)^\alpha z_1^{\alpha-1}}{1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha}\Big]\Big[\frac{\sigma\gamma(1-\mu)^\alpha z_1^\alpha}{1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha} - 1\Big] s_1 \\
&= \frac{\varrho}{\kappa s_1+1} - \zeta - 2\xi z_1 + \alpha z_1^{-1}\Big[\frac{\gamma(1-\mu)^\alpha z_1^\alpha}{1+\gamma\sigma(1-\mu)^\alpha z_1^\alpha}\Big]\Big(\frac{\sigma\delta-\beta}{\beta}\Big)\frac{\beta}{\delta}\big(\frac{\varrho z_1}{1+\kappa s} - \zeta z_1 - \xi z_1^2\big) \\
&= \frac{\varrho}{\kappa s_1+1} - \zeta - 2\xi z_1 + \alpha z_1^{-1}\frac{\delta}{\beta}\Big(\frac{\sigma\delta-\beta}{\beta}\Big)\frac{\beta}{\delta}\big(\frac{\varrho z_1}{1+\kappa s} - \zeta z_1 - \xi z_1^2\big) \\
&= \frac{\varrho}{\kappa s_1+1} - \zeta - 2\xi z_1 - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)\big(\frac{\varrho}{1+\kappa s} - \zeta - \xi z_1\big) \\
&= \big(\frac{\varrho}{\kappa s_1+1}\big) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)\big(\frac{\varrho}{\kappa s_1+1}\big) - \Big[(\zeta+\xi z_1) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\zeta+\xi z_1)\Big] - \xi z_1.
\end{aligned}$$

Define $s^*(\kappa) = \kappa s_1 = \frac{-(\vartheta_1\kappa+\vartheta_2)+\sqrt{(\vartheta_1\kappa+\vartheta_2)^2+4\kappa\vartheta_2(\varrho z_1-\vartheta_1)}}{2\vartheta_2}$. It is clear that s^* increases with respect to κ, $\lim_{\kappa\to 0} s^*(\kappa) = 0$, and $\lim_{\kappa\to\infty} s^*(\kappa) =: s^*_\infty = \frac{\varrho}{\zeta+\xi z_1} - 1 > 0$. Define $\operatorname{tr}^* J_{E_2}:(0,s^*_\infty)\mapsto\mathbb{R}$ as follows $\operatorname{tr}^* J_{E_2}(s^*) = (\frac{\varrho}{s^*+1}) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\frac{\varrho}{s^*+1}) - \Big[(\zeta+\xi z_1) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\zeta+\xi z_1)\Big] - \xi z_1$. It is clear that $\operatorname{tr}^* J_{E_2}$ decreases with respect to s^*, $\lim_{s^*\to 0}\operatorname{tr}^* J_{E_2}(s^*) = \varrho - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)\varrho - \Big[(\zeta+\xi z_1) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\zeta+\xi z_1)\Big] - \xi z_1$, and $\lim_{s^*\to s^*_\infty}\operatorname{tr}^* J_{E_2}(s^*) = -\xi z_1 < 0$. If $\varrho - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)\varrho - \Big[(\zeta+\xi z_1) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\zeta+\xi z_1)\Big] - \xi z_1 < 0$, then $\operatorname{tr}^* J_{E_2}(s^*) < 0$ for any s^*, which means $\operatorname{tr} J_{E_2} < 0$ for any κ. If $\varrho - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)\varrho - \Big[(\zeta+\xi z_1) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\zeta+\xi z_1)\Big] - \xi z_1 > 0$, then there is a unique $s^*_H \in (0, s^*_\infty)$ such that $\operatorname{tr}^* J_{E_2}(s^*) > 0$ on $(0, s^*_H)$, and $\operatorname{tr}^* J_{E_2}(s^*) < 0$ on (s^*_H, s^*_∞), which means there is a unique $\kappa_H \in (0,\infty)$ such that $\operatorname{tr} J_{E_2}(\kappa) > 0$ on $(0,\kappa_H)$, and $\operatorname{tr} J_{E_2}(\kappa) < 0$ on (κ_H,∞), where κ_H satisfies the following equation:

$$\frac{-(\vartheta_1\kappa_H+\vartheta_2)+\sqrt{(\vartheta_1\kappa_H+\vartheta_2)^2+4\kappa_H\vartheta_2(\varrho z_1-\vartheta_1)}}{2\vartheta_2} = -\frac{A+\alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\varrho)-\varrho}{A} \tag{20}$$

where $A = (1-\alpha(\frac{\beta-\sigma\delta}{\beta}))(\zeta+\xi z_1) + \xi z_1$.

Theorem 7. *Suppose that $\beta > \sigma\delta$ and $\varrho > \zeta + \xi z_1$.*

1. *If $\varrho - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)\varrho - \Big[(\zeta+\xi z_1) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\zeta+\xi z_1)\Big] - \xi z_1 < 0$, the unique equilibrium E_2 is locally asymptotically stable.*
2. *If $\varrho - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)\varrho - \Big[(\zeta+\xi z_1) - \alpha\Big(\frac{\beta-\sigma\delta}{\beta}\Big)(\zeta+\xi z_1)\Big] - \xi z_1 > 0$, there exists a unique $\kappa_H \in (0,\infty)$ such that E_2 is unstable when $\kappa < \kappa_H$, and locally asymptotically stable when $\kappa > \kappa_H$, where κ_H are defined in Equation (20). In addition, system (5) undergoes a Hopf bifurcation at E_2 when $\kappa = \kappa_H$, where κ_H is defined in (20).*

Proof of Theorem 7. From Theorem 6, there is a unique positive equilibrium E_2 if $\beta > \sigma\delta$ and $\varrho > \zeta + \xi z_1$. Recall that $\det J_{E_2} > 0$. Hence, E_2 may be either a focus or node, and its stability is determined by the sign of $\text{tr}\, J_{E_2}$. If $\varrho - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)\varrho - \left[(\zeta + \xi z_1) - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)(\zeta + \xi z_1)\right] - \xi z_1 < 0$, then $\text{tr}\, J_{E_2} < 0$ for any κ, and E_2 is always locally asymptotically stable. If $\varrho - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)\varrho - \left[(\zeta + \xi z_1) - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)(\zeta + \xi z_1)\right] - \xi z_1 > 0$, then E_2 is unstable when $\kappa < \kappa_H$, and locally asymptotically stable when $\kappa > \kappa_H$. Furthermore, $\text{tr}\, J_{E_2} = 0$ when $\kappa = \kappa_H$, and the eigenvalues of J_{E_2} are $r = \pm i\sqrt{\det J_{E_2}}$. Let $r = o(\kappa) \pm i\lambda(\kappa)$ be the roots of $r^2 - \text{tr}\, J_{E_2} r + \det J_{E_2} = 0$ when κ near κ_H, then $o(\kappa) = \frac{\text{tr}\, J_{E_2}}{2}$. We have

$$o'(\kappa) = \frac{1}{2}(1 - \alpha(\frac{\beta - \sigma\delta}{\beta}))(\frac{-\varrho s^{*'}(\kappa)}{(s^*(\kappa)+1)^2}). \tag{21}$$

Since $s^{*'}(\kappa_H) > 0$, $o'(\kappa) \neq 0$, as a result, the transversality condition is satisfied and system (5) undergoes a Hopf bifurcation at E_2 when $\kappa = \kappa_H$. □

We must compute the normal form close to the Hopf bifurcation point using κ as the bifurcation parameter in order to ascertain the properties of the bifurcation. The following truncated normal form has been calculated by Du et al. in [43] using the steps in [51].

$$\begin{aligned} \rho' &= (\kappa - \kappa_H)o'(\kappa_H)\rho + a(\kappa_H)\rho^3 + O(|\kappa - \kappa_H|^2\rho, |\kappa - \kappa_H|\rho^3, \rho^5), \\ \varphi' &= \lambda(\kappa_H) + (\kappa - \kappa_H)\lambda'(\kappa_H) + b(\kappa_H)\rho^2 + O(|\kappa - \kappa_H|^2, |\kappa - \kappa_H|\rho^2, \rho^4). \end{aligned} \tag{22}$$

Recalling that $o'(\kappa) < 0$, the properties of Hopf bifurcation are determined by $a(\kappa_H)$, which can be computed by (25) in Section 6.

Theorem 8. *If $\beta > \sigma\delta$, $\varrho > \zeta + \xi z_1$, and $\varrho - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)\varrho - \left[(\zeta + \xi z_1) - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)(\zeta + \xi z_1)\right] - \xi z_1 > 0$, system (5) undergoes a Hopf bifurcation at E_2 when $\kappa = \kappa_H$.*

1. *If $a(\kappa_H) > 0$, the bifurcation periodic solution is unstable, and it is bifurcating from E_2 as κ increases and passes κ_H.*
2. *If $a(\kappa_H) < 0$, the bifurcation periodic solution is orbitally asymptotically stable, and it is bifurcating from E_2 as κ decreases and passes κ_H.*

6. Direction of Hpof Bifurcation with κ as Bifurcation Parameter

When $\kappa = \kappa_H$, we have $\text{tr}\, J_{E_2} = 0$, and $\pm i\lambda(\kappa_H) = \pm i\sqrt{\det J_{E_2}}$ are the eigenvalues of the Jacobian matrix at (z_1, s_H). Let $\hat{z} = z - z_1$ and $\hat{s} = s - s_H$, and system (5) becomes

$$\begin{aligned} \frac{d\hat{z}}{dt} &= b_{11}\hat{z} + b_{12}\hat{s} + X_1(\hat{z}, \hat{s}), \\ \frac{d\hat{s}}{dt} &= b_{21}\hat{z} + b_{22}\hat{s} + X_2(\hat{z}, \hat{s}). \end{aligned} \tag{23}$$

where

$$\begin{aligned} X_1(\hat{z}, \hat{s}) &= \frac{1}{2}u_{20}\hat{z}^2 + u_{11}\hat{z}\hat{s} + \frac{1}{2}u_{02}\hat{s}^2 + \frac{1}{6}u_{30}\hat{z}^3 + \frac{1}{2}u_{21}\hat{z}^2\hat{s} + \frac{1}{2}u_{12}\hat{z}\hat{s}^2 + \frac{1}{6}u_{03}\hat{s}^3, \\ X_2(\hat{z}, \hat{s}) &= \frac{1}{2}v_{20}\hat{z}^2 + v_{11}\hat{z}\hat{s} + \frac{1}{2}v_{02}\hat{s}^2 + \frac{1}{6}v_{30}\hat{z}^3 + \frac{1}{2}v_{21}\hat{z}^2\hat{s} + \frac{1}{2}v_{12}\hat{z}\hat{s}^2 + \frac{1}{6}v_{03}\hat{s}^3, \end{aligned}$$

and

$$\begin{aligned} b_{11} &= -\zeta + \frac{\alpha\gamma^2\sigma s_H(1-\mu)^{2\alpha} z_1^{2\alpha-1}}{(\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1)^2} - \frac{\alpha\gamma s_H(1-\mu)^\alpha z_1^{\alpha-1}}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1} - 2\xi z_1 + \frac{\varrho}{\kappa s_H + 1}, \\ b_{12} &= -\frac{\gamma(1-\mu)^\alpha z_1^\alpha}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha + 1} - \frac{\kappa\varrho z_1}{(\kappa s_H + 1)^2}, \end{aligned}$$

$$
\begin{aligned}
b_{21} =& \frac{\alpha\beta\gamma s_H(1-\mu)^\alpha z_1^{\alpha-1}}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^2},\\
b_{22} =& \frac{\beta\gamma(1-\mu)^\alpha z_1^\alpha}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1}-\delta,\\
u_{20} =& -2\xi-\frac{2\alpha^2\gamma^3\sigma^2 s_H(1-\mu)^{3\alpha}z_1^{3\alpha-2}}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^3}+\frac{\alpha^2\gamma^2\sigma s_H(1-\mu)^{2\alpha}z_1^{2\alpha-2}}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^2}-\frac{(\alpha-1)\alpha\gamma s_H(1-\mu)^\alpha z_1^{\alpha-2}}{\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1}\\
&+\frac{\alpha(2\alpha-1)\gamma^2\sigma s_H(1-\mu)^{2\alpha}z_1^{2\alpha-2}}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^2},\\
u_{11} =& -\frac{\alpha\gamma(1-\mu)^\alpha z_1^{\alpha-1}}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^2}-\frac{\kappa\varrho}{(\kappa s_H+1)^2},\\
u_{02} =& \frac{2\kappa^2\varrho z_1}{(\kappa s_H+1)^3},\\
u_{30} =& -\frac{\alpha\gamma s_H(1-\mu)^\alpha z_1^{\alpha-3}\left(3\alpha\left(\gamma^2\sigma^2(1-\mu)^{2\alpha}z_1^{2\alpha}-1\right)+2\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^2\right)}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^4}\\
&-\frac{\alpha\gamma s_H(1-\mu)^\alpha z_1^{\alpha-3}\left(\alpha^2\left(\gamma^2\sigma^2(1-\mu)^{2\alpha}z_1^{2\alpha}-4\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)\right)}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^4},\\
u_{21} =& \frac{\alpha\gamma(1-\mu)^\alpha z_1^{\alpha-2}\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+\alpha\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha-1\right)+1\right)}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^3},\\
u_{12} =& \frac{2\kappa^2\varrho}{(\kappa s_H+1)^3},\\
u_{03} =& -\frac{6\kappa^3\varrho z_1}{(\kappa s_H+1)^4},\\
v_{20} =& -\frac{\alpha\beta\gamma s_H(1-\mu)^\alpha z_1^{\alpha-2}\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+\alpha\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha-1\right)+1\right)}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^3},\\
v_{11} =& \frac{\alpha\beta\gamma(1-\mu)^\alpha z_1^{\alpha-1}}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^2},\\
v_{02} =& 0,\\
v_{30} =& \frac{\alpha\beta\gamma s_H(1-\mu)^\alpha z_1^{\alpha-3}\left(3\alpha\left(\gamma^2\sigma^2(1-\mu)^{2\alpha}z_1^{2\alpha}-1\right)+2\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^2\right)}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^4}\\
&+\frac{\alpha\beta\gamma s_H(1-\mu)^\alpha z_1^{\alpha-3}\left(\alpha^2\left(\gamma^2\sigma^2(1-\mu)^{2\alpha}z_1^{2\alpha}-4\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)\right)}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^4},\\
v_{21} =& -\frac{\alpha\beta\gamma(1-\mu)^\alpha z_1^{\alpha-2}\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+\alpha\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha-1\right)+1\right)}{\left(\gamma\sigma(1-\mu)^\alpha z_1^\alpha+1\right)^3},\\
v_{12} =& 0,\\
v_{03} =& 0.
\end{aligned}
$$

Now, let $z=\hat{z}$, $s=\frac{1}{\lambda(\kappa_H)}(b_{11}\hat{z}+b_{12}\hat{s})$; then, system (23) becomes

$$
\begin{aligned}
\frac{dz}{dt} &= -\lambda(\kappa_H)s+G_1(z,s),\\
\frac{ds}{dt} &= \lambda(\kappa_H)+G_2(z,s).
\end{aligned}
\tag{24}
$$

where

$$F(z,s) = X_1(z, -\frac{b_{11}z + \lambda(\kappa_H)s}{b_{12}}),$$
$$G(z,s) = -\frac{1}{\lambda(\kappa_H)}(b_{11}X_1(z, -\frac{b_{11}z + \lambda(\kappa_H)s}{b_{12}})) + b_{12}X_2(z, -\frac{b_{11}z + \lambda(\kappa_H)s}{b_{12}})).$$

From [51], $a(\kappa_H)$ in (22) can be obtained by

$$a(\kappa_H) = \frac{1}{16}[F_{zzz} + F_{zss} + G_{zzs} + G_{sss}] + \frac{1}{16\lambda(\kappa_H)}[F_{zs}(F_{zz} + F_{ss}) - G_{zs}(G_{zz} + G_{ss}) - F_{zz}G_{zz} + F_{ss}G_{ss}] \quad (25)$$

7. Examples and Simulations

Example 4. Choose $\varrho = 0.81016$, $\zeta = 0.50763$, $\xi = 0.47125$, $\gamma = 0.49284$, $\mu = 0.2$, $\alpha = 0.461621$, $\sigma = 0.30155$, $\beta = 0.51521$, and $\delta = 0.11491$. Then, $\beta > \sigma\delta$, $z_1 = 0.2609$, and, $\varrho > \zeta + \xi z_1$. When κ changes, there is a unique positive equilibrium $E_2(z_1, s_1)$. Since $\varrho - \alpha(\frac{\beta-\sigma\delta}{\beta})\varrho - [(\zeta + \xi z_1) - \alpha(\frac{\beta-\sigma\delta}{\beta})(\zeta + \xi z_1)] - \xi z_1 = -0.0207 < 0$, thus, from Theorem 7, E_2 is locally asymptotically stable (see Figure 6).

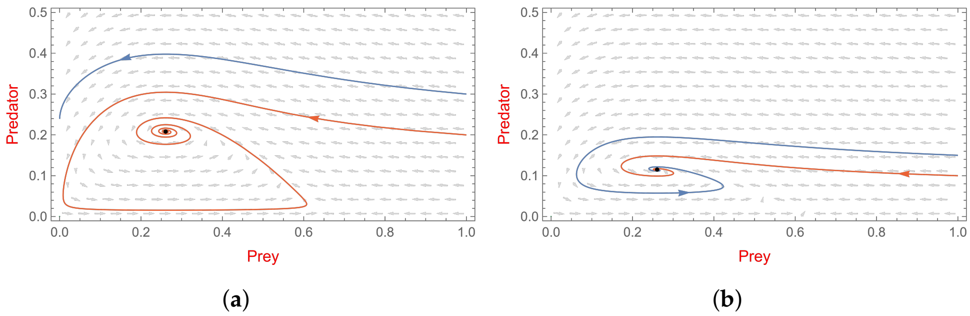


Figure 6. In Example 4, the stability of E_2 is unaffected by the variation in the value of k; E_2 is locally asymptotically stable. (**a**) $\kappa = 0.01$ (**b**) $\kappa = 0.99$.

Example 5. Choose $\varrho = 0.91016$, $\zeta = 0.50763$, $\xi = 0.47125$, $\gamma = 0.49284$, $\mu = 0.2$, $\alpha = 0.461621$, $\sigma = 0.30155$, $\beta = 0.51521$, and $\delta = 0.11491$. Then, $\beta > \sigma\delta$, $z_1 = 0.2609$, and, $\varrho > \zeta + \xi z_1$. When κ changes, there is a unique positive equilibrium $E_2(z_1, s_1)$. Since $\varrho - \alpha(\frac{\beta-\sigma\delta}{\beta})\varrho - [(\zeta + \xi z_1) - \alpha(\frac{\beta-\sigma\delta}{\beta})(\zeta + \xi z_1)] - \xi z_1 = 0.0362 > 0$, thus, from Theorem 7, there exists a unique $\kappa_H = 0.297765$ such that E_2 is unstable when $\kappa < \kappa_H$ (see Figure 7), and locally asymptotically stable when $\kappa > \kappa_H$ (see Figure 8). In addition, system (5) undergoes a Hopf bifurcation at E_2 when $\kappa = \kappa_H$ (see Figure 9a). Actually, by using the procedures in Section 6, we can determine $a(\kappa_H) = -0.134214 < 0$; therefore, from Theorem 8, the bifurcation periodic solution is orbitally asymptotically stable (see Figure 9b), and it is bifurcating from E_2 as κ decreases and passes κ_H. There is a unique limit cycle that appears when $\kappa < \kappa_H$ (see Figure 9c–e) and, through a loop of heteroclinic orbits, the limit cycle vanishes when κ decreases to $\kappa_{het} = 0.14714$ (see Figure 9f). The figures were drawn by Wolfram Mathematica [52].

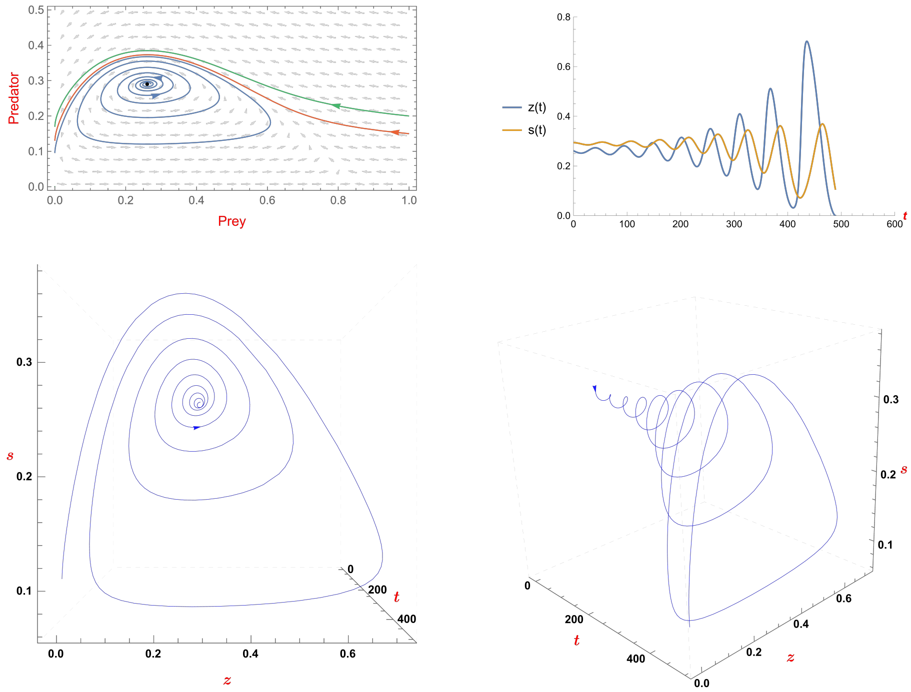


Figure 7. Choose $\varrho = 0.91016$, $\zeta = 0.50763$, $\xi = 0.47125$, $\gamma = 0.49284$, $\mu = 0.2$, $\alpha = 0.461621$, $\sigma = 0.30155$, $\beta = 0.51521$, and $\delta = 0.11491$. E_2 is unstable when $\kappa = 0.125 < \kappa_H$.

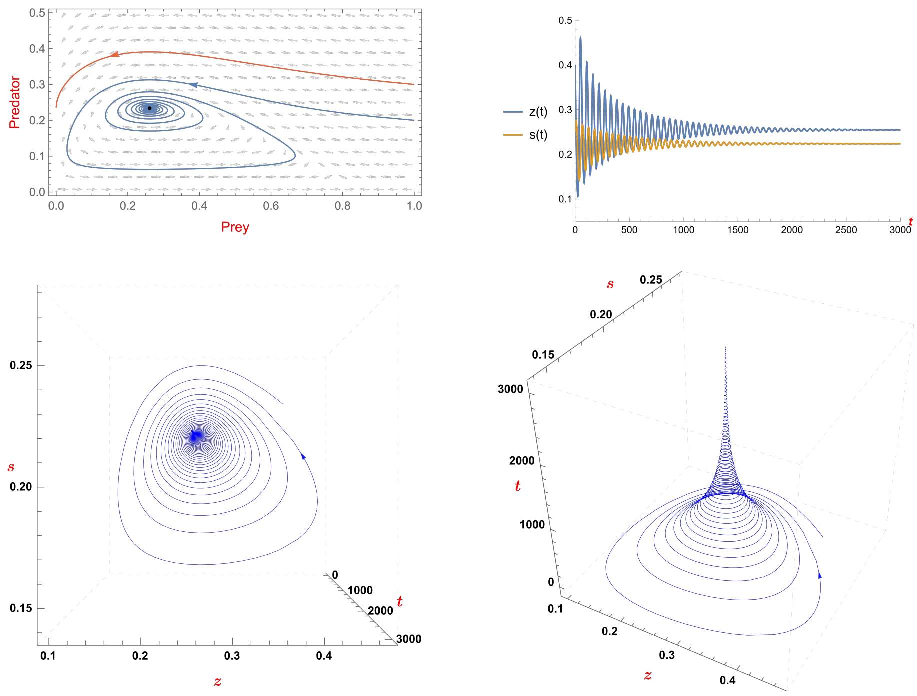


Figure 8. Choose $\varrho = 0.91016$, $\zeta = 0.50763$, $\xi = 0.47125$, $\gamma = 0.49284$, $\mu = 0.2$, $\alpha = 0.461621$, $\sigma = 0.30155$, $\beta = 0.51521$, and $\delta = 0.11491$. E_2 is locally asymptotically stable when $\kappa = 0.4226 > \kappa_H$.

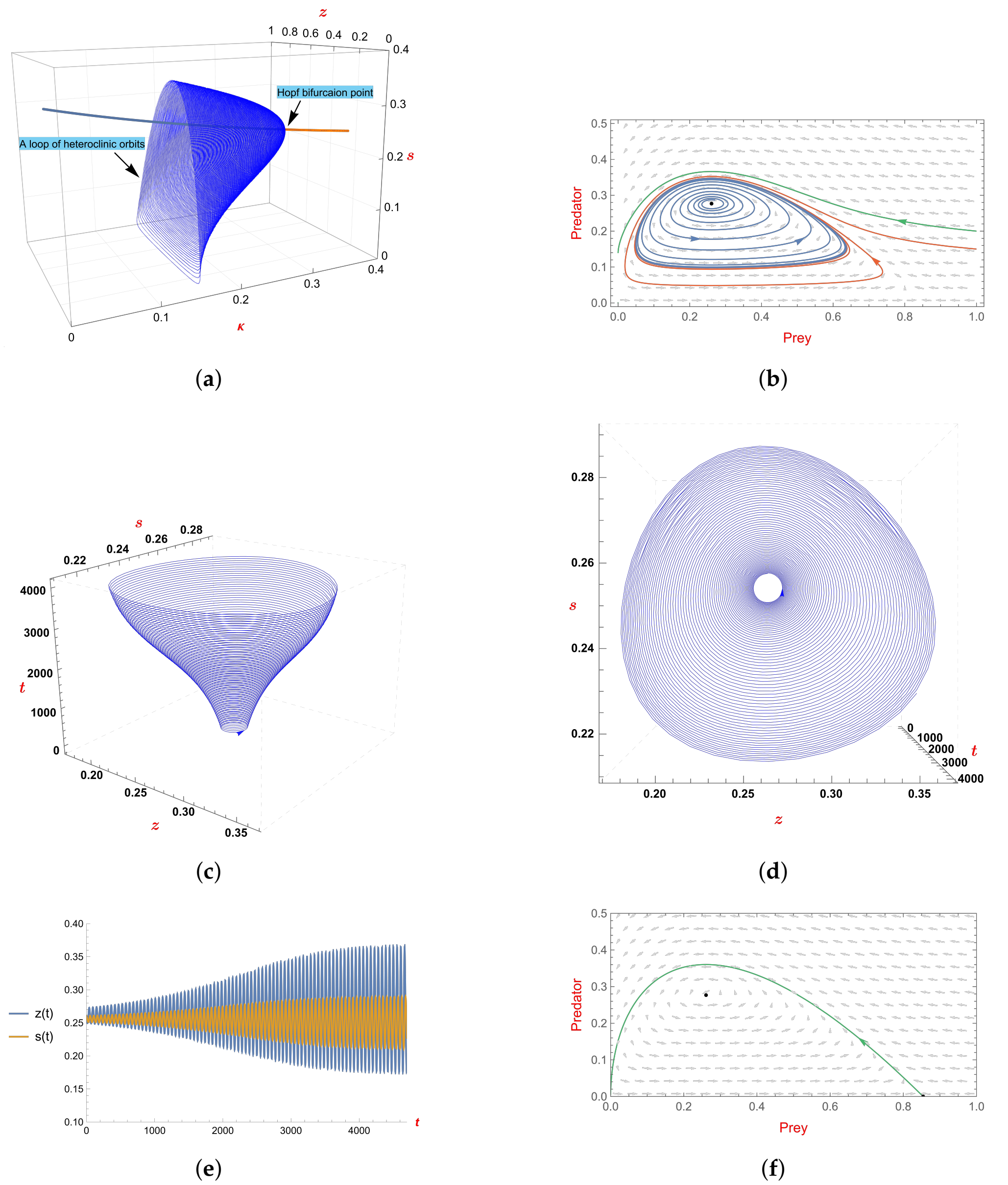


Figure 9. Choose $\varrho = 0.91016$, $\zeta = 0.50763$, $\xi = 0.47125$, $\gamma = 0.49284$, $\mu = 0.2$, $\alpha = 0.461621$, $\sigma = 0.30155$, $\beta = 0.51521$, and $\delta = 0.11491$. (**a**) Bifurcation diagram of system (5) in Example 5. (**b**) The bifurcation periodic solution is orbitally asymptotically stable. (**c**,**d**) There is a unique limit cycle that appears when $\kappa = 0.28 < \kappa_H$. (**e**) Dynamics of system 5 in Example 5 when $\kappa = 0.28$. (**f**) When $\kappa = \kappa_{het} = 0.14714$, there is a loop of heteroclinic orbits.

8. Discussion

A predator–prey system including group defense in the prey, the fear factor, and the refuge is proposed and investigated in this paper. The main goal of this study is to find the answer to the following question: how do group defense, the fear factor, and the refuge affect the qualitative dynamics of the model? According to the model presented in this paper, the functional response is classified as type IV, and it is nondifferentiable on the s-axis. The functional response ultimately increases (when $z \geq \frac{1}{1-\mu}$) as the efficiency of aggregation for prey increases, and it decreases as the refuge capacity increases. We found the following dynamic behaviors in system (5):

1. According to Theorem 2, when $\delta > \beta\gamma(1-\mu)^{\alpha}(\frac{\varrho}{\xi})^{\alpha}$, the predator population is non-persistent, i.e., the predator population falls into decay if the per capita death rate of the predator is greater than a constant $\theta = \beta\gamma(1-\mu)^{\alpha}(\frac{\varrho}{\xi})^{\alpha}$ that depends on several

parameters. Note that this θ decreases as the capacity of a refuge at t increases, and θ increases (decreases) as the value of the efficiency of aggregation for prey increases if $\frac{(1-\mu)\varrho}{\xi} > 1$ ($\frac{(1-\mu)\varrho}{\xi} < 1$).

2. In system (5), there is a maximum of three equilibria, including a positive one. The trivial steady state E_0 always exists, the predator-free steady state exists when $\varrho > \zeta$, and system (5) has a unique positive equilibrium when $\beta > \sigma\delta$ and $\varrho > \zeta + \xi(\frac{\delta}{\beta\gamma(1-\mu)^\alpha - \sigma\delta\gamma(1-\mu)^\alpha})^{\frac{1}{\alpha}}$.
3. Because of the z^α term, the Jacobian matrix is indeterminate at the origin. Therefore, it is impossible to carry out a stability analysis by simply looking at its eigenvalues. We used the definition of stability to prove that if $\varrho < \zeta$, then $E_0(0,0)$ is stable, and if $\varrho > \zeta$, then $E_0(0,0)$ is unstable.
4. If $\varrho - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)\varrho - \left[(\zeta+\xi z_1) - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)(\zeta+\xi z_1)\right] - \xi z_1 < 0$, the coexistence state of system (5) is stable and the alteration in the fear factor's value has no bearing on this stability.
5. If $\varrho - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)\varrho - \left[(\zeta+\xi z_1) - \alpha\left(\frac{\beta-\sigma\delta}{\beta}\right)(\zeta+\xi z_1)\right] - \xi z_1 > 0$, we examine the Hopf bifurcation at the unique positive equilibrium. When the fear factor's value decreases, the limit cycle appears when the fear factor's value is less than κ_H, and it disappears when the fear factor's value is equal to κ_{het} through a loop of heteroclinic orbits.

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