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Fast Linear Canonical Transform for Nonequispaced Data

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Abstract: The investigations of the discrete and fast linear canonical transform (LCT) are becoming one of the hottest research topics in modern signal processing and optics. Among them, the fast calculation of LCT for non-uniform data is one of key problems. Focus on this problem, a new fast algorithm of the LCT has been proposed in this paper firstly by interpolation and approximation theory. The proposed algorithms can calculate quickly the LCT of the data, whether the input or output data is uniform. Secondly, the complexity and precision of derived algorithms have been analyzed for different situations. Finally, the experimental results are presented to verify the correctness of the obtained results.

Keywords: linear canonical transform; fast fourier transform; interpolation; approximation theory; linear canonical series

1. Introduction

Linear canonical transform which is generated by second-order differential operators, is a four-parameter class of linear integral transform [1,2],

$$F^A(u) = \begin{cases} \int_{-\infty}^{\infty} x(t)K_A(t,u)dt, & b \neq 0 \\ \sqrt{(d)}e^{i\frac{cdu^2}{2}}x(du), & b = 0 \end{cases}$$

where

$$K_A(t,u) = \frac{1}{i2\pi b}e^{i\frac{a}{2b}t^2 - i\frac{tu}{b} + i\frac{d}{2b}u^2}$$

$A = [a,b;c,d], ad - bc = 1, F^A(u)$ is LCT of the function $x(t)$. The LCT integral kernels are Green functions of quadratic Hamiltonians that can be found in [3]. The research on LCT was first proposed by Collins (1970) and Moshinsky (1971) [1,2]. It includes many special cases, such as, the Fourier transform (FT), the fractional Fourier transform (FRFT), the Fresnel transform, the Lorentz transform and scaling transform. The class of LCTs are important in signal processing [4,5], computational and applied mathematics [6,7], optics [8] and quantum mechanics [9]. Significant applications of LCT in signal processing include radar system analysis, filter design, pattern recognition, image watermarking and so on [10,11]. Basic theories of LCT have been developed that include convolution theorems, sampling theorems, and uncertainty principles [12–14]. The numerical approximation of the LCT is of importance in modeling first-order optical systems and many signal processing applications. Therefore, the discrete and fast algorithms of the LCT are one of the most important issues in practical applications.

After the continuous LCT has been introduced, therefore, the definition and fast implementation of the discrete linear canonical transform (DLCT) have been widely considered by many researchers [15–28]. The existing algorithms can be divided into the following categories. The one is the operator decomposition type, which decomposes an arbitrary LCT operator into its special cases that have a fast algorithm [17,18,29]. The second is the eigenvector decomposition type, which computes the LCT of a function by using the eigenfunctions of the LCT [24,27]. The third is split basis algorithm, which decomposes

the discrete transform matrix of LCT into smaller matrices iteratively [20,23]. All these algorithms can effectively and rapidly calculate the entire spectrum in the LCT domain. In recent years, the new LCT algorithms have been proposed, which can realize local spectrum analysis of signals [30–32]. In addition to the above-mentioned algorithms, various other fast algorithms have been proposed in [33–39]. These research results provide a good basis for the further development of the DLCT toward meeting the requirements of practical applications. Many aspects of the fast methods of the DLCT still need to be studied. To the best of our knowledge, the existing discrete algorithms are required that both input and output data are uniform sampling. However, for certain applications, the input or output data is nonuniform. In these cases, the fast DLCT will be lost.

For overcome the aforementioned problems, in this paper, we present a set of fast algorithms for computing nonuniform DLCT, namely,

$$f(t_n) = \frac{1}{\sqrt{-i2\pi b}} \sum_{m=0}^{N-1} \alpha_m e^{-i\frac{a}{2b}t_n^2 + i\frac{u_m t_n}{b} - i\frac{d}{2b}u_m^2} \tag{1}$$

where $n = 0, 1, \cdots N$, $f(t_n) \in \mathbf{C}$, $\alpha_m \in \mathbf{C}$, $t_n \in [-b\pi, b\pi]$, and $u_m \in [-N/2, N/2 - 1]$. According to the sampling of the t_n and u_m, we will operate under the following assumptions

- Uniform samples and non integer frequencies: In Equation (1) the samples are equi-spaced, i.e., $t_j = j2\pi|b|/N$, and the frequencies $u_0, \cdots, u_{N-1}$ are non integer. This corresponds to evaluating a generalized linear canonical series at equispaced points.
- Nonuniform samples and integer frequencies: In Equation (1), $t_0, \cdots, t_{N-1}$ are nonequispaced points in $[-b\pi, b\pi]$ and the frequencies $u_0, \cdots, u_{N-1}$ are integers.
- Nonuniform samples and non-integer frequencies: In Equation (1), $t_0, \cdots, t_{N-1}$ are nonequispaced points in $[-b\pi, b\pi]$ and the frequencies $u_0, \cdots, u_{N-1}$ are non-integers. This is the fully nonuniform transform and corresponds to evaluating a generalize linear canonical series at nonequispaced points.

To develop various nonuniform fast LCT(NFLCT) for above issues, one has to exploit a nonzero working precision of $0 < \epsilon < 1$ and makes careful approximations. The approximation properties of the various approaches may be obtained by considering how they perform on the linear canonical modes.

The rest of the paper is organized as follows: In Section 2, a brief review of the related preliminaries is presented which are used in the design of the algorithms. The Section 3 is main results of this paper. In this section, we give an exact statement of the problem and introduce some notation that is used. The algorithms are derived. Some numerical examples are presented in Section 4 to illustrate the preference of the schemes. Finally, conclusions are drawn in Section 5.

2. Preliminary

The linear canonical series (LCS) is a generalized form of Fourier series (FS), which can reveal the mixed time and frequency components of signals. The basis function of LCS is defined as [40]

$$\varphi_{A,n}(t) = \sqrt{\frac{i}{T}} e^{-i\frac{a}{2b}t^2 - i\frac{d}{2b}(n2\pi b/T)^2 + i\frac{t}{b}(n2\pi b/T)} \tag{2}$$

where $n = -\infty, \cdots, -1, 0, 1, \cdots, +\infty$. Thus, $\{\cdots, \varphi_{A,-1}(t), \varphi_{A,0}(t), \varphi_{A,1}(t), \cdots\}$ construct an orthonormal basis. It can be observed that every basis function is a chirp function with chirp rate $-a/b$, which is an aperiodic function. Therefore, the LCS is only applicable to finite-length function. The LCS expansion of the finite-length function $x(t)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} C_{A,n} \varphi_{A,n}(t) = \sum_{n=-\infty}^{\infty} C_{A,n} \sqrt{\frac{i}{T}} e^{-i\frac{a}{2b}t^2 - i\frac{d}{2b}(n2\pi b/T)^2 + i\frac{t}{b}(n2\pi b/T)} \tag{3}$$

$$C_{A,n} = \sqrt{\frac{-i}{T}} \int_{-T/2}^{T/2} x(t) e^{i\frac{a}{2b}t^2 + i\frac{d}{2b}(n2\pi b/T)^2 - i\frac{t}{b}(n2\pi b/T)} \mathrm{d}t \tag{4}$$

where $t \in [-T/2, T/2]$ and $C_{A,n}$ are called LCS expansion coefficients with the parameter matrix A. The LCS expansion coefficients are computed by the inner product of the function and chirp basis function. The relationship between LCS and LCT is that the LCS expansion coefficients are the sampled values of LCT, by

$$C_{A,n} = \sqrt{\frac{2\pi b}{T}} F^A[x(t)](n\frac{2\pi b}{T}) \tag{5}$$

The well-known FS is just a special case of LCS for the parameter matrix $A = [0, 1; -1, 0]$. In addition, we also presented some well results to be used in the remainder of the paper [41].

Lemma 1. *For any real $\alpha > 0$ and complex z,*

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} e^{zt} dt = \sqrt{\frac{\pi}{\alpha}} e^{z^2/4\alpha} \tag{6}$$

Lemma 2. *For any real $\alpha > 0$ and $r > 0$,*

$$\int_{r}^{\infty} e^{-\alpha t^2} dt < \frac{e^{-\alpha r^2}}{2\alpha r} \tag{7}$$

In the next section, the main results will be derived based on above facts.

3. Main Results

3.1. Exact Statement of the Problem and Information Description of the Algorithm

3.1.1. Exact Statement of the Problem

In the reminder of this paper, we will operate under the following assumptions:

1. $u = \{u_0, u_1, \cdots, u_{N-1}\}$ and $t = \{t_0, t_1, \cdots, t_{N-1}\}$ are finite sequences of real numbers.
2. $u_m \in [-N/2, N/2 - 1]$ for $m = 0, 1, \cdots, N - 1$.
3. $t_j \in [-b\pi, b\pi]$ for $j = 0, 1, \cdots, N - 1$.
4. $\boldsymbol{\alpha} = \{\alpha_0, \cdots, \alpha_{N-1}\}$, $\mathbf{f} = \{f_{-N/2}, \cdots, f_{N/2-1}\}$, $\boldsymbol{\beta} = \{\beta_{-N/2}, \cdots, \beta_{N/2-1}\}$, $\mathbf{g} = \{g_0, g_1, \cdots, g_{N-1}\}$, $\boldsymbol{\gamma} = \{\gamma_0, \cdots, \gamma_{N-1}\}$ and $\{h_0, h_1, \cdots, h_{N-1}\}$ are finite sequences of complex numbers.

We will consider the problems of applying and inverting the LCT and its transpose. We are interested in the transformations $F, G: C^N \to C^N$ and their inverse defined by formulate

$$f_j = F(\boldsymbol{\alpha})_j = \sum_{k=0}^{N-1} \alpha_k e^{-i\frac{a}{2b}(\frac{2b\pi j}{N})^2 + i\frac{u_k}{b}\frac{2\pi j}{N} - i\frac{d}{2b}u_k^2} \tag{8}$$

for $j = -N/2, \cdots, N/2 - 1$, and

$$g_j = G(\mathbf{g})_j = \sum_{k=-N/2}^{N/2-1} \beta_k e^{-i\frac{a}{2b}t_j^2 + i\frac{kt_j}{b} - i\frac{d}{2b}k^2} \tag{9}$$

For $j = 0, 1, \cdots, N - 1$, we will also consider the more general transformation $H: C^N \to C^N$ defined by the formula

$$h_j = H(\boldsymbol{\gamma})_j = \sum_{k=0}^{N-1} \gamma_k e^{-i\frac{a}{2b}t_j^2 + i\frac{u_k t_j}{b} - i\frac{d}{2b}u_k^2} \tag{10}$$

More formally, we consider that give $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ to find $F^A(\boldsymbol{\alpha}), G^A(\boldsymbol{\beta}), H^A(\boldsymbol{\gamma})$ respectively.

3.1.2. Information Description of the Algorithm

In this section, we give information outlines of algorithms for above problems. The algorithms are based on the following principal steps. Any function $e^{-ia/(2b)t^2+i\beta t/b-id/(2b)\beta^2}$ can be accurately represented on any finite interval on the real line using a small number of terms of the $e^{\alpha t^2}e^{-ia/(2b)t^2+ikt/b-id/(2b)k^2}$ and this number of terms of q is independent of the value β. For the efficient calculation Equations (8)–(10), the following two ways will be used,

- to approximate each $e^{-iat^2/2b+iu_m t/b-idu_m^2/2b}$ in term of a q-term LCS;
- to approximate the value of a LCS at each t_n in terms of values at the nearest q uniformly-spaced nodes.

The interpolation between equispaced and nonequispaced sets of points can thus be performed in $O(Nq)$ operations.

3.2. Derivation of Algorithms

3.2.1. Relevant Facts from Approximation Theory

The principal tool of this paper is a detailed analysis of the LCS of functions $\phi : [-b\pi, b\pi] \to \mathbf{C}$ given by the formula

$$\phi(t) = e^{-\alpha t^2}e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2} \tag{11}$$

where $\alpha > 1/2$ and β are real numbers. The lemmas and theorems are presented in the following.

Theorem 1. *The functions* $\phi(t) = e^{-\alpha t^2}e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2}$, $t \in (-b\pi, b\pi)$ *can be approximated by LCS, the error of the approximation can be obtained by the following inequality*

$$\left| \phi(t) - \sum_{k=-\infty}^{\infty} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \right| \tag{12}$$
$$< e^{-\alpha b^2\pi^2}\left[(1 + \frac{1}{\pi^2 b^2})(\frac{8\alpha(1+4b)}{9} + 2b\pi + 2) + 1 \right]$$

where $k = -\infty, \cdots, \infty$

$$\rho_{A,k} = \sqrt{\frac{-i}{2b\pi}}e^{i\frac{d}{2b}(k^2-\beta^2)}\sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\beta-k)^2}{4b^2\alpha}} \tag{13}$$

a, b, c, d are linear canonical parameters, $\alpha > 1/2$ and β are any real , $q \geq 4\alpha b^2\pi$ is even integer.

Proof. The kth LCS coefficient for $\phi(t)$ is denoted by $\sigma_{A,k}$,

$$\phi(t) = \sqrt{\frac{i}{2b\pi}}\sum_{k=-\infty}^{\infty}\sigma_{A,k}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \tag{14}$$

where $t \in (-b\pi, b\pi)$,

$$\sigma_{A,k} = \sqrt{\frac{-i}{2b\pi}}\int_{-b\pi}^{b\pi}e^{i\frac{a}{2b}t^2-i\frac{kt}{b}+i\frac{d}{2b}k^2}\phi(t)dt \tag{15}$$

For any real $\alpha > 0$ and complex z, we have the following equality

$$\int_{-\infty}^{\infty}e^{-\alpha t^2}e^{zt}dt = \sqrt{\frac{\pi}{\alpha}}e^{z^2/4\alpha} \tag{16}$$

thus

$$\sigma_{A,k} = \sqrt{\frac{-i}{2b\pi}} \int_{-b\pi}^{b\pi} e^{-\alpha t^2} e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2} e^{i\frac{a}{2b}t^2 - i\frac{kt}{b} + i\frac{d}{2b}k^2} dt$$

$$= \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} \left[\int_{-\infty}^{\infty} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt - \int_{-\infty}^{-b\pi} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt \right.$$

$$\left. - \int_{b\pi}^{\infty} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt \right]$$

$$= \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} + \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} \left[\int_{\infty}^{b\pi} e^{-\alpha t^2} e^{-i\frac{(\beta-k)t}{b}} dt \right.$$

$$\left. - \int_{b\pi}^{\infty} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt \right]$$

$$= \rho_{A,k} - \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} 2 \int_{b\pi}^{\infty} e^{-\alpha t^2} \cos[(\beta-k)t/b] dt \tag{17}$$

Rearranging Equation (17), we obtain

$$\sigma_{A,k} - \rho_{A,k} = -\sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} 2 \int_{b\pi}^{\infty} e^{-\alpha t^2} \cos[(\beta-k)t/b] dt \tag{18}$$

Owing to

$$\left| 2 \int_{b\pi}^{+\infty} e^{-\alpha t^2} \cos[(\beta-k)t/b] dt + e^{-\alpha b^2\pi^2} \int_{-b\pi}^{b\pi} e^{i\frac{(\beta-k)t}{b}} dt \right|$$

$$\leq 2 \int_{b\pi}^{+\infty} e^{-\alpha t^2} dt + 2b\pi e^{-\alpha b^2\pi^2} < 2b\pi e^{-\alpha b^2\pi^2} \left(\frac{1}{2\alpha b^2\pi^2} + 1 \right)$$

$$< 2b\pi e^{-\alpha b^2\pi^2} \left(\frac{1}{b^2\pi^2} + 1 \right) \tag{19}$$

and integrating by parts, we have

$$2 \int_{b\pi}^{+\infty} e^{-\alpha t^2} \cos[(\beta-k)t/b] dt$$

$$= -\frac{2b}{\beta-k} e^{-\alpha b^2\pi^2} \sin[(\beta-k)\pi] + \frac{4\alpha b}{\beta-k} \int_{b\pi}^{+\infty} t e^{-\alpha t^2} \sin[(\beta-k)t/b] dt \tag{20}$$

After rearranging the terms in Equation (20) and integrating by parts again, we obtain

$$\left| 2 \int_{b\pi}^{+\infty} e^{-\alpha t^2} \cos[(\beta-k)t/b] dt + \frac{2b e^{-\alpha b^2\pi^2}}{\beta-k} \sin[(\beta-k)\pi] \right|$$

$$= \left| \frac{4b\alpha}{\beta-k} \int_{b\pi}^{+\infty} t e^{-\alpha t^2} \sin[(\beta-k)t/b] dt \right|$$

$$\leq \frac{4b^2\alpha}{(\beta-k)^2} \left(b\pi e^{-\alpha b^2\pi^2} + \int_{b\pi}^{+\infty} e^{-\alpha t^2} dt + \int_{b\pi}^{+\infty} t \cdot 2\alpha t e^{-\alpha t^2} dt \right) \tag{21}$$

$$< \frac{4b^2\alpha}{(\beta-k)^2} \left(2b\pi e^{-\alpha b^2\pi^2} + 2 \int_{b\pi}^{+\infty} e^{-\alpha t^2} dt \right) < \frac{4b^2\alpha e^{-\alpha\pi^2 b^2}}{(\beta-k)^2} \left(2b\pi + \frac{2}{2\alpha b\pi} \right)$$

$$= \frac{8b^3\pi\alpha e^{-\alpha b^2\pi^2}}{(\beta-k)^2} \left(1 + \frac{1}{b^2\pi^2} \right)$$

Therefore, we can obtain the following inequalities

$$\left| 2 \int_{b\pi}^{+\infty} e^{-\alpha t^2} \cos[(\beta - k)t]dt + e^{-\alpha \pi^2 b^2} \int_{-b\pi}^{b\pi} e^{i(\beta - k)t/b} dt \right|$$
$$< \frac{8b^3 \pi \alpha e^{-\alpha b^2 \pi^2}}{(\beta - k)^2} (1 + \frac{1}{b^2 \pi^2}) \tag{22}$$

Due to Equations (19) and (22), we have the inequalities

$$\left| \sigma_{A,k} - \rho_{A,k} - \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2 - \beta^2)} \cdot e^{-\alpha \pi^2 b^2} \int_{-b\pi}^{b\pi} e^{i(\beta - k)t/b} dt \right|$$
$$< \left| \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2 - \beta^2)} \right| \cdot 2b\pi e^{-\alpha b^2 \pi^2} (\frac{1}{b^2 \pi^2} + 1) = \sqrt{2b\pi} e^{-\alpha b^2 \pi^2} (1 + \frac{1}{b^2 \pi^2}) \tag{23}$$

$$\left| \sigma_{A,k} - \rho_{A,k} - \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2 - \beta^2)} \cdot e^{-\alpha \pi^2 b^2} \int_{-b\pi}^{b\pi} e^{i(\beta - k)t/b} dt \right|$$
$$< \frac{4b^2 \alpha \sqrt{2b\pi} e^{-\alpha b^2 \pi^2}}{(\beta - k)^2} (1 + \frac{1}{b^2 \pi^2}) \tag{24}$$

for any $t \in (-b\pi, b\pi)$, combination of Equations (14), (23), and (24), we have

$$\left| \phi(t) - \sum_{k=-\infty}^{+\infty} \rho_{A,k} \sqrt{\frac{-i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} - e^{-\alpha b^2 \pi^2} e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2} \right|$$
$$= \left| \sum_{k=-\infty}^{+\infty} (\sigma_{A,k} - \rho_{A,k}) \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} - e^{-\alpha b^2 \pi^2} e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2} \right|$$
$$< \left| \sum_{k, |\beta - k| \geq |b\pi|} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \frac{4b^2 \alpha \sqrt{2b\pi} e^{-\alpha b^2 \pi^2}}{(\beta - k)^2} (1 + \frac{1}{b^2 \pi^2}) \right|$$
$$+ \left| \sum_{k, |\beta - k| < |b\pi|} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \sqrt{2b\pi} e^{-\alpha b^2 \pi^2} (1 + \frac{1}{b^2 \pi^2}) \right| \tag{25}$$
$$< \sum_{k, |\beta - k| \geq |b\pi|} \frac{4b^2 \alpha e^{-\alpha b^2 \pi^2}}{(k - \beta)^2} (1 + \frac{1}{b^2 \pi^2}) + 2|b\pi| e^{-\alpha b^2 \pi^2} (1 + \frac{1}{b^2 \pi^2})$$
$$< 4\alpha b^2 e^{-\alpha b^2 \pi^2} (1 + \frac{1}{b^2 \pi^2}) \times 2 \sum_{k=|b\pi|}^{\infty} \frac{1}{k^2} + 2(b\pi + 1) e^{-\alpha b^2 \pi^2} (1 + \frac{1}{b^2 \pi^2})$$

Due to

$$\sum_{k=|b\pi|}^{\infty} \frac{1}{k^2} < \frac{1}{|b\pi|^2} + \int_{|b\pi|}^{\infty} \frac{1}{t^2} dt = \frac{1}{|b\pi|^2} + \frac{1}{|b\pi|} < \frac{1 + 4b}{9b^2} \tag{26}$$

and substituting Equation (26) into Equation (25), we obtain

$$\left| \phi(t) - \sum_{k=-\infty}^{\infty} \rho_{A,k} \sqrt{\frac{-i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{b}k^2} - e^{-\alpha b^2 \pi^2} e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2} \right|$$
$$< e^{-\alpha b^2 \pi^2} (1 + \frac{1}{\pi^2 b^2})[\frac{8\alpha(1 + 4b)}{9} + 2b\pi + 2] \tag{27}$$

Then, we make use of the triangle inequality and Equation (27) to obtain

$$
\left| \phi(t) - \sum_{k=-\infty}^{\infty} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \right|
$$
$$
< e^{-\alpha b^2 \pi^2} \left[(1 + \frac{1}{\pi^2 b^2}) \big(\frac{8\alpha(1+4b)}{9} + 2b\pi + 2 \big) + 1 \right]
\tag{28}
$$

Thus, the proof of the Theorem 1 is complemented. □

According to Theorem 1, functions $e^{-\alpha t^2} e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2}$ can be approximated by linear canonical series whose coefficients are given analytically, and the error of the approximation decreases exponentially as α increases.

The coefficients $\rho_{A,k}$ in Equation (13) have a peak at $k = [\beta]$ ($[\star]$ is the nearest integer to $\star$), and decay exponentially as $k \to \pm\infty$. We keep only the $q+1$ largest coefficients, where the integer q is chosen such as

$$
q \geq 4\alpha b^2 \pi
\tag{29}
$$

thus

$$
e^{(-q/2)^2/4b^2} \leq e^{-\alpha b^2 \pi^2}
\tag{30}
$$

The following theorem provides a method for approximating functions $\phi(t)$ defined in Equation (11) by a $q+1$ term series, and estimates the truncation error under the conditions Equation (29).

Theorem 2. *The functions $\phi(t) = e^{-\alpha t^2} e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2}, t \in (-b\pi, b\pi)$ can be approximated by $q+1$ term linear canpnical series, and the truncation error can be obtained by the following inequality*

$$
\left| \phi(t) - \sum_{k=[\beta]-q/2}^{[\beta]+q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \right|
$$
$$
< e^{-\alpha b^2 \pi^2} \left[(1 + \frac{1}{b^2 \pi^2}) \big(\frac{8\alpha(1+4b)}{9} + 2b\pi + 2 \big) + 1 + \frac{1}{b\sqrt{2\pi}} (1 + \frac{1}{\pi}) \right]
\tag{31}
$$

where $\rho_{A,k}$ is defined by Equation (13), a, b, c, d are linear canonical parameters, $\alpha > 1/2$ and β are any real, and q is an even integer such that $q \geq 4\alpha b^2 \pi$.

Proof. For any $t \in (-b\pi, b\pi)$,

$$
\left| \phi(t) - \sum_{k=[\beta]-q/2}^{[\beta]+q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \right|
$$
$$
< \left| \phi(t) - \sum_{k=-\infty}^{\infty} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \right|
$$
$$
+ \left| \sum_{k>[\beta]+q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \right|
$$
$$
+ \left| \sum_{k<[\beta]-q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \right|
\tag{32}
$$

Based on Equation (13) and the triangle inequality, we have

$$
\left| \sum_{k>[\beta]+q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \right|
$$
$$
< \sqrt{\frac{1}{2b\pi}} \sum_{k=[\beta]+q/2+1}^{\infty} \frac{1}{\sqrt{2b\pi}} \left| e^{i\frac{d}{2b}(k^2-\beta^2)} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \right|
$$
$$
= \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=[\beta]+q/2+1}^{\infty} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} < \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=q/2}^{\infty} e^{-\frac{k^2}{4b^2\alpha}}
\tag{33}
$$

$$
\left| \sum_{k<[\beta]-q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \right|
$$
$$
< \sqrt{\frac{1}{2b\pi}} \sum_{k=-\infty}^{[\beta]-q/2-1} \frac{1}{\sqrt{2b\pi}} \left| e^{i\frac{d}{2b}(k^2-\beta^2)} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \right|
$$
$$
= \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=[\beta]-q/2-1}^{\infty} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} < \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=q/2}^{\infty} e^{-\frac{k^2}{4b^2\alpha}}
\tag{34}
$$

Because of real $\alpha > 0$ and $r > 0$, we obtain

$$
\int_r^{\infty} e^{-\alpha t^2} dt < \frac{e^{-\alpha r^2}}{2\alpha r}
\tag{35}
$$

Thus,

$$
\sum_{k=q/2}^{\infty} e^{-k^2/4b^2\alpha} < e^{-(q/2)^2/4b^2\alpha} + \int_{q/2}^{\infty} e^{-t^2/4b^2\alpha} dt < e^{-\alpha b^2\pi^2}\left(1+\frac{1}{\pi}\right)
\tag{36}
$$

Substituting Equation (36) into Equations (33) and (34), we have

$$
\left| \sum_{k>[\beta]+q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{dk^2}{2b}} \right| + \left| \sum_{k<[\beta]-q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{dk^2}{2b}} \right|
$$
$$
< \frac{1}{b\sqrt{\alpha\pi}} e^{-\alpha b^2\pi^2}\left(1+\frac{1}{\pi}\right)
\tag{37}
$$

Substituting Equations (28), (37) into Equation (32), we obtain

$$
\left| \phi(t) - \sum_{k=[\beta]-q/2}^{[\beta]+q/2} \rho_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \right|
$$
$$
< e^{-\alpha b^2\pi^2}\left[\left(1+\frac{1}{\pi^2 b^2}\right)\left(\frac{8\alpha(1+4b)}{9}+2b\pi+2\right)+1+\frac{1}{b\sqrt{\alpha\pi}}\left(1+\frac{1}{\pi}\right) \right]
$$
$$
< e^{-\alpha b^2\pi^2}\left[\left(1+\frac{1}{\pi^2 b^2}\right)\left(\frac{8\alpha(1+4b)}{9}+2b\pi+2\right)+1+\frac{1}{b\sqrt{2\pi}}\left(1+\frac{1}{\pi}\right) \right]
\tag{38}
$$

□

The following corollary describes that the function $e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2}$ can be approximated by using a series of $q+1$ terms.

Corollary 1. *Supposed that the conditions of Theorem 2 are satisfied. Then, multiplying both sides of Equation (31) by $e^{\alpha t^2}$, we obtain*

$$\left| e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2} - e^{\alpha t^2} \sum_{k=[\beta]-q/2}^{[\beta]+q/2} \sqrt{\frac{i}{2b\pi}} \rho_{A,k} e^{-i\frac{a}{2b}t^2 + i\frac{kt}{b} - i\frac{d}{2b}k^2} \right|$$

$$< e^{-\alpha b^2 \pi^2 (1-1/m^2)} \left[(1 + \frac{1}{\pi^2 b^2})(\frac{8\alpha(1+4b)}{9} + 2b\pi + 2) + 1 + \frac{1}{b\sqrt{2\pi}}(1 + \frac{1}{\pi}) \right] \tag{39}$$

where $t \in [-b\pi/m, b\pi/m]$, $m \geq 2$ is an integer.

Finally, Theorem 3 makes use of a simple linear scaling to generalize the inequality Equation (39) from $[-b\pi/m, b\pi/m]$ to $[-s, s]$,

Theorem 3. *Let $\alpha > 1/2$, β, $s > 0$ be real numbers and let $m \geq 2$, $q \geq 4\alpha\pi$ be integers. Then, for any $t \in [-s, s]$*

$$\left| e^{-i\frac{a}{2b}t^2 + i\frac{\beta t}{b} - i\frac{d}{2b}\beta^2} - e^{\alpha(\frac{b\pi t}{ms})^2} \sum_{k=[ms\beta/b\pi]-q/2}^{[ms\beta/b\pi]+q/2} \rho'_{A,k} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b'}t^2 + ik\frac{\pi}{ms}t - i\frac{dk^2}{2b}(\frac{b\pi}{ms})^2} \right|$$

$$< e^{-\alpha b^2 \pi^2 (1-1/m^2)} \left[(1 + \frac{1}{\pi^2 b^2})(\frac{8\alpha(1+4b)}{9} + 2b\pi + 2) + 1 + \frac{1}{b\sqrt{2\pi}}(1 + \frac{1}{\pi}) \right] \tag{40}$$

where

$$\rho'_{A,k} = \sqrt{\frac{-i}{2b\alpha}} e^{i\frac{d}{2b}(\frac{b\pi}{ms})^2(k^2 - (ms\beta/b\pi)^2)} e^{-\frac{(ms\beta/b\pi - k)^2}{4b^2\alpha}} \tag{41}$$

Corollary 2. *If $a = d = 0, b = 1, c = -1$ in Theorem 3, and the conditions of Theorem 3 are satisfied. Then, for any $t \in [-s, s]$*

$$\left| e^{i\beta t} - e^{\alpha(\frac{\pi t}{ms})^2} \sum_{k=[ms\beta/\pi]-q/2}^{[ms\beta/\pi]+q/2} \rho_k \sqrt{\frac{i}{2\pi}} e^{ik\frac{\pi}{ms}t} \right|$$

$$< e^{-\alpha\pi^2(1-\frac{1}{m^2})} \left[(1 + \frac{1}{\pi^2})(\frac{32\alpha}{9} + 2\pi + 2) + 1 + \frac{1}{\sqrt{2\pi}}(1 + \frac{1}{\pi}) \right] \tag{42}$$

where ρ_k is defined by

$$\rho_k = \sqrt{\frac{-i}{2\alpha}} e^{-(\beta ms/\pi - k)^2/4\alpha} \tag{43}$$

3.2.2. Implementation of Algorithms

In this subsection, the notations to be used for the detailed algorithm descriptions are introduced. For an integer $m \geq 2$ and a real number $\alpha > 0$, we will define a real number $\epsilon > 0$ by

$$\epsilon = e^{-\alpha b^2 \pi^2 (1-1/m^2)} \left[(1 + \frac{1}{\pi^2 b^2})(\frac{8\alpha(1+4b)}{9} + 2b\pi + 2) + 1 + \frac{1}{b\sqrt{2\pi}}(1 + \frac{1}{\pi}) \right] \tag{44}$$

and we will denote by q the smallest even natural number such that

$$q \geq 4\pi b^2 \alpha \tag{45}$$

For an integer m and a set of real number u_k, we will denote by μ_k the neatest integer to mu_k for $k = 0, 1, \cdots, N - 1$,

$$\rho'_{A,k,n} = \sqrt{\frac{-i}{2b\alpha\pi}} e^{i\frac{d}{2bm^2}[(\mu_n + k)^2 - (mu_n)^2]} e^{-\frac{[mu_n - (\mu_n + k)]^2}{4b^2\alpha}} \tag{46}$$

for $n = 0, 1, \cdots, N-1$, and $k = -q/2, \cdots, q/2$.

Corollary 3. *Setting $s = b\pi$ in Theorem 3, we obtain that*

$$\left| e^{-i\frac{a}{2b}t^2 + i\frac{u_n t}{b} - i\frac{d}{2b}u_n^2} - e^{\alpha(\frac{t}{m})^2} \sum_{k=-q/2}^{q/2} \rho'_{A,k,n} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i(k+\mu_n)\frac{t}{mb} - i\frac{d(\mu_n+k)^2}{2bm^2}} \right| < \epsilon \quad (47)$$

for any $n = 0, 1, \cdots, N-1$, and any $t \in [-b\pi, b\pi]$, where ϵ is defined by Equation (44).

For a given set of complex numbers α_n, we will denote by τ_l the unique set of complex coefficients such that

$$\sum_{n=0}^{N-1} \alpha_n e^{-i\frac{a}{2b}t^2 + i\frac{u_n t}{b} - i\frac{d}{2b}u_n^2}$$

$$= \sum_{n=0}^{N-1} \alpha_n e^{\alpha(\frac{t}{m})^2} \sum_{k=-q/2}^{q/2} \rho'_{A,k,n} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i(k+\mu_n)\frac{t}{mb} - i\frac{d(\mu_n+k)^2}{2bm^2}} \quad (48)$$

$$= e^{-i\frac{a}{2b}t^2} e^{\alpha(\frac{t}{m})^2} \sqrt{\frac{i}{2b\pi}} \sum_{l=-mN/2}^{mN/2-1} \tau_l e^{iklt/bm}$$

so that

$$\tau_l = \sum_{n,k,\mu_n+k=l} \alpha_n e^{-i\frac{d(\mu_n+k)^2}{2bm^2}} \rho'_{A,k,n} \quad (49)$$

We denote by T_j a set of complex numbers defined by the formula

$$T_j = \sum_{l=-mN/2}^{mN/2-1} \tau_l e^{i\frac{2\pi l j}{mN}} \quad (50)$$

for $j = -mN/2, \cdots, mN/2 - 1$. Furthermore, the $\widetilde{f}_j$ is denoted by the formula

$$\widetilde{f}_j = e^{-i\frac{a}{2b}(\frac{2b\pi j}{N})^2} e^{\alpha(\frac{2b\pi j}{mN})^2} \sqrt{\frac{i}{2b\pi}} T_j \quad (51)$$

for $j = -N/2, \cdots, N/2$.

Remark 1. *Combing Equations (47)–(51) with the triangle inequality, we see that*

$$\left| f_j - \widetilde{f}_j \right| < \frac{1}{\sqrt{2b\pi}} \epsilon \sum_{n=0}^{N-1} |\alpha_n| \quad (52)$$

for $j = -N/2, \cdots, N/2$, where f_j is defined by Equation (8).

Thus, the implements of NFLCT for Equation (8) is Algorithm 1.

For an integer m and a set of real of numbers t_j, we will denote by v_j the nearest integer to $t_j mN/(2\pi)$ for $j = 0, 1, \cdots, N-1$, and by $Q_{j,k}$ a set of real numbers defined by the formula

$$Q_{j,k} = \frac{1}{2\pi\sqrt{\alpha}} e^{-[t_j mN/2\pi - (v_j+k)]^2/4\alpha} \quad (53)$$

Algorithm 1 NFLCT for fast computation of Equation (8).

Input parameter $A = [a, b; c, d]$, the $\{u_0, u_1, \cdots, u_{N-1}\}, \{\alpha_0, \cdots, \alpha_{N-1}\}$.
Choose precision ϵ α and $q = \lceil 4\pi b^2 \alpha \rceil$
for $n = 0 : N - 1$
Compute μ_n, the nearest integer to mu_n
   for $k = -q/2 : q/2$
Calculate $\rho'_{A,k,n}$ and τ_{μ_n+k} according to
Equation (46) and Equation (49), respectively   end
end
Comment: Evaluate τ_k Fourier Seres at uniform sampling in $[-bm\pi, bm\pi]$
using inverse FFT of size mN
Calculate T_j according Equation (50), for $j = -mN/2, \cdots, mN/2 - 1$
Comment: Scale the values at those points which lie in $[-b\pi, b\pi]$
for $j = -N/2 : N/2 - 1$
$\widetilde{f}_j = T_j e^{-ia/(2b)(2b\pi j/N)^2} e^{\alpha(2b\pi j/mN)^2} \sqrt{\frac{i}{2b\pi}} T_j$
end
Output: Approximate values $\widetilde{f}_j, j = -N/2, \cdots, N/2 - 1$.
The total complexity is $2Nq + mN/2 \log N + 3N$

Corollary 4. *Setting $s = N/2$ in Corollary 2, we obtain that*

$$\left| e^{it_j n} - e^{\alpha(\frac{2\pi n}{mN})^2} \sum_{k=-q/2}^{q/2} Q_{j,k} e^{in(k+v_j)2\pi/Nm} \right| < \epsilon' \tag{54}$$

for $j = 0, 1, \cdots, N - 1$ and $n \in [-N/2, N/2]$, where ϵ is defined by

$$\epsilon' = e^{-\alpha\pi^2(1-\frac{1}{m^2})}(5\alpha + 28/3) \tag{55}$$

For a given set of complex numbers $\{\beta_n\}$, the $\{\beta'_n\}$ is a set of complex numbers defined by the formula

$$\beta'_n = \beta_n e^{-i\frac{d}{2b}n^2 + \alpha(2\pi n/mN)^2} \tag{56}$$

for $n = -N/2, \cdots, N/2 - 1$, and by U_l a set of complex numbers defined by the formula

$$U_l = \sum_{n=-N/2}^{N/2-1} \beta'_n e^{i2\pi nl/mN} \tag{57}$$

for $l = -mN/2, \cdots, mN/2$. Furthermore, taking account of the periodicity of l, we will denote by $\{\widetilde{g}_j\}$ another set of complex numbers defined by the formula

$$\widetilde{g}_j = e^{-i\frac{a}{2b}(bt_j)^2} \sum_{l=-q/2}^{q/2} Q_{j,l} U_{v_j+l} \tag{58}$$

for $j = 0, 1, \cdots, N - 1$

Remark 2. *Combining Equations (54)–(58) with the triangle inequality, we obtain that*

$$\left| g_j - \widetilde{g}_j \right| < \epsilon' \sum_{n=0}^{N-1} |\beta_n| \tag{59}$$

for $j = 0, 1, \cdots, N - 1$, where $\{g_j\}$ are defined by Equation (9).

Thus, the implements of NFLCT for Equation (9) is Algorithm 2.

Algorithm 2 NFLCT for fast computation of Equation (9).

Input parameter $A = [a, b; c, d]$, the $\{-N/2, \cdots, N/2 - 1\}$, $\{\beta_{-N/2}, \cdots, \beta_{N/2-1}\}$,
$\{t_0, \cdots, t_{N-1}\}$.
Choose precision ϵ α and $q = \lceil 4\pi b^2 \alpha \rceil$
for $j = 0 : N - 1$
Compute v_j, the nearest integer to $t_j mN/(2\pi)$
Calculate $\beta'_{j-N/2} = \beta_{j-N/2} e^{-id/2b(j-N/2)^2 + \alpha[(2\pi(j-N/2))/(mN)]^2}$ according to Equation (56)
  for $k = -q/2 : q/2$
Calculate $Q_{j,k}$ according to Equation (53).
   end
end
Comment: Evaluate Fourier series at uniform sampling in $[-bm\pi, bm\pi]$
using inverse FFT of size mN
Calculate $U_l = \sum_{n=-N/2}^{N/2} \beta'_n e^{i2\pi nl/mN}$, for $l = -mN/2, \cdots, mN/2 - 1$
Comment: Calculate approximate values at desired points in terms of the values at
equispaced points $[-b\pi, b\pi]$
for $j = 0 : N - 1$
for $l = -q/2 : q/2$
$\widetilde{g}_j = \widetilde{g}_j + Q_{j,l} U_{v_j+l}$
end
Output: Approximate values $\widetilde{g}_j = e^{-ia/(2b)(bt_j)^2} \widetilde{g}_j, j = -N/2, \cdots, N/2 - 1$.
The total complexity is $Nq + mN/2 \log_2 N + 3N$

Corollary 5. *Setting $d = N/2$ in Corollary 2, we obtain that*

$$\left| e^{int_j/m} - e^{\alpha\left(\frac{2\pi n}{Nm^2}\right)^2} \sum_{l=-q/2}^{q/2} Q_{j,l} e^{i2\pi n \frac{v_j+l}{m^2 N}} \right| < \epsilon \tag{60}$$

for any $j = 0, 1, \cdots, N - 1$, and any $n \in [-mN/2, mN/2 - 1]$, ϵ is defined by Equation (44).

For a given set of complex numbers $\{\gamma_k\}$, the $\{v_j\}$ is the unique set of complex coefficients such that

$$\sum_{n=0}^{N-1} \gamma_n \sum_{k=-q/2}^{q/2} e^{\alpha\left(\frac{t}{m}\right)^2} \sum_{k=-q/2}^{q/2} \rho'_{A,k,n} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i(k+\mu_n)\frac{t}{mb} - i\frac{d(\mu_n+k)^2}{2bm^2}}$$

$$= e^{-i\frac{a}{2b}t^2} e^{\alpha\left(\frac{t}{m}\right)^2} \sqrt{\frac{i}{2b\pi}} \sum_{l=-mN/2}^{mN/2-1} v_l e^{ilt/bm} \tag{61}$$

so that

$$v_l = \sum_{n,k,\eta_n+k=l} \gamma_n e^{-i\frac{d(\mu_n+k)^2}{2bm^2}} \rho'_{A,k,n} \tag{62}$$

Here, we take $t = t/b$, then the Equation (61) can be rewritten as

$$\sum_{n=0}^{N-1} \gamma_n \sum_{k=-q/2}^{q/2} e^{\alpha\left(\frac{t}{m}\right)^2} \sum_{k=-q/2}^{q/2} \rho'_{A,k,n} \sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2 + i(k+\mu_n)\frac{t}{mb} - i\frac{d(\mu_n+k)^2}{2bm^2}}$$

$$= e^{-i\frac{a}{2b}(tb)^2} e^{\alpha\left(\frac{tb}{m}\right)^2} \sqrt{\frac{i}{2b\pi}} \sum_{l=-mN/2}^{mN/2-1} v_l e^{ilt/m} \tag{63}$$

where $t \in [-\pi, \pi]$. The $\{V_l\}$ is a set of complex numbers defined by the formula

$$V_l = \sum_{k=-mN/2}^{mN/2-1} v_k e^{\alpha(2\pi k/m^2 N)} e^{2\pi i k l/m^2 N} \tag{64}$$

for $l = -m^2 N/2, \cdots, m^2 N/2 - 1$.

Furthermore, the $\left\{\widetilde{h}_j\right\}$ is another set of complex numbers defined by the formula

$$\widetilde{h}_j = \sqrt{\frac{i}{2b\pi}} e^{\alpha(bt_j/m)^2} e^{-i\frac{a}{2b}(t_j b)^2} \sum_{l=-q/2}^{q/2} Q_{j,l} V_{v_j+l} \tag{65}$$

for $j = 0, 1, \cdots, N-1$. Thus, the implement of NFLCT for Equation (10) is Algorithm 3.

Algorithm 3 NFLCT for fast computation of Equation (10).

Input parameter $A = [a, b; c, d]$, the $\{u_0, u_1, \cdots, u_{N-1}\}$, $\{\alpha_0, \cdots, \alpha_{N-1}\}$.
Choose precision ϵ α and $q = \lceil 4\pi b^2 \alpha \rceil$
for $n = 0 : N-1$
Compute μ_n, the nearest integer to mu_n
   for $k = -q/2 : q/2$
Calculate $\rho'_{A,k,n}$ and τ_{μ_n+k} according to Equation (46) and Equation (49), respectively.
  end
end
Comment: Evaluate τ_k Fourier Seres at uniform sampling in $[-bm\pi, bm\pi]$
using inverse FFT of size mN
Calculate T_j according Equation (50), for $j = -mN/2, \cdots, mN/2 - 1$
Comment: Scale the values at those points which lie in $[-b\pi, b\pi]$
for $j = -N/2 : N/2 - 1$
$\widetilde{f}_j = e^{-ia/(2b)(2b\pi j/N)^2} e^{\alpha(2b\pi j/mN)^2} \sqrt{\frac{i}{2b\pi}} T_j$
end
Output: Approximate values $\widetilde{f}_j, j = -N/2, \cdots, N/2 - 1$.
The total complexity is $2Nq + mN\log N + 3N$

---

4. Simulations

In this section, some numerical examples are given to support our theoretical analysis in the above section. All tests of numerical examples are implemented in Matlab R2016a. Two measures of precision are selected for each algorithm,

$$E_\infty = \max_{0 \leq j \leq N-1} |\widetilde{f}_j - f_j| / \sum_{j=0}^{N-1} |\alpha_j| \tag{66}$$

$$E_2 = \sqrt{\sum_{j=0}^{N-1} |\widetilde{f}_j - f_j|^2 / \sum_{j=0}^{N-1} |f_j|^2} \tag{67}$$

where α is the input data, f is the result of a direct computation, and $\widetilde{f}_j$ is the result of computation by proposed methods.

Example 1. *Here we consider the transformation $F^A : C^N \to C^N$ of Problem 1, as defined Equation (8). In this example, $j = -N/2, \cdots, N/2 - 1$, $u_0, \cdots, u_{N-1}$ were randomly distributed on the interval $[-N/2, N/2]$, and $\alpha_0, \cdots, \alpha_{N-1}$ were generated randomly on the unit square in the complex plane defined as*

$$0 \leq Re(z) \leq 1, \tag{68}$$

We take $a = 2, b = 1, c = 3, d = 2; \alpha = 0.5993, q = 14$.

We applied Algorithm 1 and direct method to this problem respectively. For different N, we loop Algorithm 1 20 times, the error mean are presented in Table 1. The results show that the precision is almost independent the length of input data.

Table 1. For different N, the error comparison between direct method and Algorithm 1.

N	64	128	256	512	1024
E_∞	0.0149	0.0084	0.0037	0.0024	9.7624×10^{-4}
E_2	0.0536	0.0433	0.0271	0.0253	0.0141

Example 2. *The Problem 2 as defined Equation (9) is considered for $j = 0, \cdots, N - 1$. In this example, $t_0, \cdots, t_{N-1}$ were randomly distributed on the interval $[-b\pi, b\pi]$, and $\beta_k = e^{-i2k^2 + i3m_k}$, $k = -N/2, \cdots, N/2 - 1$, m_k were distributed randomly on the interval $[-N/2, N/2 - 1]$.*

We take the parameters $a = 2, b = 1, c = 7, d = 4$, the interpolate factors $m = 2$, the terms of LCS $q = 10, \alpha = 0.646$. For $N = 64$, we loop our algorithm 20 times, the results are presented in Figure 1. It shows that the Algorithm 2 has almost the same effective as the direct method. For different N, the results of E_∞ and E_2 are showed in Table 2. It suggested that the precision of the Algorithm 2 is independent N. For different α and q, the errors are presented in Table 3. It shows that the lager α and q is, the more accuracy of the Algorithm 2 is, and the rate of accuracy improvement is decreasing. The complexity of the Algorithm 2 dependents q and N. For $m = 2$ and different q, the calculation is plotted in Figure 2. Therefore, we should not choose too big q according to the accuracy and complexity.

Table 2. Errorcomparison between the direct summation Equation (9) and the Algorithm 2 in case of difference N.

N	64	128	256	512
E_∞	2.1569×10^{-6}	2.0019×10^{-6}	2.1367×10^{-6}	2.0761×10^{-6}
E_2	2.1113×10^{-6}	2.2353×10^{-6}	2.2271×10^{-6}	2.0740×10^{-6}

Table 3. For fixed $N = 64$, error comparison between the direct summation Equation (9) and the Algorithm 2 in case of difference α and q.

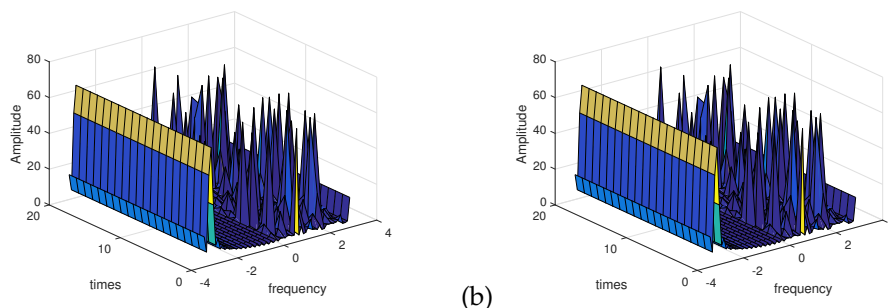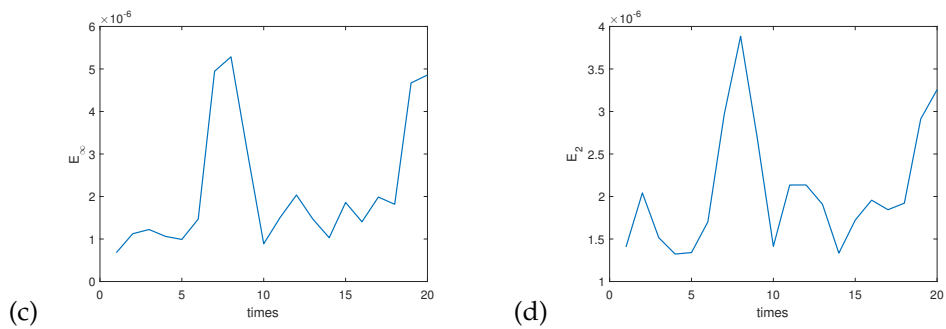| (α, q)	(0.246, 4)	(0.446, 6)	(0.646, 10)	(0.846, 12)
E_∞	0.0016	2.8818×10^{-4}	2.4913×10^{-6}	1.0767×10^{-6}
E_2	0.0023	2.5009×10^{-4}	2.2433×10^{-6}	1.1986×10^{-6}

(a)   (b)

Figure 1. *Cont.*

Figure 1. Twenty independent experiments. Amplitude (**a**) obtained by directly summation Equation (10); (**b**) obtained by Algorithm 2. Error of amplitude between the direct summation and Algorithm 2 (**c**) E_∞; (**d**) E_2.
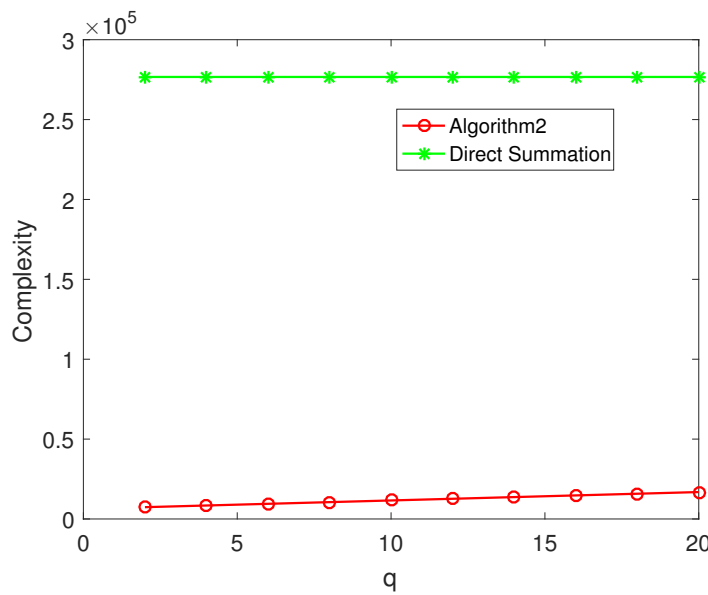

Figure 2. When $m = 2, N = 526$, complexity comparison between Algorithm 2 and direct summation for different q.

Example 3. *Here we consider the transformation $G : C^N \to C^N$ of the Problem 3 as defined Equation (10). In this example, $t_0, \cdots, t_{N-1}$ were randomly distributed on the interval $[-b\pi, b\pi]$, $u_0, \cdots, u_{N-1}$ were randomly distributed on the interval $[-N/2, N/2]$, and*

$$\gamma_k = 2e^{i0.4u_k^2 + i2u_k} + e^{i0.4u_k^2 + i4u_k} + e^{i0.4u_k^2 - i4u_k} \tag{69}$$

where $k = 0, 1 \cdots, N - 1, a = 0.234, b = 1.5, c = -0.5835, d = 0.5333$; interpolation factor $m = 2$; the terms of the linear canonical series $q = 8$ and the constant $\alpha = 0.245$.

For $N = 512$, the Figure 3 shows the amplitude and error for this problem by direct summation Equation (10) and Algorithm 3, respectively. It suggested that the Algorithm 3 has the almost same performance as the direct method. For different N, the Table 4 shows the error E_∞ and E_2 between direct method and Algorithm 3 by cyclic algorithms 20 times. The results show that the precision of Algorithm 3 is almost independent of the length of input data.
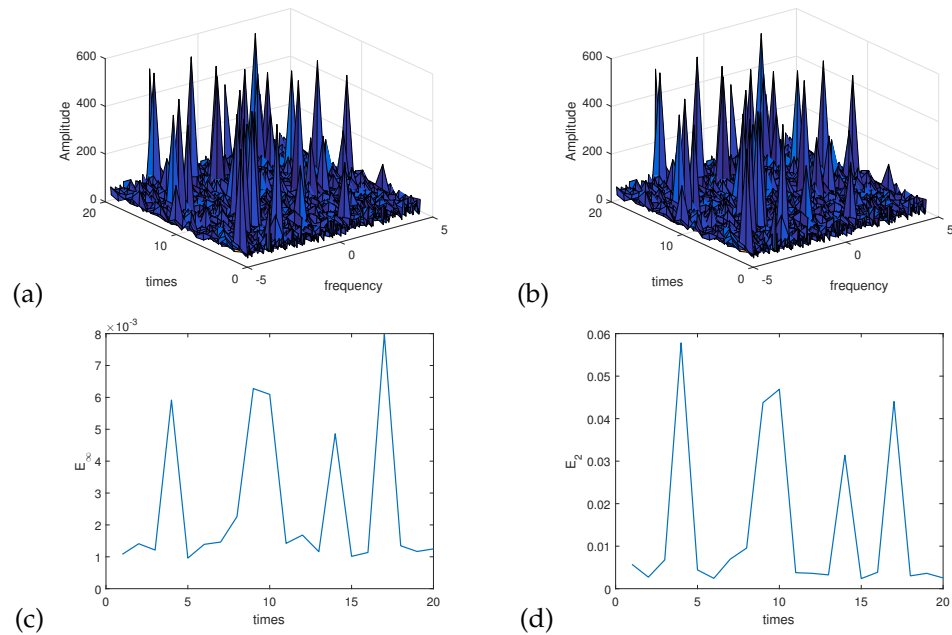
Figure 3. Twenty independent experiments. Amplitude (**a**) obtained by directly summation Equation (10); (**b**) obtained by Algorithm 3. Error of amplitude between the direct summation and Algorithm 3 (**c**) E_∞; (**d**) E_2.

Table 4. The error E_∞ and E_2 between direct method and Algorithm 3 by cyclic algorithms 20 times.

N	64	128	256	512	1024
E_∞	0.0089	0.0033	0.0025	0.0014	0.0014
E_2	0.0343	0.0166	0.0162	0.0067	0.0102

From the above three examples, the errors produced by Algorithms 1–3 are comparable with those produced by the corresponding direct methods. we can see that the numerical results coincide with the theoretical analyses which show the high efficiency of the new method.

The results of this paper can be generalized in the following ways: the algprithms of 1, 2 and 3 will allow the efficient application of linear transform $F_1, G_1, H_1 : C^N \to C^M$ defined by

$$F_1(\boldsymbol{\alpha})_j = \sum_{k=0}^{N-1} \alpha_k e^{-i\frac{a}{2b}(\frac{2b\pi j}{N})^2 + i\frac{u_k}{b}\frac{2\pi j}{N} - i\frac{d}{2b}u_k^2} \tag{70}$$

for $j = -M/2, \cdots, M/2 - 1$, and

$$G_1(\mathbf{g})_j = \sum_{k=-N/2}^{N/2-1} \beta_k e^{-i\frac{a}{2b}t_j^2 + i\frac{kt_j}{b} - i\frac{d}{2b}k^2} \tag{71}$$

For $j = 0, 1, \cdots, M - 1$, we will also consider the more general transformation $H : C^N \to C^N$ defined by the formula

$$H_1(\boldsymbol{\gamma})_j = \sum_{k=0}^{N-1} \gamma_k e^{-i\frac{a}{2b}t_j^2 + i\frac{u_k t_j}{b} - i\frac{d}{2b}u_k^2} \tag{72}$$

The algorithms ofthis paper also assume that $u_k \in (-N/2, N/2), t_j \in (-b\pi, b\pi)$. Other distributions can be handle by partitioning the vectors u and t, treating each partition separately and finally combining the results.

5. Conclusions

In this paper, we have described three algorithms for computing DLCT for nonequispaced data based on the interpolation formulae and approximation theory, which transform a function values from equispace to nonequispaced points. The complexity and precision of derived algorithms are also presented. The simulation shows that the derived approach is effective for computing nonuniform DLCT. The proposed algorithm can be viewed as generalizations of discrete linear canonical transform, and will have a broad range of applications in many branches of mathematics, science and engineering.

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