fractal and fractional

Brief Report

Unlimited Sampling Theorem Based on Fractional Fourier Transform

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Abstract: The recovery of bandlimited signals with high dynamic range is a hot issue in sampling research. The unlimited sampling theory expands the recordable range of traditional analog-to-digital converters (ADCs) arbitrarily, and the signal is folded back into a low dynamic range measurement, avoiding the saturation problem. Since the non-bandlimited signal in the Fourier domain cannot be directly applied to its existing theory, the non-bandlimited signal in the Fourier domain may be bandlimited in the fractional Fourier domain. Therefore, this brief report studies the unlimited sampling problem of high dynamic non-bandlimited signals in the Fourier domain based on the fractional Fourier transform. Firstly, a mathematical signal model for unlimited sampling is proposed. Secondly, based on this mathematical model, the annihilation filtering method is used to estimate the arbitrary folding time. Finally, a novel fractional Fourier domain unlimited sampling theorem is obtained. The theory proves that, based on the folding characteristics of the self-reset ADC, the number of samples is not affected by the modulo threshold, and any folding time can be handled.

Keywords: Fourier transform; fractional Fourier transform; unlimited sampling theorem; nonlinear modulus mapping

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1. Introduction

In signal processing, sampling [1–3] is the primary task faced in the process of digitizing the signal. Since Shannon's sampling theorem was proposed [4], sampling theory has been developed for more than 70 years, and its theoretical results [5–8] are so rich that it has become one of the research hotspots in the field of signal processing. From a practical standpoint, point-wise samples of the function are obtained using the analog-to-digital converter (ADC), but the ADC has a limited dynamic range $[-\lambda, \lambda]$ [9]. Whenever the signal exceeds a certain preset threshold λ, the ADC will saturate, and the aliases signal will be clipped due to clipping, we refer to these ADCs as clipped ADCs or traditional ADCs (C-ADC) [10,11]. Since most signals in practical applications are not limited by broadband, the dynamic range is very wide. Therefore, recovering the signal from the tailored version of the C-ADC is an inaccurate inverse problem. Generally, recovery methods almost alleviate the effects of clipping at the expense of oversampling. To solve these problems, so self-reset ADC (S-ADC) was proposed [12–14]. Each time the input signal reaches the upper (lower) saturation limit, these S-ADCs will be reset to the other corresponding thresholds, which allows the S-ADC to reset rather than saturate, resulting in analog sampling. When the signal reaches the upper (lower) threshold point, it will fold backward (front) by an integer multiple of 2λ. This phenomenon is equivalent to modulo arithmetic on the input signal, which is very helpful for processing high dynamic range signals.

Because of the S-ADC's ability to process high dynamic range signals, Bandari et al. recently made the first pioneering contribution [15]. He proposed the unlimited sampling theorem and developed the first provable reflector of the guaranteed algorithm. Similar to Shannon's sampling theorem, the unlimited sampling theorem proves that the bandlimited signal can be recovered from analog sampling as long as it meets a certain sampling density criterion and is not affected by the ADC threshold. In this way, the results allow the perfect recovery of the bandlimited function, whose amplitude exceeds the ADC threshold by orders of magnitude. The results [15] have led to a lot of follow-up work, and the theoretical research of unlimited sampling has gradually enriched [16–20]. In this modular sampling framework, many scholars have studied the sampling and reconstruction of bandlimited functions and smooth functions under different backgrounds. The paper is shown that the bandlimited function is uniquely characterized by modular samples under certain conditions [21]. Ordentlich et al. studied the recovery of quantization modulus samples by using edge information [22]. Musa et al. gave the modulus sampling theory of the S-ADC sparse signal [23]. The unlimited sampling method based on wavelet is suitable for general smooth signals, not limited to bandlimited signals [24]. The paper is mainly applicable to bandlimited signals in the Fourier domain on unlimited sampling method [25]. Most of the unlimited sampling frameworks are based on the bandlimited signals, but there are few articles on non-bandlimited signals. For various applications of the non-bandlimited signal models, the original results are not directly applicable. Therefore, it is very necessary to study the unlimited sampling theory of non-bandlimited signals in the Fourier domain.

As a general form of the Fourier transform (FT), the fractional Fourier transform (FRFT) extends dimensionality of traditional FT-based spectral analysis [6,26,27]. The FRFT has an additional degree of freedom compared to the FT, which makes it more flexible and suitable for non-stationary signals. By transforming α from 0 to $\frac{\pi}{2}$, the FRFT will be able to fully characterize the signal at the transition from time to frequency, overcoming the limitation of the FT only to perform frequency analysis. Furthermore, unlike conventional FT, which uses a complex exponential signal as the basis function, the FRFT utilizes a chirp signal as the basis function, which implies that a signal that is not bandlimited in the Fourier domain can be bandlimited in the fractional Fourier domain. Therefore, it is very meaningful to study the sampling theorem under the unlimited sampling framework based on the FRFT.

The FRFT can expand the signal range applicable to traditional sampling theory. Since the non-bandlimited signal in the Fourier domain cannot be directly applied to its existing theory, the non-bandlimited signal in the Fourier domain may be bandlimited in the fractional Fourier domain. Therefore, we propose a method based on the fractional Fourier domain bandlimited signal reconstruction theory under the framework of unlimited sampling to make up for the shortcomings of the Fourier domain reconstruction theory. To enrich the content of the unlimited sampling framework, we extend the theory of the unlimited sampling framework in the Fourier domain to the fractional Fourier domain. Our main work is to perform a modular operation on folds introduced by modular nonlinearity in the fractional Fourier domain and to estimate the fractional spectrum of instantaneous folding time to obtain a new sampling theorem in the fractional Fourier domain.

The main contributions of this brief report are summarized as follows. Firstly, the folding formula of non-linear and modulus mapping in the fractional Fourier domain is introduced, and a mathematical signal model of unlimited sampling for the FRFT is proposed. Secondly, based on this mathematical model, the fractional spectrum of the arbitrary folding time is estimated using the annihilation filtering method. Finally, a novel fractional Fourier domain unlimited sampling theorem is obtained. It is shown that the sampling theorem is independent of the modulo threshold and that it can be applied to arbitrary folding times.

This report is organized as follows. In Section 2, we briefly describe the FRFT and unlimited sampling theorem in the Fourier domain. In Section 3, a mathematical signal model for unlimited sampling is proposed in the fractional Fourier domain. In Section 4, a novel unlimited sampling theorem in the fractional Fourier domain is proposed. In Section 5, relevant applications of the proposed method are provided. We conclude this report in Section 6.

2. Preliminaries

2.1. Fractional Fourier Transform

Definition 1. *The FRFT of a signal $x(t) \in L^2(\mathbb{R})$ with an angle α is defined as [27–30]*

$$X_\alpha(u) = \mathcal{F}_\alpha[x(t)](u) \triangleq \int_{-\infty}^{+\infty} x(t)K_\alpha(u,t)dt, \tag{1}$$

where $\mathcal{F}_\alpha$ is the FRFT operator, u stands for fractional frequency, $K_\alpha(u,t)$ denotes the kernel function of the FRFT

$$K_\alpha(u,t) = \begin{cases} A_\alpha e^{i(\frac{\cot\alpha}{2}t^2 - \csc\alpha ut + \frac{\cot\alpha}{2}u^2)}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k-1)\pi \end{cases} \tag{2}$$

where $A_\alpha \triangleq \sqrt{\frac{1-i\cot\alpha}{2\pi}}$, the rotation angle of the FRFT is expressed as $\alpha = \frac{p\pi}{2}$, and p is the order of the FRFT. The domain $0 < \alpha < \frac{\pi}{2}$ are called fractional Fourier domains in [28], and this definition is also adopted in this report.

The FRFT can be understood as the rotation of the time-frequency plane. The essence of the FRFT of a signal is to decompose the signal with the chirp signal $K_\alpha(u,t)$ as the basis function. According to the FRFT of the signal $x(t)$, it can be determined whether it is bandlimited in the fractional Fourier domain.

The FRFT has linear transform additivity, namely

$$\mathcal{F}_{\alpha+\beta}[x(t)](u) = \mathcal{F}_\alpha[x(t)](u) \cdot \mathcal{F}_\beta[x(t)](u) = X_\alpha(u) \cdot X_\beta(u). \tag{3}$$

It can be seen that the inverse transform of the FRFT relative to the α angle is the FRFT with the parameter $-\alpha$ angle, we have

$$x(t) = \mathcal{F}_{-\alpha}\{X_\alpha(u)\} = \int_{-\infty}^{+\infty} X_\alpha(u)K_{-\alpha}(u,t)du, \tag{4}$$

when $\alpha = -\frac{\pi}{2}$, the FRFT degenerates to the traditional inverse FT; when $\alpha = \frac{\pi}{2}$, the FRFT degenerates to traditional FT, $X_{\frac{\pi}{2}}(u) = \int_{-\infty}^{+\infty} x(t)e^{-i2\pi ut}dt$; when $\alpha = 0$, the FRFT degenerates to an identity transformation, $X_0(u) = x(t)$; when $\alpha = \pi$, the FRFT degenerates to the inversion of the signal with respect to the time axis, $X_\pi(u) = x(-t)$.

Definition 2. *A signal $x(t)$ is called Ω_α bandlimited signal in the fractional Fourier domain, which means*

$$X_\alpha(u) = 0, \qquad |\, u \,| > \Omega_\alpha, \tag{5}$$

where Ω_α is called the bandwidth of signal $x(t)$ in the fractional Fourier domain. It has been shown that if a nonzero signal is bandlimited in the αth fractional Fourier domain, it can't be bandlimited in the fractional Fourier domain with another angle β, where $\beta \neq \pm\alpha + n\pi$ for any integer n [26].

2.2. Unlimited Sampling Theorem in the Fourier domain

Definition 3. *The central modulo operation is defined by the mapping [15]*

$$\mathscr{M}_\lambda : g \longmapsto 2\lambda\left(\left[\left[\frac{g}{2\lambda} + \frac{1}{2}\right]\right] - \frac{1}{2}\right), \qquad [[g]] \overset{def}{=} g - \lfloor g \rfloor, \tag{6}$$

where $[[g]]$ defines the fractional part of input signal g and $\lambda > 0$ is the ADC threshold. Note that Equation (6) is a nonlinear modulus mapping, which converts a smooth function into a *discontinuous function. It is equivalent to a centered modulo operation since* $\mathscr{M}_\lambda(g) \equiv g \mod 2\lambda$. *By implementing the mapping Equation (6), it is clear that out-of-range amplitudes are folded back into the dynamic range* $[-\lambda, \lambda]$.

Let's review some important conclusions in [16,25].

Lemma 1 ((Modular decomposition property) [16])**.** *Let* $g \in \mathcal{B}_\Omega$*, where* $\mathcal{B}_\Omega$ *denotes the space of* σ*-bandlimited functions, and* $\mathscr{M}_\lambda(\cdot)$ *be defined in Equation (6) with* λ *is a fixed, positive constant. Then, the bandlimited function* $g(t)$ *admits a decomposition*

$$g(t) = z(t) + \varepsilon_g(t), \tag{7}$$

$$\varepsilon_g(t) = 2\lambda \sum\nolimits_{m \in \mathbb{Z}} e[m] 1_{\mathcal{D}_m}(t), \quad e[m] \in \mathbb{Z}, \tag{8}$$

where $z(t) = \mathscr{M}_\lambda(g(t))$, $\varepsilon_g(t)$ *is a simple function, and* $\cup_{m \in \mathbb{Z}} \mathcal{D}_m = \mathbb{R}$ *is a partition of the real line into intervals* $\mathcal{D}_m$.

The process of solving discontinuities is very critical. Lemma 1 just proves this problem. Each bandlimited function, whether continuous or discrete, can be decomposed into the sum of the modular function and the stepwise residual of the simple function. Observe that the output function $z(t)$ is the difference between $g(t)$ and a piecewise constant function $\varepsilon_g(t)$.

Theorem 1 ((Unlimited sampling theorem in the Fourier domain) [25])**.** *Let* $g \in \mathcal{B}_\Omega$ *be a* τ*-periodic function. Suppose that we are given Q modulo samples of where* $y[k] = \mathscr{M}_\lambda(g(kT))$ *folded at most M times. Then a sufficient condition for recovery of* $g(t)$ *from* $y[k]$ *(up to a constant) is that,* $T \leq \dfrac{\tau}{Q}$ *and* $Q \geq 2\left(\dfrac{\Omega\sigma}{2\pi} + M + 1\right)$.

Unlike the traditional FT, which uses a complex exponential signal as the basis function, the FRFT uses a chirp signal as the basis function. This connotation determines that a non-bandlimited signal in the Fourier domain may be bandlimited in the fractional Fourier domain. Therefore, the unlimited sampling study of non-bandlimited signals in the Fourier domain can be transformed into the theoretical study of bandlimited signals in the fractional Fourier domain. The next step is to study the unlimited sampling theory of bandlimited signals in the fractional Fourier domain.

3. Mathematical Model for Unlimited Sampling with FRFT

In this Section, we will study the unlimited sampling theory of bandlimited signals in the fractional Fourier domain. Here we make the following symbolic regulations: the sets of real, integer, and complex-valued numbers are denoted by $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{C}$, respectively.

3.1. Mathematical Signal Model

Based on the periodic signal model in the Fourier domain, this report proposes a periodic signal model in the fractional Fourier domain, so mathematically $x(t)$ can be represented as follows.

Let $x(t)$ be a Ω_α bandlimited function satisfies $x(t) = x(t + \sigma)$, $\forall t \in \mathbb{R}$. Then $x(t)$ has a fractional Fourier series (FRFS) expansion

$$x(t) = \sum_{|w| \leq R} \widehat{X}_\alpha(w) \Phi_{-\alpha}(w, t), \tag{9}$$

where $\widehat{X}_\alpha(w)$ is FRFS coefficient, and

$$\Phi_\alpha(w,t) = \sqrt{\frac{\sin\alpha - i\cos\alpha}{\sigma}} e^{i(\frac{\cot\alpha}{2}t^2 - \csc(\alpha)wu_0t + \frac{\cot\alpha}{2}w^2u_0^2)}, \tag{10}$$

where $R = \left\lceil \frac{\Omega_\alpha}{u_0} \right\rceil$, $u_0 = \frac{2\pi\sin\alpha}{\sigma}$.

The discrete-time representation $x(nT_s), n \in \mathbb{Z}$ of the signal $x(t)$ can be obtained by uniformly sampling at intervals of T_s. The discrete-time FRFT of the α angle of the discrete-time signal $x(t)$ is defined as follows

$$X_{\alpha,s} = \mathcal{F}_\alpha[x(nT_s)](u) \triangleq \sum_{n=-\infty}^{+\infty} x(nT_s)K_\alpha(u, nT_s), \tag{11}$$

where K_α is given by Equation (2), and T_s is the sampling period.

Let sampling function $x(t)$ obtains Q modular samples at the sampling rate T in the interval $t \in \left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right]$, then FRFS coefficients of $x(t)$ in the fractional Fourier domain have a form

$$\widehat{X}_\alpha(w) = \begin{cases} \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} x(t)\Phi_\alpha(w,t)dt, & w \in \mathbb{E}_{R,Q}, \\ 0, & w \in \mathbb{I}_Q \backslash \mathbb{E}_{R,Q}, |w| > \Omega_\alpha, \end{cases} \tag{12}$$

where $\mathbb{I}_Q = \{0, 1, \cdots, Q-1\}$ denote the set of Q contiguous integers, and $\mathbb{E}_{R,Q}$ is given by

$$\mathbb{E}_{R,Q} = [0, R] \cup [Q-R, Q-1], \quad |\mathbb{E}_{R,Q}| = 2R+1. \tag{13}$$

Remark 1. *The well-known Fourier series (FS) is just a special case of the FRFS for* $\alpha = \frac{\pi}{2}$*, please see [25]. In order to solve for* $\widehat{X}_\alpha(w)$ *in (12), we must require* $Q \geq 2R+1$*. Because of* $QT = \sigma$*, so* $T \leq \frac{\sigma}{Q} \leq \frac{\sigma}{2R+1}$.

The hypothesis of periodic functions in our report only provides a practical method for recovering signals from the folding measurements below. However, when the signal is aperiodic, the theoretical reconstruction guarantees that the aperiodic signal can also be expanded by the discrete-time FRFT, but additional requirements are required for sampling samples, and this report will not expand in detail.

3.2. Nonlinear Modulus Mapping

This report uses the definition and properties of generalized modular non-linear mapping in Equation (6); this phenomenon is equivalent to modulo arithmetic on the input function. According to Lemma 1, it gives the following form

$$v_x(t) = x(t) - \mathscr{M}_\lambda(x(t)) =: \textstyle\sum_{m\in\mathbb{Z}} c[m]\mathbf{1}_{[t_m, t_{m+1}]}(t), \tag{14}$$

where $c[m] \in \mathbb{R}$, $\mathbf{1}_{[t_a,t_b]}$ is the indicator function on $[t_a, t_b]$, and $t_m \in \left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right]$ denotes the folding instants with $t_a < t_b$. Obviously, the output function $\mathscr{M}_\lambda(x(t))$ is the difference between $x(t)$ and a residual function $v_x(t)$. The sampling process to obtain modulo samples of a function is outlined in Figure 1. The [16] requires that the correlation coefficient of the residual function $v_x(t)$ is an integer multiple of 2λ, while [25] does not need to make assumptions about its correlation coefficient.

Without loss of the generality, we make the following symbolic regulations:

1. Let $f[k] \overset{\text{def}}{=} x(kT)$, $h[k] \overset{\text{def}}{=} \mathscr{M}_\lambda(x(kT))$, $v[k] \overset{\text{def}}{=} v_x(kT)$, then

$$f[k] = h[k] + v[k]. \tag{15}$$

2. Let $\Delta^N f = \Delta^{N-1}(\Delta f)$ denote the Nth difference operator with $\Delta f = f[k+1] - f[k]$, $\overline{f}[k] \overset{\text{def}}{=} \Delta f[k]$, $\overline{h}[k] \overset{\text{def}}{=} \Delta h[k]$, and $\overline{v}[k] \overset{\text{def}}{=} \Delta v_x(kT)$, then

$$\overline{f}[k] = \overline{h}[k] + \overline{v}[k] = \overline{h}[k] + \sum\nolimits_{m\in M} c[m]\delta(kT - t_m), \quad k \in \mathbb{I}_Q, \tag{16}$$

where δ denotes the Dirac distribution, $c[m]$ are unknown weights, t_m are unknown fold instant, and the size of the set M depends on the dynamic range of the signal relative to the threshold λ.

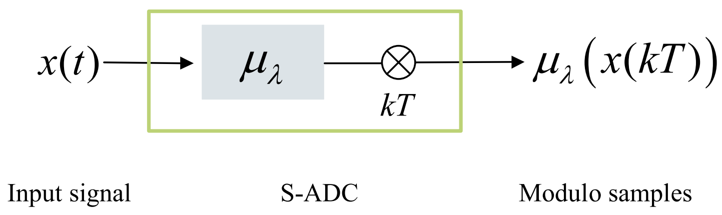


Figure 1. Unlimited sampling architecture for obtaining modulo samples.

Similar to the phase unwrapping theory, we can obtain the following fact from Itoh's condition [31]. When the max-norm of the first-order finite difference of the samples is bounded by 2λ or $|f[k+1] - f[k]| \leq 2\lambda$, the first-order finite difference operator on the modular sequence can be reversed operation to restore.

Equation (16) is written as the fractional Fourier domain

$$\overline{H}_\alpha[n] = \begin{cases} \overline{F}_\alpha[n] - \overline{V}_\alpha[n], & n \in \mathbb{E}_{R,Q-1} \\ -\overline{V}_\alpha[n], & n \in \mathbb{I}_{Q-1} \backslash \mathbb{E}_{R,Q-1} \end{cases} \tag{17}$$

where $\overline{F}_\alpha$, $\overline{H}_\alpha$, and $\overline{V}_\alpha$ are the FRFT of $\overline{f}$, $\overline{h}$, and $\overline{v}$, respectively. At the same time, the discrete FRFT form of $\overline{h}[k]$ is given

$$\begin{aligned} \overline{H}_\alpha[n] &= \sum\nolimits_{k\in\mathbb{I}_{Q-1}} \overline{h}[k] K_\alpha(n,k) \\ &= \sum\nolimits_{k\in\mathbb{I}_{Q-1}} A_\alpha \overline{h}[k]\, e^{i\left(\frac{\cot\alpha}{2}k^2 - \csc(\alpha)\overline{u}_0 kn + \frac{\cot\alpha}{2}\overline{u}_0^2 n^2\right)}, \end{aligned} \tag{18}$$

where $A_\alpha \triangleq \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and $\overline{u}_0 = \dfrac{2\pi\sin\alpha}{Q-1}$. When $\alpha = \dfrac{\pi}{2}$, this transform is the discrete FT, see [25] for details.

If we want to recover $f[k]$, we must solve $v[k]$, then Equation (15) is transformed into solving Equation (16), and the key to solving Equation (17) is to find the value of the unknown folding instant $\{c[m], t_m\}_{m\in\mathbb{Z}}$. Using Equations (10) and (16), we can obtain

$$\begin{aligned} \overline{V}_\alpha[n] &= \sum\nolimits_{k\in\mathbb{I}_{Q-1}} \sum\nolimits_{m\in M} c[m]\delta(kT - t_m)\Phi_\alpha(n,kT) \\ &= \sum\nolimits_{m\in M} c[m] A_\alpha e^{i\left(\frac{\cot\alpha}{2T^2}t_m^2 - \frac{\csc(\alpha)\overline{u}_0 n}{T}t_m + \frac{\cot\alpha}{2}\overline{u}_0^2 n^2\right)}, \end{aligned} \tag{19}$$

where $A_\alpha \triangleq \sqrt{\frac{1-i\cot\alpha}{2\pi}}$, and M denotes the size of the set depends on the dynamic range of the signal relative to the threshold λ. When $\alpha = \dfrac{\pi}{2}$, this transform is the discrete FT [25]. The estimation of the unknown parameters in Equation (19) is called the spectral estimation problem [32,33].

4. Unlimited Sampling Theorem in the Fractional Fourier Domain

4.1. Computing the Folding Instants

If we want to recover $v[k]$, we must find the value of the unknown folding instant $\{c[m], t_m\}_{m\in\mathbb{Z}}$. Equation (19) is the spectral estimation problem. The commonly used spectrum estimation methods are annihilation filter (AF) [32,34], ESPRIT [35], MUSIC [36], etc. Among them, the AF is the most commonly used method in many theoretical analyses and practical applications. In principle, the signal reconstruction process is to use the obtained set of moments or fractional Fourier coefficients of the input signal to solve a spectrum problem to achieve an accurate estimation of the unknown parameter $\{c[m], t_m\}_{m\in\mathbb{Z}}$. For convenience, Equation (19) is written as follows

$$\overline{V}_\alpha[n] = A_\alpha \underbrace{e^{i\frac{\cot\alpha}{2}\overline{u}_0^2 n^2}}_{\kappa(n)} \left(\underbrace{\sum\nolimits_{m\in M} \underbrace{c[m]e^{i\frac{\cot\alpha}{2T^2}t_m^2}}_{\chi_m} \cdot \underbrace{e^{-i\csc(\alpha)\frac{\overline{u}_0}{T}nt_m}}_{\varsigma_m^n}}_{\Im(n)} \right), \tag{20}$$

where $\Im(n) = \sum_{m\in M} \chi_m \varsigma_m^n$. Since the formula is very complicated, we rewrite the formula $\overline{V}_\alpha[n] = A_\alpha \kappa(n)\Im(n)$. Because the part of $\kappa(n)$ does not contain unknown parameters, we separately perform an annihilation filter to get $\{c[m], t_m\}$, and finally bring in Equation (20), and get $\overline{v}$ through inverse FRFT.

First, let's analyze $\Im(n)$ in detail below,

$$\Im(n) = \sum\nolimits_{m\in M} \chi_m \varsigma_m^n. \tag{21}$$

Equation (21) is a classic spectrum estimation problem, which can be handled by an annihilation filter. It is known from the [32] that we can accurately estimate the unknown parameters χ_m and ς_m from $2K$ continuous non-zero measured values $\Im(n)$. The following is divided into two parts to solve separately.

(1) Construct the filter $\{\Gamma[\vartheta]\}_{\vartheta=0,1,\cdots,M}$ so that its zero point is the parameter

$$\varsigma_m = \left\{ e^{-i\frac{\csc(\alpha)\overline{u}_0}{T}t_\vartheta} \right\}_{\vartheta=0}^{\vartheta=M-1},$$

then the z transform of the filter can be expressed as

$$\Gamma[z] = \prod_{m=0}^{M-1}(1-\varsigma_m z^{-1}) = \sum_{\vartheta=0}^{M} \Gamma[\vartheta]z^{-\vartheta}. \tag{22}$$

It can be seen that the root of the polynomial is the parameter ς_m. Therefore, this report convolutes it directly, so it has

$$\begin{aligned}
(\Gamma * \Im)[n] &= \sum_{\vartheta=0}^{M} \Gamma[\vartheta]\Im[n-\vartheta] \\
&= \sum_{\vartheta=0}^{M}\sum_{m=0}^{M-1} c[m]e^{i\frac{\cot\alpha}{2T^2}t_m^2} \cdot \Gamma[\vartheta] \cdot e^{-i\frac{\csc(\alpha)\cdot(n-\vartheta)\overline{u}_0}{T}t_m} \\
&= \sum_{m=0}^{M-1} c[m]e^{i\frac{\cot\alpha}{2T^2}t_m^2} \underbrace{\sum_{\vartheta=0}^{M} \Gamma[\vartheta]e^{i\frac{\csc(\alpha)\cdot\vartheta\overline{u}_0}{T}t_m}}_{\Gamma[\varsigma_m]} e^{-i\frac{\csc(\alpha)\cdot n\overline{u}_0}{T}t_m} = 0.
\end{aligned} \tag{23}$$

We write Equation (23) in the form of matrix-vector to obtain

$$\begin{bmatrix} \Im[M-1] & \Im[M-2] & \cdots & \Im[0] \\ \Im[M] & \Im[M-1] & \cdots & \Im[1] \\ \vdots & \vdots & \vdots & \vdots \\ \Im[N-1] & \Im[N-2] & \cdots & \Im[N-M] \end{bmatrix} \begin{bmatrix} \Gamma[1] \\ \Gamma[2] \\ \vdots \\ \Gamma[M] \end{bmatrix} = -\begin{bmatrix} \Im[M] \\ \Im[M+1] \\ \vdots \\ \Im[N] \end{bmatrix}, \tag{24}$$

where $\Im = \left[\Im[0], \Im[1], \cdots, \Im[M] \right]^T$, $\Im[M] = 1$. The unique solution can be obtained $\Gamma[\vartheta], \vartheta = 1, 2, \cdots, M$, and the instantaneous folding time $\{t_m\}_{m\in\mathbb{Z}}$ can be obtained.

(2) To estimate the amplitude parameter χ_m, extract M continuous values from the known coefficient $\Im[n]$, that is to say, $m = 0, 1, \cdots, M$, and write the $\Im(n) = \sum_{m=0}^{M-1} \chi_m \varsigma_m^n$ in the form of a matrix-vector

$$U\chi = \Im, \tag{25}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \varsigma_0 & \varsigma_1 & \cdots & \varsigma_{M-1} \\ \vdots & \vdots & \vdots & \vdots \\ \varsigma_0^{M-1} & \varsigma_1^{M-1} & \cdots & \varsigma_{M-1}^{M-1} \end{bmatrix} \begin{bmatrix} e^{i\frac{\cot\alpha}{2T^2}t_0^2} & & & \\ & e^{i\frac{\cot\alpha}{2T^2}t_1^2} & & \\ & & \ddots & \\ & & & e^{i\frac{\cot\alpha}{2T^2}t_{M-1}^2} \end{bmatrix} \cdot \begin{bmatrix} c[0] \\ c[1] \\ \vdots \\ c[M-1] \end{bmatrix} = \begin{bmatrix} \Im[0] \\ \Im[1] \\ \vdots \\ \Im[M-1] \end{bmatrix}, \tag{26}$$

where U is a vandermonde matrix, it is a matrix whose columns are geometric series. For any integer $a, b = 0, 1, \cdots, M-1$ ($a \neq b$) satisfy $U_a \neq U_b$, and U is non-singular, at this time, Equation (25) has a unique solution. It needs to be emphasized here that we generally use the least squares method to obtain an estimate of the amplitude information.

Through the above two steps, the instantaneous folding time $\{c[m], t_m\}$ can be obtained. we bring $\{c[m], t_m\}$ into Equation (20), and get $\overline{v}$ through the inverse FRFT.

4.2. Unlimited Sampling Theorem in the Fractional Fourier Domain

Through the above research content, we have found the value of the unknown folding instant $\{c[m], t_m\}_{m\in\mathbb{Z}}$. If $v[k]$ is known, we can infer $v[k]$ from $h[k]$ and recover $f[k]$ from $h[k]$. The unlimited sampling theorem in the fractional Fourier domain is given below.

Theorem 2 (Unlimited sampling theorem in the fractional Fourier domain)**.** *Let $x(t)$ be a Ω_α bandlimited function satisfies $x(t) = x(t+\sigma)$, $\forall t \in \mathbb{R}$, and $h[k] = \mathscr{M}_\lambda(x(kT))$ folded at most M times. Then a sufficient condition for recovery of $x(t)$ from $h[k]$, is that $T \leq \frac{\sigma}{Q}$ and $Q \geq 2\left(\frac{\Omega_\alpha \sigma}{2\pi \sin\alpha} + M + 1\right)$, where M is known.*

Proof of Theorem 2. From the unlimited sampling part of the fractional Fourier domain in Equations (17) and (20), we can get the

$$\begin{bmatrix} \Im[M-1] & \Im[M-2] & \cdots & \Im[0] \\ \Im[M] & \Im[M-1] & \cdots & \Im[1] \\ \vdots & \vdots & \vdots & \vdots \\ \Im[N-1] & \Im[N-2] & \cdots & \Im[N-M] \end{bmatrix} \begin{bmatrix} \Gamma[1] \\ \Gamma[2] \\ \vdots \\ \Gamma[M] \end{bmatrix} = -\begin{bmatrix} \Im[M] \\ \Im[M+1] \\ \vdots \\ \Im[N] \end{bmatrix},$$

that is, $\Gamma[\vartheta]$.

From

$$(\Gamma * \Im)[n] = 0,$$

we can get the root of ς_m.

Bring Equation (19) into Equation (17), we obtain

$$\begin{aligned} \overline{F}_\alpha[n] &= \overline{H}_\alpha[n] + \overline{V}_\alpha[n] \\ &= \sum\nolimits_{k \in \mathbb{I}_{Q-1}} A_\alpha \overline{h}[k]\, e^{i\left(\frac{\cot\alpha}{2}k^2 - \csc(\alpha)\cdot\overline{u}_0 kn + \frac{\cot\alpha}{2}\overline{u}_0^2 n^2\right)} \\ &+ \sum\nolimits_{m \in M} c[m] A_\alpha e^{i\left(\frac{\cot\alpha}{2T^2}t_m^2 - \frac{\csc(\alpha)\cdot\overline{u}_0 n}{T}t_m + \frac{\cot\alpha}{2}\overline{u}_0^2 n^2\right)}. \end{aligned} \tag{27}$$

Using the least square method to estimate $c[m]$, and get v_k in Equation (15). we develop a method that allows for inferring $v[k]$ from $h[k]$.

Based on Theorem 1, we get the sampling density criterion, and the following conclusions can be drawn

$$|\mathbb{I}_{Q-1} \backslash \mathbb{E}_{R,Q-1}| = Q - 2R - 2 \geq 2M, \tag{28}$$

where M is known.

Due $QT = \sigma$, $R = \left\lceil \frac{\Omega_\alpha}{u_0} \right\rceil$, $u_0 = \frac{2\pi \sin\alpha}{\sigma}$, we have

$$T = T_{FRFT} \leq \frac{\sigma}{2(R + M + 1)} = \frac{\sigma}{2(\lceil \Omega_\alpha \sigma / 2\pi \sin\alpha \rceil + M + 1)}. \tag{29}$$

Proof completed. □

Remark 2. *After performing M folds, Equation (29) can guarantee the restoration and reconstruction of folding moment $\{c[m], t_m\}_{m=0}^{M-1}$. Theorem 2 turns out that the unlimited sampling theorem has nothing to do with the modulus threshold and can handle arbitrary folding time. When $\alpha = \frac{\pi}{2}$, see [25].*

Remark 3. *Indeed, the choice of the optimal angle α of the FRFT must take into account the actual application: if the modulation frequency is known, the optimal estimate can be obtained under the angle α determined by the modulation frequency; if the modulation frequency is unknown, it can be estimated in advance using the instants frequency estimation method, and then the estimated modulation frequency is used to determine the angle α.*

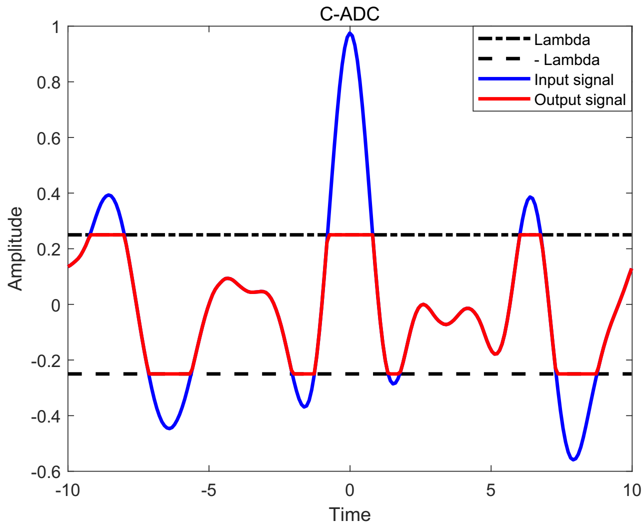
Remark 4. *The proposed method only needs to compute the first-order difference and then isolate the nonlinearity-induced folding in the FRFT domain. The [16] needs to compute the N-order difference and recover the high-order difference from the model sample. In comparison to [16], the number of differentiations is much lower in the proposed method.*

The unlimited sampling theorem proves that the non-bandlimited signal in the Fourier domain based on the FRFT can be recovered from analog sampling as long as it meets Equation (29), whose amplitude exceeds the ADC threshold by orders of magnitude. It is particularly important that the signal is not affected by the ADC threshold.

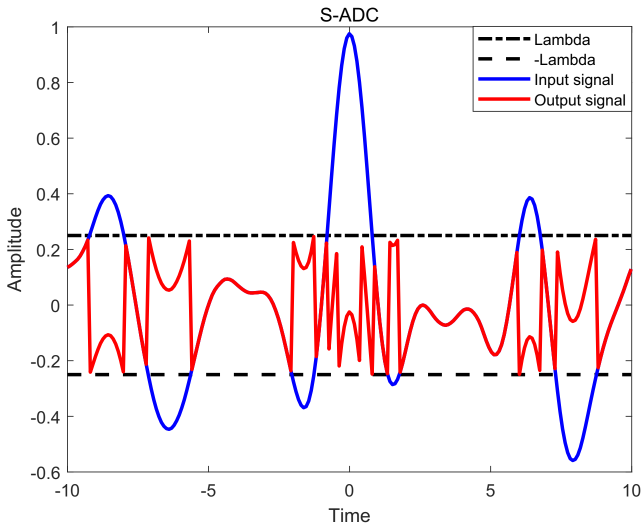
5. Potential Application

5.1. Self-Reset ADC

The standard process of digitizing a signal involves bandwidth limiting by an annihilation filter, followed by sampling with an ADC. In practice, the C-ADC has a limited dynamic range $[-\lambda, \lambda]$; if the input signal exceeds this range, the signal is clipped. The clipping problem is a serious problem that manifests itself in the form of non-linear artifacts in audio-visual data and applications involving the sampling of biomedical data. The S-ADCs are rapidly developing with advances in ADC design. The theoretical concept of the S-ADCs has been in the literature since as early as the late 1970s, and physical implementations have only been in development since the early 2000s. One of the earliest references was by Rhee and Joo [37], where the authors proposed the S-ADC in the context of CMOS image sensors. When the upper or lower saturation threshold is reached, it resets to the corresponding threshold so that subsequent changes are captured even if the saturation limit is exceeded. The conceptual difference between the C-ADC and S-ADC is shown in Figure 2. The aim of developing the S-ADC is to enable the dynamic range of natural images in real-world applications to exceed the range that the C-ADC can handle. This capability is critical not only for consumer photography, but also for life sciences and bioimaging.



(a) Input signal and output signal in C−ADC



(b) Input signal and output signal in S−ADC

Figure 2. Conceptual difference between the C−ADC and S−ADC. (**a**) Input signal and output signal in C−ADC; (**b**) input signal and output signal in S−ADC.

The work presented in this study revolves around the theoretical aspects of unlimited sampling. Based on the good properties of the FRFT, we propose a novel fractional Fourier domain sampling theorem that isolates the folds introduced by the modulus nonlinearity in the fractional Fourier domain, which can deal with the non-idealities and uncertainties introduced by the hardware while ensuring a low sampling rate. It is shown in theory that, based on the folding properties of the S-ADC, we can recover some signals up to multiples of the ADC threshold λ and can deal with arbitrarily long folding times.

- In the practical application of the S-ADC, the proposed method can satisfy the condition that λ is unknowable, so it can resist any non-ideality.
- The proposed method only needs to calculate the first-order differential, which is especially useful for the S-ADC in case of errors.
- The proposed mathematical model in this study has a certain possibility in S-ADC. The proposed signals in this report are periodic bandlimited signals in the fractional Fourier domain, and this limitation reflects a practical limitation. Typically, instead of sampling on an ideal real line, the signal is sampled at finite intervals.

Based on the folding characteristics of the S-ADC hardware, the core of the proposed method is a mathematical model of the FRFT unlimited sampling signal. The folding introduced by the modulus nonlinearity can be isolated in the fractional Fourier domain, leading to frequency estimation problems. An annihilating filter estimation method is used to deal with arbitrarily close folding instants. In future work, we are committed to combining theory and practice, and have achieved breakthrough results in both hardware and algorithms.

5.2. Future Directions

- Based on the mathematical conclusions obtained in this report, we will study a series of simulation experiments with the proposed theory. Some practical application ADC examples using the proposed unlimited sampling theorem will also be investigated in the future.
- In the future, we will study the optimal fractional order angle α selection method in sampling theory and tell readers how to determine the optimal parameter α of the FRFT in the reconstruction of the analog signal from its sampled signal.
- The broader signal transformation domain is also a topic worthy of our attention in the future. Extending our results to a wider domain of transforms, such as linear canonical transform, linear canonical wavelet transform, and canonical S-transform, etc., is a very interesting follow-up question.

6. Conclusions

In this report, we study the sampling theorem of bandlimited signals in the fractional Fourier domain based on the unlimited sampling framework of modulo measurement. Our main work is to perform modular operations in the fractional Fourier domain with the folding introduced by modular nonlinearity, and then to deal with the problem of fractional spectrum estimation. It turns out that the unlimited sampling theorem has nothing to do with the modulus threshold and can handle arbitrary folding time. In future work, we are committed to combining theory and practice and have achieved breakthrough results in both hardware and algorithms.

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