fractal and fractional

Article

On Fractional Integral Inequalities of Riemann Type for Composite Convex Functions and Applications

Miguel Vivas-Cortez [1], Muzammil Mukhtar [2], Iram Shabbir [2], Muhammad Samraiz [3,*] and Muhammad Yaqoob [2]

1 Escuela de Ciencias Físicas y Matemáticas, Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Av. 12 de octubre 1076, Apartado, Quito 17-01-2184, Ecuador; mjvivas@puce.edu.ec
2 Department of Mathematics, The Islamia University of Bahawalpur, Bahawalnagar Campus, Bahawalnagar 62300, Pakistan; muzammilmukhtar3@gmail.com or muzammil_mukhtar@iub.edu.pk (M.M.); erum.feas@gmail.com (I.S.); myaqoob653@gmail.com (M.Y.)
3 Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan
* Correspondence: muhammad.samraiz@uos.edu.pk or msamraizuos@gmail.com

Abstract: In this study, we apply fractional calculus on certain convex functions and derive many novel mean-type inequalities by employing fractional calculus and convexity theory. In order to investigate fractional mean inequalities, we first build an identity in this study. Then, with its help, we derive many mean-type inequalities and estimate the error of HH inequality using a generalized version of RL-fractional integrals and certain classes of convex functions. The results obtained are validated by taking specific functions. Many mean-type inequalities that exist in the literature are generalized by the main results of this study.

Keywords: mean inequalities; fractional integral; Hölder's inequality; Minkowski inequality

MSC: 26A33; 35J05

Citation: Vivas-Cortez, M.; Mukhtar, M.; Shabbir, I.; Samraiz, M.; Yaqoob, M. On Fractional Integral Inequalities of Riemann Type for Composite Convex Functions and Applications. *Fractal Fract.* **2023**, *7*, 345. https://doi.org/10.3390/fractalfract7050345

Academic Editor: Ricardo Almeida

Received: 26 February 2023
Revised: 27 March 2023
Accepted: 31 March 2023
Published: 22 April 2023

1. Introduction

In recent times, fractional calculus has gained much attention due to fractional calculus having received a lot of attention recently as a result of its widespread use in contemporary sciences. Numerous real-world issues have been resolved using fractional theory, including estimating blood alcohol level [1], modelling in continuum mechanics [2], controlling hexapod robots [3], determining non-local continua [4], administrating drug dosages [5], forming memory through membranes [6], determining semilunar heat valve vibrations [7], diffusing water in sand [8], determining the growth rate and population of poisonous bacteria [9], analyzing social and economic cycles [10], and many others [11–13].

Mathematical inequalities span many fields of science. Their importance has increased greatly due to their widespread use, especially in estimating the solutions of bounded value problems. Inequalities and convexity are very closely related. An important inequality that describes and reveals this convexity is known as the Hermite Hadamard (HH) inequality [14], which is given as

Theorem 1. *Let $\xi : U \subseteq R \to R$ be a convex function, then the following inequality holds.*

$$\xi\left(\frac{p_1+p_2}{2}\right) \leq \frac{1}{p_2-p_1}\int_{p_1}^{p_2} \xi(\delta)d\delta \leq \frac{\xi(p_1)+\xi(p_2)}{2}.$$

The HH inequality is formulated for different interesting situations such as those in [15,16] for generalized fractional operators and those in [17–19] via different convexities. The convexities in certain types of inequalities can also be seen in [20,21].

Fractional operators provide more generalized and advance results even where ordinary calculus fails by reducing many constraints. Their growing use has attracted many scientists towards this non-integer order calculus. Farid et al. [22] applied RL fractional integrals to study the HH inequality with convexity for monotonic functions.

To generalize and extend the results, we apply the (k, s)-Riemann integral [23] and find a special identity that helps us to obtain many mean-type inequalities already known from the literature. We also estimate the error of mean-type inequalities. The main results of our study are more advanced and generalized from the pre-existing literature. Moreover, the results of [24–26] are connected with that from our work.

This paper is organize in the following way. Section 2 states some important related preliminaries. Section 3 consists of a generalized identity and the theorems based on this established identify generalize the estimates presented in [22]. In Section 4, some conclusions are presented.

2. Preliminaries

Here, we present some basic facts that are needed to understand the main results.

Definition 1. *A function $\xi : [m, n] \to (-\infty, \infty)$ is a convex function if it satisfies the inequality;*

$$\xi(\delta x + (1-\delta)y) \leq \delta\xi(x) + (1-\delta)\xi(y),$$

where $0 \leq \delta \leq 1$ and $x, y \in [m, n]$.

The next definition describes the criteria for a certain function to be convex with respect to some other function with a strictly monotone property.

Definition 2 ([27])**.** *The function ξ is convex with respect to strictly the monotone function ϱ if $(\xi \circ \varrho^{-1})$ is a convex function.*

One can read many different aspects of the HH inequality and its generalizations in [28–30]. A lot of interesting applications have encouraged mathematicians to investigate its different versions. The HH inequality for a function satisfying Definition 2 can be seen in the next theorem.

Theorem 2 ([31])**.** *Suppose that $I_1, I_2 \subset (-\infty, +\infty)$, $\varrho : [m, n] \subset I_2 \to (-\infty, \infty)$ is a strictly monotone function and $\xi : [m, n] \subset I_1 \to (-\infty, \infty)$ is convex with respect to ϱ. Then,*

$$\xi\left(\varrho^{-1}\left(\frac{\varrho(m) + \varrho(n)}{2}\right)\right) \leq \frac{1}{\varrho(n) - \varrho(m)} \int_{\varrho(m)}^{\varrho(n)} \xi\left(\varrho^{-1}(\delta)\right) d\delta \leq \frac{\xi(m) + \xi(n)}{2}.$$

The classical gamma function is extended in the next definition.

Definition 3 ([32])**.** *The k-Gamma function symbolized as Γ_k is defined as*

$$\Gamma_k(w) = \int_0^\infty e^{-\frac{\delta^k}{k}} \delta^{w-1} d\delta \text{ where } w \in \mathbb{C} \text{ and } Re(w) > 0.$$

The k-Gamma function has a similar property to that of the classical gamma function i.e., $\delta\Gamma_k(\delta) = \Gamma_k(\delta + k)$.

The Riemann–Liouville fractional integrals (which we simply call RL-fractional integrals) are defined as follows:

Definition 4 ([33]). *The left-sided and right-sided RL-fractional integrals of f having an order $\beta > 0$ are described as*

$$I^{\beta}_{m^+} f(x) = \frac{1}{\Gamma(\beta)} \int_m^x (x-\delta)^{\beta-1} f(\delta) d\delta, \qquad x > m,$$
$$I^{\beta}_{n^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^n (\delta-x)^{\beta-1} f(\delta) d\delta, \qquad x < n.$$

The next definition presents one of the extended versions of the RL-fractional integral given in [34].

Definition 5. *The left- and right-sided (k,s)-RL-fractional integrals of a function f of the order $\beta > 0$ are defined as follows:*

$$ {}^s_k I^{\beta}_{m^+} f(x) = \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_m^x \left(x^{s+1} - \delta^{s+1} \right)^{\frac{\beta}{k}-1} \delta^s f(\delta) d\delta, \qquad x > m,$$

$$ {}^s_k I^{\beta}_{n^-} f(x) = \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_x^n \left(\delta^{s+1} - x^{s+1} \right)^{\frac{\beta}{k}-1} \delta^s f(\delta) d\delta, \qquad x < n.$$

3. Main Results

In this section, we present many mean-type inequalities obtained by applying the extended version of the RL-fractional integral "(k,s)-RL-fractional integral" and certain classes of the convex functions.

The basic identity essential for our main results is obtained in the following lemma, which is further used in estimating the error of the HH inequality.

Lemma 1. *Let the two real mappings ξ and ϱ be defined on $[m,n]$ with $m < n$ such that ϱ has the strictly monotone property. If $\xi \circ (\varrho^{s+1})^{-1}$ is differentiable and $(\xi \circ (\varrho^{s+1})^{-1})' \in L[m,n]$, then the following equation holds:*

$$\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2\left(\varrho^{s+1}(n) - \varrho^{s+1}(m)\right)^{\frac{u}{k}}} \left({}^s_k I^u_{\varrho^{s+1}(m)^+} \xi(n) + {}^s_k I^u_{\varrho^{s+1}(n)^-} \xi(m) \right)$$
$$= \frac{\varrho^{s+1}(n) - \varrho^{s+1}(m)}{2} \int_0^1 \left((1-\delta)^{\frac{u}{k}} - \delta^{\frac{u}{k}} \right) \left(\xi \circ (\varrho^{s+1})^{-1} \right)' \left(\delta\varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n) \right) d\delta. \qquad (1)$$

Proof. By evaluating the integral, we have

$$\begin{aligned}
&\int_0^1 (1-\delta)^{\frac{u}{k}} \left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \left(\delta \varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n)\right) d\delta \\
&= \frac{(1-\delta)^{\frac{u}{k}} \left(\xi \circ (\varrho^{s+1})^{-1}\right) \left(\delta \varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n)\right)}{\varrho^{s+1}(m) - \varrho^{s+1}(n)} \Bigg|_o^1 \\
&+ \frac{u}{k} \int_0^1 \frac{(1-\delta)^{\frac{u}{k}-1} \left(\xi \circ (\varrho^{s+1})^{-1}\right) (\delta \varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n)) d\delta}{\varrho^{s+1}(m) - \varrho^{s+1}(n)} \\
&= \frac{\xi(n)}{\varrho^{s+1}(n) - \varrho^{s+1}(m)} - \frac{\frac{u}{k}}{\varrho^{s+1}(n) - \varrho^{s+1}(m)} \\
&\times \int_0^1 (1-\delta)^{\frac{u}{k}-1} \left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right) \left(\delta \varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n)\right) d\delta \\
&= \frac{\xi(n)}{\varrho^{s+1}(n) - \varrho^{s+1}(m)} - \frac{\frac{u}{k}}{(\varrho^{s+1}(n) - \varrho^{s+1}(m))^{\frac{u}{k}+1}} \\
&\times \int_{\varrho^{s+1}(m)}^{\varrho^{s+1}(n)} \left(z^{s+1} - \varrho^{s+1}(m)\right)^{\frac{u}{k}-1} \left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right) (z^{s+1}) z^s (s+1) dz \\
&= \left(\frac{\xi(n)}{\varrho^{s+1}(n) - \varrho^{s+1}(m)}\right) - \frac{(s+1)^{\frac{u}{k}} \Gamma_k(u+k)}{\left(\varrho^{s+1}(n) - \varrho^{s+1}(m)\right)^{\frac{u}{k}+1}} \left({}_k^s I^u_{\varrho^{s+1}(n)^-} \xi(m)\right).
\end{aligned} \tag{2}$$

Similarly, integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1 \delta^{\frac{u}{k}} \left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \left(\delta \varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n)\right) d\delta \\
&= \frac{-\xi(n)}{\varrho^{s+1}(n) - \varrho^{s+1}(m)} + \frac{(s+1)^{\frac{u}{k}} \Gamma_k(u+k)}{\left(\varrho^{s+1}(n) - \varrho^{s+1}(m)\right)^{\frac{u}{k}+1}} \left({}_k^s I^u_{\varrho^{s+1}(m)^+} \xi(n)\right).
\end{aligned} \tag{3}$$

Substituting (2) and (3) in (1) gives the desired result. □

Lemma 1 helps us to derive the following error estimation of the HH inequality given in Theorem 2.

Theorem 3. *Let the two real mappings ξ and ϱ be defined on $[m, n]$ with $m < n$ such that ϱ has the strictly monotone property. If $\xi \circ (\varrho^{s+1})^{-1}$ is differentiable and $(\xi \circ (\varrho^{s+1})^{-1})' \in L[m, n]$, then the inequality*

$$\begin{aligned}
&\left| \frac{\xi(m) + \xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}} \Gamma_k(u+k)}{2\left(\varrho^{s+1}(n) - \varrho^{s+1}(m)\right)^{\frac{u}{k}}} \left({}_k^s I^u_{\varrho^{s+1}(m)^+} \xi(n) + {}_k^s I^u_{\varrho^{s+1}(n)^-} \xi(m)\right) \right| \\
&\leq \frac{|\varrho^{s+1}(n) - \varrho^{s+1}(m)|}{2\left(\frac{u}{k} + 1\right)} \left(1 - \frac{1}{2^{\frac{u}{k}}}\right) \left(\left| \left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \left(\varrho^{s+1}(m)\right) \right| + \left| \left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \left(\varrho^{s+1}(n)\right) \right| \right),
\end{aligned} \tag{4}$$

holds when $|(\xi \circ (\varrho^{s+1})^{-1})'|$ is convex.

Proof. By utilizing Lemma 1 with simple calculus, the above inequality can be described by

$$\begin{aligned}
&\left| \frac{\xi(m) + \xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}} \Gamma_k(u+k)}{2\left(\varrho^{s+1}(n) - \varrho^{s+1}(m)\right)^{\frac{u}{k}}} \left({}_k^s I^u_{\varrho^{s+1}(m)^+} \xi(n) + {}_k^s I^u_{\varrho^{s+1}(n)^-} \xi(m)\right) \right| \\
&\leq \frac{|\varrho^{s+1}(n) - \varrho^{s+1}(m)|}{2} \int_0^1 \left| (1-\delta)^{\frac{u}{k}} - \delta^{\frac{u}{k}} \right| \left| \left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \left(\delta \varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n)\right) \right| d\delta.
\end{aligned} \tag{5}$$

Since $\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'$ is convex, using this fact on the right-hand side of (5) will imply the following:

$$\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2}-\frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2\left(\varrho^{s+1}(n)-\varrho^{s+1}(m)\right)^{\frac{u}{k}}}\left({}_k^sI^u_{\varrho^{s+1}(m)^+}\xi(n)+{}_k^s\,I^u_{\varrho^{s+1}(n)^-}\xi(m)\right)\right|\\
&\leq\frac{\left|\varrho^{s+1}(n)-\varrho^{s+1}(m)\right|}{2}\int_0^1\left|(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right|\\
&\times\left(\delta\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(m)\right|+(1-\delta)\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(n)\right|\right)d\delta\\
&\leq\frac{\left|\varrho^{s+1}(n)-\varrho^{s+1}(m)\right|}{2}\int_0^{\frac{1}{2}}\left|(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right|\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\left(\delta\varrho^{s+1}(m)+(1-\delta)\varrho^{s+1}(n)\right)\right|d\delta\\
&+\int_{\frac{1}{2}}^1\left|(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right|\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\left(\delta\varrho^{s+1}(m)+(1-\delta)\varrho^{s+1}(n)\right)\right|d\delta\\
&=\frac{\left|\varrho^{s+1}(n)-\varrho^{s+1}(m)\right|}{2}\left[\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(m)\right|\int_0^{\frac{1}{2}}\left(\delta(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}+1}\right)d\delta\right.\\
&+\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(n)\right|\int_0^{\frac{1}{2}}\left((1-\delta)^{\frac{u}{k}+1}-\delta^{\frac{u}{k}}(1-\delta)\right)d\delta\\
&+\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(m)\right|\int_{\frac{1}{2}}^1\left(\delta^{\frac{u}{k}+1}-\delta(1-\delta)^{\frac{u}{k}}\right)d\delta\\
&\left.+\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(n)\right|\int_{\frac{1}{2}}^1\left(\delta^{\frac{u}{k}}(1-\delta)-(1-\delta)^{\frac{u}{k}+1}d\delta\right)\right].
\end{aligned}$$

Thus, we can obtain our desired result after a few computations. □

In the next examples, we discuss the applications of Theorem 3 for some specific functions.

Example 1. *By substituting $\varrho^{s+1}(x)=e^x$ in (4), we have the inequality*

$$\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2}-\frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2\left(e^n-e^m\right)^{\frac{u}{k}}}\left({}_k^sI^u_{e^m+}\xi(n)+{}_k^sI^u_{e^n-}\xi(m)\right)\right|\\
&\leq\frac{|e^n-e^m|}{2(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(e^{-m}|\xi'(m)|+e^{-n}|\xi'(n)|\right).
\end{aligned}$$

Example 2. *By substituting $\varrho^{s+1}(x)=e^x$ and $\dfrac{u}{k}=1$ in (4), we have the inequality*

$$\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2}-\frac{k(s+1)}{2(e^n-e^m)}\left({}_k^sI^u_{e^m+}\xi(n)+{}_k^sI^u_{e^n-}\xi(m)\right)\right|\\
&\leq\frac{|e^n-e^m|}{8}\left(e^{-m}|\xi'(m)|+e^{-n}|\xi'(n)|\right).
\end{aligned}$$

Example 3. *By substituting* $\varrho^{s+1}(x) = \frac{1}{x}$ *in (4), we obtain the following:*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2}\left(\frac{mn}{n-m}\right)^{\frac{u}{k}}\left({}^s_kI^u_{\frac{1}{m}-}\xi\circ g\left(\frac{1}{n}\right) + {}^s_kI^u{}_{\frac{1}{n}+}\xi\circ g\left(\frac{1}{m}\right)\right)\right|$$
$$\leq \frac{|m-n|}{2|mn|(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(m^2|\xi'(m)| + n^2|\xi'(n)|\right),$$

where $g(\delta) = \frac{1}{\delta}$.

Example 4. *By substituting* $\varrho^{s+1}(x) = \frac{1}{x}$ *and* $\frac{u}{k} = 1$ *in (4), we obtain*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{k(s+1)mn}{n-m}\int_{\frac{1}{n}}^{\frac{1}{m}}(\xi\circ g)(\delta)d\delta\right| \leq \frac{|m-n|}{8|mn|}\left(m^2|\xi'(m)| + n^2|\xi'(n)|\right),$$

where $g(\delta) = \frac{1}{\delta}$.

Example 5. *By substituting* $\varrho^{s+1}(x) = x^r$, $r \neq 0$ *in (4), we obtain the inequality*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{r^{\frac{u}{k}}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(n^r-m^r)^{\frac{u}{k}}}\left({}^{r,s}_kI^u_{m+}\xi(\delta) + {}^{r,s}_kI^u_{n-}\xi(\delta)\right)\right|$$
$$\leq \frac{|n^r-m^r|}{2|r|(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(m^{1-r}|\xi'(m)| + n^{1-r}|\xi'(n)|\right).$$

Example 6. *By substituting* $\varrho^{s+1}(x) = x^r$, $r \neq 0$ *and* $\frac{u}{k} = 1$ *in (4), we obtain*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{k(s+1)(r)}{2(n^r-m^r)}\int_n^m \delta^{r-1}f(\delta)d\delta\right| \leq \frac{|n^r-m^r|}{8|r|}\left(m^{1-r}|\xi'(m)| + n^{1-r}|\xi'(n)|\right).$$

Example 7. *By substituting* $\varrho^{s+1}(x) = \ln(x)$ *in (4), we have the inequality*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2\big(\ln(n)-\ln(m)\big)^{\frac{u}{k}}}\left({}^s_kI^u_{\ln(m)+}\xi(n) + {}^s_kI^u_{\ln(n)-}\xi(m)\right)\right|$$
$$\leq \frac{|\ln(n)-\ln(m)|}{2(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(m|\xi'(m)| + n|\xi'(n)|\right).$$

Example 8. *By substituting* $\varrho^{s+1}(x) = \ln(x)$ *and* $\frac{u}{k} = 1$ *in (4), we obtain*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{k(s+1)}{ln(n)-ln(m)}\int_m^n \frac{\xi(\delta)}{\delta}d\delta\right| \leq \frac{ln(n)-ln(m)}{8}\left(m|\xi'(m)| + n|\xi'(n)|\right).$$

Theorem 4. *Let the two real mappings* ξ *and* ϱ *be defined on* $[m,n]$ *with* $m < n$ *such that* ϱ *has the strictly monotone property. If* $\xi\circ(\varrho^{s+1})^{-1}$ *is differentiable and* $(\xi\circ(\varrho^{s+1})^{-1})' \in L[m,n]$*, then the inequality*

$$\left|\frac{\xi(m)+\xi(n)}{2}-\frac{\left(s+1\right)^{\frac{u}{k}}\Gamma_k\left(u+k\right)}{2(\varrho^{s+1}(n)-\varrho^{s+1}(m))^{\frac{u}{k}}}\left({}_k^sI^u_{\varrho^{s+1}(m)^+}\xi(n)+{}_k^sI^u_{\varrho^{s+1}(n)^-}\xi(m)\right)\right|$$
$$\leq\frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{2^{\frac{1}{q}}(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right) \tag{6}$$
$$\times\left(\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'(\varrho^{s+1}(m))\right|^q+\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'(\varrho^{s+1}(n))\right|^q\right)^{\frac{1}{q}}$$

holds whenever $\left|\left(\xi\circ(\varrho^{s+1})^{-1}\right)'\right|^q$ *is convex for* $q\geq 1$.

Proof. If $q=1$, then the above inequality can be obtained by using basic calculus in Lemma 1 and $\left|\left(\xi\circ(\varrho^{s+1})^{-1}\right)'\right|^q$ is convex.

Let $q>1$. The right-hand side of Lemma 1, along with the power mean inequality and basic calculus, then implies the following inequality:

$$\left|\frac{\xi(m)+\xi(n)}{2}-\frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(\varrho^{s+1}(n)-\varrho^{s+1}(m))^{\frac{u}{k}}}\left({}_k^sI^u_{\varrho^{s+1}(m)^+}\xi(n)+{}_k^sI^u_{\varrho^{s+1}(n)^-}\xi(m)\right)\right|$$
$$\leq\frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{2}$$
$$\times\left(\int_0^1\left|(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right|\right. \tag{7}$$
$$\left.\times\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\left(\delta\varrho^{s+1}(m)+(1-\delta)\varrho^{s+1}(n)\right)\right|^q d\delta\right)^{\frac{1}{q}}.$$

Consider

$$\int_0^1\left|(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right|d\delta=\int_0^{\frac{1}{2}}\left((1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}d\delta\right)+\int_{\frac{1}{2}}^1\left(\delta^{\frac{u}{k}}-(1-\delta)^{\frac{u}{k}}d\delta\right)$$
$$=\frac{2}{(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right). \tag{8}$$

Since $\left|(\xi\circ(\varrho^{s+1})^{-1})'\right|^q$ is convex,

$$\int_0^1\left|(1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right|\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'(\delta\varrho^{s+1}(m)+(1-\delta)\varrho^{s+1}(n))\right|^q d\delta$$
$$\leq\int_0^{\frac{1}{2}}\left((1-\delta)^{\frac{u}{k}}-\delta^{\frac{u}{k}}\right)\left(\delta\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(m)\right|^q+(1-\delta)\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(n)\right|^q\right)d\delta \tag{9}$$
$$+\int_{\frac{1}{2}}^1\left(\delta^{\frac{u}{k}}-(1-\delta)^{\frac{u}{k}}\right)\left(\delta\left|\left(\xi\circ\varrho^{-1}\right)'\varrho^{s+1}(m)\right|^q+(1-\delta)\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(n)\right|^q\right)d\delta.$$

This can also be written as

$$
\begin{aligned}
&= \left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \varrho^{s+1}(m)\right|^q \left(\int_o^{\frac{1}{2}} \left(\delta(1-\delta)^{\frac{u}{k}} - \delta^{\frac{u}{k}+1}\right)d\delta + \int_{\frac{1}{2}}^1 \delta\left(\delta^{\frac{u}{k}} - (1-\delta)^{\frac{u}{k}}\right)d\delta\right) \\
&+ \left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \varrho^{s+1}(n)\right|^q \left(\int_o^{\frac{1}{2}} \left((1-\delta)^{\frac{u}{k}+1} - \delta^{\frac{u}{k}}(1-\delta)\right)d\delta + \right. \\
&\left.\int_{\frac{1}{2}}^1 \left(\delta^{\frac{u}{k}}(1-\delta) - (1-\delta)^{\frac{u}{k}+1}\right)d\delta\right) \\
&= \left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \varrho^{s+1}(m)\right|^q \frac{1}{(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right) + \\
&\left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)' \varrho^{s+1}(n)\right|^q \frac{1}{(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right).
\end{aligned} \tag{10}
$$

Using (8)–(10) in (7) and some computations gives us our desired result. □

In the next examples, we discuss the applications of Theorem 4 for some specific functions.

Example 9. *By substituting $\varrho^{s+1}(x) = e^x$ in (6), we obtain*

$$
\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(e^n - e^m)^{\frac{u}{k}}}\left({}_k^s I^u_{e^{m}+}\xi(n) + {}_k^s I^u_{e^{n}-}\xi(m)\right)\right| \\
&\leq \frac{|e^n - e^m|}{2^{\frac{1}{q}}(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right)\left(\left|e^{-m}\xi'(m)\right|^q + \left|e^{-n}\xi'(n)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Example 10. *By substituting $\varrho^{s+1}(x) = x$ in (6), we obtain*

$$
\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(m-n)^{\frac{u}{k}}}\left({}_k^s I^u_{m+}\xi(n) + {}_k^s I^u_{n-}\xi(m)\right)\right| \\
&\leq \frac{|m-n|}{2^{\frac{1}{q}}(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right)\left(|\xi'(m)|^q + |\xi'(n)|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Example 11. *By substituting $\varrho^{s+1}(x) = \dfrac{1}{x}$ in (6), we obtain*

$$
\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2}\left(\frac{mn}{n-m}\right)^{\frac{u}{k}}\left({}_k^s I^{\frac{u}{k}}_{\frac{1}{m}-}\xi \circ g\left(\frac{1}{n}\right) + {}_k^s I^{\frac{u}{k}}_{\frac{1}{n}-}\xi \circ g\left(\frac{1}{m}\right)\right)\right| \\
&\leq \frac{|m-n|}{2^{\frac{1}{q}}|mn|(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right)\left(m^{2q}\left|\xi'\varrho^{s+1}(m)\right|^q + n^{2q}\left|\xi'\varrho^{s+1}(n)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Example 12. *By substituting $\varrho^{s+1}(x) = \ln(x)$ in (6), we obtain*

$$
\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(\ln n - \ln m)^{\frac{u}{k}}}\left({}_k^s I^u_{\ln m^+}\xi(n) + {}_k^s I^u_{\ln n^-}\xi(m)\right)\right| \\
&\leq \frac{|\ln(n) - \ln(m)|}{2^{\frac{1}{q}}(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right)\left(m^q\left|\xi'\varrho^{s+1}(m)\right|^q + n^q\left|\xi'\varrho^{s+1}(n)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Example 13. *By substituting $\varrho^{s+1}(x) = x^r$ where $r \neq 0$ in (6), we obtain*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{r^{\frac{u}{k}}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(n^r - m^r)^{\frac{u}{k}}}\left({}_k^sI^u_{m^+}\xi(n)+{}_k^sI^u_{n^-}\xi(m)\right)\right|$$
$$\leq \frac{|n^r - m^r|}{2^{\frac{1}{q}}|r|(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(m^{(1-r)q}\left|\xi'\varrho^{s+1}(m)\right|^q + n^{(1-r)q}\left|\xi'\varrho^{s+1}(n)\right|^q\right).$$

The next theorem is proved with the help of the following lemma.

Lemma 2 ([35]). *For $0 < \beta < 1$ and $0 \leq m < n$, we have*

$$\left|m^\beta - n^\beta\right| \leq \left(n - m\right)^\beta. \tag{11}$$

Theorem 5. *Let the two real mappings ξ and ϱ be defined on $[m, n]$ with $m < n$ such that ϱ has the strictly monotone property. If $\xi \circ (\varrho^{s+1})^{-1}$ is differentiable and $(\xi \circ (\varrho^{s+1})^{-1})' \in L[m, n]$, then the inequality*

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(\varrho^{s+1}(n) - \varrho^{s+1}(m))^{\frac{u}{k}}}\left({}_k^sI^u_{\varrho^{s+1}(m)^+}\xi(n) + {}_k^sI^u_{\varrho^{s+1}(n)^-}\xi(m)\right)\right|$$
$$\leq \frac{|\varrho^{s+1}(n) - \varrho^{s+1}(m)|}{2^{1+\frac{1}{q}}(\frac{up}{k}+1)^{\frac{1}{p}}}\left(\left|\left(\xi\circ(\varrho^{s+1})^{-1}\right)'\varrho^{s+1}(m)\right|^q + \left|\left(\xi\circ(\varrho^{s+1})^{-1}\right)'\varrho^{s+1}(n)\right|^q\right)^{\frac{1}{q}} \tag{12}$$

holds whenever $\left|(\xi \circ (\varrho^{s+1})^{-1})'\right|^q$, where $q > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, is convex.

Proof. Using Hölder's inequality and simple calculus on the right-hand side of Lemma 1 gives us

$$\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(\varrho^{s+1}(n)-\varrho^{s+1}(m))^{\frac{u}{k}}}\left({}_k^sI^u_{\varrho^{s+1}(m)^+}\xi(n) + {}_k^sI^u_{\varrho^{s+1}(n)^-}\xi(m)\right)\right|$$
$$\leq \frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{2}\left(\int_0^1\left|(1-\delta)^{\frac{u}{k}} - \delta^{\frac{u}{k}}\right|^p\right)^{\frac{1}{p}} \tag{13}$$
$$\times\left(\int_0^1\left|\left(\xi\circ(\varrho^{s+1})^{-1}\right)'(\delta\varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n))\right|^q d\delta\right)^{\frac{1}{q}},$$

We apply Lemma 2 and obtain

$$\int_0^1\left|(1-\delta)^{\frac{u}{k}} - \delta^{\frac{u}{k}}\right|^p d\delta \leq \int_0^1 (1-2\delta)^{\frac{up}{k}} d\delta$$
$$\int_0^{\frac{1}{2}}(1-2\delta)^{\frac{up}{k}} d\delta + \int_{\frac{1}{2}}^1 (2\delta-1)^{\frac{up}{k}} d\delta = \frac{1}{\frac{up}{k}+1}.$$

Now, using the fact that $\left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)'\right|^q$ is convex, we obtain

$$\begin{aligned}
&\int_0^1 \left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)'\left(\delta\varrho^{s+1}(m) + (1-\delta)\varrho^{s+1}(n)\right)\right|^q d\delta \\
&\leq \int_0^1 \left(\delta\left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(m)\right|^q + (1-\delta)\left|\left(\xi \circ \left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(n)\right|^q\right) d\delta \\
&= \frac{1}{2}\left(\left|\left(\xi \circ (\varrho^{s+1})^{-1}\right)'\varrho^{s+1}(m)\right|^q + \left|\left(\xi \circ (\varrho^{s+1})^{-1}\right)'\varrho^{s+1}(n)\right|^q\right).
\end{aligned}$$

Hence we obtain our desired result (12) after making the computations above. □

In the next examples, we discuss the applications of Theorem 5 for some specific functions.

Example 14. *By substituting $\varrho^{s+1}(x) = e^x$ in (12), we obtain*

$$\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(e^n - e^m)^{\frac{u}{k}}}\left({}^s_kI^u_{e^m+}\xi(n) + {}^s_kI^u_{e^n-}\xi(m)\right)\right| \\
&\leq \frac{|e^n - e^m|}{2^{1+\frac{1}{q}}(\frac{up}{k}+1)^{\frac{1}{p}}}\left(\left|e^m\xi'(m)\right|^q + \left|e^n\xi'(n)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Example 15. *By substituting $\varrho^{s+1}(x) = \dfrac{1}{x}$ in (12), we obtain the inequality*

$$\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)(mn)^{\frac{u}{k}}}{2(n-m)^{\frac{u}{k}}}\left({}^s_kI^u_{\frac{1}{m}-}(\xi \circ g)(\frac{1}{n}) + {}^s_kI^u_{\frac{1}{n}-}(\xi \circ g)(\frac{1}{m})\right)\right| \\
&\leq \frac{|m-n|}{2^{1+\frac{1}{q}}(\frac{up}{k}+1)^{\frac{1}{p}}|mn|}\left(m^{2q}\left|\xi'(m)\right|^q + n^{2q}\left|\xi'(n)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $g(t) = \frac{1}{t}$.

Example 16. *By substituting $\varrho^{s+1}(x) = x^r$ where $r \neq 0$ in (12), we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{r^{\frac{u}{k}}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(n^r - m^r)}\left({}^s_kI^u_{m+}\xi(n) + {}^s_kI^u_{n-}\xi(m)\right)\right| \\
&\leq \frac{|n^r - m^r|}{2^{1+\frac{1}{q}}|r|(\frac{up}{k}+1)^{\frac{1}{p}}}\left(m^{(1-r)q}\left|\xi'(m)\right|^q + n^{(1-r)q}\left|\xi'(n)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Example 17. *By substituting $\varrho^{s+1}(x) = \ln(x)$ in (12), we obtain*

$$\begin{aligned}
&\left|\frac{\xi(m)+\xi(n)}{2} - \frac{(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{2(\ln n - \ln m)^{\frac{u}{k}}}\left({}^s_kI^u_{\ln m+}\xi(n) + {}^s_kI^u_{\ln n-}\xi(m)\right)\right| \\
&\leq \frac{\ln(n) - \ln(m)}{2^{1+\frac{1}{q}}(\frac{up}{k}+1)^{\frac{1}{p}}}\left(m^q\left|\xi'(m)\right|^q + n^q\left|\xi'(n)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

The error estimation of the inequality (2) can be studied with the help of the following identity.

Lemma 3. *The equality*

$$\begin{aligned}
&\frac{2^{\frac{u}{k}-1}\left(s+1\right)^{\frac{u}{k}}\Gamma_k\left(u+k\right)}{\left(\varrho^{s+1}(n)-\varrho^{s+1}(m)\right)^{\frac{u}{k}}}\left({}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}^+}\xi(n)+{}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}^-}\xi(m)\right)\\
&-\xi\left(\left(\varrho^{s+1}\right)^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)\right)\\
&=\frac{\varrho^{s+1}(n)-\varrho^{s+1}(m)}{4}\left(\int_0^1\delta^{\frac{u}{k}}\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\frac{\varrho^{s+1}(m)\delta}{2}+\left(\frac{2-\delta}{2}\right)\varrho^{s+1}(n)\right)d\delta\\
&-\int_0^1\delta^{\frac{u}{k}}\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\left(\frac{\varrho^{s+1}(m)(2-\delta)}{2}+\left(\frac{\delta}{2}\right)\varrho^{s+1}(n)\right)d\delta\Bigg).
\end{aligned} \tag{14}$$

holds under the same assumptions as in Lemma 1.

Proof. Integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1\delta^{\frac{u}{k}}\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\left(\frac{\varrho^{s+1}(m)\delta}{2}\right)+\left(\frac{2-\delta}{2}\right)\left(\varrho^{s+1}(n)\right)d\delta\\
&=\frac{\delta^{\frac{u}{k}}\left(\xi\circ(\varrho^{s+1})^{-1}\right)\frac{\varrho^{s+1}(m)\delta}{2}+\left(\frac{2-\delta}{2}\right)\varrho^{s+1}(n)}{\frac{\varrho^{s+1}(m)-\varrho^{s+1}(n)}{2}}\Bigg|_0^1\\
&-\frac{u}{k}\int_0^1\delta^{\frac{u}{k}-1}\frac{\left(\xi\circ(\varrho^{s+1})^{-1}\right)\frac{\varrho^{s+1}(m)\delta}{2}+\left(\frac{2-\delta}{2}\right)\varrho^{s+1}(n)}{\frac{\varrho^{s+1}(m)-\varrho^{s+1}(n)}{2}}d\delta\\
&=-\frac{2\xi(\varrho^{s+1})^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)}{(\varrho^{s+1}(n)-\varrho^{s+1}(m))}\\
&+\left(\frac{2^{\frac{u}{k}}}{\varrho^{s+1}(n)-\varrho^{s+1}(m)}\right)\int_0^1\left(\delta^{\frac{u}{k}}\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)\varrho^{s+1}(m)\frac{\delta}{2}+\left(\frac{2-\delta}{2}\right)\varrho^{s+1}(n)\right)d\delta.
\end{aligned} \tag{15}$$

The change in variable results in the following:

$$=-\frac{2\xi\left(\left(\varrho^{s+1}\right)^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)\right)}{\left(\varrho^{s+1}(n)-\varrho^{s+1}(m)\right)}+\frac{2^{\frac{u}{k}+1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\varrho^{s+1}(n)-\varrho^{s+1}(m)^{\frac{u}{k}}}\left({}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}^+}\xi(n)\right).$$

Similarly,

$$\begin{aligned}
&\int_0^1\delta^{\frac{u}{k}}\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\left(\frac{\varrho^{s+1}(m)(2-\delta)}{2}+\left(\frac{\delta}{2}\right)\varrho^{s+1}(n)\right)d\delta=\frac{2\xi\left((\varrho^{s+1})^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)\right)}{(\varrho^{s+1}(n)-\varrho^{s+1}(m))}\\
&-\frac{2^{\frac{u}{k}}}{\varrho^{s+1}(n)-\varrho^{s+1}(m)}\int_0^1\left(\delta^{\frac{u}{k}-1}\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)\left(\frac{\varrho^{s+1}(m)\delta}{2}\right)+\left(\frac{2-\delta}{2}\right)\varrho^{s+1}(n)\right)d\delta\\
&=\frac{2\xi\left((\varrho^{s+1})^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)\right)}{(\varrho^{s+1}(n)-\varrho^{s+1}(m))}+\frac{2^{\frac{u}{k}+1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\varrho^{s+1}(n)-\varrho^{s+1}(m)^{\frac{u}{k}}}\left({}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}^+}\xi(m)\right).
\end{aligned} \tag{16}$$

The identity (14) is established by using (15) and (16). □

The following error estimate of the HH inequality is obtained with the help of Lemma 3.

Theorem 6. *Let the assumptions of Theorem 3 be true. Then, the following inequality*

$$\begin{aligned}
&\left|\frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{(\varrho^{s+1}(n)-\varrho^{s+1}(m))^{\frac{u}{k}}}\left({}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}+}\xi(n)+{}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}-}\xi(m)\right)\right.\\
&\left.-\xi\left(\left(\varrho^{s+1}\right)^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)\right)\right| \qquad (17)\\
&\leq\frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{4(\frac{u}{k}+1)}\left(\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(m)\right|+\left|\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\varrho^{s+1}(n)\right|\right).
\end{aligned}$$

holds.

Proof. By using the fact that $\left|(\xi\circ\left(\varrho^{s+1}\right)^{-1})'\right|$ is convex and basic calculus, Lemma 3 implies the following:

$$\begin{aligned}
&\left|\frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{(\varrho^{s+1}(n)-\varrho^{s+1}(m))^{\frac{u}{k}}}\left({}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}+}\xi(n)+{}_k^sI^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}-}\xi(m)\right)\right.\\
&\left.-\xi\left(\left(\varrho^{s+1}\right)^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)\right)\right|\\
&\leq\frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{4(\frac{u}{k}+1)}\left(\int_0^1\left|\delta^{\frac{u}{k}}\left(\xi\circ\left(\varrho^{s+1}\right)^{-1}\right)'\left(\frac{\varrho^{s+1}(m)\delta}{2}\right)+\left(\frac{2-\delta}{2}\right)\varrho^{s+1}(n)\right|d\delta\right.\\
&\left.+\int_0^1\left|\delta^{\frac{u}{k}}(\xi\circ(\varrho^{s+1})^{-1})'\left(\frac{\varrho^{s+1}(m)(2-\delta)}{2}\right)+(\frac{\delta}{2})\varrho^{s+1}(n)\right|d\delta\right)\\
&\leq\frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{4(\frac{u}{k}+1)}\left(\left(\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(m)\right|+\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(n)\right|\right)\int_0^1\delta^{\frac{u}{k}}d\delta\right).
\end{aligned}$$

Thus, a few calculations give (17). □

In the next examples, we discuss the applications of Theorem 6 for some specific functions.

Example 18. *Putting $\varrho^{s+1}(x)=e^x$ in (17), we obtain the inequality*

$$\begin{aligned}
&\left|\frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{e^n-e^m}\left({}_k^sI^u_{(\frac{e^m+e^n}{2})+}\xi(n)+{}_k^sI^u_{(\frac{e^m+e^n}{2})-}\xi(m)-\xi\left(ln\left(\frac{e^m+e^n}{2}\right)\right)\right)\right|\\
&\leq\frac{|e^n-e^m|}{4(\frac{u}{k}+1)}\left(\left|e^m\xi'(m)\right|+\left|e^n\xi'(n)\right|\right).
\end{aligned}$$

Example 19. *Putting $\varrho^{s+1}(x)=\frac{1}{x}$ in (17), we obtain the following:*

$$\begin{aligned}
&\left|\frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)(mn)^{\frac{u}{k}}}{(n-m)^{\frac{u}{k}}}\left({}_k^sI^u_{(\frac{m+n}{2mn})-}\xi\circ g\left(\frac{1}{n}\right)+{}_k^sI^u_{(\frac{m+n}{2mn})+}\xi\circ g\left(\frac{1}{m}\right)\right)\right.\\
&\leq\frac{(n-m)}{4(\frac{u}{k}+1)|mn|}\left(m^2\left|\xi'(m)\right|+n^2\left|\xi'(n)\right|\right),
\end{aligned}$$

where $g(\delta)=\dfrac{1}{\delta}$.

Example 20. *Putting $\varrho^{s+1}(x) = x^r$ where $r \neq 0$ in (17), we obtain the following inequality:*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)(r)^{\frac{u}{k}}}{(n^r - m^r)^{\frac{u}{k}}} \left(\left({}^s_k I^u_{\left(\frac{m^r+n^r}{2}\right)^{\frac{1}{r}}+} \xi(n) + {}^s_k I^u_{\left(\frac{m^r+n^r}{2}\right)^{\frac{1}{r}}-} \xi(m) \right) - \xi\left(\frac{m^r+n^r}{2} \right)^{\frac{1}{r}} \right) \right|$$
$$\leq \frac{|n^r - m^r|(\frac{u}{k}+2)}{4(\frac{u}{k}+1)|r|} \left(m^{(1-r)}|\xi'(m)| + n^{(1-r)}|\xi'(n)| \right).$$

Example 21. *Putting $\varrho^{s+1}(x) = \ln(x)$ in (17), we obtain the inequality*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\ln(n) - \ln(m)} \left({}^s_k I^u_{\left(\frac{\ln(m)+\ln(n)}{2}\right)+} \xi(n) + {}^s_k I^u_{\left(\frac{ln(m)+ln(n)}{2}\right)-} \xi(m) - \xi\left(exp\left(\frac{\ln(m)+\ln(n)}{2} \right) \right) \right) \right|$$
$$\leq \frac{|\ln(n) - \ln(m)|}{4(\frac{u}{k}+1)} \left(m|\xi'(m)| + n|\xi'(n)| \right).$$

Next, we give another theorem.

Theorem 7. *Let the assumptions of Theorem 4 be true. Then, the following inequality*

$$\begin{aligned} &\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\left(\varrho^{s+1}(n)-\varrho^{s+1}(m)\right)^{\frac{u}{k}}} \left({}^s_k I^u_{\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)+} \xi(n) + {}^s_k I^u_{\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)-} \xi(m) \right) \right. \\ &\left. -\xi\left(\varrho^{s+1}\right)^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2} \right) \right| \\ &\leq \frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}} \left[\left(\left| (\xi \circ (\varrho^{s+1})^{-1})'\varrho^{s+1}(m) \right|^q \left(\frac{u}{k}+1 \right) \right. \right. \\ &\left. + \left| (\xi \circ (\varrho^{s+1})^{-1})'\varrho^{s+1}(n) \right|^q \left(\frac{u}{k}+3 \right) \right)^{\frac{1}{q}} \\ &\left. + \left(\left| (\xi \circ (\varrho^{s+1})^{-1})'\varrho^{s+1}(m) \right|^q \left(\frac{u}{k}+3 \right) + \left| (\xi \circ (\varrho^{s+1})^{-1})'\varrho^{s+1}(n) \right|^q \left(\frac{u}{k}+1 \right) \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{18}$$

holds.

Proof. If $q = 1$, then the above inequality can be obtained using basic calculus in Lemma 3 and $\left|(\xi \circ (\varrho^{s+1})^{-1})'\right|^q$ is convex.

Now, suppose that $q > 1$. The right-hand side of Lemma 3, along with the power mean inequality and simple calculus, imply the following inequality:

$$\begin{aligned} &\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{(\varrho^{s+1}(n) - \varrho^{s+1}(m))^{\frac{u}{k}}} \left({}^s_k I^u_{\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)+} \xi(n) + {}^s_k I^u_{\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)-} \xi(m) \right) \right. \\ &\left. - \xi\left(\left(\varrho^{s+1}\right)^{-1}\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2} \right) \right) \right| \\ &\leq \frac{|\varrho^{s+1}(n) - \varrho^{s+1}(m)|}{4} \left(\int_0^1 \delta^{\frac{u}{k}} d\delta \right)^{1-\frac{1}{q}} \\ &\times \left(\int_0^1 \left(\left| \delta^{\frac{u}{k}} \left(\xi \circ (\varrho^{s+1})^{-1} \right)' \left(\frac{\varrho^{s+1}(m)\delta}{2} + \left(\frac{2-\delta}{2} \right) \varrho^{s+1}(n) \right) \right|^q d\delta \right. \right. \\ &\left. + \int_0^1 \left| \delta^{\frac{u}{k}} \left(\xi \circ (\varrho^{s+1})^{-1} \right)' \left(\frac{\varrho^{s+1}(m)(2-\delta)}{2} + \left(\frac{\delta}{2} \right) \varrho^{s+1}(n) \right) \right|^q d\delta \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq \frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{4(\frac{u}{k}+1)^{1-\frac{1}{q}}}$$

$$\times\left[\left(\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(m)\right|^q\int_0^1\frac{\delta^{\frac{u}{k}+1}}{2}d\delta+\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(n)\right|^q\int_0^1\frac{(2-\delta)\delta^{\frac{u}{k}}}{2}d\delta\right)^{1-\frac{1}{q}}\right.$$

$$\left.+\left(\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(m)\right|^q\int_0^1\frac{(2-\delta)\delta^{\frac{u}{k}}}{2}d\delta+\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(n)\right|^q\int_0^1\frac{\delta^{\frac{u}{k}+1}}{2}d\delta\right)^{1-\frac{1}{q}}\right]$$

$$=\frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}}\left[\left(\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(m)\right|^q\left(\frac{u}{k}+1\right)+\right.\right.$$

$$\left.\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(n)\right|^q\left(\frac{u}{k}+3\right)\right)^{\frac{1}{q}}$$

$$\left.+\left(\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(m)\right|^q\left(\frac{u}{k}+3\right)+\left|(\xi\circ(\varrho^{s+1})^{-1})'\varrho^{s+1}(n)\right|^q\left(\frac{u}{k}+1\right)\right)^{\frac{1}{q}}\right].$$

□

In the next examples, we discuss the applications of Theorem 7 for some specific functions.

Example 22. *By substituting* $\varrho^{s+1}(x)=e^x$ *in (18), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{e^n-e^m}\left({}_k^sI^u_{(\frac{e^m+e^n}{2})^+}\xi(n)+{}_k^sI^u_{(\frac{e^m+e^n}{2})^-}\xi(m)-\xi\left(ln\left(\frac{e^m+e^n}{2}\right)\right)\right)\right|$$

$$\leq\frac{|e^n-e^m|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}}$$

$$\times\left[\left(|e^m\xi'(m)|^q\left(\frac{u}{k}+1\right)+|e^n\xi'(n)|^q\left(\frac{u}{k}+3\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(|e^m\xi'(m)|^q\left(\frac{u}{k}+3\right)+|e^n\xi'(n)|^q\left(\frac{u}{k}+1\right)\right)^{\frac{1}{q}}\right].$$

Example 23. *By substituting* $\varrho^{s+1}(x)=\dfrac{1}{x}$ *in (18), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)(mn)^{\frac{u}{k}}}{(n-m)^{\frac{u}{k}}}\left({}_k^sI^u_{(\frac{m+n}{2mn})^-}\xi\circ g\left(\frac{1}{n}\right)+{}_k^sI^u_{(\frac{m+n}{2mn})^+}\xi\circ g\left(\frac{1}{m}\right)\right)\right|$$

$$\leq\frac{|m-n|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}|mn|}$$

$$\times\left[\left(m^{2q}|\xi'(m)|^q\left(\frac{u}{k}+1\right)+n^{2q}|\xi'(n)|^q\left(\frac{u}{k}+3\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(m^{2q}|\xi'(m)|^q\left(\frac{u}{k}+3\right)+n^{2q}|\xi'(n)|^q\left(\frac{u}{k}+1\right)\right)^{\frac{1}{q}}\right],$$

where $g(\nu)=\frac{1}{\nu}$.

Example 24. *By substituting $\varrho^{s+1}(x) = x^r$ where $r \neq 0$ in (18), we obtain the inequality*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)(r)^{\frac{u}{k}}}{(n^r - m^r)^{\frac{u}{k}}} \left({}^s_k I^u_{\left(\frac{m^r+n^r}{2}\right)^{\frac{1}{r}}+} \xi(n) + {}^s_k I^u_{\left(\frac{m^r+n^r}{2}\right)^{\frac{1}{r}}-} \xi(m) \right) - \xi\left(\frac{m^r+n^r}{2}\right)^{\frac{1}{r}} \right|$$
$$\leq \frac{|n^r - m^r|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}|r|}$$
$$\times \left[\left(m^{q(1-r)}|\xi'(m)|^q \left(\frac{u}{k}+1\right) + n^{q(1-r)}|\xi'(n)|^q \left(\frac{u}{k}+3\right) \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(m^{q(1-r)}|\xi'(m)|^q \left(\frac{u}{k}+3\right) + n^{q(1-r)}|\xi'(n)|^q \left(\frac{u}{k}+1\right) \right)^{\frac{1}{q}} \right].$$

Example 25. *By substituting $\varrho^{s+1}(x) = \ln(x)$ in (18), we obtain*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\ln(n) - \ln(m)} \left({}^s_k I^u_{\left(\frac{\ln(m)+\ln(n)}{2}\right)+} \xi(n) + {}^s_k I^u_{\left(\frac{\ln(m)+\ln(n)}{2}\right)-} \xi(m) - \xi\left(exp\left(\frac{\ln(m)+\ln(n)}{2} \right) \right) \right) \right|$$
$$\leq \frac{|\ln(n) - \ln(m)|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}}$$
$$\times \left[\left(m^q|\xi'(m)|^q \left(\frac{u}{k}+1\right) + n^q|\xi'(n)|^q \left(\frac{u}{k}+3\right) \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(m^q|\xi'(m)|^q \left(\frac{u}{k}+3\right) + n^q|\xi'(n)|^q \left(\frac{u}{k}+1\right) \right)^{\frac{1}{q}} \right].$$

Theorem 8. *The following inequality*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\left(\varrho^{s+1}(n)-\varrho^{s+1}(m)\right)^{\frac{u}{k}}} \left({}^s_k I^u_{\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)+} \xi(n) + {}^s_k I^u_{\left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}\right)-} \xi(m) \right) \right.$$
$$\left. -\xi\left(\left(\varrho^{s+1}\right)^{-1} \left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2} \right) \right) \right| \tag{19}$$
$$\leq \frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}} \left(\left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(m) \right| + \left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(n) \right| \right).$$

holds under the same assumptions as in Theorem 5.

Proof. By using Hölder's inequality and basic calculus in Lemma 3, we obtain

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\left(\varrho^{s+1}(n)-\varrho^{s+1}(m)\right)^{\frac{u}{k}}} \left({}^s_k I^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}+} \xi(n) + {}^s_k I^u_{\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2}-} \xi(m) \right) \right.$$
$$\left. -\xi\left((\varrho^{s+1})^{-1} \left(\frac{\varrho^{s+1}(m)+\varrho^{s+1}(n)}{2} \right) \right) \right|$$
$$\leq \frac{|\varrho^{s+1}(n)-\varrho^{s+1}(m)|}{4} \left(\int_0^1 \delta^{\frac{up}{k}} d\delta \right)^{\frac{1}{p}} \tag{20}$$
$$\times \left[\int_0^1 \left| (\xi \circ (\varrho^{s+1})^{-1})' \left(\frac{\varrho^{s+1}(m)\delta}{2} \right) + \left(\frac{2-\delta}{2} \right) \varrho^{s+1}(n) \right| d\delta \right.$$
$$\left. + \int_0^1 \left| \delta^{\frac{u}{k}} (\xi \circ (\varrho^{s+1})^{-1})' \left(\frac{\varrho^{s+1}(m)\left(2-\delta\right)}{2} \right) + \left(\frac{\delta}{2} \right) \varrho^{s+1}(n) \right|^q d\delta \right].$$

As $|(\xi \circ (\varrho^{s+1})^{-1})'|$ is convex, the right-hand side of the inequality (20) takes the following form:

$$\leq \frac{|\varrho^{s+1}(n) - \varrho^{s+1}(m)|}{4(\frac{up}{k}+1)^{\frac{1}{p}}} \left(\left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(m) \right|^q \int_0^1 \frac{\delta}{2} d\delta + \right.$$

$$\left. \left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(n) \right|^q \int_0^1 \left(\frac{2-\delta}{2} \right) d\delta \right)^{\frac{1}{q}}$$

$$+ \left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(m) \right|^q \int_0^1 \left(\frac{2-\delta}{2} \right) d\delta + \left(\left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(n) \right) \right|^q \int_0^1 \frac{\delta}{2} d\delta \right)^{\frac{1}{q}}$$

$$\leq \frac{|\varrho^{s+1}(n) - \varrho^{s+1}(m)|}{4(\frac{up}{k}+1)^{\frac{1}{p}}} \left[\left(\frac{|(\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(m)|^q + 3|(\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(n)|^q}{4} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{3|(\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(m)|^q + |(\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(n)|^q}{4} \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{|\varrho^{s+1}(n) - \varrho^{s+1}(m)|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}} \left(\left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(m) \right| + \left| (\xi \circ (\varrho^{s+1})^{-1})' \varrho^{s+1}(n) \right| \right).$$

In the above computations, we use $m^p + n^q \leq (m+n)^q$ where $q > 1$ and $m, n \geq 0$ to obtain our desired result. □

In the next examples, we discuss the applications of Theorem 8 for some specific functions.

Example 26. *Putting $\varrho^{s+1}(x) = e^x$ in (19), we obtain*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}} \Gamma_k(u+k)}{e^n - e^m} \left({}_k^s I^u_{(\frac{e^m+e^n}{2})^+} \xi(n) + {}_k^s I^u_{(\frac{e^m+e^n}{2})^-} \xi(m) - \xi\left(ln\left(\frac{e^m+e^n}{2} \right) \right) \right| \right.$$

$$\leq \frac{|e^n - e^m|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}} \left(e^m |\xi'(m)| + e^n |\xi'(n)| \right).$$

Example 27. *By substituting $\varrho^{s+1}(x) = \dfrac{1}{x}$ in (19), we obtain the inequality*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}} \Gamma_k(u+k)(mn)^{\frac{u}{k}}}{(n-m)^{\frac{u}{k}}} \left({}_k^s I^u_{(\frac{m+n}{2mn})^-} \xi \circ g\left(\frac{1}{n} \right) + {}_k^s I^u_{(\frac{m+n}{2mn})^+} \xi \circ g\left(\frac{1}{m} \right) \right) \right.$$

$$\leq \frac{|m-n|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}|mn|} \left(m^2 |\xi'(m)| + n^2 |\xi'(n)| \right),$$

where $g(\delta) = \dfrac{1}{\delta}$.

Example 28. *By using $\varrho^{s+1}(x) = x^r$ in (19), we obtain*

$$\left| \frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}} \Gamma_k(u+k)(r)^{\frac{u}{k}}}{(n^r - m^r)^{\frac{u}{k}}} \left(\left({}_k^s I^u_{(\frac{m^r+n^r}{2})^{\frac{1}{r}+}} \xi(n) + {}_k^s I^u_{(\frac{m^r+n^r}{2})^{\frac{1}{r}-}} \xi(m) \right) - \xi\left(\frac{m^r+n^r}{2} \right)^{\frac{1}{r}} \right) \right|$$

$$\leq \frac{|n^r - m^r|}{4^{1-\frac{1}{p}}|r|(\frac{up}{k}+1)^{\frac{1}{p}}} \left(m^{1-r} |\xi'(m)| + n^{1-r} |\xi'(n)| \right).$$

Example 29. *Putting* $\varrho^{s+1}(x) = \ln(x)$ *in (19), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}(s+1)^{\frac{u}{k}}\Gamma_k(u+k)}{\ln(n)-\ln(m)}\left({}_{k}^{s}I^{u}_{\left(\frac{\ln(m)+\ln(n)}{2}\right)^{+}}\xi(n) + {}_{k}^{s}I^{u}_{\left(\frac{\ln(m)+\ln(n)}{2}\right)^{-}}\xi(m) - \xi\left(exp\left(\frac{\ln(m)+\ln(n)}{2}\right)\right)\right)\right|$$
$$\leq \frac{|\ln(n)-\ln(m)|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}}\left(m|\xi'(m)|+n|\xi'(n)|\right).$$

4. Concluding Remarks

Several generalized novel mean-type inequalities were obtained via certain convex functions using fractional calculus and convexity theory. Using the generalized form of the RL-fractional integral, namely the (k,s)-RL-fractional integral, we established various mean-type inequalities for strictly monotone functions in this article. The main results were formulated after first establishing a special identity. Through different choices of strictly monotone functions, the corresponding inequalities were also produced. The findings of this article can be reduced to that of [22] by taking $k = 1$ and $s = 0$. The obtained results are hopefully helpful in the field of modified scientific disciplines.

Author Contributions: Conceptualization, M.S. and M.M.; methodology, I.S. and M.Y.; software, M.S. and M.M.; validation, M.M., M.V.-C. and M.Y.; formal analysis, M.S. and I.S.; investigation, M.M., M.S., I.S. and M.V.-C.; data curation, M.Y., M.M. and I.S.; writing—original draft preparation, M.S. and M.M.; writing—review and editing, M.S., M.M. and I.S.; visualization, M.M.; supervision, M.M. and M.S.; project administration, M.V.-C. and M.S., funding acquisition, M.V.-C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Almeida, R.; Bastos, N.R.O.; Monteiro, M.T.T. An optimization method for the best fractional order to estimate real data analysis. In Proceedings of the 15th International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2015, Rota Cadiz, Spain, 3–7 July 2015.
2. Drapaca, C.S.; Sivaloganathan, S. A Fractional model of continuum mechanics. *J. Elast.* **2012**, *107*, 105–123. [CrossRef]
3. Machado, J.A.T.; Silva, M.F.; Barbosa, R.S.; Jesus, I.S.; Reis, C.M.; Marcos, M.G.; Galhano, A.F. Some applications of fractional calculus in engineering. *Math. Probl. Eng.* **2010**, *2010*, 639801. [CrossRef]
4. Failla, G.; Zingales, M. Advanced materials modelling via fractional calculus: Challenges and perspectives. *Phil.Trans. R. Soc. A* **2020**, *378*, 20200050. [CrossRef] [PubMed]
5. Sopasakis, P.; Sarimveis, H.; Macheras, P.; Dokoumetzidis, A. Fractional calculus in pharmacokinetics. *J. Pharmacokinet Pharmacodyn.* **2018**, *45*, 107–125. [CrossRef] [PubMed]
6. Cesarone, F.; Caputo, M.; Cametti, C. Memory formalism in the passive diffusion across a biological membrane. *J. Membr. Sci.* **2004**, *250*, 79–84. [CrossRef]
7. ElShahed, E. A fractional calculus model of semilunar heart valve vibrations. In Proceedings of the International Mathematica Symposium, London, UK, 7–11 July 2003.
8. Iaffaldano, G.; Caputo, M.; Martino, S. Experimental and theoretical memory diffusion of water in the sand. *Hydrol. Earth Syst. Sci.* **2006**, *10*, 93–100. [CrossRef]
9. Jumarie, G. New stochastic fractional models for malthusian growth, the poissonian birth process and optimal management of populations. *Math. Comput. Model.* **2006**, *44*, 231–254. [CrossRef]
10. Caputo, M. *The Role of Memory in Modeling Social and Economic Cycles of Extreme Events. A Handbook of Alternative Theories of Public Economics*; Edward Elgar Publishing: Cheltenham, UK, 2014; pp. 245–259.
11. Magin, R.L. *Fractional Calculus in Bio-Engineering*; Begell House Publishers: Danbury, CT, USA, 2006.
12. Temur, Z. Kalanov vector calculus and Maxwells equations: Logic errors in mathematics and electrodynamics. *Open J. Math. Sci.* **2020**, *4*, 343–355.
13. Ghiasi, E.K.; Noeiaghdam, S. Truncating the series expansion for unsteady velocity-dependent Eyring-Powell fluid. *Eng. Appl. Sci. Lett.* **2020**, *3*, 28–34. [CrossRef]
14. Hadamard, J. Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appliquées* **1893**, 171–216. Available online: http://eudml.org/doc/234668 (accessed on 25 February 2023).

15. Wu, S.; Iqbal, S.; Aamir, M.; Samraiz, M.; Younus, A. On some Hermite–Hadamard inequalities involving k-fractional operators. *J. Inequal. Appl.* **2021**, *2021*, 32. [CrossRef]
16. Baleanu, D.; Samraiz, M.; Perveen, Z.; Iqbal, S.; Nisar, K.S.; Rahman, G. Hermite-Hadamard-Fejer type inequalities via fractional integral of a function concerning another function. *AIMS Math.* **2021**, *6*, 4280–4295. [CrossRef]
17. Kang, S.M.; Farid, G.; Nazeer, W.; Tariq, B. Hadamard and Fejér–Hadamard inequalities for extended generalized fractional integrals involving special functions. *J. Inequal. Appl.* **2018**, *2018*, 119. [CrossRef] [PubMed]
18. Kwun, Y.C.; Farid, G.; Ullah, S.; Nazeer, W.; Mahreen, K.; Kang, S.M. Inequalities for a unified integral operator and associated results in fractional calculus. *IEEE Access* **2019**, *7*, 126283–126292. [CrossRef]
19. Kwun, Y.C.; Saleem, M.S.; Ghafoor, M.; Nazeer, W.; Kang, S.M. Hermite–Hadamard-type inequalities for functions whose derivatives are ξ-convex via fractional integrals. *J. Inequalities Appl.* **2019**, *44*, 1–16. [CrossRef]
20. Samraiz, M.; Nawaz, F.; Abdalla, B.; Abdeljawad, T.; Rahman, G.; Iqbal, S. Estimates of trapezium-type inequalities for h-convex functions with applications to quadrature formulae. *AIMS Press* **2021**, *6*, 7625–7648. [CrossRef]
21. Samraiz, M.; Nawaz, F.; Iqbal, S.; Abdeljawad, T.; Rahman, G.; Nisar, K.S. Certain mean-type fractional integral inequalities via different convexities with applications. *J. Inequal. Appl.* **2020**, *2020*, 208. [CrossRef]
22. Farid, G.; Pećarić, J.; Nonlaopon, K. Inequalities for fractional Riemann–Liouville integrals of certain class of convex functions. *Adv. Contin. Discret. Model.* **2022**, *2022*, 8. [CrossRef]
23. Priya, M.; Uthayakumar, R. Analytical properties of (k,s)-Riemann–Liouville fractional integral and its fractal dimension. *J. Anal.* **2021**, *2021*, 1391–1402. [CrossRef]
24. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
25. İşcan, I.; Wu, S. Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. *Appl. Math. Comput.* **2014**, *238*, 237–244. [CrossRef]
26. Toplu, T.; Set, E.; İşcan, I.; Maden, S. Hermite–Hadamard-type inequalities for p-convex functions via Katugampola fractional integrals. *Facta Univ. Ser. Math. Inform.* **2019**, *34*, 149–164. [CrossRef]
27. İşcan, I. Hermite–Hadamard type inequalities for harmonically convex functions. *Hacet. J. Math. Stat.* **2014**, *43*, 935–942. [CrossRef]
28. Farid, G.; Nazeer, W.; Saleem, M.S.; Mehmood, S.; Kang, S.M. Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications. *Mathematics* **2018**, *6*, 248. [CrossRef]
29. Kwun, Y.C.; Farid, G.; Nazeer, W.; Ullah, S.; Kang, S.M. Generalized Riemann-Liouville k-fractional integrals associated with ostrowski type inequalities and error bounds of Hadamard inequalities. *IEEE Access* **2018**, *6*, 64946–64953. [CrossRef]
30. Yang, X.; Farid, G.; Nazeer, W.; Yussouf, M.; Chu, Y.M.; Dong, C. Fractional generalized Hadamard and Fejér-Hadamard inequalities for m-convex functions. *AIMS Math.* **2020**, *5*, 6325–6340. [CrossRef]
31. Turhan, S.; Kunt, M.; İşcan, I. Hermite–Hadamard type inequalities for M_ϕ A-convex functions. *Int. J. Math. Model.Comp.* **2020**, *10*, 57–75.
32. Diaz, R.; Pariguan, E. On hypergeometric function and pochhammer k-symbol. *Divulg. Math.* **2007**, *15*, 179–192.
33. Kilbas, A.A.; Srivastava, H.M.; Trujilo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies: New York, NY, USA, 2006.
34. Sarıkaya, M.Z.; Dahmani, Z.; Kiris, M.E.; Ahmad, F. (k,s)-Riemann-Liouville fractional integral and applications. *Hacet. J. Math. Stat.* **2016**, *45*, 77–89. [CrossRef]
35. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series: Elementary Functions*; Gordon and Breach: New York, NY, USA, 1986; Volume 1.