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Octonion Special Affine Fourier Transform: Pitt's Inequality and the Uncertainty Principles

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Abstract: The special affine Fourier transform (SAFT) is an extended version of the classical Fourier transform and incorporates various signal processing tools which include the Fourier transforms, the fractional Fourier transform, the linear canonical transform, and other related transforms. This paper aims to introduce a novel octonion special affine Fourier transform ($\mathbb{O}-$SAFT) and establish several classes of uncertainty inequalities for the proposed transform. We begin by studying the norm split and energy conservation properties of the proposed ($\mathbb{O}-$SAFT). Afterwards, we generalize several uncertainty relations for the ($\mathbb{O}-$SAFT) which include Pitt's inequality, Heisenberg–Weyl inequality, logarithmic uncertainty inequality, Hausdorff–Young inequality, and local uncertainty inequalities. Finally, we provide an illustrative example and some possible applications of the proposed transform.

Keywords: quaternion special affine Fourier transform (QSAFT); octonion; octonion Fourier transform ($\mathbb{O}-$OFT); octonion special affine Fourier transform ($\mathbb{O}-$SAFT); uncertainty principle

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1. Introduction

In the era of communication and information technology, the classical transform, better known as the Fourier transform (FT) in the hypercomplex setting, sees significant demand. It works on the principle of not losing the spectral relations while treating multi-channel signals as an algebraic whole. In the present era, many hypercomplex FTs have been studied via different approaches, such as [1,2]. An interesting development in hypercomplex FTs is its applications in modern day aspects, such as watermarking, color image processing, pattern recognition, image filtering, edge detection, and many more [3–7]. The leading hypercomplex FT is the quaternion Fourier transform (QFT). Due to the applications of QFTs in optics and signal processing, they are the most vigorous. In [8,9], the authors discussed the properties of the QFT and the wide range of its applications. After the generalized quaternion Fourier transform (QFT), the quaternion linear canonical transform (QLCT) is the most effective in signal processing due to its extra parameters, see [10–13]. With the passage of time, QLCT with four parameters has been extended to a six-parameter transform, which is commonly known as quaternion special affine Fourier transform (QSAFT). Owing to its time-shifting and frequency-modulation parameters, the QSAFT has gained momentum over classical QLCT for its potential to be used in image and signal processing, see [14–18].

In hypercomplex signal processing, the eighth-ordered Cayley–Dickson algebra, which is also called octonion algebra, has received remarkable attention. Hahn and Snopek, in 2011 [19], initiated the study of the octonion Fourier transform ($\mathbb{O}-$FT). Onwards, $\mathbb{O}-$FT turns to be a powerful point in research in the field of modern signal and image processing. Various properties and related uncertainty principles associated with $\mathbb{O}-$FT have been studied in [20–23]. Recently, Gao and Li [24] developed octonion linear canonical transform ($\mathbb{O}-$LCT) as a generalized version of $\mathbb{O}-$FT. They examined certain vital properties such as the inversion formula, isometry, and Riemann–Lebesgue lemma. Furthermore, they proved Heisenberg's and Donoho–Stark's uncertainty principles. Later on, the authors of [25] introduced the octonion spectrum of 3D short-time LCT signals, where they formulated several new classes of uncertainty inequalities. Thus, in the field of multidimensional hypercomplex signals, the door has been opened by successively obtaining the generalizations of $\mathbb{O}-$FT and $\mathbb{O}-$LCTs. Recently, Dar and Bhat [26] introduced Wigner distribution in the framework of octonion LCT as a generalization of $\mathbb{O}-$LCT and studied some properties and established Heisenberg's inequality, Logarithmic uncertainty inequality, and it associated Hausdorff–Young inequality. As an aligned motivation of the generalizations of $\mathbb{O}-$FT and $\mathbb{O}-$LCTs, we deduce the octonion special affine Fourier transform ($\mathbb{O}-$SAFT) and provide uncertainty principles for the proposed octonion special affine Fourier transform $\mathbb{O}-$SAFT.

The main purposes of this study are described in a nutshell as follows:

- To propose a novel integral transform coined as octonion special affine Fourier transform ($\mathbb{O}-$SAFT).
- To discuss the primarily characterizations of our obtained transform including the closed-form representation and its relation with QSAFT, inversion formula, and energy conservation.
- To establish Pitt's inequality for the proposed transform.
- To derive Hausdorff–Young inequality associated with the $\mathbb{O}-$SAFT.
- To formulate the logarithmic uncertainty inequality, Heisenberg–Weyl inequality, and local uncertainty inequality associated with the octonion special affine Fourier transform ($\mathbb{O}-$SAFT).

This paper is formulated as follows: Section 2 collects some preliminaries and basic notations. We define the $\mathbb{O}-$SAFT and determine its properties in Section 3. Section 4 is devoted to the development of a series of uncertainty inequalities including the Pitt's inequality, logarithmic uncertainty inequality,Heisenberg–Weyl inequality, Hausdorff–Young inequality and local uncertainty inequality associated with the $\mathbb{O}-$SAFT. In Section 5, we provide an illustrative example and possible applications of the proposed transform. We conclude this study by summarizing our obtained results and extracting important remarks in Section 6.

2. Preliminaries

2.1. Octonion Algebra

The octonion algebra, $\mathbf{O}$, is constructed from the eighth-order Cayley–Dickson algebra. Accordingly, a hypercomplex number $o \in \mathbf{O}$ is written as an ordered pair of quaternions $q_0, q_1 \in \mathbf{H}$ [27]

$$\begin{aligned} o &= (q_0, q_1) = ((z_0, z_1), (z_2, z_3)) \\ &= q_0 + q_1 \cdot \lambda_4 = (z_0 + z_1 \cdot \lambda_2) + (z_2 + z_3 \cdot \lambda_2) \cdot \lambda_4 \end{aligned} \tag{1}$$

equivalently

$$o = s_o + \sum_{i=1}^{7} s_i \lambda_i = s_0 + s_1\lambda_1 + s_2\lambda_2 + s_3\lambda_3 + s_4\lambda_4 + s_5\lambda_5 + s_6\lambda_6 + s_7\lambda_7. \tag{2}$$

We summarize the properties in Table 1 [21].

The conjugation of an octonion looks like

$$\bar{o} = s_0 - s_1\lambda_1 - s_2\lambda_2 - s_3\lambda_3 - s_4\lambda_4 - s_5\lambda_5 - s_6\lambda_6 - s_7\lambda_7 \tag{3}$$

Therefore, the norm is defined by $|o| = \sqrt{o\bar{o}}$ and $|o|^2 = \sum_{i=o}^{7} s^i$. Additionally, $|o_1 o_2| = |o_1||o_2|, \forall o_1, o_2 \in \mathbf{O}$.

Table 1. Multiplication Rules in Octonion Algebra.

$\cdot$	1	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
1	1	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
λ_1	λ_1	-1	λ_3	$-\lambda_2$	λ_5	$-\lambda_4$	$-\lambda_7$	λ_6
λ_2	λ_2	$-\lambda_3$	-1	λ_1	λ_6	λ_7	$-\lambda_4$	$-\lambda_5$
λ_3	λ_3	λ_2	$-\lambda_1$	-1	λ_7	$-\lambda_6$	λ_5	$-\lambda_4$
λ_4	λ_4	$-\lambda_5$	$-\lambda_6$	$-\lambda_7$	-1	λ_1	λ_2	λ_3
λ_5	λ_5	λ_4	$-\lambda_7$	λ_6	$-\lambda_1$	-1	$-\lambda_3$	λ_2
λ_6	λ_6	λ_7	λ_4	$-\lambda_5$	$-\lambda_2$	λ_3	-1	$-\lambda_1$
λ_7	λ_7	$-\lambda_6$	λ_5	λ_4	$-\lambda_3$	$-\lambda_2$	λ_1	-1

Using (1), we have

$$o = a + b\lambda_4 \tag{4}$$

where $a = s_0 + s_1\lambda_1 + s_2\lambda_2 + s_3\lambda_3$ and $b = s_4 + s_5\lambda_1 + s_6\lambda_2 + s_7\lambda_3$ are both in $\mathbf{H}$. The following lemma follows directly by a simple verification.

Lemma 1 ([21])**.** *For any $a, b \in \mathbf{H}$, we have (1) $\lambda_4 a = \bar{a}\lambda_4$; (2) $\lambda_4(a\lambda_4) = -\bar{a}$; $(3)(a\lambda_4)\lambda_4 = -a$; (4) $a(b\lambda_4) = (ba)\lambda_4$; (5) $(a\lambda_4)b = (a\bar{b})\lambda_4$; $(6)(a\lambda_4)(b\lambda_4) = -\bar{b}a$.*

It is obvious that, for an octonion $a + b\lambda_4, a, b \in \mathbf{H}$, we have

$$\overline{a + b\lambda_4} = \bar{a} - b\lambda_4 \tag{5}$$

and

$$|a + b\lambda_4|^2 = |a|^2 + |b|^2. \tag{6}$$

An octonion-valued function $f : \mathbf{R}^3 \longrightarrow \mathbf{O}$ has the following explicit form

$$\begin{aligned} f(x) &= f_0 + f_1(x)\lambda_1 + f_2(x)\lambda_2 + f_3(x)\lambda_3 + f_4(x)\lambda_4 \\ &\quad + f_5(x)\lambda_5 + f_6(x)\lambda_6 + f_7(x)\lambda_7 \\ &= f_0 + f_1\lambda_1 + (f_2 + f_3\lambda_1)\lambda_2 \\ &\quad + [f_4 + f_5\lambda_1 + (f_6 + f_7\lambda_1)\lambda_2]\lambda_4 \\ &= g(x) + h(x)\lambda_4 \end{aligned} \tag{7}$$

where each $f_i(x)$ is a real valued function, $g, h \in \mathbf{H}$ are as in (1) and $x = (x_1, x_2, x_3) \in \mathbf{R}^3$. For each octonion-valued function $f(x)$ over $\mathbf{R}^3$ and $1 \leq p < \infty$, the L^p-norm of f is defined by

$$\|f\|_p^p = \int_{\mathbf{R}^3} |f(x)|^p dx. \tag{8}$$

For $p = \infty$, then the L^∞ norm is given by

$$\|f\|_\infty = esssup_{x \in \mathbf{R}^3}|f(x)|. \tag{9}$$

2.2. Special Affine Fourier Transform

The special affine Fourier transform (SAFT) [28] of any function $f : \mathbf{R} \longrightarrow \mathbf{O}$ with respect to the matrix parameter $A = (a, b, c, d, \tau, \eta)$ is defined as

$$\mathbb{S}_A[f](w) = \int_{\mathbf{R}} f(x)\Lambda_A^i(x, w)dx. \tag{10}$$

with

$$\Lambda_A(x,w) = \frac{1}{\sqrt{2\pi|b|}} \times e^{\frac{i}{2b}\left[ax^2-2x(w-\tau)-2w(d\tau-b\eta)+d(w^2+\tau^2)-\frac{\pi}{2}\right]}, b \neq 0 \tag{11}$$

Additionally, the special affine Fourier transform in the quaternion [29] setting is given as

Let $A_s = \begin{bmatrix} a_s & b_s| & \tau_s \\ c_s & d_s| & \eta_s \end{bmatrix} \in \mathbf{R}^{2\times 2}$ be a matrix parameter satisfying $\det(A_s) = 1$, for $s = 1,2$. Then, the QSAFT of signal $f \in L^2(\mathbf{R}^2, \mathbf{H})$ is defined by

$$^{o}\mathbb{S}^{i,j}_{A_1,A_2}[f](w) = \int_{\mathbf{R}^2} f(x)\Lambda^i_{A_1}(x_1,w_1)\Lambda^j_{A_2}(x_2,w_2)\mathrm{d}x, \tag{12}$$

where $w = (w_1,w_2), x = (x_1,x_2) \in \mathbf{R}^2$, and the kernel signals $\Lambda^i_{A_1}(x_1,w_1)$, $\Lambda^j_{A_2}(x_2,w_2)$ are, respectively, given by

$$\Lambda^i_{A_1}(x_1,w_1) = \frac{1}{\sqrt{2\pi b_1 i}} e^{\frac{i}{2b_1}\left[a_1x_1^2-2x_1(w_1-\tau_1)-2w_1(d_1\tau_1-b_1\eta_1)+d_1(w_1^2+\tau_1^2)\right]}, b_1 \neq 0 \tag{13}$$

$$\Lambda^j_{A_2}(x_2,w_2) = \frac{1}{\sqrt{2\pi b_2 j}} e^{\frac{j}{2b_2}\left[a_2x_2^2-2x_2(w_2-\tau_2)-2w_2(d_2\tau_2-b_2\eta_2)+d_2(w_2^2+\tau_2^2)\right]}, b_2 \neq 0 \tag{14}$$

3. Octonionic Special Affine Fourier Transform

This section introduces the definition of the proposed transform, the octonion special affine Fourier transform ($\mathbb{O}-$SAFT). Consequently, we study some important properties of $\mathbb{O}-$SAFT, such as closed-form representation, inversion formula, split of norm, and energy conservation. In order to establish the fundamental properties for the proposed transform, we shall revisit the definitions of the octonion Fourier transform ($\mathbb{O}-$FT) [30] and the octonion linear canonical transform($\mathbb{O}-$LCT) [24]. Let us begin with definition of $\mathbb{O}-$FT.

3.1. Octonion Fourier Transform

Let $\lambda_i, i = 1,2\ldots,7$ denote the imaginary units in Cayley–Dickson algebra of octonions, then for an octonion-valued function $f \in L^1(\mathbf{R}, \mathbf{O})$, the one-dimensional $\mathbb{O}-$FT [30] is given by

$$\mathcal{F}_{\lambda_4}\{f\}(w) = \int_{\mathbf{R}} f(x)e^{-\lambda_4 2\pi xw}dx, \tag{15}$$

with inversion

$$f(x) = \mathcal{F}^{-1}_{\lambda_4}\{\mathcal{F}_{\lambda_4}\{f\}\}(x) = \int_{\mathbf{R}} \mathcal{F}_{\lambda_4}\{f\}(w)e^{\lambda_4 2\pi xw}dx, \tag{16}$$

For the octonion-valued function $f \in L^1(\mathbf{R}^3, \mathbf{O}) \cap L^2(\mathbf{R}^3, \mathbf{O})$, the three-dimensional $\mathbb{O}-$FT [23,24] is defined as

$$\begin{aligned}&\mathcal{F}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\\&=\int_{\mathbf{R}^3} f(x)e^{-\lambda_1 2\pi x_1 w_1}e^{-\lambda_2 2\pi x_2 w_2}e^{-\lambda_4 2\pi x_3 w_3}dx,\end{aligned} \tag{17}$$

with inversion

$$\begin{aligned}f(x) &= \mathcal{F}^{-1}_{\lambda_1,\lambda_2,\lambda_4}\{\mathcal{F}_{\lambda_1,\lambda_2,\lambda_4}\{f\}\}(x)\\&= \int_{\mathbf{R}^3} \mathcal{F}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)e^{-\lambda_1 2\pi x_1 w_1}\\&\qquad \times e^{-\lambda_2 2\pi x_2 w_2}e^{-\lambda_4 2\pi x_3 w_3}dw,\end{aligned} \tag{18}$$

where $w = (w_1, w_2, w_3), \quad x = (x_1, x_2, x_3) \in \mathbf{R}^3$.

It is worth mentioning that in the previous integrals (16) and (18), the multiplications were performed from the left to the right as a result of the non-associativity property of the octonion. Moreover, in (17), the order of imaginary units is not accidental, see [20].

3.2. Octonion Linear Canonical Transform

Recently Gao, W.B and Li, B.Z [24] proposed a linear canonical transform in the octonion domain, which they called octonion linear canonical transform ($\mathbb{O}-$LCT):

For $f \in L^1(\mathbf{R}^3, \mathbf{O})$, the one-dimensional $\mathbb{O}-$LCT with respect to the unimodular matrix $A = (a, b, c, d)$ is given by

$$\mathcal{L}^A_{\lambda_4}\{f\}(w) = \int_{\mathbf{R}} f(x)\Lambda^{\lambda_4}_A(x, w)dx, \tag{19}$$

where

$$K^{\lambda_4}_A(x, w) = \frac{1}{\sqrt{2\pi|b|}}e^{\frac{\lambda_4}{2b}\left[ax^2 - 2xw - + dw^2 - \frac{\pi}{2}\right]}, \quad b \neq 0$$

and the inversion formula

$$f(x) = \int_{\mathbf{R}} \mathcal{L}^A_{\lambda_4}\{f\}(w)\Lambda^{-\lambda_4}_A(x, w)dx, \tag{20}$$

where $\Lambda^{-\lambda_4}_A(x, w) = \Lambda^{\lambda_4}_{A^{-1}}(w, x)$ and $A^{-1} = (d, -b, -c, a)$.

Letting $f \in L^1(\mathbf{R}^3, \mathbf{O}) \cap L^2(\mathbf{R}^3, \mathbf{O})$ be an octonion-valued function, it is found that the three-dimensional $\mathbb{O}-$LCT with respect to the matrix parameter $A_k = (a_k, b_k, c_k, d_k)$ satisfying $det(A_k) = 1, k = 1, 2, 3$ is defined as

$$\begin{aligned}&\mathcal{L}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\\&= \int_{\mathbf{R}^3} f(x)K^{\lambda_1}_{A_1}(x_1, w_1)K^{\lambda_2}_{A_1}(x_2, w_2)K^{\lambda_4}_{A_3}(x_3, w_3)dx\end{aligned} \tag{21}$$

where $x = (x_1, x_2, x_3)$, $w = (w_1, w_2, w_3)$,

$$\Lambda^{\lambda_1}_{A_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi|b_1|}}e^{\frac{\lambda_1}{2b_1}\left[a_1x_1^2 - 2x_1w_1 + d_1w_1^2 - \frac{\pi}{2}\right]}, \quad b_1 \neq 0$$

$$\Lambda^{\lambda_2}_{A_2}(x_2, w_2) = \frac{1}{\sqrt{2\pi|b_2|}}e^{\frac{\lambda_2}{2b_2}\left[a_2x_2^2 - 2x_2w_2 + d_2w_2^2 - \frac{\pi}{2}\right]}, \quad b_2 \neq 0$$

and

$$\Lambda_{A_3}^{\lambda_4}(x_3, w_3) = \frac{1}{\sqrt{2\pi|b_3|}} e^{\frac{\lambda_4}{2b_3}\left[a_3x_3^2 - 2x_3w_3 + d_3w_3^2 - \frac{\pi}{2}\right]}, \quad b_3 \neq 0.$$

Now, we are in a position to define the octonion special affine Fourier transform ($\mathbb{O}-$SAFT).

Definition 1 (One-dimensional $\mathbb{O}-$SAFT). *Let $f \in L^1(\mathbf{R}, \mathbf{O})$, then the one-dimensional $\mathbb{O}-$SAFT with respect to a unimodular matrix parameter $A = (a, b, c, d, \tau, \eta)$ is defined as follows:*

$$ {}^{o}\mathbb{S}^{A}{}_{\lambda_4}\{f\}(w) = \int_{\mathbf{R}} f(x)\Lambda_A^{\lambda_4}(x, w)dx, \tag{22}$$

where

$$\Lambda_A^{\lambda_4}(x, w) = \frac{1}{\sqrt{2\pi|b|}} \times e^{\frac{\lambda_4}{2b}\left[ax^2 - 2x(w-\tau) - 2w(d\tau - b\eta) + d(w^2+\tau^2) - \frac{\pi}{2}\right]}, \quad b \neq 0. \tag{23}$$

The one-dimensional $\mathbb{O}-$SAFT of a signal $f \in L^1(\mathbf{R}, \mathbf{O})$ can be rewritten in terms of one-dimensional $\mathbb{O}-$FT (15) as

$$ {}^{o}\mathbb{S}^{A}{}_{\lambda_4}\{f\}(w) = \frac{1}{\sqrt{2\pi|b|}} \mathcal{F}_{\lambda_4}\left\{f(x)e^{\lambda_4[\frac{a}{2b}x^2 + \frac{1}{b}x\tau]}\right\} \left(\frac{w}{2\pi|b|}\right) e^{\lambda_4[\frac{d}{2b}(w^2+\tau^2) + \frac{w}{b}(b\eta - d\tau) - \frac{\pi}{4}]} \tag{24}$$

where $b \neq 0$.

Theorem 1. *Let f be an octonion-valued signal such that ${}^{o}\mathbb{S}^{A}{}_{\lambda_4}\{f\} \in L^1(\mathbf{R}, \mathbf{O})$. Then, the inversion formula of the one-dimensional $\mathbb{O}-$SAFT is*

$$f(x) = \int_{\mathbf{R}} {}^{o}\mathbb{S}^{A}{}_{\lambda_4}\{f\}(w)\overline{\Lambda_A^{\lambda_4}(x, w)}dw, \tag{25}$$

where $\overline{\Lambda_A^{\lambda_4}(x, w)} = \Lambda_A^{-\lambda_4}(x, w)$ and $b \neq 0$.

Proof. By applying (24) and (16) and following the procedure of Theorem 1 [24], the proof of the theorem follows. □

By replacing the complex unit i in the ordinary special affine Fourier transform with the imaginary units in the octonions, the three-dimensional octonion special affine Fourier transform ($\mathbb{O}-$SAFT) could be defined as follows:

Definition 2 (Three-dimensional $\mathbb{O}-$SAFT). *Let $A_k = \begin{bmatrix} a_k & b_k & | & \tau_k \\ c_k & d_k & | & \eta_k \end{bmatrix}$ be a matrix parameter such that a_k, b_k, c_k, d_k, p_k, $q_k \in \mathbf{R}$ and $a_kd_k - b_kc_k = 1$, for $k = 1, 2, 3$. The three-dimensional $\mathbb{O}-$SAFT of an octonion-valued signal f over $\mathbf{R}^3$ is given by*

$$ {}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w) = \int_{\mathbf{R}^3} f(x)\Lambda_{A_1}^{\lambda_1}(x_1, w_1)\Lambda_{A_1}^{\lambda_2}(x_2, w_2)\Lambda_{A_3}^{\lambda_4}(x_3, w_3)dx \tag{26}$$

where $x = (x_1, x_2, x_3)$, $w = (w_1, w_2, w_3)$, *and multiplication in the above integrals is performed from right to left. The kernels are given by*

$$\Lambda_{A_1}^{\lambda_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi|b_1|}} \times e^{\frac{\lambda_1}{2b_1}\left[a_1x_1^2 - 2x_1(w_1-\tau_1) - 2w_1(d_1\tau_1 - b_1\eta_1) + d_1(w_1^2+\tau_1^2) - \frac{\pi}{2}\right]}, \quad b_1 \neq 0,$$

$$\Lambda_{A_2}^{\lambda_2}(x_2, w_2) = \frac{1}{\sqrt{2\pi|b_2|}} \times e^{\frac{\lambda_2}{2b_2}\left[a_2x_2^2 - 2x_2(w_2-\tau_2) - 2w_2(d_2\tau_2 - b_2\eta_2) + d_2(w_2^2+\tau_2^2) - \frac{\pi}{2}\right]}, \quad b_2 \neq 0$$

and

$$\Lambda_{A_3}^{\lambda_4}(x_3, w_3) = \frac{1}{\sqrt{2\pi|b_3|}} \times e^{\frac{\lambda_4}{2b_3}\left[a_3x_3^2 - 2x_3(w_3-\tau_3) - 2w_3(d_3\tau_3 - b_3\eta_3) + d_3(w_3^2+\tau_3^2) - \frac{\pi}{2}\right]}, \quad b_3 \neq 0\,.$$

Note that for $b_k = 0, k = 1,2,3$, the $\mathbb{O}-$SAFT boils down to chirp multiplication. For the sake of brevity, we always set $b_k \neq 0$ in the paper unless stated otherwise.

Additionally, note that the kernels with imaginary units $\lambda_1, \lambda_2, \lambda_4$ are octonion-valued and do not reduce to the quaternion cases, thus the present integral transform is more interesting and complicated.

Now, we will obtain the closed-form representation of $\mathbb{O}-$SAFT defined in (26), let us begin by setting

$$\xi_k = \frac{1}{2b_k}\left[a_kx_k^2 - 2x_k(w_k - \tau_k) - 2w_k(d_k\tau_k - b_k\eta_k) + d_k(w_k^2 + \tau_k^2) - \frac{\pi}{2}\right], \quad k = 1,2,3. \tag{27}$$

Thus,

$$\begin{aligned}
&\Lambda_{A_1}^{\lambda_1}(x_1, w_1)\Lambda_{A_2}^{\lambda_2}(x_2, w_2)\Lambda_{A_3}^{\lambda_4}(x_3, w_3) \\
&= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} e^{\lambda_1\xi_1}e^{\lambda_2\xi_2}e^{\lambda_4\xi_3} \\
&= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} (c_1 + \lambda_1 s_1)(c_2 + \lambda_2 s_2)(c_3 + \lambda_4 s_3) \\
&= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} (c_1c_2c_3 + s_1c_2c_3\lambda_1 + c_1s_2c_3\lambda_2 \\
&+ s_1s_2c_3\lambda_3 + c_1c_2s_3\lambda_4 + s_1c_2s_3\lambda_5 + c_1s_2s_3\lambda_6 + s_1s_2s_3\lambda_7),
\end{aligned} \tag{28}$$

where $c_k = \cos\xi_k$ and $s_k = \sin\xi_k$, $\quad k = 1,2,3$.

Corresponding to the closed-form representation QLCT, we obtain a closed-form representation of $\mathbb{O}-$SAFT by using (28) in (26) in the form of the following lemma:

Lemma 2 (Closed-form representation)**.** *The $\mathbb{O}-$SAFT of three-dimensional signal $f : \mathbf{R}^3 \longrightarrow \mathbf{O}$ has the following closed-form representation:*

$$\begin{aligned}
{}^{O}\mathbb{S}_{\lambda_1,\lambda_2,\lambda_3}^{A_1,A_2,A_3}\{f\}(w) &= \Phi_0(w) + \Phi_1(w) + \Phi_2(w) + \Phi_3(w) \\
&+ \Phi_4(w) + \Phi_5(w) + \Phi_6(w) + \Phi_7(w)
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\Phi_0(w) &= \int_{\mathbf{R}^3} f_{eee}(x)\frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}}c_1c_2c_3dx,\\
\Phi_1(w) &= \int_{\mathbf{R}^3} f_{oee}(x)\frac{\lambda_1}{2\pi\sqrt{2\pi|b_1b_2b_3|}}s_1c_2c_3dx,\\
\Phi_2(w) &= \int_{\mathbf{R}^3} f_{eoe}(x)\frac{\lambda_2}{2\pi\sqrt{2\pi|b_1b_2b_3|}}c_1s_2c_3dx,\\
\Phi_3(w) &= \int_{\mathbf{R}^3} f_{ooe}(x)\frac{\lambda_3}{2\pi\sqrt{2\pi|b_1b_2b_3|}}s_1s_2c_3dx,\\
\Phi_4(w) &= \int_{\mathbf{R}^3} \frac{\lambda_4}{2\pi\sqrt{2\pi|b_1b_2b_3|}}f_{eeo}(x)c_1c_2s_3dx,\\
\Phi_5(w) &= \int_{\mathbf{R}^3} f_{oeo}(x)\frac{\lambda_5}{2\pi\sqrt{2\pi|b_1b_2b_3|}}s_1c_2s_3dx,\\
\Phi_6(w) &= \int_{\mathbf{R}^3} f_{eoo}(x)\frac{\lambda_6}{2\pi\sqrt{2\pi|b_1b_2b_3|}}c_1s_2s_3dx,
\end{aligned}$$

and

$$\Phi_7(w) = \int_{\mathbf{R}^3} f_{ooo}(x)\frac{\lambda_7}{2\pi\sqrt{2\pi|b_1b_2b_3|}}s_1s_2s_3dx,$$

and the functions $f_{xyz}, x, y, z \in \{e, o\}$ *are eight components of* f *of different parities with respect to the appropriate variables, for example,* $f_{oeo}(x)$ *is odd with respect to* x_1*, even with respect to* x_2*, and odd with respect to* x_3*.*

Under suitable conditions, the original octonion-valued signal f can be reconstructed from $\mathbb{O}-$SAFT by its inverse transform.

Theorem 2 (Inversion for 3D $\mathbb{O}-$SAFT)**.** *If* $f, {}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\} \in L^1(\mathbf{R}^3, \mathbf{O})$*, then* f *can be reconstructed by the formula*

$$\begin{aligned}
f(x) &= \int_{\mathbf{R}^3} {}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\Lambda^{-\lambda_4}_{A_3}(x_1, w_1)\\
&\quad\times\Lambda^{-\lambda_2}_{A_2}(x_2, w_2)\Lambda^{-\lambda_1}_{A_1}(x_3, w_3)dw,
\end{aligned}$$

for almost $x \in \mathbf{R}^3$*.*

Proof. Follows from [24] Theorem 2. □

For the clarity of the formulas, we denote $x_{l,m,n} = (lx_1, mx_2, nx_3)$, $l, m, n \in \{+, -\}$, i.e., $x_{+-+} = (x_1, -x_2, x_3)$ and denote the even and odd parts of a function $f(x)$ by $f_e(x)$ and $f_o(x)$, where $f_e = (f(x_{+++}) + f(x_{++-}))/2$, which is only even in the third variable x_3. Similarly, $f_o = (f(x_{+++}) - f(x_{++-}))/2$.

Next, we show that the norm of $\mathbb{O}-$SAFT splits into four norms of quaternion functions. This Lemma is important as the equations presented in the proof will be used in the subsequent sections.

Lemma 3 (Relation with QSAFT)**.** *If* $f = g + h\lambda_4 \in L^2$*, then*

$$\begin{aligned}
\|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}\|_2^2 &= \frac{1}{2\pi|b_3|}\Big(\|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}\|_2^2 + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_o\}\|_2^2\\
&\quad+\|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_e\}\|_2^2 + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_o\}\|_2^2\Big)
\end{aligned}$$

Proof. As $\Lambda_{A_1}^{-\lambda_1}(x_1,w_1)\Lambda_{A_2}^{-\lambda_2}(x_2,w_2)\in\mathbf{H}$, from [21], we obtain

$$\begin{aligned}
&{}^{o}\mathbb{S}_{\lambda_1,\lambda_2,\lambda_4}^{A_1,A_2,A_3}\{f\}(w)\\
&=\int_{\mathbf{R}^3} g(x)\Lambda_{A_1}^{\lambda_1}(x_1,w_1)\Lambda_{A_2}^{\lambda_2}(x_2,w_2)\Lambda_{A_3}^{\lambda_4}(x_3,w_3)dx\\
&+\int_{\mathbf{R}^3} h(x)\Lambda_{A_1}^{-\lambda_1}(x_1,w_1)\Lambda_{A_2}^{-\lambda_2}(x_2,w_2)\lambda_4\Lambda_{A_3}^{\lambda_4}(x_3,w_3)dx\\
&=\frac{1}{\sqrt{2\pi|b_3|}}\int_{\mathbf{R}^3} g_e(x)\Lambda_{A_1}^{\lambda_1}(x_1,w_1)\Lambda_{A_2}^{\lambda_2}(x_2,w_2)c_3dx\\
&+\frac{1}{\sqrt{2\pi|b_3|}}\int_{\mathbf{R}^3} h_o(x)\Lambda_{A_1}^{-\lambda_1}(x_1,w_1)\Lambda_{A_2}^{-\lambda_2}(x_2,w_2)s_3dx\\
&+\Bigg(\frac{1}{\sqrt{2\pi|b_3|}}\int_{\mathbf{R}^3} h_e(x)\Lambda_{A_1}^{-\lambda_1}(x_1,w_1)\Lambda_{A_2}^{-\lambda_2}(x_2,w_2)c_3dx\\
&-\frac{1}{\sqrt{2\pi|b_3|}}\int_{\mathbf{R}^3} g_o(x)\Lambda_{A_1}^{\lambda_1}(x_1,w_1)\Lambda_{A_2}^{\lambda_2}(x_2,w_2)s_3dx\Bigg)\lambda_4.
\end{aligned}\tag{30}$$

From (30), it is clear that $\mathbb{O}-$SAFT can be divided into four QSAFTs. Thus, the norm of $\mathbb{O}-$SAFT splits into four norms of quaternion functions as follows:

$$\begin{aligned}
&\|{}^{o}\mathbb{S}_{\lambda_1,\lambda_2,\lambda_4}^{A_1,A_2,A_3}\{f\}\|_2^2\\
&=\frac{1}{2\pi|b_3|}\left\|\int_{\mathbf{R}^3} g_e(x)\Lambda_{A_1}^{\lambda_1}(x_1,w_1)\Lambda_{A_2}^{\lambda_2}(x_2,w_2)c_3dx\right\|_2^2\\
&+\frac{1}{2\pi|b_3|}\left\|\int_{\mathbf{R}^3} h_o(x)\Lambda_{A_1}^{-\lambda_1}(x_1,w_1)\Lambda_{A_2}^{-\lambda_2}(x_2,w_2)s_3dx\right\|_2^2\\
&+\frac{1}{2\pi|b_3|}\left\|\int_{\mathbf{R}^3} h_e(x)\Lambda_{A_1}^{-\lambda_1}(x_1,w_1)\Lambda_{A_2}^{-\lambda_2}(x_2,w_2)c_3dx\right\|_2^2\\
&+\frac{1}{2\pi|b_3|}\left\|\int_{\mathbf{R}^3} g_o(x)\Lambda_{A_1}^{\lambda_1}(x_1,w_1)\Lambda_{A_2}^{\lambda_2}(x_2,w_2)s_3dx\right\|_2^2
\end{aligned}\tag{31}$$

where the equality is based on the fact that f_e and f_o are orthogonal in the L^2 inner product.

$$\begin{aligned}
\|{}^{o}\mathbb{S}_{\lambda_1,\lambda_2,\lambda_4}^{A_1,A_2,A_3}\{f\}\|_2^2 \quad = \quad &\frac{1}{2\pi|b_3|}\Big(\|{}^{o}\mathbb{S}_{\lambda_1,\lambda_2}^{A_1,A_2}\{g_e\}\|_2^2+\|{}^{o}\mathbb{S}_{\lambda_1,\lambda_2}^{A_1,A_2}\{h_o\}\|_2^2\\
&+\|{}^{o}\mathbb{S}_{\lambda_1,\lambda_2}^{A_1,A_2}\{h_e\}\|_2^2+\|{}^{o}\mathbb{S}_{\lambda_1,\lambda_2}^{A_1,A_2}\{g_o\}\|_2^2\Big)
\end{aligned}$$

which completes the proof. □

Again, denoting the quaternion form of f by $f=g+h\lambda_4$, we have

$$\begin{aligned}
\|f\|_2^2 &= \|g+h\lambda_4\|_2^2\\
&= \|g\|_2^2+\|h\|_2^2\\
&= \|g_e\|_2^2+\|g_o\|_2^2+\|h_e\|_2^2+\|h_o\|_2^2
\end{aligned}\tag{32}$$

Now, by the Plancherel theorem for the QSAFT, we have

$$\left\|\int_{\mathbf{R}^3} g_e(x)\Lambda_{A_1}^{\lambda_1}(x_1,w_1)\Lambda_{A_2}^{\lambda_2}(x_2,w_2)c_3dx\right\|_2^2=\|g_e\|_2^2\tag{33}$$

Similar results hold for the remaining three norms.

On applying (32) and (33) in Lemma 3, we have proved the following energy conservation relation for $\mathbb{O}-$SAFT.

Theorem 3 (Energy conservation)**.** *Let $f: \mathbf{R}^3 \longrightarrow \mathbf{O}$ be a continuous and square integrable octonion-valued signal function. Then we have*

$$\|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|^2 = \frac{1}{2\pi|b_3|}\|f\|_2^2. \tag{34}$$

4. Uncertainty Inequalities for $\mathbb{O}-$SAFT

In this section, we establish various types of inequalities for $\mathbb{O}-$SAFT.

4.1. Hausdorff–Young Inequality for $\mathbb{O}-$SAFT

In this subsection we will establish Hausdorff–Young inequality, which is very important in signal processing. This inequality will be helpful for researchers in establishing Shannon's entropy uncertainty relation.

Lemma 4 (Hausdorff–Young inequality QSAFT [31])**.** *For $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2,}\{f\}(w)\|_q \le (2\pi)^{\frac{1}{q}-\frac{1}{p}}|b_1 b_2|^{\frac{1}{q}-\frac{1}{2}}\|f(x)\|_p. \tag{35}$$

Theorem 4 (Hausdorff–Young inequality $\mathbb{O}-$SAFT)**.** *For $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|_q \le (2\pi)^{\frac{1}{2q}-\frac{1}{p}}|b_1 b_2|^{\frac{1}{q}-\frac{1}{2}}|b_3|^{-\frac{1}{2q}}\|f(x)\|_p. \tag{36}$$

Proof. From (30), we obtain

$$\begin{aligned}
&\|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|_q \\
&= \frac{1}{(2\pi|b_3|)^{\frac{1}{2q}}}\Big(\|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|_q + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_o\}(w)\|_q \\
&+ \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_e\}(w)\|_q + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_o\}(w)\|_q\Big)
\end{aligned}$$

Now applying Lemma 4 to the right-hand side of the above equation, we obtain

$$\begin{aligned}
&\|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|_q \\
&\le \frac{1}{(2\pi|b_3|)^{\frac{1}{2q}}}(2\pi)^{\frac{1}{q}-\frac{1}{p}}|b_1 b_2|^{\frac{1}{q}-\frac{1}{2}}\big(\|g_e(x)\|_p + \|g_o(x)\|_p \\
&+ \|h_e(x)\|_p + \|h_o(x)\|_p\big). \\
&= (2\pi)^{\frac{1}{2q}-\frac{1}{p}}|b_1 b_2|^{\frac{1}{q}-\frac{1}{2}}|b_3|^{-\frac{1}{2q}}\|f(x)\|_p
\end{aligned}$$

where the last equality follows from (32).

This completes the proof. □

4.2. Pitt's Inequality and the Logarithmic Uncertainty Principle

We demonstrate the sharp Pitt's inequality for the $\mathbb{O}-$SAFT and establish the associated logarithmic uncertainty inequality in the current section. To prove the Pitt's inequality, we mainly depend on the QSAFT. Thus, we begin by presenting the following Pitt's inequality for the QSAFT.

Lemma 5 (Pitt's inequality for QSAFT [18])**.** *For $f \in \mathcal{S}(\mathbf{R}^2, \mathbf{H})$, and $0 \le \alpha < 2$,*

$$\int_{\mathbf{R}^2}\left|\frac{w}{b}\right|^{-\alpha}\left|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{f(x)\}(w)\right|^2 dw \le \frac{C_\alpha}{4\pi^2}\int_{\mathbf{R}^2}|x|^\alpha|f(x)|^2\,dx. \tag{37}$$

with $C_\alpha := \frac{4\pi^2}{2^\alpha}[\Gamma(\frac{2-\alpha}{4})/\Gamma(\frac{2+\alpha}{4})]^2$ *and* $\Gamma(.)$ *is the Gamma function and* $\mathcal{S}(\mathbf{R}^2, \mathbf{H})$ *denotes the Schwartz space.*

Theorem 5 (Pitt's inequality for the $\mathbb{O}-$SAFT)**.** *For* $f \in \mathcal{S}(\mathbf{R}^3, \mathbf{O})$ *and* $0 \le \alpha < 3$ *and under the assumptions of Lemma 5, we have*

$$\begin{aligned} &\int_{\mathbf{R}^3} \left|\frac{w}{b}\right|^{-\alpha} \left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_3}\{f(x)\}(w)\right|^2 dw \\ &\le \frac{C_\alpha}{8\pi^3|b_3|}\int_{\mathbf{R}^3}|x|^\alpha|f(x)|^2dx. \end{aligned} \tag{38}$$

Proof. We have split $\mathbb{O}-$SAFT into four QSAFTs in Lemma 3, therefore

$$\begin{aligned} &\int_{\mathbf{R}^3} \left|\frac{w}{b}\right|^{-\alpha} \left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_3}\{f(x)\}(w)\right|^2 dw \\ &= \frac{1}{2\pi|b_3|}\left(\int_{\mathbf{R}^3}\left|\frac{w}{b}\right|^{-\alpha}|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)|^2dw \right.\\ &+\int_{\mathbf{R}^3}\left|\frac{w}{b}\right|^{-\alpha}|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_o\}(w)|^2dw \\ &+\int_{\mathbf{R}^3}\left|\frac{w}{b}\right|^{-\alpha}|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_e\}(w)|^2dw \\ &\left.+\int_{\mathbf{R}^3}\left|\frac{w}{b}\right|^{-\alpha}|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)|^2dw\right) \end{aligned} \tag{39}$$

By the Pitt's inequality for QSAFT (37), we have

$$\begin{aligned} &\int_{\mathbf{R}^3} \left|\frac{w}{b}\right|^{-\alpha} \left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e(x)\}(w)\right|^2 dw \\ &\le \frac{C_\alpha}{4\pi^2}\int_{\mathbf{R}^3}|x|^\alpha|f(x)|^2\,dx. \end{aligned}$$

Similar inequalities hold for the remaining three terms. By collecting all and inserting in (39), we obtain

$$\begin{aligned} &\int_{\mathbf{R}^3} \left|\frac{w}{b}\right|^{-\alpha} \left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_3}\{f(x)\}(w)\right|^2 dw \\ &\le \frac{C_\alpha}{8\pi^3|b_3|}\left(\int_{\mathbf{R}^3}|x|^\alpha|g_e|^2\,dx + \int_{\mathbf{R}^3}|x|^\alpha|g_o(x)|^2\,dx\right.\\ &\left.+\int_{\mathbf{R}^3}|x|^\alpha|h_e(x)|^2\,dx + \int_{\mathbf{R}^3}|x|^\alpha|h_o(x)|^2\,dx\right) \\ &= \frac{C_\alpha}{8\pi^3|b_3|}\int_{\mathbf{R}^3}|x|^\alpha\left(|g_e(x)|^2+|g_o(x)|^2+|h_e(x)|^2+|h_o(x)|^2\right)dx. \\ &= \frac{C_\alpha}{8\pi^3|b_3|}\int_{\mathbf{R}^3}|x|^\alpha|f(x)|^2dx \end{aligned}$$

where the last equality occurs because of (32).

This completes the proof. □

Here, C_α cannot be smaller any more. It is equal to the ordinary complex and the quaternion cases. Thus, the inequality is sharp. If $\alpha = 0$, it changes to equality; at $\alpha = 0$, differentiating the sharp Pitt's inequalities lead to the following logarithmic uncertainty inequality for the $\mathbb{O}-$SAFT.

Theorem 6 (Logarithmic uncertainty principle for the $\mathbb{O}-$SAFT)**.** *Let $f \in \mathcal{S}(\mathbf{R}^3, \mathbf{O})$, then the following inequality is satisfied:*

$$2\pi|b_3| \int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \left|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_3}\{f\}(w)\right|^2 dw + \int_{\mathbf{R}^2} \ln|x||f(x)|^2 dx \geq D \int_{\mathbf{R}^2} |f(x)|^2 dx \tag{40}$$

with $D = \ln(2) + \Gamma'(\frac{1}{2})/\Gamma(\frac{1}{2})$.

Proof. Following the procedure of Theorem 4.11 in [18], we will obtain the desired result. □

Alternatively, we can prove the logarithmic uncertainty principle for the $\mathbb{O}-$SAFT from the Logarithmic uncertainty principle for the QSAFT [18].

Lemma 6 (Logarithmic uncertainty principle for the QSAFT [18])**.** *Let $f \in \mathcal{S}(\mathbf{R}^2, \mathbf{H})$, then*

$$\int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \left|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{f\}(w)\right|^2 dw + \int_{\mathbf{R}^2} \ln|x||f(x)|^2 dx \geq D \int_{\mathbf{R}^2} |f(x)|^2 dx \tag{41}$$

with $D = \ln(2) + \Gamma'(\frac{1}{2})/\Gamma(\frac{1}{2})$.

Proof of Theorem 6. By Lemma 3, $\mathbb{O}-$SAFT can be written in split quaternion form as

$$\begin{aligned}
&\|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|^2 \\
&= \frac{1}{2\pi|b_3|}\Big(\|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|^2 + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_o\}(w)\|^2 \\
&+ \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_e\}(w)\|^2 + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_o\}(w)\|^2\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|^2 dw \\
&= \frac{1}{2\pi|b_3|}\int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \Big(\|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|^2 + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_o\}(w)\|^2 \\
&+ \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_e\}(w)\|^2 + \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_o\}(w)\|^2\Big) dw.
\end{aligned}$$

This implies that

$$\begin{aligned}
&2\pi|b_3| \int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|^2 dw \\
&= \int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|^2 dw \\
&+ \int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_o\}(w)\|^2 dw + \int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \\
&\times \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{h_e\}(w)\|^2 dw + \int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^{o}\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_o\}(w)\|^2 dw
\end{aligned} \tag{42}$$

Additionally, by virtue of (32), we can write

$$\begin{aligned}\int_{\mathbf{R}^2} \ln|x||f(x)|^2dx &= \int_{\mathbf{R}^2} \ln|x||g_e(x)|^2dx + \int_{\mathbf{R}^2} \ln|x||g_o(x)|^2dx \\ &+ \int_{\mathbf{R}^2} \ln|x||h_e(x)|^2dx + \int_{\mathbf{R}^2} \ln|x||h_o(x)|^2dx. \end{aligned} \quad (43)$$

By logarithmic uncertainty principle for the QSAFT given in Lemma 6, we get

$$\begin{aligned}&\int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|^2 + \int_{\mathbf{R}^2} \ln|x||g_e(x)|^2dx \\ &\geq D\int_{\mathbf{R}^2} |g_e(x)|^2dx. \end{aligned} \quad (44)$$

Similarly,

$$\begin{aligned}&\int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|^2 + \int_{\mathbf{R}^2} \ln|x||g_e(x)|^2dx \\ &\geq D\int_{\mathbf{R}^2} |g_o(x)|^2dx, \end{aligned} \quad (45)$$

$$\begin{aligned}&\int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|^2 + \int_{\mathbf{R}^2} \ln|x||g_e(x)|^2dx \\ &\geq D\int_{\mathbf{R}^2} |h_e(x)|^2dx. \end{aligned} \quad (46)$$

$$\begin{aligned}&\int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{g_e\}(w)\|^2 + \int_{\mathbf{R}^2} \ln|x||g_e(x)|^2dx \\ &\geq D\int_{\mathbf{R}^2} |h_o(x)|^2dx. \end{aligned} \quad (47)$$

Collecting all Equations (44)–(47) and making use of (42) and (43), we obtain the desired result

$$\begin{aligned}&2\pi|b_3| \int_{\mathbf{R}^2} \ln\left|\frac{w}{b}\right| \left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_3}\{f\}(w)\right|^2 dw \\ &+ \int_{\mathbf{R}^2} \ln|x||f(x)|^2dx \geq D\int_{\mathbf{R}^2} |f(x)|^2dx. \end{aligned}$$

This completes the proof. □

4.3. *Heisenberg–Weyl Inequality*

To establish Heisenberg–Weyl inequality for $\mathbf{O}-$SAFT, we first need the Heisenberg–Weyl inequality for QSAFT.

Theorem 7 (Heisenberg–Weyl inequality for QSAFT [18])**.** *Let $f \in L^2(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$ be the quaternion-valued signal, then*

$$\left(\int_{\mathbb{R}^2} x^2|f(x)|^2dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left\{\frac{w}{2\pi b_1 b_2}\right\}^2 \left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[f](w)\right|^2 dw\right)^{\frac{1}{2}} \geq \frac{1}{4\pi}\int_{\mathbb{R}^2} |f(x)|^2dx. \quad (48)$$

We are now ready to establish the Heisenberg–Weyl inequality for the proposed $\mathbf{O}-$SAFT.

Theorem 8 (Heisenberg–Weyl inequality for $\mathbf{O}-$SAFT)**.** *Let $f \in L^2(\mathbb{R}^3, \mathbb{O}) \cap L^2(\mathbb{R}^3, \mathbb{O})$ be the octonion-valued signal, then the following inequality holds:*

$$\int_{\mathbf{R}^2} x^2|f(x)|^2dx \int_{\mathbb{R}^2} \left\{\frac{w}{2\pi b_1 b_2}\right\}^2 \left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}[f](w)\right|^2 dw \geq \frac{1}{32\pi^3|b_3|}\left(\int_{\mathbb{R}^2} |f(x)|^2dx\right)^2. \quad (49)$$

Proof. From this, we have

$$\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}[f](w)\right|^2 dw$$
$$= \frac{1}{2\pi|b_3|}\left(\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[g_e](w)\right|^2 dw + \int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[g_o](w)\right|^2 dw\right. \tag{50}$$
$$\left. + \int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[h_e](w)\right|^2 dw + \int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[h_o](w)\right|^2 dw\right).$$

Equation (50) yields

$$2\pi|b_3|\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}[f](w)\right|^2 dw$$
$$= \int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[g_e](w)\right|^2 dw + \int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[g_o](w)\right|^2 dw \tag{51}$$
$$+ \int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[h_e](w)\right|^2 dw + \int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[h_o](w)\right|^2 dw.$$

Additionally, by virtue of (32), we can write

$$\int_{\mathbf{R}^2} x^2|f(x)|^2dx = \int_{\mathbf{R}^2} x^2|g_e(x)|^2dx + \int_{\mathbf{R}^2} x^2|g_o(x)|^2dx + \int_{\mathbf{R}^2} x^2|h_e(x)|^2dx + \int_{\mathbf{R}^2} x^2|h_o(x)|^2dx. \tag{52}$$

□

By the Heisenberg–Weyl inequality for QSAFT given by (48), we have

$$\int_{\mathbb{R}^2} x^2|g_e(x)|^2dx\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[g_e](w)\right|^2 dw \geq \left(\frac{1}{4\pi}\int_{\mathbb{R}^2}|g_e(x)|^2dx\right)^2, \tag{53}$$

$$\int_{\mathbb{R}^2} x^2|g_o(x)|^2dx\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[g_o](w)\right|^2 dw \geq \left(\frac{1}{4\pi}\int_{\mathbb{R}^2}|g_o(x)|^2dx\right)^2. \tag{54}$$

Additionally,

$$\int_{\mathbb{R}^2} x^2|h_e(x)|^2dx\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[h_e](w)\right|^2 dw \geq \left(\frac{1}{4\pi}\int_{\mathbb{R}^2}|h_e(x)|^2dx\right)^2, \tag{55}$$

$$\int_{\mathbb{R}^2} x^2|h_o(x)|^2dx\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}[h_o](w)\right|^2 dw \geq \left(\frac{1}{4\pi}\int_{\mathbb{R}^2}|h_o(x)|^2dx\right)^2, \tag{56}$$

Collecting (53)–(56), we obtain

$$2\pi|b_3|\int_{\mathbb{R}^2}\left\{\frac{w}{2\pi b_1 b_2}\right\}^2\left|{}^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}[f](w)\right|^2 dw\int_{\mathbf{R}^2} x^2|f(x)|^2dx \geq \left(\frac{1}{4\pi}\int_{\mathbb{R}^2}|f(x)|^2dx\right)^2. \tag{57}$$

On further simplifying, (57) gives the desired result.

4.4. Local Uncertainty Principle for $\mathbb{O}-$SAFT

We establish the local uncertainty principle for $\mathbb{O}-$SAFT in this subsection.

Lemma 7. *[Local uncertainty principle QSAFT [31]] (1) For $0 < \alpha < 1$ and for all $f \in L^2(\mathbf{R}^2, \mathbf{H})$, there is a constant M_α and all measurable sets $E \subset \mathbf{R}^3$ that hold*

$$\int_{bE} |^{o}\mathbb{S}^{\lambda_1,\lambda_2}_{A_1,A_2}\{f\}(w)|^2 dw \le M_\alpha |E|^\alpha \| |x|^\alpha f\|_2^2. \tag{58}$$

(2) If $\alpha > 1$,and for all $f \in L^2(\mathbf{R}^2, \mathbf{H})$, there is a constant M_α and all measurable set $E \subset \mathbf{R}^3$ that holds

$$\int_{bE} |^{o}\mathbb{S}^{\lambda_1,\lambda_2}_{A_1,A_2}\{f\}(w)|^2 dw \\ \le M_\alpha |b_1 b_2|^{\alpha-\frac{1}{\alpha}} |E|^\alpha \|f\|_2^{2-2\alpha} \| |x|^\alpha f\|_2^{\frac{2}{\alpha}}, \tag{59}$$

$$M_\alpha = \begin{cases} \frac{(1+\alpha^2)}{\alpha^{2\alpha}}(2-2\alpha)^{\alpha-2} & 0<\alpha<1 \\ \frac{\pi}{\alpha\Gamma(1/2)}\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)(\alpha-1)^\alpha\left(1-\frac{1}{\alpha}\right)^{-1} & \alpha>1 \end{cases}$$

Theorem 9 (Local UP for $\mathbb{O}-$SAFT)**.** *(1) For $0 < \alpha < 1$ and for all $f \in L^2(\mathbf{R}^3, \mathbf{O})$, there is a constant M_α and all measurable sets $E \subset \mathbf{R}^3$ that hold*

$$\int_{bE} |^{o}\mathbb{S}^{\lambda_1,\lambda_2}_{A_1,A_2}\{f\}(w)|^2 dw \le \frac{1}{2\pi|b_3|} M_\alpha |E|^\alpha \| |x|^\alpha f\|_2^2. \tag{60}$$

(2) If $\alpha > 1$, and for all $f \in L^2(\mathbf{R}^3, \mathbf{O})$, there is a constant M_α and all measurable sets $E \subset \mathbf{R}^3$ that hold

$$\int_{bE} |^{o}\mathbb{S}^{\lambda_1,\lambda_2}_{A_1,A_2}\{f\}(w)|^2 dw \\ \le \frac{1}{2\pi|b_3|} M_\alpha |b_1 b_2|^{\alpha-\frac{1}{\alpha}} |E|^\alpha \|f\|_2^{2-2\alpha} \| |x|^\alpha f\|_2^{\frac{2}{\alpha}}, \tag{61}$$

$$M_\alpha = \begin{cases} \frac{(1+\alpha^2)}{\alpha^{2\alpha}}(2-2\alpha)^{\alpha-2} & 0<\alpha<1 \\ \frac{\pi}{\alpha\Gamma(1/2)}\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)(\alpha-1)^\alpha\left(1-\frac{1}{\alpha}\right)^{-1} & \alpha>1 \end{cases}$$

Proof. By the splitting of $\mathbb{O}-$SAFT in (30), we have

$$\begin{aligned} &^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w) \\ &= \frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbf{R}^3} g_e(x) \Lambda^{\lambda_1}_{A_1}(x_1, w_1) \Lambda^{\lambda_2}_{A_2}(x_2, w_2) c_3 dx \\ &+ \frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbf{R}^3} h_o(x) \Lambda^{-\lambda_1}_{A_1}(x_1, w_1) \Lambda^{-\lambda_2}_{A_2}(x_2, w_2) s_3 dx \\ &+ \left(\frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbf{R}^3} h_e(x) \Lambda^{-\lambda_1}_{A_1}(x_1, w_1) \Lambda^{-\lambda_2}_{A_2}(x_2, w_2) c_3 dx \right. \\ &\left. - \frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbf{R}^3} g_o(x) \Lambda^{\lambda_1}_{A_1}(x_1, w_1) \Lambda^{\lambda_2}_{A_2}(x_2, w_2) s_3 dx \right) \lambda_4. \end{aligned}$$

By setting $f_m(x) = g_e(x_{+++}) + h_o(x_{+-+})\lambda_2$ and $f_n(x)x = h_e(x_{+++}) - g_o(x_{+-+})\lambda_2$, then $\mathbb{O}-$SAFT can be written as the combination of two QSAFTs as

$$\|^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|^2$$
$$= \frac{1}{2\pi|b_3|}\|^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{f_m\}(w)\|^2 + \|^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{f_n\}(w)\|^2 \tag{62}$$

Now, by Lemma 7 we have

$$\int_{bE} |^o\mathbb{S}^{\lambda_1,\lambda_2}_{A_1,A_2}\{f_j\}(w)|^2 dw \le M_\alpha |E|^\alpha \||x|^\alpha f_j\|_2^2, \quad j = m, n$$

Therefore from (62) and (63), we have

$$\int_{bE} \|^o\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\|^2 dw$$
$$= \frac{1}{2\pi|b_3|}\int_{bE}$$
$$\left(\|^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{f_m\}(w)\|^2 dw + \|^o\mathbb{S}^{A_1,A_2}_{\lambda_1,\lambda_2}\{f_n\}(w)\|^2 dw\right)$$
$$\le \frac{1}{2\pi|b_3|} M_\alpha |E|^\alpha \left(\||x|^\alpha f_m\|^2 + \||x|^\alpha f_n\|_2^2\right)$$
$$= \frac{1}{2\pi|b_3|} M_\alpha |E|^\alpha \||x|^\alpha f\|_2^2 \tag{63}$$

which completes the proof.

Similarly we can easily prove (61). □

5. Possible Applications and Illustrative Example

In this section we first present the possible applications of the $\mathbb{O}-$SAFT and then we give the numerical illustration of the proposed transform.

Recently, the theory of Quaternion SAFT has proliferated into the domain of multidimensional signals. It is a applied in colour image processing, watermarking, image filtering, edge detection, and pattern recognition [16,17,32]. On the other-side $\mathbb{O}-$SAFT is one of the hot research topics in the modern signal processing community, due to its extra free parameters compared to the $\mathbb{O}-$LCT. The $\mathbb{O}-$SAFT can be applied in mathematical physics, especially electromagnetics, filter design, color image analysis and processing, electrodynamics, gravity theory, and quantum physics [33]. Apart from this, the $\mathbb{O}-$SAFT can be applied to analyze various water-marking techniques. Additionally, it can be used in the recovery of band-limited octonionic-valued signals in the $\mathbb{O}-$SAFT domain [34] .

Furthermore, our uncertainty principles could play an important role in the time-frequency analysis in the $\mathbb{O}-$SAFT space and have some applications in signal recovery. The Heisenberg–Weyl inequality for $\mathbb{O}-$SAFT defines a lower bound for the product of a signal spread and its bandwidth. From [35,36], we see that the uncertainty principles can be used for the estimation of bandwidth.

Now, we will present a numerical illustration to show the correctness of the proposed transform. Consider the the function

$$f(x_1, x_2, x_3)$$
$$= \begin{cases} e^{-(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2)} & \text{if } |x_1| < \frac{1}{2}, |x_2| < \frac{1}{2}, |x_3| < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ are positive real constants.

Then, $\mathbb{O}-$SAFT of f and is given by

$$
\begin{aligned}
&{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}\{f\}(w)\\
&=\int_{\mathbf{R}^3} f(x)\frac{1}{\sqrt{2\pi|b_1|}}\\
&\times e^{\frac{\lambda_1}{2b_1}\left[a_1x_1^2-2x_1(w_1-\tau_1)-2w_1(d_1\tau_1-b_1\eta_1)+d_1(w_1^2+\tau_1^2)-\frac{\pi}{2}\right]}\\
&\times\frac{1}{\sqrt{2\pi|b_2|}}e^{\frac{\lambda_2}{2b_2}\left[a_2x_2^2-2x_2(w_2-\tau_2)-2w_2(d_2\tau_2-b_2\eta_2)+d_2(w_2^2+\tau_2^2)-\frac{\pi}{2}\right]}\\
&\times\frac{1}{\sqrt{2\pi|b_3|}}e^{\frac{\lambda_4}{2b_3}\left[a_3x_3^2-2x_3(w_3-\tau_3)-2w_3(d_3\tau_3-b_3\eta_3)+d_3(w_3^2+\tau_3^2)-\frac{\pi}{2}\right]}dx\\
&=\frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}}\int_{u_1-\frac{1}{2}}^{u_1+\frac{1}{2}}\int_{u_2-\frac{1}{2}}^{u_2+\frac{1}{2}}\int_{u_3-\frac{1}{2}}^{u_3+\frac{1}{2}}\\
&\times e^{\frac{\lambda_1}{2b_1}\left[a_1x_1^2-2x_1(w_1-\tau_1)-2w_1(d_1\tau_1-b_1\eta_1)+d_1(w_1^2+\tau_1^2)-\frac{\pi}{2}\right]-\alpha_1x_1^2}\\
&\times e^{\frac{\lambda_2}{2b_2}\left[a_2x_2^2-2x_2(w_2-\tau_2)-2w_2(d_2\tau_2-b_2\eta_2)+d_2(w_2^2+\tau_2^2)-\frac{\pi}{2}\right]-\alpha_2x_2^2}\\
&\times e^{\frac{\lambda_4}{2b_4}\left[a_3x_3^2-2x_3(w_3-\tau_3)-2w_3(d_3\tau_3-b_3\eta_3)+d_3(w_3^2+\tau_3^2)-\frac{\pi}{2}\right]-\alpha_2x_3^2}\\
&\times dx_1dx_2dx_3\\
&=\frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}}\int_{u_1-\frac{1}{2}}^{u_1+\frac{1}{2}}e^{\frac{\lambda_1}{2b_1}\left[x_1^2(a_1-2\lambda_1b_1\alpha_1)-2x_1(w_1-\tau_1\right]}dx_1\\
&\times e^{\frac{\lambda_1}{2b_1}\left[-2w_1(d_1\tau_1^2-b_1\eta_1)+d_1(w_1^2+\tau_1^2)-\frac{\pi}{2}\right]}\\
&\times\int_{u_2-\frac{1}{2}}^{u_2+\frac{1}{2}}e^{\frac{\lambda_2}{2b_2}\left[x_2^2(a_2-2\lambda_2b_2\alpha_2)-2x_2(w_2-\tau_2)\right]}dx_2\\
&\times e^{\frac{\lambda_2}{2b_2}\left[-2w_2(d_2\tau_2^2-b_2\eta_2)+d_2(w_2^2+\tau_2^2)-\frac{\pi}{2}\right]}\\
&\times\int_{u_3-\frac{1}{2}}^{u_3+\frac{1}{2}}e^{\frac{\lambda_4}{2b_3}\left[x_3^2(a_3-2\lambda_4b_3\alpha_3)-2x_3(w_3-\tau_3)\right]}dx_3\\
&\times e^{\frac{\lambda_4}{2b_3}\left[-2w_3(d_3\tau_3^2-b_3\eta_3)+d_3(w_3^2+\tau_3^2)-\frac{\pi}{2}\right]}
\end{aligned}
$$

Let us choose $\alpha_1=\frac{-\lambda_1a_1}{2b_1}$, $\alpha_2=\frac{-\lambda_2a_2}{2b_2}$ and $\alpha_3=\frac{-\lambda_4a_3}{2b_3}$ we have

$$
\begin{aligned}
&{}^{o}\mathbb{S}^{A_1,A_2,A_3}_{\lambda_1,\lambda_2,\lambda_4}f(w)\\
&=\frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}}\int_{u_1-\frac{1}{2}}^{u_1+\frac{1}{2}}e^{\frac{\lambda_1x_1(w_1-\tau_1)}{b_1}}dx_1\\
&\times e^{\frac{\lambda_1}{2b_1}\left[-2w_1(d_1\tau_1^2-b_1\eta_1)+d_1(w_1^2+\tau_1^2)-\frac{\pi}{2}\right]}\\
&\times\int_{u_2-\frac{1}{2}}^{u_2+\frac{1}{2}}e^{\frac{\lambda_2x_2(w_2-\tau_2)}{b_2}}dx_2e^{\frac{\lambda_2}{2b_2}\left[-2w_2(d_2\tau_2^2-b_2\eta_2)+d_2(w_2^2+\tau_2^2)-\frac{\pi}{2}\right]}\\
&\times\int_{u_3-\frac{1}{2}}^{u_3+\frac{1}{2}}e^{\frac{\lambda_4x_3(w_3-\tau_3)}{b_3}}dx_3e^{\frac{\lambda_3}{2b_3}\left[-2w_3(d_3\tau_3^2-b_3\eta_3)+d_3(w_3^2+\tau_3^2)-\frac{\pi}{2}\right]}\\
&=\frac{1}{2\pi\sqrt{2\pi|b_1b_2|}}\frac{b_1}{\lambda_1(w_1-\tau_1)}\left(e^{\frac{\lambda_1x_1(w_1-\tau_1)}{b_1}}\right)_{u_1-\frac{1}{2}}^{u_1+\frac{1}{2}}\\
&\times e^{\frac{\lambda_1}{2b_1}\left[-2w_1(d_1\tau_1^2-b_1\eta_1)+d_1(w_1^2+\tau_1^2)-\frac{\pi}{2}\right]}\\
&\times\frac{b_2}{\lambda_2(w_2-\tau_2)}\left(e^{\frac{\lambda_2x_2(w_2-\tau_2)}{b_2}}\right)_{u_2-\frac{1}{2}}^{u_2+\frac{1}{2}}\\
&\times e^{\frac{\lambda_2}{2b_2}\left[-2w_2(d_2\tau_2^2-b_2\eta_2)+d_2(w_2^2+\tau_2^2)-\frac{\pi}{2}\right]}\\
&\times\frac{b_3}{\lambda_4(w_3-\tau_3)}\left(e^{\frac{\lambda_3x_3(w_3-\tau_3)}{b_3}}\right)_{u_3-\frac{1}{2}}^{u_3+\frac{1}{2}}\\
&\times e^{\frac{\lambda_4}{2b_3}\left[-2w_3(d_3\tau_3^2-b_3\eta_3)+d_3(w_3^2+\tau_3^2)-\frac{\pi}{2}\right]}.
\end{aligned}
$$

6. Conclusions

In this paper, we propose the novel $\mathbb{O}-$SAFT, based on the association between the $\mathbb{O}-$SAFT and the QSAFT via split norm. We have established some basic properties of the proposed transform, including the inversion formula and energy conservation. Finally, the uncertainty inequalities for the $\mathbb{O}-$SAFT, such as logarithmic uncertainty inequality, Hausdorff–Young inequality, and local uncertainty, are obtained. We intend, in the near future, to investigate the physical influence and engineering applications resulting from the current study. Additionally, we propose to study convolution and correlation theorems for the $\mathbb{O}-$SAFT.

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Abbreviations

The following abbreviations are used in this manuscript:

SAFT	Special affine Fourier transform
$\mathbb{O}-$SAFT	Octonion special affine Fourier transform
OFT	Octonion Fourier transform

References

1. Brackx, F.; Hitzer, E.; Sangwine, S. History of quaternion and Clifford-Fourier transforms and wavelets. In *Quaternion and Clifford Fourier Transforms and Wavelets*; Trends in Mathematics; Birkhäuser: Basel, Switzerland, 2013; Volume 27, pp. XI–XXVII.
2. De Bie, H. Fourier transforms in Clifford analysis. In *Operator Theory*; Alpay, D., Ed.; Springer: Basel, Switzerland, 2015.
3. Snopek, K.M. The study of properties of n-d analytic signals and their spectra in complex and hypercomplex domains. *Radio Eng.* **2012**, *21*, 29–36.
4. Pei, S.C.; Chang, J.H.; Ding, J.J. Color pattern recognition by quaternion correlation. In Proceedings of the IEEE International Conference Image Processing, Thessaloniki, Greece, 7–10 October 2001; pp. 894–897.
5. Sangwine, S.J.; Ell, T.A. Colour image filters based on hypercomplex convolution. *IEEE Proc. Vis. Image Signal Process.* **2000**, *49*, 89–93. [CrossRef]
6. Witten, B.; Shragge, J. Quaternion-based signal processing, stanford exploration project. Presented at the 2006 SEG Annual Meeting, New Orleans, LA, USA, 1–6 October 2006; pp. 2862–2866.
7. Took, C.C.; Mandic, D.P. The quaternion LMS algorithm for adaptive filtering of hypercomplex processes. *IEEE Trans. Signal Process.* **2009**, *57*, 1316–1327. [CrossRef]
8. Bülow, T.; Sommer, G. The hypercomplex signal-a novel extensions of the analytic signal to the multidimensional case. *IEEE Trans. Signal Process.* **2001**, *49*, 2844–2852. [CrossRef]
9. Bas, P.; LeBihan, N.; Chassery, J.M. Color image water marking using quaternion Fourier transform. In Proceedings of the IEEE International Conference on Acoustics Speechand Signal and Signal Processing, ICASSP, Hong Kong, China, 6–10 April 2003; pp. 521–524.
10. Yang, Y.; Kou, K.I. On uncertainty principles for hypercomplex signals in the linear canonical transform domains. *Signal Process.* **2014**, *95* , 67–75. [CrossRef]
11. Zhang, Y.N.; Li, B.Z. Novel uncertainty principles for two-sided quaternion linear canonical transform. *Adv. Appl. Clifford Algebr.* **2018**, *28*, 15. [CrossRef]
12. Kou, K.I.; Ou, J.; Morais, J. Uncertainty principles associated with quaternionic linear canonical transforms. *Math. Meth. Appl. Sci.* **2016**, *39*, 2722–2736. [CrossRef]

13. Gao, W.B.; Li, B.Z. Uncertainty principle for the two-sided quaternion windowed linear canonical transform. *Circuits Syst. Signal Process.* **2022**, *41*, 1324–1348.
14. Roopkumar, R. One dimensional quaternion special special affine Fourier transform. *Adv. Appl. Clifford Algebr.* **2021**, *31*, 73. [CrossRef]
15. Bhat, M.Y.; Dar, A.H. The algebra of 2D Gabor quaternionic special affine Fourier transform and uncertainty principles. *J. Anal.* **2022**, *30*, 637–649. [CrossRef]
16. Bhat, M.Y.; Dar, A.H. Donoho Starks and Hardys Uncertainty Principles for the Shortotime Quaternion Special Affine Fourier Transform. *arXiv* **2021**, arXiv:2110.02754. [CrossRef]
17. Bhat, M.Y.; Dar, A.H. Convolution and Correlation Theorems for Wigner-Ville Distribution Associated with the Quaternion Special Affine Fourier Transform. *Signal Image Video Process.* **2022**, *16*, 1235–1242. [CrossRef]
18. Haoui, E.Y.; Hitzer, E. Generalized uncertainty principles associated with the quaternionic special affine Fourier transform. *Complex Var. Elliptic Equ.* **2022**, *67*, 2102–2121. [CrossRef]
19. Hahn, S.; Snopek, K. The unified theory of n-dimensional complex and hyper- complex analytic signals. *Bull. Polish Acad. Sci. Tech. Sci.* **2011**, *59*, 167–181.
20. Blaszczyk, L.; Snopek, K. Octonion Fourier transform of real-valued functions of three variables-selected properties and examples. *Signal Process.* **2017**, *136*, 29–37. [CrossRef]
21. Lian, P. The octonionic Fourier transform: Uncertainty relations and convolution. *Signal Process.* **2019**, *164*, 295–300. [CrossRef]
22. Blaszczyk, L. A generalization of the octonion Fourier transform to 3-D octonion-valued signals: Properties and possible applications to 3-D LTI partial differential systems. *Multidimens. Syst. Signal Process.* **2020**, *31*, 1227–1257. [CrossRef]
23. Blaszczyk, L. Discrete octonion Fourier transform and the analysis of discrete 3-D data. *Comput. Appl. Math.* **2020**, *39*, 329. [CrossRef]
24. Gao, W.B.; Li, B.Z. The octonion linear canonical transform. *Signal Process.* **2021**, *188*, 108233. [CrossRef]
25. Bhat, M.Y.; Dar, A.H. Octonion spectrum of 3D short-time LCT signals.*Opt. Int. J. Light Electron Opt.* **2022**, *261*, 169156. [CrossRef]
26. Dar, A.H.; Bhat, M.Y. Wigner distribution and associated uncertainty principles in the framework of octonion linear canonical transform. *Opt. Int. J. Light Electron Opt.* **2023**, *272*, 170213. [CrossRef]
27. Conway, J.H.; Smith, D.A. *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*; A K Peters Ltd.: Natick, MA, USA, 2003.
28. Xu, S.; Chai, Y.; Hu, Y.; Jiang, C.; Li, Y. Reconstruction of digital spectrum from periodic nonuniformly sampled signals in special affine Fourier transform domain. *Opt. Commun.* **2015**, *348*, 59–65. [CrossRef]
29. Kou, K.I.; Morais, J.; Zhang, Y. Generalized prolate spheroidal wave functions for offset linear canonical transform in clifford analysis. *Math. Meth. App. Sci.* **2013**, *36*, 1028–1041. [CrossRef]
30. Heredia, C.J.; Garcia, E.A.; Espinosa, C.V. One dimensional octonion Fourier transform. *J. Math. Control. Sci. Appl.* **2021**, *7*, 91–106.
31. Zhu, X.; Zheng, X. Uncertainty principles for the two-sided offset quaternion linear canonical transform. *Math. Methods Appl. Sci.* **2021**, *44*, 14236–14255. [CrossRef]
32. Bhat, M.Y.; Dar, A.H. The Two-Sided Short-time Quaternionic Offset Linear Canonical Transform and associated Convolution and Correlation. *Math. Methods Appl. Sci.* **2023**. [CrossRef]
33. Li, C.; Li, B.; Xiao, L.; Hu, Y.; Tian, L. A watermarking method based on hypercom-plex Fourier transform and visual attention. *J. Inf. Comput. Sci.* **2012**, *9*, 4485–4492.
34. Zhao, H.; Wang, R.Y.; Song, D.P. Recovery of bandlimited signals in linear canonical transform domain from noisy samples. *Circuits Syst. Signal Process.* **2014**, *33*, 1997–2008. [CrossRef]
35. Gao, W.B.; Li, B.Z. Uncertainty principles for the short-time linear canonical transform of complex signals. *Digit. Signal Process.* **2021**, *111*, 102953. [CrossRef]
36. Stern, A. Uncertainty principles in linear canonical transform domains and some of their implications in optics. *J. Opt. Soc. Am. A Opt. Image Sci. Vis.* **2008**, *25*, 647–652. [CrossRef]