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Quasi-Synchronization and Dissipativity Analysis for Fractional-Order Neural Networks with Time Delay

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Abstract: The objective of this research is to examine the global dissipativity and quasi-synchronization of fractional-order neural networks (FNNs). A global dissipativity criterion is established through the creation of an appropriate Lyapunov function, together with some fractional-order inequality techniques. Additionally, the issue of quasi-synchronization for drive-response FNNs is investigated using linear state feedback control. The study reveals the synchronization error converges to a bounded region by choosing an appropriate control parameter. Finally, the effectiveness of the obtained works are validated through three numerical examples.

Keywords: fractional-order neural networks; quasi-synchronization; dissipativity; linear feedback control

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Citation: Liu, Y.; Zhang, C.; Li, M. Quasi-Synchronization and Dissipativity Analysis for Fractional-Order Neural Networks with Time Delay. *Fractal Fract.* **2023**, *7*, 364. https://doi.org/10.3390/fractalfract7050364

Academic Editor: Gani Stamov

Received: 5 April 2023
Revised: 26 April 2023
Accepted: 27 April 2023
Published: 28 April 2023

1. Introduction

Neural networks possess quick computing speeds, robust associative capabilities, adaptability, tolerance to faults, and self-organizing skills. They have found extensive use in many areas [1], with a vast potential for research. A multitude of scholars have explored this field, proposing many neural network models [2,3]. In recent decades, fractional calculus operators have been widely applied in neural networks due to their features of memory and non-locality. Fractional neural networks have achieved many excellent results, such as those in [4,5].

The swift advancement in fractional derivative theory has led to the creation of numerous fractional-order models utilizing fractional-order differential equations [6–11]. Compared to the traditional inter-order cases, fractional-order models exhibit consistent heredity and memory across various processes. In [10], using fractional calculus could accurately characterize the dynamical characteristics of pyramidal neurons, as recent studies have shown. As a result, the use of fractional derivatives has garnered increasing attention in recent years. Additionally, experiments have shown that the capacitor used in electronic circuits exhibits fractional-order characteristics. The corresponding voltage–current relation can be acquired as $i(s) = C\frac{d^{\varsigma}V(s)}{ds^{\varsigma}} \triangleq CD_s^{\varsigma}V(s)$ [9,12], leading to the establishment of fractional neural networks (FNNs). Here, ς represents the order of the capacitors. In comparison to traditional inter-order neural networks, FNNs are more advantageous and meaningful in emulating the behavior of neurons in the brain. In practical applications, time delays are an inescapable aspect of communication channels, which may result in oscillation or chaos. For example, if time delays are selected as bifurcation parameters, the issues related to the stability of a system at the Hopf bifurcation point in complex-valued FNNs have been addressed applying Laplace transforms and the theory of differential equations with non-integer orders [13]. As a result, it is imperative to study the dynamics of FNNs with delays in a comprehensive manner. Numerous impressive outcomes, such as global stabilization [14], stability [15], and synchronization [16], have been extensively documented.

Dynamical analyses of neural networks are an important precondition for designing and practically applying them. Particularly, dissipativity is an extension of Lyapunov stability. It provides a framework for analyzing the energy flow and dissipation in a system, which is crucial for understanding the system's behavior over time. Dissipativity has been successfully applied in various fields to describe system performance, such as control theory, robotics, and power systems, among others. Moreover, it has been shown that dissipativity can be employed to investigate the synchronization and anti-synchronization of delayed FNNs, which is important for control systems [17–19]. Moreover, achieving synchronization in FNNs is a critical yet difficult task. Synchronization refers to the phenomenon where two systems exhibit identical manifestation, which can be achieved through external excitation or coupling. However, complete synchronization, when there are parameter mismatches, may not be achieved. Put differently, a non-zero synchronization error can always exist. Then, quasi-synchronization is often considered more practical and feasible than achieving complete synchronization. Since then, a significant amount of research has been dedicated to the subject of quasi-synchronization.

The two most critical challenges for quasi-synchronization, as indicated in [20,21], are designing a straightforward yet powerful controller and estimating the error of synchronization. To achieve quasi-synchronization, it is essential to consider the practical stability issue and obtain the error bound, which are also significant challenges. Nevertheless, conventional analysis and control methods, which are appropriate for systems with integer order, cannot be directly implemented on FNNs. The reason why the previous control methods and techniques are not directly applicable to FNNs is that FNNs are modeled by a set of differential equations with fractional order, this non-integer order introduces extra complexity to the system dynamics, which makes it difficult to utilize traditional analysis and control techniques. Time delay is also considered. Hence, deriving criteria for quasi-synchronization and dissipativity with time delay and fractional-order differential equations is a pressing issue. To achieve this goal, two core problems must be solved: (1) how to deal with quasi-synchronization and dissipativity and (2) how to maintain the synchronization error within a narrow range through control.

To address the previous discussion and motivations, this investigation focuses on the quasi-synchronization and dissipativity of delayed FNNs. The primary contributions can be outlined as follows:

(1) The fractional-order Lyapunov method is applied in the investigation of the quasi-synchronization and dissipativity issues of delayed FNNs, which provides a new approach for analyzing these types of networks.
(2) By employing fractional-order inequalities and a suitable Lyapunov function, a universal dissipativity criterion is derived. Additionally, the application of linear feedback control is employed to establish sufficient conditions for achieving quasi-synchronization in FNNs, which further contributes to the understanding of the synchronization behavior of these networks.
(3) By selecting suitable control parameters, it is possible to regulate the synchronization error bound within a relatively small range. This outcome has practical implications for designing controllers for FNNs. Furthermore, this study's results demonstrate that this research can alleviate the overly cautious nature of previous work, indicating the potential of this approach to advance the field of network analysis.

In Table 1, the notation used in this paper is described.

Table 1. Notation and descriptions.

Notation	Description
$\|\| \cdot \|\|_2$	The 2-norm
$diag(v_1, v_2, \ldots, v_n)$	A diagonal matrix
$A > 0$ (or $A < 0$)	A is positive definite (or negative definite)

2. Preliminaries and Problem Formulation

This section delineates the different definitions and lemmas that will be utilized later in the paper. The Caputo fractional derivative is the derivative of choice in this study.

Definition 1 ([22]). *The Caputo fractional derivative of function $\psi(t)$ is presented as follows:*

$$ {}_{t_0}D_t^{\varsigma}\psi(t) = \frac{1}{\Gamma(1-\varsigma)}\int_{t_0}^{t}(t-\tau)^{-\varsigma}\psi'(\tau)d\tau, $$

where $t \geq t_0$, $0 < \varsigma < 1$, Gamma function $\Gamma(\varsigma) = \int_0^{+\infty} \sigma^{-t} t^{\varsigma-1} dt$, and $\psi(t)$ is a function.

Lemma 1 ([23]). *Let differentiable functions $\phi(t) \in R^n$ be*

$$ \frac{1}{2}{}_{t_0}D_t^{\varsigma}\phi^T(t)P\phi(t) \leq \phi^T(t)P_{t_0}D_t^{\varsigma}\phi(t) \qquad \forall\varsigma \in (0,1], $$

where $P \in R^{n\times n} > 0$.

Lemma 2 ([24]). *For given vectors $\phi, \varphi \in R^n$ and a constant $\gamma > 0$, it yields*

$$ 2\phi^T\varphi \leq \gamma\phi^T\phi + \gamma^{-1}\varphi^T\varphi. $$

Take the following delayed FNNs into account:

$$ {}_0D_t^{\varsigma}\phi(t) = -C\phi(t) + Q\eta(\phi(t)) + R\xi(\phi(t-\tau)) + I(t), \tag{1} $$

where $\phi(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_n(t)]^T$, $C = diag(c_1, c_2, \ldots, c_n)$, $Q = (Q_{\iota v})_{n\times n}$, $R = (R_{\iota v})_{n\times n}$, $\eta(\phi(t)) = [\eta_1(\phi_1(t)), \eta_2(\phi_2(t)), \ldots, \eta_n(\phi_n(t))]^T$, $\xi(\phi(t-\tau)) = [\xi_1(\phi_1(t-\tau)), \xi_2(\phi_2(t-\tau)), \ldots, \xi_n(\phi_n(t-\tau))]^T$, $I(t) = [I_1(t), I_2(t), \ldots, I_n(t)]^T$, $\vartheta_\phi(s) = [\vartheta_{\phi 1}(s), \vartheta_{\phi 2}(s), \ldots, \vartheta_{\phi n}(s)]^T$.

Definition 2 ([25]). *A system is considered to be dissipative, as defined by system (1), if a compact set $S \subset R^n$ exists such that for all $\phi_0 \in R^n$, there exists a $T > 0$ such that when $t \geq t_0 + T$, $\phi(t, t_0, \phi_0) \subset S$. If this condition is met, then S is commonly referred to as a globally attractive set.*

Assumption 1 ([11]). *Assume that $\eta_v(\cdot)$ and $\xi_v(\cdot)$ satisfy*

$$ |\eta_v(\alpha) - \eta_v(\delta)| \leq L_v|\alpha - \delta|, $$

$$ |\xi_v(\alpha) - \xi_v(\delta)| \leq M_v|\alpha - \delta|, $$

where $L_v, M_v > 0$, respectively. $\alpha, \delta \in R$ and $v = 1, \ldots, n$.

3. Global Dissipativity of Delayed FNNs

The primary focus of this section is on the global dissipativity of FNNs (1). The estimation of the globally attractive set is carried out in detail using an appropriate Lyapunov function.

Theorem 1. *Under Assumption 1, if*

$$ 2\underline{c} - 2\sigma(Q)L_{\max} - \gamma_1\sigma(R)M_{\max} - \gamma_2 - \gamma_1^{-1}\sigma(R)M_{\max} > 0, $$

then system (1) is a dissipative system and $S = \left\{\phi(t) : ||\phi(t)||_2 \leq \sqrt{\frac{N}{\lambda - v}}\right\}$, where $\lambda = 2\underline{c} - 2\sigma(Q)L_{\max} - \gamma_1\sigma(R)M_{\max} - \gamma_2$, $v = \gamma_1^{-1}\sigma(R)M_{\max}$, $N = \gamma_2^{-1}||I^*||_2^2$, $\underline{c} = \min_{1\leq\iota\leq n}\{c_\iota\}$, $\sigma(Q) = ||Q||_2$, $\sigma(R) = ||R||_2$, $I^* = (\sup|I_1(t)|, \sup|I_2(t)|, \ldots, \sup|I_n(t)|)^T$, $L_{\max} = \max_{1\leq j\leq n}\{L_v\}$, $M_{\max} = \max_{1\leq j\leq n}\{M_v\}$.

Proof. Choose a Lyapunov candidate function as

$$V(t) = \phi^T(t)\phi(t).$$

Based on Lemma 1–Lemma 2 and Assumption 1, it has

$$\begin{aligned}
{}_0D_t^{\varsigma}V(t) &\leq 2\phi^T(t){}_0D_t^{\varsigma}\phi(t)\\
&= 2\phi^T(t)\big[-C\phi(t) + Q\eta(\phi(t)) + R\xi(\phi(t-\tau)) + I(t)\big]\\
&= -2\phi^T(t)C\phi(t) + 2\phi^T(t)Q\eta(\phi(t)) + 2\phi^T(t)R\xi(\phi(t-\tau)) + 2\phi^T(t)I(t)\\
&\leq -2\underline{c}||\phi(t)||_2^2 + 2||\phi(t)||_2||Q||_2||\eta(\phi(t))||_2 + 2||\phi(t)||_2||R||_2||\xi(\phi(t-\tau))||_2\\
&\quad + 2||\phi(t)||_2||I(t)||_2\\
&\leq -2\underline{c}||\phi(t)||_2^2 + 2||Q||_2L_{\max}||\phi(t)||_2^2 + ||R||_2M_{\max}(\gamma_1||\phi(t)||_2^2 + \gamma_1^{-1}||\phi(t-\tau)||_2^2)\\
&\quad + (\gamma_2||\phi(t)||_2^2 + \gamma_2^{-1}||I(t)||_2^2)\\
&= (-2\underline{c} + 2\sigma(Q)L_{\max} + \gamma_1\sigma(R)M_{\max} + \gamma_2)||\phi(t)||_2^2\\
&\quad + \gamma_1^{-1}\sigma(R)M_{\max}||\phi(t-\tau)||_2^2 + \gamma_2^{-1}||I^*||_2^2\\
&= -(2\underline{c} - 2\sigma(Q)L_{\max} - \gamma_1\sigma(R)M_{\max} - \gamma_2)V(\phi(t))\\
&\quad + \gamma_1^{-1}\sigma(R)M_{\max}V(\phi(t-\tau)) + \gamma_2^{-1}||I^*||_2^2.
\end{aligned}$$

Let $\lambda = 2\underline{c} - 2\sigma(Q)L_{\max} - \gamma_1\sigma(R)M_{\max} - \gamma_2$, $v = \gamma_1^{-1}\sigma(R)M_{\max}$, $N = \gamma_2^{-1}||I^*||_2^2$; then, we obtain

$$\begin{aligned}
{}_0D_t^{\varsigma}V(t) &\leq -\lambda V(\phi(t)) + vV(\phi(t-\tau)) + N\\
&\leq -\lambda V(\phi(t)) + v\sup_{t-\tau\leq s\leq t} V(\phi(s)) + N.
\end{aligned}$$

In addition, $\lambda - v > 0$. Then, based on the fractional Halanay inequality [26], one has

$$V(\phi(t)) \leq \frac{N}{\lambda - v}, \qquad t \to +\infty,$$

i.e.,

$$||\phi(t)||_2 \leq \sqrt{\frac{N}{\lambda - v}}, \qquad t \to +\infty.$$

Then, system (1) is a dissipative system. □

4. Quasi-Synchronization of Delayed FNNs

For system (1), let $I(t) = 0$ and consider the response system as

$$_0D_t^{\varsigma}\varphi(t) = -C'\varphi(t) + Q'\eta(\varphi(t)) + R'\xi(\varphi(t-\tau)) + u(t), \tag{2}$$

where $\varphi(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)]^T$, $C' = diag(c_1', c_2', \ldots, c_n')$, $Q' = (Q_{\iota v}')_{n\times n}$, $R' = (R_{\iota v}')_{n\times n}$, $\eta(\phi(t)) = [\eta_1(\phi_1(t)), \eta_2(\phi_2(t)), \ldots, \eta_n(\phi_n(t))]^T$, $\xi(\phi(t-\tau)) = [\xi_1(\phi_1(t-\tau)), \xi_2(\phi_2(t-\tau)), \ldots, \xi_n(\phi_n(t-\tau))]^T$, $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T$, $\vartheta_\varphi(s) = [\vartheta_{\varphi 1}(s), \vartheta_{\varphi 2}(s), \ldots, \vartheta_{\varphi n}(s)]^T$.

Assume $\sigma(t) = \varphi(t) - \phi(t)$ and

$$u(t) = -F\sigma(t), \tag{3}$$

where $F = diag(f_1, f_2, \ldots, f_n)$.

Then, one has the error system:

$$
\begin{aligned}
{}_0D_t^{\varsigma}\sigma(t) &= -(C^{'}+F)\sigma(t) + Q^{'}[\eta(\varphi(t)) - \eta(\phi(t))] + R^{'}[\xi(\varphi(t-\tau)) - \xi(\phi(t-\tau))] \\
&\quad + (C^{'}-C)\phi(t) + (Q^{'}-Q)\eta(\phi(t)) + (R^{'}-R)\eta(\phi(t-\tau)) \\
&= -(C^{'}+F)\sigma(t) + Q^{'}\varphi(\sigma(t)) + R^{'}\varphi(\sigma(t-\tau)) + z(t),
\end{aligned} \tag{4}
$$

where $\varphi(\sigma(t)) = \eta(\varphi(t)) - \eta(\phi(t))$, $\varphi(\sigma(t-\tau)) = \xi(\varphi(t-\tau)) - \xi(\phi(t-\tau))$, $z(t) = (C^{'}-C)x(t) + (Q^{'}-Q)\eta(\phi(t)) + (R^{'}-R)\eta(\phi(t-\tau))$. The initial condition is $\sigma(s) = \phi(s), -\tau \leq s \leq 0$, where $\phi(s) = (\phi_1(s), \ldots, \phi_n(s))^T$, $\phi_\iota(s) = \vartheta_{\varphi i}(s) - \vartheta_{\phi i}(s)$, $s \in [-\tau, 0]$ and $\iota = 1, 2, \ldots, n$.

Set $\Delta C = (C - C^{'})$, $\Delta\tilde{Q} = Q^{'} - Q$, $\Delta\tilde{R} = R^{'} - R$, $L_{\max} = \max_{1\leq j\leq n}\{L_j\}$, $M_{\max} = \max_{1\leq j\leq n}\{M_j\}$. Then, we have

$$
\begin{aligned}
||z(t)||_2 &= ||(C-C^{'})x(t) + \Delta\tilde{Q}(t)\eta(\phi(t)) + \Delta\tilde{R}(t)\xi(\phi(t-\tau)) \\
&\leq ||\Delta C||_2||\phi(t)||_2 + ||\Delta\tilde{Q}||_2||\eta(\phi(t))||_2 + ||\Delta\tilde{R}||_2||\xi(\phi(t-\tau))||_2 \\
&\leq ||\Delta C||_2||\phi(t)||_2 + ||\Delta\tilde{Q}||_2 L_{\max}||\phi(t)||_2 + ||\Delta\tilde{R}||_2 M_{\max}||\phi(t-\tau)||_2
\end{aligned}
$$

where $\Delta C = diag(\Delta c_1, \Delta c_2, \ldots, \Delta c_n)$ and $\Delta c_i = c_i - c_i^{'}$, $i = 1, 2, \ldots, n$.

It is widely recognized that the boundedness of trajectories is a fundamental characteristic of chaotic systems, which is due to the inherent dissipativity of such systems. Thus, assume a constant $z > 0$ such that $||\phi(t)||_2 \leq z$ for all $t \geq -\tau$. That is, for a constant z^*, it yields

$$
||z(t)||_2 \quad \leq \quad (||\Delta C||_2 + ||\Delta\tilde{Q}||_2 L_{\max} + ||\Delta\tilde{R}||_2 M_{\max})z = z^*. \tag{5}
$$

Next, the quasi-synchronization criterion is presented.

Theorem 2. *Under Assumption 1, if*

$$
2(\underline{c}^{'} + \underline{f}) - 2\sigma(Q^{'})L_{\max} - \gamma_1\sigma(R^{'})M_{\max} - \gamma_2 - \gamma_1^{-1}\sigma(R^{'})M_{\max} > 0,
$$

then the error system (4) will converge to the region $D = \left\{\sigma(t) : ||\sigma(t)||_2 \leq \sqrt{\frac{Z}{\lambda - v}}\right\}$, *where* $\lambda = 2(\underline{c}^{'} + \underline{f}) - 2\sigma(Q^{'})L_{\max} - \gamma_1\sigma(R^{'})M_{\max} - \gamma_2$, $v = \gamma_1^{-1}\sigma(R^{'})M_{\max}$, $Z = \gamma_2^{-1}(z^*)^2$, $\underline{c}^{'} = \min_{1\leq i\leq n}\{c_i^{'}\}$, $\underline{f} = \min_{1\leq i\leq n}\{f_i\}$, $\sigma(Q^{'}) = ||Q^{'}||_2$, $\sigma(R^{'}) = ||R^{'}||_2$, $L_{\max} = \max_{1\leq j\leq n}\{L_\nu\}$, $M_{\max} = \max_{1\leq j\leq n}\{M_\nu\}$.

Proof. Take the Lyapunov function

$$
V(t) = \sigma^T(t)\sigma(t). \tag{6}
$$

From Lemma 1–Lemma 2 and Assumption 1, one has

$$\begin{aligned}
{}_0D_t^{\varsigma}V(t) &\leq 2\sigma^T(t){}_0D_t^{\varsigma}\sigma(t) \\
&= 2\sigma^T(t)\big[-(C^{'}+F)\sigma(t)+Q^{'}\varphi(\sigma(t))+R^{'}\varphi(\sigma(t-\tau))+Z(t)\big] \\
&= -2\sigma^T(t)(C^{'}+F)\sigma(t)+2\sigma^T(t)Q^{'}\varphi(\sigma(t))+2\sigma^T(t)R^{'}\varphi(\sigma(t-\tau))+2\sigma^T(t)Z(t) \\
&\leq -2(\underline{c}^{'}+\underline{f})||\sigma(t)||_2^2+2||Q^{'}||_2L_{\max}||\sigma(t)||_2^2 \\
&\quad +2||\sigma(t)||_2||R^{'}||_2M_{\max}||\sigma(t-\tau)||_2+2||\sigma(t)||_2||z(t)||_2 \\
&\leq -2(\underline{c}^{'}+\underline{f})||\sigma(t)||_2^2+2||Q^{'}||_2L_{\max}||\sigma(t)||_2^2 \\
&\quad +||R^{'}||_2M_{\max}(\gamma_1||\sigma(t)||_2^2+\gamma_1^{-1}||\sigma(t-\tau)||_2^2)+(\gamma_2||\sigma(t)||_2^2+\gamma_2^{-1}||\sigma(t-\tau)||_2^2) \\
&\leq \big[-2(\underline{c}^{'}+\underline{f})+2\sigma(Q^{'})L_{\max}+\gamma_1\sigma(R^{'})M_{\max}\big]||\sigma(t)||_2^2 \\
&\quad +\gamma_1^{-1}\sigma(R^{'})M_{\max}||\sigma(t-\tau)||_2^2+\gamma_2^{-1}(z^*)^2 \\
&= -\big[2(\underline{c}^{'}+\underline{f})-2\sigma(Q^{'})L_{\max}-\gamma_1\sigma(R^{'})M_{max}\big]V(\sigma(t)) \\
&\quad +\gamma_1^{-1}\sigma(R^{'})M_{\max}V(\sigma(t-\tau))+\gamma_2^{-1}(z^*)^2.
\end{aligned}$$

Let $\lambda = 2(\underline{c}^{'}+\underline{f})-2\sigma(Q^{'})L_{\max}-\gamma_1\sigma(R^{'})M_{\max}-\gamma_2$, $\upsilon=\gamma_1^{-1}\sigma(R^{'})M_{\max}$, $Z=\gamma_2^{-1}(z^*)^2$; then, one has

$$\begin{aligned}
{}_0D_t^{\varsigma}V(t) &\leq -\lambda V(\sigma(t))+\upsilon V(\sigma(t-\tau))+Z \\
&\leq -\lambda V(\sigma(t))+\upsilon \sup_{t-\tau\leq s\leq t} V(\sigma(s))+Z.
\end{aligned}$$

In addition, $\lambda-\upsilon>0$. Similarly, it yields

$$||\sigma(t)||_2 \leq \sqrt{\frac{Z}{\lambda-\upsilon}}, \qquad t\to+\infty,$$

i.e.,

$$D=\left\{\sigma(t):||\sigma(t)||_2\leq\sqrt{\frac{Z}{\lambda-\upsilon}}\right\}, \qquad t\to+\infty.$$

This completes the proof. □

If there are no parameter mismatches, we have

$${}_0D_t^{\varsigma}\varphi(t) = -C\varphi(t)+Q\eta(\varphi(t))+R\xi(\varphi(t-\tau))+u(t), \tag{7}$$

where $\varphi(t) = [\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)]^T$, $C = diag(c_1,c_2,\ldots,c_n)$, $Q = (Q_{\iota\upsilon})_{n\times n}$, $R=(R_{\iota\upsilon})_{n\times n}$, $\eta(\phi(t))=[\eta_1(\phi_1(t)),\eta_2(\phi_2(t)),\ldots,\eta_n(\phi_n(t))]^T$, $\xi(\phi(t-\tau))=[\xi_1(\phi_1(t-\tau)),\ \xi_2(\phi_2(t-\tau)),\ldots,\xi_n(\phi_n(t-\tau))]^T$, $u(t) = [u_1(t),u_2(t),\ldots,u_n(t)]^T$, $\vartheta_\varphi(s)=[\vartheta_{\varphi1}(s),\vartheta_{\varphi2}(s),\ldots,\vartheta_{\varphi n}(s)]^T$.

Thus, one has

$$\begin{aligned}
{}_0D_t^{\varsigma}\sigma(t) &= -(C+F)\sigma(t)+Q[\eta(\varphi(t))-\eta(\phi(t))]+R[\xi(\varphi(t-\tau))-\xi(\phi(t-\tau))] \\
&= -(C+F)\sigma(t)+Q\varphi(\sigma(t))+R\varphi(\sigma(t-\tau))
\end{aligned} \tag{8}$$

where $\varphi(\sigma(t))=\eta(\varphi(t))-\eta(\phi(t))$, $\varphi(\sigma(t-\tau))=\xi(\varphi(t-\tau))-\xi(\phi(t-\tau))$. Next, the synchronization criterion is presented.

Corollary 1. *Under Assumption 1, if*

$$2(\underline{c}+\underline{f})-2\sigma(Q)L_{\max}-\gamma_1\sigma(R)M_{\max}-\gamma_1^{-1}\sigma(R)M_{\max}>0,$$

then $D = \{\sigma(t) : ||\sigma(t)||_2 = 0\}$*, where* $\lambda = 2(\underline{c} + \underline{f}) - 2\sigma(Q)L_{\max} - \gamma_1\sigma(R)M_{\max}$*,* $v = \gamma_1^{-1}\sigma(R)M_{\max}$*,* $\underline{c} = \min_{1\le i\le n}\{c_i\}$*,* $\underline{f} = \min_{1\le i\le n}\{f_i\}$*,* $\sigma(Q) = ||Q||_2$*,* $\sigma(R) = ||R||_2$*,* $L_{\max} = \max_{1\le j\le n}\{L_\nu\}$*,* $M_{\max} = \max_{1\le j\le n}\{M_\nu\}$*. Thus, the systems (1)–(7) can achieve synchronization.*

As the FNN model (1) is a novel proposal, we first utilize the maximum absolute value to establish a criterion, which allows us to compare our method with previous approaches and demonstrate its superiority. The criterion is stated as follows.

Theorem 3. *Under Assumption 1, if*

$$\min_{1\le i\le n}\left[2(c_\iota' + f_\iota) - \sum_{\nu=1}^{n}(Q_{\iota\nu}'L_\nu + Q_{\nu\iota}'L_\iota) - \gamma_1\sum_{\nu=1}^{n}R_{\iota\nu}'M_\nu - \gamma_2\right] - \gamma_1^{-1}\max_{1\le i\le n}\left(\sum_{\nu=1}^{n}R_{\nu\iota}'M_\iota\right) > 0,$$

then $D = \left\{\sigma(t) : ||\sigma(t)||_2 \le \sqrt{\frac{Z}{\lambda - v}}\right\}$*, where* $\lambda = \min_{1\le i\le n}\left[2(c_\iota' + f_\iota) - \sum_{\nu=1}^{n}(Q_{\iota\nu}'L_\nu + Q_{\nu\iota}'L_\iota) - \gamma_1\sum_{\nu=1}^{n}R_{\iota\nu}'M_\nu - \gamma_2\right]$*,* $Z = \gamma_2^{-1}(z^*)^2$*,* $v = \gamma_1^{-1}\max_{1\le i\le n}(\sum_{\nu=1}^{n}R_{\nu\iota}'M_\iota)$*.*

Proof. Set the Lyapunov function as

$$V(t) = \sum_{\iota=1}^{n} e_i^2(t).$$

The proof of Theorem 3 can be demonstrated similarly to that of Theorem 2, and hence it is omitted here. □

Remark 1. *In Theorem 3, algebraic conditions were obtained based on the maximum absolute value method, which is similar to [27–35]. However, it should be noted that such algebraic conditions are not desirable when dealing with high-dimensional FNNs. Compared to the proposed criteria in [27–35], the verification process involved in Theorem 2 is less time consuming. As a result, the computational burden is not increased in Theorem 2.*

Remark 2. *Based on Theorem 2, quasi-synchronization can be achieved in the drive-response systems if*

$$\underline{f} > \frac{1}{2}\left[2\sigma(Q')L_{\max} + \gamma_1\sigma(R')M_{\max} + \gamma_2 + \gamma_1^{-1}\sigma(R')M_{\max}\right] - \underline{c}'.$$

On the other hand, Theorem 3 provides a different condition for achieving quasi-synchronization:

$$\underline{f} > \frac{1}{2}\left[\max_{1\le i\le n}\left(\sum_{\nu=1}^{n}(Q_{\iota\nu}'L_\nu + Q_{\nu\iota}'L_\iota) + \gamma_1\sum_{\nu=1}^{n}R_{\iota\nu}'M_\nu\right) + \gamma_2 + \gamma_1^{-1}\max_{1\le i\le n}\left(\sum_{\nu=1}^{n}R_{\nu\iota}'M_\iota\right)\right] - \underline{c}',$$

where $\underline{c}' = \min_{1\le i\le n} c_i'$ *and* $\underline{f} = \min_{1\le i\le n} f_i$*. It should be noted that Theorem 2 has the potential to enhance the results obtained in Theorem 3 to a certain degree. The advantages of Theorem 2 are discussed in Section 5.*

5. Numerical Examples

Example 1. *Take system (1), with* $n = 2$*,* $\varsigma = 0.95$*,* $\eta_\nu(\phi_\nu) = \xi_\nu(\phi_\nu) = tanh(\phi_\nu)$*,* $\nu = 1, 2$*,* $\tau = 1$*. Take* $c_1 = 2.5$*,* $c_2 = 2.5$*,* $I_1(t) = 0.5sin(t)$*,* $I_2(t) = -0.4cos(t)$*, and*

$$Q = \begin{bmatrix} 0.25 & 0.3 \\ 0.2 & -0.1 \end{bmatrix}, \quad R = \begin{bmatrix} 0.4 & 0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

Through simple calculation, it can be obtained that $L_{\max} = 1$*,* $M_{\max} = 1$*,* $\sigma(Q) = 0.3953$*, and* $\sigma(R) = 0.4472$*. By taking* $\gamma_1 = 1$ *and* $\gamma_2 = 1$*, we obtain* $N = \gamma_2^{-1}||I^*||2^2 = 0.41$*,* $\lambda = 2\underline{c} - 2\sigma(Q)Lmax - \gamma_1\sigma(R)M_{\max} - \gamma_2 = 2.7622$*,* $v = \gamma_1^{-1}\sigma(R)M_{\max} = 0.4472$*, and*

$\lambda - \upsilon = 2.315 > 0$. *This indicates that Theorem 1 is satisfied, and one has* $||\phi||_2 \leq \sqrt{\frac{N}{\lambda-\upsilon}} = 0.4208$. *Therefore, the set* $S = \phi : ||\phi||_2 \leq 0.4208$ *is called globally attractive. The simulation results with four randomly chosen initial values are presented in Figures 1 and 2. It can be observed that the FNNs exhibit dissipative behavior.*

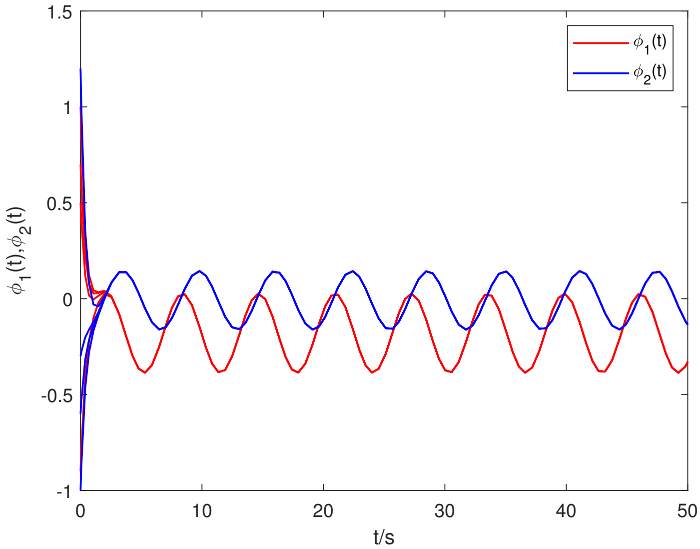

Figure 1. The trajectories of ϕ_1 and ϕ_2.

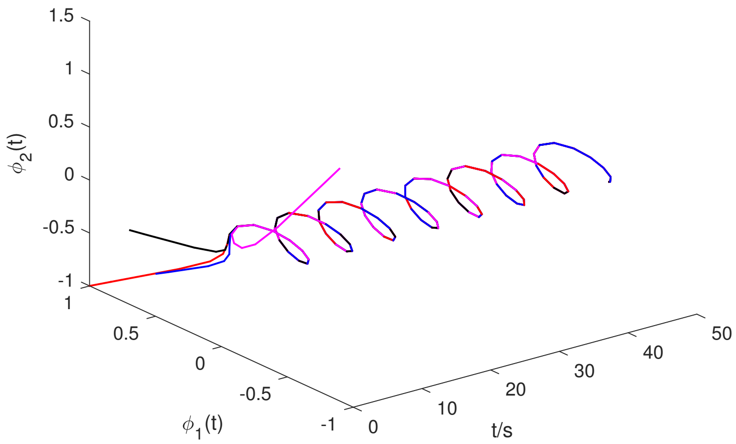

Figure 2. Phase portraits.

Example 2. *Take system (1) with* $n = 2$, $\varsigma = 0.98$, $\eta_\nu(\phi_\nu) = \xi_\nu(\phi_\nu) = tanh(\phi_\nu)$, $\nu = 1, 2$, $\tau = 0.92$. *Take* $c_1 = 4$, $c_2 = 2$, $I_1(t) = 0 = I_2(t)$, $\vartheta_\phi(s) = (-0.9, 0.6)^T$, $s \in [-0.92, 0]$, *and*

$$Q = \begin{bmatrix} 2.1 & -2.1 \\ -0.55 & 2.6 \end{bmatrix}, \quad R = \begin{bmatrix} -3.9 & -2.7 \\ -1.6 & -3.7 \end{bmatrix}.$$

Assuming this system (2) with $n = 2$, $\varsigma = 0.98$, $\eta_\nu(\phi_\nu) = \xi_\nu(\phi_\nu) = tanh(\phi_\nu)$, $\nu = 1, 2$, *through simple calculation, it can be obtained that* $L_{\max} = 1$, $M_{\max} = 1$, $\tau = 0.92$. *Take* $c_1^{'} = 3.2$, $c_2^{'} = 2.2$, $I_1(t) = 0 = I_2(t)$, $\vartheta_\varphi(s) = (-2.8, 2.4)^T$, $s \in [-0.92, 0]$, $u(t) = (u_1(t), u_2(t))^T$, *and*

$$Q = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}.$$

From Figure 3, we can obtain that $||\phi(t)||_2 \leq 1.25$, *then* $z^* = (||\Delta C||_2 + ||\Delta\tilde{Q}||_2 L_{\max} + ||\Delta\tilde{R}||_2 M_{\max})z = 6.2112$. *In Figure 4, the synchronization error* $||\sigma(t)||_2$ *is given without controller (3).*

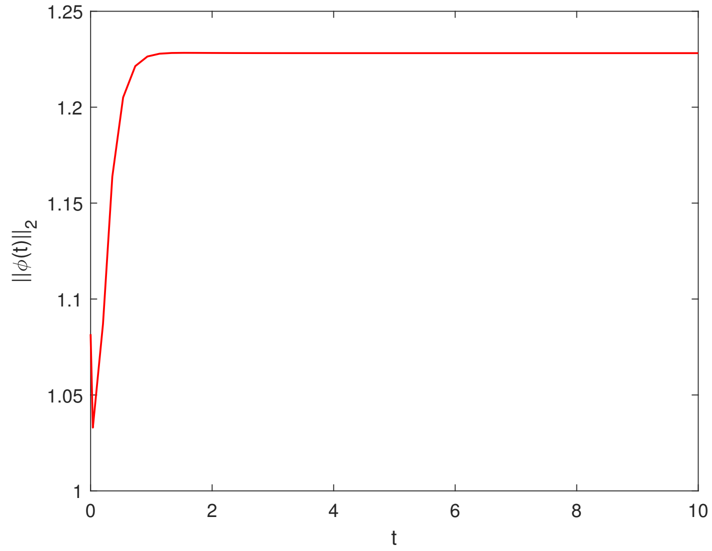

Figure 3. Evolution of $||\phi(t)||_2$ of system (1).

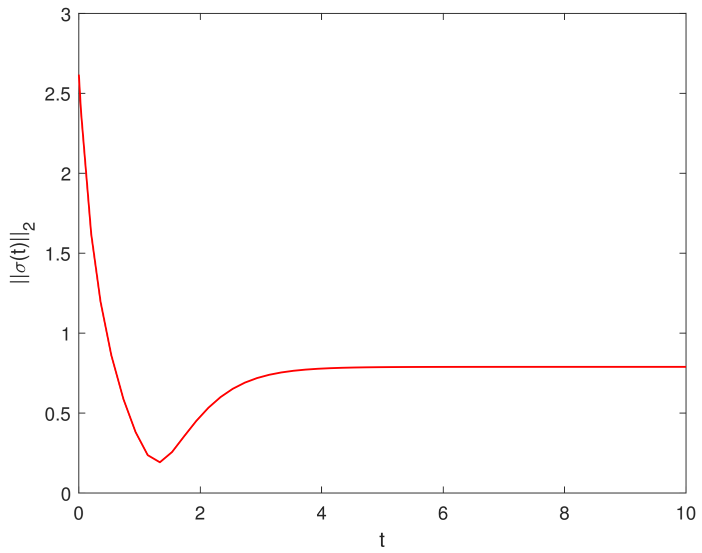

Figure 4. $||\sigma(t)||_2$ without controller.

By setting $f_1 = 20$, $f_2 = 20$, $\gamma_1 = 1$, $\gamma_2 = 19$, $\sigma(Q) = 3.8229$, $\sigma(R) = 5.9919$, $\sigma(Q^{'}) = 3.6108$, *and* $\sigma(R^{'}) = 4.4142$, *we obtain* $\lambda = 2(\underline{c}^{'} + \underline{f}) - 2\sigma(Q^{'})Lmax - \gamma_1\sigma(R^{'})M_{max}$ $-\gamma_2 = 13.7624$, $\upsilon = \gamma_1^{-1}\sigma(R^{'})M_{max} = 4.4142$, *and* $\lambda - \upsilon = 9.35 > 0$. *Hence, based on Theorem 2, the quasi-synchronization can be realized for systems (1) and (2) with* $||\sigma(t)||_2 \leq \sqrt{\frac{Z}{\lambda-\upsilon}} = 0.4660$, *which is confirmed by Figure 5.*

To demonstrate the superiority of Theorem 2 over Theorem 3, a comparison is presented below.

Table 2 shows that when $\gamma_1 = 1$ *and* $\gamma_2 = 19$, *the systems (1) and (2) achieve quasi-synchronization if*

$$\underline{f} > \frac{1}{2}\Big[2\sigma(Q^{'})L_{\max} + \gamma_1\sigma(R^{'})M_{\max} + \gamma_2 + \gamma_1^{-1}\sigma(R^{'})M_{\max}\Big] - \underline{c}^{'} = 15.325,$$

based on Theorem 2. However, according to Theorem 3, one needs to ensure that

$$\underline{f} > \frac{1}{2}\left[\max_{1\leq i\leq n}\Big(\sum_{\nu=1}^{n}(Q^{'}_{\iota\nu}L_\nu + Q^{'}_{\nu\iota}L_\iota) + \gamma_1\sum_{\nu=1}^{n}R^{'}_{\iota\nu}M_\nu\Big) + \gamma_2 + \gamma_1^{-1}\max_{1\leq i\leq n}\Big(\sum_{\nu=1}^{n}R^{'}_{\nu\iota}M_\iota\Big)\right] - \underline{c}^{'}$$
$$= 16.3.$$

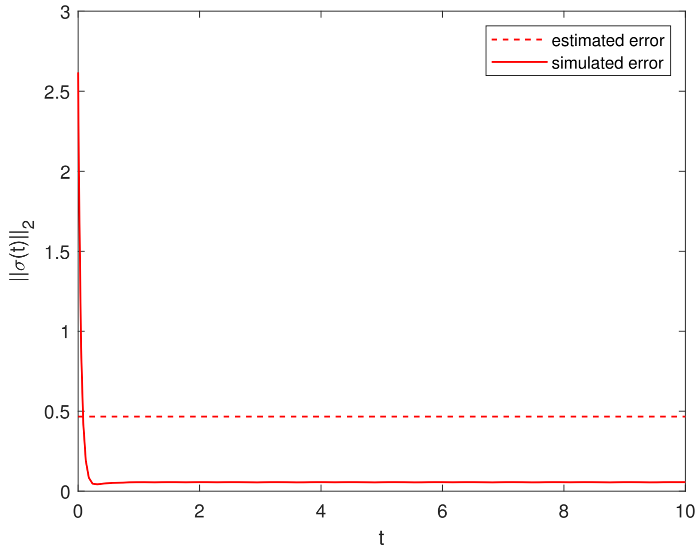

Figure 5. Estimated error and simulated error.

Table 2. Comparison of Theorem 2 and Theorem 3 for Example 2 with $\gamma_1 = 1$ and $\gamma_2 = 19$.

Method	Requirement for $\underline{f}$
Theorem 2	$\underline{f} > 15.325$
Theorem 3	$\underline{f} > 16.3$

Example 3. *Take system (1) with $n = 3$, $\varsigma = 0.85$, $\eta_\nu(\phi_\nu) = \xi_\nu(\phi_\nu) = tanh(\phi_\nu)$, $\tau = 0.8$. Set $c_1 = 2$, $c_2 = 2$, $c_3 = 2$, $\vartheta_\phi(s) = (0.8, -0.6, 0.8)^T$, $s \in [-1, 0]$, and*

$$Q = \begin{bmatrix} 2.1 & 0 & 0 \\ 0 & 5.7 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}, \quad R = \begin{bmatrix} -3.9 & 0 & 0 \\ 0 & -3.7 & 0 \\ 0 & 0 & 13 \end{bmatrix}.$$

Assuming that the response system (2) has identical parameters and initial conditions $\vartheta_\varphi(s) = (1.2, -1.5, 1.7)^T$, $s \in [-1, 0]$, the systems (1) and (2) are asynchronous, as shown in Figure 6.

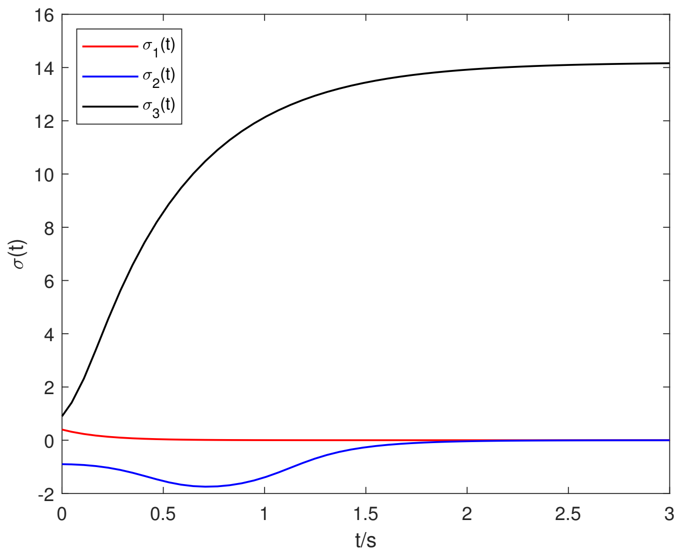

Figure 6. Time response trajectories of $\sigma(t)$.

Take $f_1 = f_2 = f_3 = 15$, $\gamma_1 = 1$, $\sigma(Q) = 5.7$, *and* $\sigma(R) = 13$. *Then, we obtain* $\lambda = 2(\underline{c} + \underline{f}) - 2\sigma(Q)L_{max} - \gamma_1\sigma(R)M_{max} = 15.3$, $v = \gamma_1^{-1}\sigma(R)M_{max} = 13$, *and* $\lambda - v = 2.3 > 0$. *Therefore, according to Corollary 1, the systems (1) and (2) achieve synchronization with the controller (3), which is verified by Figure 7.*

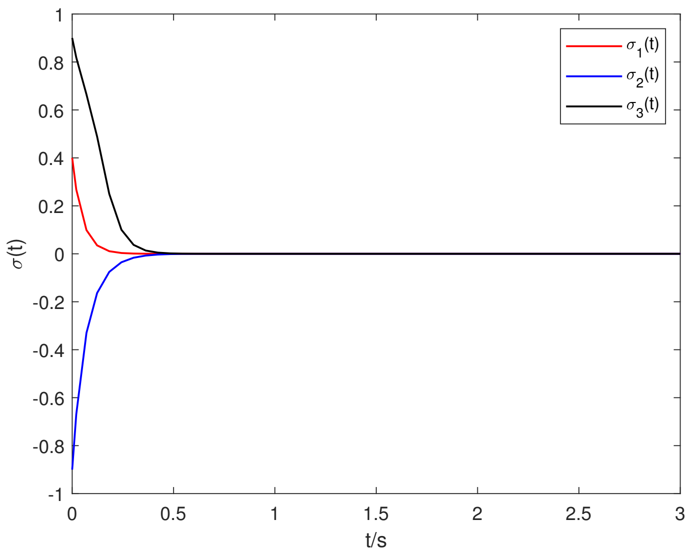

Figure 7. Time response trajectories of $\sigma(t)$ under controller (3).

6. Conclusions

The global dissipation and quasi-synchronization characteristics of FNNs are studied. To obtain sufficient criteria for quasi-synchronization and dissipativity, suitable Lyapunov functions are constructed. To extend some existing results and achieve more relaxed criteria, the 2-norm of the matrix is used. Numerical examples demonstrate the achievability of the proposed conclusions. The introduction of a memristor will cause state switching in the system. We aim to investigate the quasi-synchronization of memristor-based FNNs in the future, which remain open problems and require further investigation.

Author Contributions: Writing, Y.L. and M.L.; funding acquisition, C.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the NNSF of China under grant 61803233.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to express our thanks to the editors and reviewers.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviation

FNNs fractional-order neural networks

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