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Dynamic Properties for a Second-Order Stochastic SEIR Model with Infectivity in Incubation Period and Homestead-Isolation of the Susceptible Population

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Abstract: In this article, we analyze a second-order stochastic SEIR epidemic model with latent infectious and susceptible populations isolated at home. Firstly, by putting forward a novel inequality, we provide a criterion for the presence of an ergodic stationary distribution of the model. Secondly, we establish sufficient conditions for extinction. Thirdly, by solving the corresponding Fokker–Plank equation, we derive the probability density function around the quasi-endemic equilibrium of the stochastic model. Finally, by using the epidemic data of the corresponding deterministic model, two numerical tests are presented to illustrate the validity of the theoretical results. Our conclusions demonstrate that nations should persevere in their quarantine policies to curb viral transmission when the COVID-19 pandemic proceeds to spread internationally.

Keywords: stochastic SEIR model; stationary distribution; extinction; density function

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1. Introduction

As everyone knows, Corona-virus 2019 (COVID-19) has become a serious threat to human health and lives in recent years. On 25 January 2022, the World Health Organization (WHO) declared that nearly 50,000 new deaths were reported. As of 23 January 2022, 634 million confirmed cases and 6.6 million deaths have been reported worldwide [1]. It is worth emphasizing that social isolation has had a direct effect on the physical transmission of infectious diseases[2]. The COVID-19 outbreak clearly showed that physical protection and social isolation play a key role in controlling epidemics when vaccines or antiviral drugs for the virus are lacking. Mathematical modeling is an important promoting factor in discussing the dynamic behaviors of a disease's spread [3]. Based on the importance of social isolation, Jiao et al. [4] developed an SEIR model with infectivity in the incubation period and homestead-isolation of the susceptible population. The model is given as follows:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta(1-\theta_1)S[I+\theta_2 E] - \mu S, \\ \frac{dE}{dt} = \beta(1-\theta_1)S[I+\theta_2 E] - (\delta+\mu)E, \\ \frac{dI}{dt} = \delta E - (\gamma+\sigma+\mu)I, \\ \frac{dR}{dt} = (\gamma+\theta_3\sigma)I - \mu R, \end{cases} \tag{1}$$

where S is the numbers of susceptible people, E denotes the numbers of the exposed population, I stands for the numbers of the infected population, $R(t)$ denotes the numbers of the recovered population at time t. $\Lambda > 0$ signifies the enrolling rate, $\beta > 0$ refers to the infection rate from S to E, $0 < \theta_1 < 1$ indicates the homestead-isolation rate of the susceptible population, $0 < \theta_2 < 1$ represents the infectious effect of the exposed population in incubation period, $\mu > 0$ is the natural death rate, $\delta > 0$ is the transition rate from E to I, $\gamma > 0$ is the transition rate from I to R, $\sigma > 0$ represents the hospitalized rate of I for the disease, $\theta_3 > 0$ depicts the recurring rate of I, and $\delta > \theta_2(\gamma + \sigma + \mu)$.

It is obvious that the individual $R(t)$ does not adversely affect the dynamics of SEIR. Therefore, we will neglect the last equation of model (1), and it can be reducible to:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta(1-\theta_1)S[I+\theta_2 E] - \mu S, \\ \dfrac{dE}{dt} = \beta(1-\theta_1)S[I+\theta_2 E] - (\delta+\mu)E, \\ \dfrac{dI}{dt} = \delta E - (\gamma+\sigma+\mu)I. \end{cases} \tag{2}$$

Then, Jiao et al. [4] derived that model (1) demonstrates a disease-free equilibrium $P^0(S^0, 0, 0)$ with $S^0 = \dfrac{\Lambda}{\mu}$ and an interior endemic equilibrium $P^+(S^+, E^+, I^+)$, where

$$S^+ = \frac{(\gamma+\sigma+\mu)(\delta+\mu)}{\beta(1-\theta)[\delta+\theta_2(\gamma+\sigma+\mu)]}, E^+ = \frac{\Lambda\beta(1-\theta_1)[\delta+\theta_2(\gamma+\sigma+\mu)] - \mu(\gamma+\sigma+\mu)(\delta+\mu)}{\beta(1-\theta_1)(\delta+\theta_2(\gamma+\sigma+\mu))(\delta+\mu)},$$

$I^+ = \dfrac{\delta E^+}{\gamma+\sigma+\mu}$ with $\Lambda\beta(1-\theta_1)[\delta+\theta_2(\gamma+\sigma+\mu)] > \mu(\gamma+\sigma+\mu)(\delta+\mu)$, and the basic reproduction number is defined as $\mathfrak{R}_0 = \dfrac{\Lambda\beta(1-\theta_1)[\delta+\theta_2(\gamma+\sigma+\mu)]}{\mu(\gamma+\sigma+\mu)(\delta+\mu)}$.

In practical terms, the process of disease transmission is frequently disturbed by external factors. Mao et al. have claimed that the incorporation of a small amount of random fluctuation in the deterministic model of infectious disease can actively prevent outbreaks in the latent population [5]. Recently, several authors have considered the random fluctuation factor into biological mathematical models of infectious diseases and environmental pollution [6–10]. In [11], in view of nonlinear stochastic disturbance, Mu et al. discussed the extinction and persistence for a stochastic micro-organism flocculation model. In [12], for a stochastic staged model of the progression of HIV/AIDS with nonlinear disturbances, Jiang et al. first presented a new approach called the "stochastic p-threshold method" to remove nonlinear terms caused by second-order disturbances and obtained the unique ergodic stationary distribution of the model. In [13], Zhou et al. made use of the "p-stochastic criterion technique" together with some inequalities they presented to eliminate the effect of third-order perturbation of a stochastic model. Motivated by these works, we introduce nonlinear stochastic disturbances into model (2), which is then expressed by:

$$\begin{cases} dS = [\Lambda - \beta(1-\theta_1)S(I+\theta_2 E) - \mu S]dt + (\sigma_{11}+\sigma_{12}S)SdB_1(t), \\ dE = [\beta(1-\theta_1)S(I+\theta_2 E) - (\delta+\mu)E]dt + (\sigma_{21}+\sigma_{22}E)EdB_2(t), \\ dI = [\delta E - (\gamma+\sigma+\mu)I]dt + (\sigma_{31}+\sigma_{32}I)IdB_3(t), \end{cases} \tag{3}$$

where $\{B_i(t)\}_{t\geq 0}(i = 1, 2)$ denote mutually independent standard Brownian motions and are defined on the complete probability space $\{\Omega, \Gamma, \{\Gamma_t\}_{t\geq 0}, \mathbb{P}\}$ with an increasing and right continuous filtration $\{\Gamma_t\}_{t\geq 0}$; the positive constants $\sigma_{ij}(i = 1, 2, 3, j = 1, 2)$ are the white noise intensities.

Notably, relatively few discussions exist deriving the explicit expression of the probability density function as a result of the difficulty of solving the high-dimensional Fokker–Plank equation. In [14], Zhou et al. investigated the density function analysis of a stochastic SVI epidemic model with a half-saturated incidence rate for the first time. In light of this work, some researchers are looking into the properties of stochastic epidemic models [2,6,15–20]. However, no authors have studied the exact expression of the density function of model (3) yet, which is the key work of the present paper. In addition, the aforementioned "stochastic p-threshold method" is not completely suitable for the corresponding deterministic case with a complicated basic reproduction number when researching the unique ergodic stationary distribution of a stochastic epidemic disease model. Therefore, another aim of this paper is to improve the "stochastic p-threshold method" and explore the unique ergodic stationary distribution and extinction for model (3).

The remainder of this paper is arranged in the following manner. In Section 2, we investigate the stationary distribution for model (3). In Section 3, we give adequate conditions of extinction for model (3). Section 4 derives a precise formula for the probability density function of model (3). In Section 5, two examples are presented to demonstrate the practicality of our conclusions.

2. Existence of Ergodic Stationary Distribution

In the whole paper, let $\mathbb{R}^n$ be an n-dimensional Euclidean space: $\mathbb{R}^n_+ = \{(y_1, \dots, y_n) \in \mathbb{R}^n | y_k > 0, 0 \leq k \leq n\}, l_1 \wedge l_2 \wedge \dots \wedge l_n = \min\{l_1, l_2, \dots, l_n\}, h_1 \vee h_2 \vee \dots \vee h_n = \max\{h_1, h_2, \dots, h_n\}$. If $B_{s \times t}$ is a real matrix, let B^T be the transpose matrix of B. When $s = t$, B^{-1} represents the inverse matrix of B.

Lemma 1. *For any initial data $(S_0, E_0, I_0) \in \mathbb{R}^3_+$, there is a unique positive solution $(S(t), E(t), I(t))$ of model (3) on $t \geq 0$, and the solution will remain in $\mathbb{R}^3_+$ with a probability of one (a.s.).*

The proof of Lemma 1 is given in the following Appendix A.

Consider $Y(t)$ to be a regular time-homogeneous Markov process in $\mathbb{R}^d$ satisfying the following SDE:

$$dY(t) = f(Y(t))dt + \sum_{l=1}^{k} g_l(Y(t))dB_l(t). \tag{4}$$

Its diffusion matrix is:

$$A(y) = (\hat{a}_{ij}(y)), \;\; \hat{a}_{ij}(y) = \sum_{l=1}^{k} g_l^i(y) g_l^j(y).$$

Lemma 2. *([17,21,22]). Assume that there exists a bounded domain $\mathbb{D} \subset \mathbb{R}^d$ with the regular boundary Γ and*
(A_1) $\exists\, L > 0$, such that $\Sigma^d_{i,j} = \hat{a}_{ij}(y)\hat{\xi}_i\hat{\xi}_j, y \in \mathbb{D}, \hat{\xi} \in \mathbb{R}^d$;
(A_2) $\exists\, C^2$-function $V(y) \geq 0$ and $\exists\, \varpi > 0$ such that $\mathcal{L}V \leq -\varpi$, where $\mathcal{L}$ denotes the Itô's differential operator defined in (45) of Ref. [13].
Then, system (4) exists a unique ergodic stationary distribution $\pi(\cdot)$ for any $\mathbb{R}^d_+$, such that

$$\mathbb{P}_y\Big\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(Y(t))dt = \int_{\mathbb{R}^d} f(y)\pi(dy) \Big\} = 1,$$

and let $f(\cdot)$ be an integrable function with respect to the measure $\pi(\cdot)$.

Lemma 3. *If $x \geq 0$, then*

$$x^n \geq \Big(\frac{n-1}{2(n-2)} x^{n-2} - \frac{1}{2(n-2)} \Big)(x^2 + 1),$$

where $n \geq 3$ and the sign of inequality holds if and only if $x = 1$.

The proof of Lemma 3 is given in Appendix A.

Theorem 1. *Assume that $\mathfrak{R}_0^S > 1$, where*

$$\mathfrak{R}_0^S = \frac{\beta(1-\theta_1)\Lambda\delta}{\left(\mu + \frac{\sigma_{11}^2}{2} + 2\sqrt{\Lambda\sigma_{11}\sigma_{12}} + 2\sqrt[3]{\Lambda^2\sigma_{12}^2}\right)\left(\gamma + \sigma + \mu + \frac{\sigma_{31}^2}{2}\right)\left(\delta + \mu + \frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\Lambda^2\sigma_{22}^2}\right)} + \frac{\beta(1-\theta_1)\theta_2\Lambda}{\left(\mu + \frac{\sigma_{11}^2}{2} + 2\sqrt{\Lambda\sigma_{11}\sigma_{12}} + 2\sqrt[3]{\Lambda^2\sigma_{12}^2}\right)\left(\delta + \mu + \frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\Lambda^2\sigma_{22}^2}\right)};$$

then, for any initial data $(S(0), E(0), I(0)) \in \mathbb{R}_+^3$, model (3) possesses a unique ergodic stationary distribution.

The proof of Theorem is given in Appendix A.

Remark 1. *By combining setting up constant $C_1 + C_2$ in V_1 with "stochastic p-threshold method", we successfully proved Theorem 1. The new method can make up for the deficiency of the common "stochastic p-threshold method" to a certain extent.*

Remark 2. *We can see that $\mathfrak{R}_0^S = \mathfrak{R}_0$ when $\sigma_{ij}^2 \equiv 0 (i = 1,2,3, j = 1,2)$ in model (3).*

Remark 3. *From $\mathfrak{R}_0^S$ in Theorem 1, we can come to the conclusion that θ_1 plays an active role in the prevention and control of an infectious disease.*

3. Extinction

Note that

$$d\widetilde{A} = \left(\Lambda - \mu\widetilde{A}\right)dt + \left(\sigma_{11}\widetilde{A} + \sigma_{12}\widetilde{A}^2\right)dB_1(t), \tag{5}$$

with the same initial data $\widetilde{A}(0) = S(0)$. According to Ref. [23], model (5) exists as a stationary solution with the density

$$\pi(x) = Qx^{-2-\frac{2(2\Lambda\sigma_{12}+\mu\sigma_{11})}{\sigma_{11}^3}}(\sigma_{11}+\sigma_{12}x)^{-2+\frac{2(2\Lambda\sigma_{12}+\mu\sigma_{11})}{\sigma_{11}^3}} \exp\left(-\frac{2\Lambda}{\sigma_{11}(\sigma_{11}+\sigma_{12}x)}\right)\left(\frac{\Lambda}{x} + \frac{2\Lambda\sigma_{12}+\mu\sigma_{11}}{\sigma_{11}}\right), x \in (0, +\infty),$$

where

$$Q = \frac{1}{\int_0^\infty x^{-2-\frac{2(2\Lambda\sigma_{12}+\mu\sigma_{11})}{\sigma_{11}^3}}(\sigma_{11}+\sigma_{12}x)^{-2+\frac{2(2\Lambda\sigma_{12}+\mu\sigma_{11})}{\sigma_{11}^3}} \exp\left(-\frac{2\Lambda}{\sigma_{11}(\sigma_{11}+\sigma_{12}x)}\right)\left(\frac{\Lambda}{x} + \frac{2\Lambda\sigma_{12}+\mu\sigma_{11}}{\sigma_{11}}\right)dx}$$

is a constant such that $\int_0^\infty \pi(x)dx = 1$. Define

$$\mathfrak{R}_0^E = \mathfrak{R}_0\rho_0 \int_0^\infty \left|x - \frac{\Lambda}{\mu}\right|\pi(x)dx - \frac{\sigma_{22}^2 \wedge \sigma_{32}^2}{4},$$

where $\rho_0 = \dfrac{\beta(1-\theta_1)}{\rho_1 \wedge \rho_2}$.

Theorem 2. *Let $(S(t), E(t), I(t))$ be the solution to model (3) with any initial data $(S_0, E_0, I_0) \in \mathbb{R}_+^3$. If $\mathfrak{R}_0 < 1, \mathfrak{R}_0^E < 0$, then*

$$\limsup_{t\to\infty} \frac{1}{t}\ln(\rho_1 E(t) + \rho_2 I(t)) \le \mathfrak{R}_0^E < 0, \quad a.s.,$$

and

$$\lim_{t\to\infty} E(t) = 0, \lim_{t\to\infty} I(t) = 0, \text{ a.s.},$$

where the positive constants ρ_1, ρ_2 *are defined as follows:*

$$\rho_1 = \mathfrak{R}_0,\ \rho_2 = \frac{\Lambda\beta(1-\theta_1)}{\mu(\gamma+\sigma+\mu)}.$$

The proof of Theorem 2 is given in Appendix A.

Remark 4. *In view of* $\mathfrak{R}_0^E$ *in Theorem 2, a larger* θ_1 *is favorable to restraining the number of infected individuals and eradicating the disease at the beginning of its outbreak.*

4. Density Function Analysis of Model (3)

Here, we discuss the representation of the local probability density function of model (3) under only linear stochastic perturbations.

Let $x_1 = \ln S, x_2 = \ln E, x_3 = \ln I$. For Equation (3), we have:

$$\begin{cases} dx_1 = \Big(\Lambda e^{-x_1} - \beta(1-\theta_1)(e^{x_3} + \theta_2 e^{x_2}) - \mu_1\Big)dt + \sigma_{11} dB_1(t), \\ dx_2 = \Big(\beta(1-\theta_1)e^{x_1}(e^{x_3} + \theta_2 e^{x_2})e^{-x_2} - (\delta+\mu_2)\Big)dt + \sigma_{21} dB_2(t), \\ dx_3 = (\delta e^{x_2 - x_3} - (\gamma+\sigma+\mu_3))dt + \sigma_{31} dB_3(t), \end{cases} \tag{6}$$

where $\mu_i = \mu + \dfrac{\sigma_{i1}^2}{2}, i = 1,2,3$. Assume that $\widehat{\mathfrak{R}_0^S} > 1$; then, the unique quasi-endemic equilibrium $E^* = (S^*, E^*, I^*) = (e^{x_1^*}, e^{x_2^*}, e^{x_3^*})$ satisfies:

$$\begin{cases} \Lambda e^{-x_1^*} - \beta(1-\theta_1)(e^{x_3^*} + \theta_2 e^{x_2^*}) - \mu_1 = 0, \\ \beta(1-\theta_1)e^{x_1}(e^{x_3^*} + \theta_2 e^{x_2^*})e^{-x_2^*} - (\delta+\mu_2) = 0, \\ \delta e^{x_2^*} e^{-x_3^*} - (\gamma+\sigma+\mu_3) = 0, \end{cases}$$

where

$$S^* = \frac{(\gamma+\sigma+\mu_3)(\delta+\mu_2)}{\beta(1-\theta_1)[\delta+\theta_2(\gamma+\sigma+\mu_3)]} > 0, E^* = \frac{\mu_1(\delta+\mu_2)(\gamma+\sigma+\mu_3)(\widehat{\mathfrak{R}_0^S}-1)}{\beta(1-\theta_1)[\delta+\theta_2(\gamma+\sigma+\mu_3)](\delta+\mu_2)} > 0,$$
$$I^* = \frac{\delta E^*}{\gamma+\sigma+\mu_3} > 0, \widehat{\mathfrak{R}_0^S} = \frac{\Lambda\beta(1-\theta_1)[\delta+\theta_2(\gamma+\sigma+\mu_3)]}{\mu_1(\delta+\mu_2)(\gamma+\sigma+\mu_3)}.$$

Evidently, (S^*, E^*, I^*) coincides with the stable endemic equilibrium point (S^+, E^+, I^+) of the deterministic model (1) when $\sigma_{i1} = 0, i = 1,2,3, \widehat{\mathfrak{R}_0^S} = \mathfrak{R}_0^S$ if $\sigma_{11} = \sigma_{22} = 0$.

Let $(y_1, y_2, y_3) = (x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*)$, where $x_1^* = \ln S^*, x_2^* = \ln E^*, x_3^* = \ln I^*$. The linearized model of Equation (6) is as follows:

$$\begin{cases} dy_1 = (-a_{11}y_1 - a_{12}y_2 - a_{13}y_3)dt + \sigma_1 dB_1(t), \\ dy_2 = (a_{21}y_1 - a_{22}y_2 + a_{23}y_3)dt + \sigma_2 dB_2(t), \\ dy_3 = (a_{32}y_2 - a_{33}y_3)dt + \sigma_3 dB_3(t), \end{cases} \tag{7}$$

where $a_{11} = \Lambda e^{-x_1^*} > 0, a_{12} = \beta(1-\theta_1)\theta_2 e^{x_2^*} > 0, a_{13} = \beta(1-\theta_1)e^{x_3^*} > 0$, $a_{21} = \beta(1-\theta_1)e^{x_1^*}e^{x_3^*-x_2^*} + \beta(1-\theta_1)\theta_2 e^{x_1^*} > 0, a_{22} = \beta(1-\theta_1)e^{-x_2^*}e^{x_1^*+x_3^*} > 0$, $a_{23} = \beta(1-\theta_1)e^{x_1^*-x_2^*}e^{x_3^*} > 0, a_{32} = \delta e^{x_2^*-x_3^*} > 0, a_{33} = \delta e^{x_2^*-x_3^*} > 0$. Furthermore, letting $Y = (y_1, y_2, y_3)^T$, $\Lambda = diag(\sigma_1, \sigma_2, \sigma_3)$, and $B(t) = (B_1(t), B_2(t), B_3(t))^T$, then model (7) can be rewritten into $dY = AYdt + \Lambda dB(t)$, where

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ a_{21} & -a_{22} & a_{23} \\ 0 & a_{32} & -a_{33} \end{pmatrix}.$$

Lemma 4. *([15]) For $G_0^2 + A_0\Sigma_0 + \Sigma_0 A_0^T = 0$, where $G_0 = \mathrm{diag}(1,0,0)$,*

$$A_0 = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{8}$$

$$\Sigma_0 = \begin{pmatrix} \frac{a_2}{2(a_1a_2-a_3)} & 0 & -\frac{1}{2(a_1a_2-a_3)} \\ 0 & \frac{1}{2(a_1a_2-a_3)} & 0 \\ -\frac{1}{2(a_1a_2-a_3)} & 0 & \frac{a_1}{2(a_1a_2-a_3)} \end{pmatrix}.$$

If $a_1 > 0, a_3 > 0$ and $a_1a_2 - a_3 > 0$, then matrix Σ_0 is positive definite.

Lemma 5. *([15]) For $G_0^2 + B_0\theta_0 + \theta_0 B_0^T = 0$, where $G_0 = \mathrm{diag}(1,0,0)$,*

$$B_0 = \begin{pmatrix} -b_1 & -b_2 & -b_3 \\ 1 & 0 & 0 \\ 0 & 0 & b_{33} \end{pmatrix}. \tag{9}$$

If $b_1 > 0$ and $b_2 > 0$, then θ_0 is semi-positive definite, which takes the form

$$\theta_0 = \begin{pmatrix} \frac{1}{2b_1} & 0 & 0 \\ 0 & \frac{1}{2b_1b_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

Theorem 3. *Let $Y = (y_1, y_2, y_3)$ be a solution to model (3) with any initial data $(y_1(0), y_2(0), y_3(0)) \in \mathbb{R}_3$. If $\widehat{\mathfrak{R}_0^S} > 1, r_1r_2 - r_3 > 0$, where $r_1 = a_{11} + a_{22} + a_{32}, r_2 = a_{11}a_{32} + a_{11}a_{22} + a_{12}a_{21}, r_3 = a_{13}a_{21}a_{32} + a_{32}a_{12}a_{21}$, then there exists a unique density function $\varphi(S, E, I)$ around the quasi-stable equilibrium point E^*, where*

$$\varphi(S,E,I) = (2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}}e^{-\frac{1}{2}(\ln\frac{S}{S^*},\ln\frac{E}{E^*},\ln\frac{I}{I^*})\Sigma^{-1}(\ln\frac{S}{S^*},\ln\frac{E}{E^*},\ln\frac{I}{I^*})^T},$$

in which $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$ is positive definite. Σ_1 is described in (A28), Σ_2 is depicted in (A30) and, for different cases, Σ_3 is described in (A32) and (A33).

The proof of Theorem 3 is given in Appendix A.

Remark 5. *If $\widehat{\mathfrak{R}_0^S} > 1$, Theorem 3 shows that all of the distributions of the subpopulations $S(t)$, $E(t)$ and $I(t)$ will separately converge to the corresponding stationary marginal distributions $\mu_1(S), \mu_2(E), \mu_3(I)$ of $\varpi(\cdot)$ as $t \to \infty$. By letting $\Sigma = (\rho_{ij})_{3\times 3}$, we then combine Theorem 3 to obtain that the distribution $\mu_1(S)$ around S^* has an approximately log-normal density function $I_1(S)$. Moreover, the distribution $\mu_2(E)$ around E^* has an approximately log-normal density function $I_2(E)$, and the distribution $\mu_3(I)$ around I^* has an approximately log-normal density function $I_3(I)$, where*

$$I_1(S) = \frac{1}{S\sqrt{2\pi\rho_{11}}}e^{-\frac{(\ln S - \ln S^*)^2}{2\rho_{11}}}, \quad I_2(E) = \frac{1}{E\sqrt{2\pi\rho_{22}}}e^{-\frac{(\ln E - \ln E^*)^2}{2\rho_{22}}},$$

$$I_3(I) = \frac{1}{I\sqrt{2\pi\rho_{33}}}e^{-\frac{(\ln I - \ln I^*)^2}{2\rho_{33}}}.$$

5. Numerical Tests

Now we present two numerical tests and several figures to support our findings.

Numerical Test 1. To make use of model (3) to investigate the spread of COVID-19, in system (1), we use the same data as in Ref. [4]: $\Lambda = 10, \mu = 0.3, \sigma = 0.2, \beta = 0.2, \gamma = 0.2, \delta = 0.3, \theta_1 = 0.7, \theta_2 = 0.1$. This implies that the P^+ of model (1) is globally asymptotically stable. Now, let $\sigma_{11} = 0.01, \sigma_{12} = 0.0008, \sigma_{21} = 0.01, \sigma_{22} = 0.002, \sigma_{31} = 0.019$ and $\sigma_{32} = 0.02$. According to our calculation, $\mathfrak{R}_0^S = 1.0661 > 1$. From Theorem 1, system (3) has a unique ergodic stationary distribution (see Figure 1).

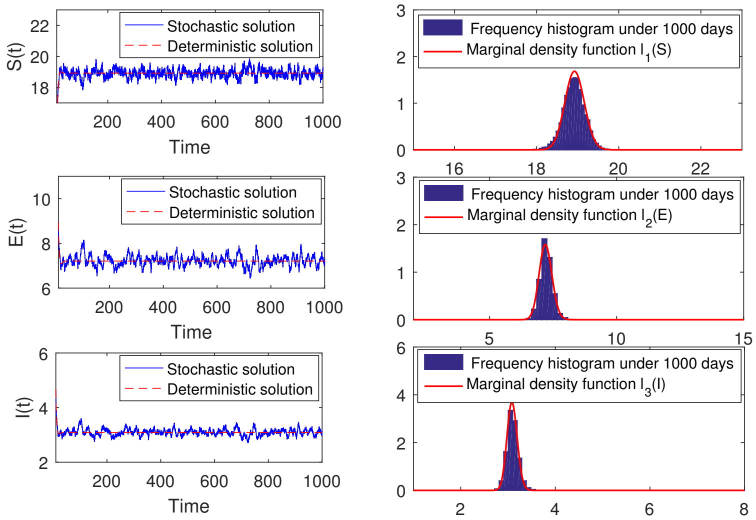

Figure 1. The left-hand column presents the solutions of the deterministic model (2) and the stochastic model (3), where $\Lambda = 10, \mu = 0.3, \sigma = 0.2, \beta = 0.2, \gamma = 0.2, \delta = 0.3, \theta_1 = 0.7, \theta_2 = 0.1, \sigma_{11} = 0.01, \sigma_{12} = 0.0008, \sigma_{21} = 0.01, \sigma_{22} = 0.002, \sigma_{31} = 0.019, \sigma_{32} = 0.02$. Initial data: $S(0) = 100$, $E(0) = 15$, $I(0) = 20$. The right-hand column presents the frequency histograms and the marginal densities $I_1(S), I_2(E)$ and $I_3(I)$ of the approximate density function $\varphi(S, E, I)$.

Additionally, let $\sigma_{11} = 0.01, \sigma_{21} = 0.01, \sigma_{31} = 0.019, \sigma_{12} = \sigma_{22} = \sigma_{32} = 0$; we derive the equilibrium point $P^* = (18.925, 7.201, 3.086), \widehat{\mathfrak{R}_0^S} = 1.761, r_1 r_2 - r_3 = 1.024 > 0$. Then, the criteria of Theorem 3 hold. According to Theorem 3, one can see that

$$\Sigma = \begin{pmatrix} 0.0001567 & -0.0000231 & 0.0000905 \\ -0.0000231 & 0.0005228 & -0.0005173 \\ 0.0000905 & -0.0005173 & 0.0007306 \end{pmatrix}.$$

$\varphi(S, E, I)$ of model (3) is deduced as

$$\varphi(S, E, I) = 16174 e^{-\frac{1}{2}(\ln\frac{S}{18.925}, \ln\frac{E}{7.201}, \ln\frac{I}{3.086})\Sigma^{-1}(\ln\frac{S}{18.925}, \ln\frac{E}{7.201}, \ln\frac{I}{3.086})^T},$$

and

$$I_1(S) = \frac{31.8776}{S}e^{-\frac{(\ln S - 2.9405)^2}{0.0003134}}, \quad I_2(E) = \frac{17.4523}{E}e^{-\frac{(\ln E - 1.9742)^2}{0.0010456}},$$

$$I_3(I) = \frac{14.7632}{I}e^{-\frac{(\ln I - 1.1269)^2}{0.0014612}}.$$

The right-hand column in Figure 1 presents the frequency histograms and the marginal densities.

Numerical Test 2. The same data are presented in Ref. [4]: $\Lambda = 10, \mu = 0.3, \sigma = 0.2, \beta = 0.2, \gamma = 0.2, \delta = 0.3, \theta_1 = 0.9, \theta_2 = 0.1$. This shows that the disease-free equilibrium P^0 of model (2) is globally asymptotically stable. Now, set the stronger white noise intensities $\sigma_{11} = 0.5, \sigma_{12} = 0.0001, \sigma_{21} = 2.1, \sigma_{22} = 0.9487, \sigma_{31} = 0.01, \sigma_{32} = 0.99$. Then, $\mathfrak{R}_0 = 0.5873$, $Q = 1/16340590$, $\int_0^\infty \left| x - \frac{\Lambda}{\mu} \right| \pi(x) dx = 16.696$, $\mathfrak{R}_0^E = 0.908 < 1$. According to Theorem 2, the disease will die out (see Figure 2).

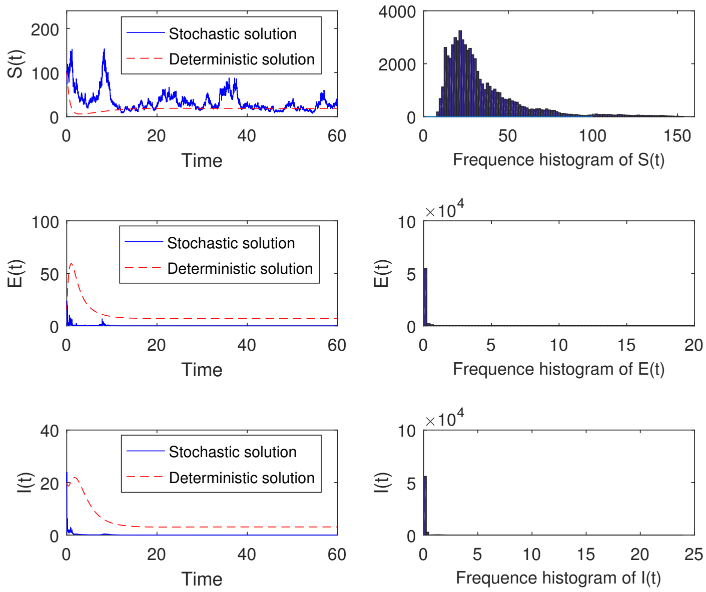

Figure 2. The solutions of the deterministic model (2) and the stochastic model (3), where $\Lambda = 10, \mu = 0.3, \sigma = 0.2, \beta = 0.2, \gamma = 0.2, \delta = 0.3, \theta_1 = 0.9, \theta_2 = 0.1, \sigma_{11} = 0.5, \sigma_{12} = 0.0001, \sigma_{21} = 2.1, \sigma_{22} = 0.9487, \sigma_{31} = 0.01, \sigma_{32} = 0.99$. Initial data $S(0) = 100, E(0) = 15, I(0) = 20$. The right column displays the histogram of the probability density functions of S, E, I populations.

6. Conclusions and Discussion

This paper investigates a second-order stochastic SEIR model with infectivity in the incubation period and homestead-isolation of the susceptible population. Firstly, we study the existence of a unique ergodic stationary distribution of model (3), which shows that the disease will last for a long time. Secondly, we obtain the corresponding probability density function with respect to the stationary solution to model (3). Finally, the sufficient conditions for disease's extinction in model (3) are obtained.

However, because of the limitation of our mathematical approaches and the complexity of model (3), we have no ability to obtain a reasonable result concerning the threshold of disease eradication. In addition, we shall investigate the corresponding dynamics of the stochastic model with regime switching. We therefore leave these problems for our future work.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Proof of Lemma 1. Let $V : \mathbb{R}^3_+ \longrightarrow \mathbb{R}_+$:

$$V(S,E,I) = \left(\frac{S^p}{p} - 1 - \log S\right) + \left(\frac{E^p}{p} - 1 - \log E\right) + \left(\frac{I^p}{p} - 1 - \log I\right),$$

where $0 < p < 1$. By means of Itô's formula [24–28], we derive:

$$\begin{aligned}\mathcal{L}\left(\frac{S^p}{p} - 1 - \log S\right) =& S^{p-1}(\Lambda - \beta(1-\theta_1)S(I+\theta_2 E) - \mu S) - \frac{1}{2}(1-p)S^p(\sigma_{11}+\sigma_{12}S)^2 \\ & - \frac{1}{S}(\Lambda - \beta(1-\theta_1)S(I+\theta_2 E) - \mu S) + \frac{1}{2}(\sigma_{11}+\sigma_{12}S)^2.\end{aligned} \tag{A1}$$

Analogously, we compute:

$$\begin{aligned}\mathcal{L}\left(\frac{E^p}{p} - 1 - \log E\right) =& E^{p-1}(\beta(1-\theta_1)S(I+\theta_2 E) - (\delta+\mu)E) - \frac{1}{2}(1-p)E^p(\sigma_{21}+\sigma_{22}E)^2 \\ & - \frac{1}{E}(\beta(1-\theta_1)S(I+\theta_2 E) - (\delta+\mu)E) + \frac{1}{2}(\sigma_{21}+\sigma_{22}E)^2,\end{aligned} \tag{A2}$$

$$\begin{aligned}\mathcal{L}\left(\frac{I^p}{p} - 1 - \log I\right) =& I^{p-1}(\delta E - (\gamma+\sigma+\mu)I) - \frac{1}{2}(1-p)I^p(\sigma_{31}+\sigma_{32}I)^2 \\ & - \frac{1}{I}(\delta E - (\gamma+\sigma+\mu)I) + \frac{1}{2}(\sigma_{31}+\sigma_{32}I)^2.\end{aligned} \tag{A3}$$

Adding (A1)–(A3) together, we obtain:

$$\begin{aligned}\mathcal{L}V =& 3\mu + \delta + \sigma + \gamma + \frac{\sigma_{11}^2}{2} + \frac{\sigma_{21}^2}{2} + \frac{\sigma_{31}^2}{2} + \sigma_{11}\sigma_{12}S + \frac{\sigma_{12}^2 S^2}{2} + \sigma_{21}\sigma_{22}E + \frac{\sigma_{22}^2 E^2}{2} \\ & + \sigma_{31}\sigma_{32}I + \frac{\sigma_{32}^2 I^2}{2} + \frac{\Lambda}{S^{1-p}} - \frac{\Lambda}{S} - \left(\beta(1-\theta_1)(I+\theta_2 E) + \mu + \frac{1}{2}(1-p)\sigma_{11}^2\right)S^p \\ & - (1-p)\sigma_{11}\sigma_{12}S^{p+1} + \frac{\beta(1-\theta_1)SI}{E^{1-p}} - \frac{\beta(1-\theta_1)SI}{E} - \beta(1-\theta_1)\theta_2 S \\ & - \left(-\beta(1-\theta_1)\theta_2 S + (\delta+\mu) + \frac{1}{2}(1-p)\sigma_{21}^2\right)E^p - (1-p)\sigma_{21}\sigma_{22}E^{p+1} \\ & + \frac{\delta E}{I^{1-p}} - \frac{\delta E}{I} - \left((\gamma+\sigma+\mu) + \frac{1}{2}(1-p)\sigma_{31}^2\right)I^p - (1-p)\sigma_{31}\sigma_{32}I^{p+1} \\ & - \frac{1}{2}(1-p)\sigma_{12}^2 S^{p+2} - \frac{1}{2}(1-p)\sigma_{22}^2 E^{p+2} - \frac{1}{2}(1-p)\sigma_{32}^2 I^{p+2} \\ \le& 3\mu + \delta + \sigma + \gamma + \frac{\sigma_{11}^2}{2} + \frac{\sigma_{21}^2}{2} + \frac{\sigma_{31}^2}{2} + \sigma_{11}\sigma_{12}S + \frac{\sigma_{12}^2 S^2}{2} + \sigma_{21}\sigma_{22}E + \frac{\sigma_{22}^2 E^2}{2} \\ & + \sigma_{31}\sigma_{32}I + \frac{\sigma_{32}^2 I^2}{2} + \frac{\Lambda}{S^{1-p}} - \frac{\Lambda}{S} + \frac{\beta(1-\theta_1)SI}{E^{1-p}} - \frac{\beta(1-\theta_1)SI}{E} + \frac{\delta E}{I^{1-p}} - \frac{\delta E}{I} \\ & + \beta(1-\theta_1)\theta_2 S E^p - (1-p)\sigma_{11}\sigma_{12}S^{p+1} - (1-p)\sigma_{21}\sigma_{22}E^{p+1} - (1-p)\sigma_{31}\sigma_{32}I^{p+1} \\ & - \frac{1}{2}(1-p)\sigma_{12}^2 S^{p+2} - \frac{1}{2}(1-p)\sigma_{22}^2 E^{p+2} - \frac{1}{2}(1-p)\sigma_{32}^2 I^{p+2} \\ \le& 3\mu + \delta + \sigma + \gamma + \frac{\sigma_{11}^2}{2} + \frac{\sigma_{21}^2}{2} + \frac{\sigma_{31}^2}{2} + \sigma_{11}\sigma_{12}S + \frac{\sigma_{12}^2 S^2}{2} + \sigma_{21}\sigma_{22}E + \frac{\sigma_{22}^2 E^2}{2} \\ & + \sigma_{31}\sigma_{32}I + \frac{\sigma_{32}^2 I^2}{2} + K_1\Lambda + K_2\delta E + K_3\delta E - \frac{1}{2}(1-p)\sigma_{12}^2 S^{p+2} - \frac{1}{2}(1-p)\sigma_{22}^2 E^{p+2} \\ & - \frac{1}{2}(1-p)\sigma_{32}^2 I^{p+2} \\ \le& M,\end{aligned}$$

where M is a positive constant, $K_1 = \max_{S\in\mathbb{R}_+}\left\{\frac{1}{S^{1-p}} - \frac{1}{S}\right\}, K_2 = \max_{E\in\mathbb{R}_+}\left\{\frac{1}{E^{1-p}} - \frac{1}{E}\right\}$, $K_3 = \max_{I\in\mathbb{R}_+}\left\{\frac{1}{I^{1-p}} - \frac{1}{I}\right\}$. The rest of the proof is similar to Theorem 2.1 in [19], Theorem 2.1 in [5] and Lemma 1 in Lu et al. [29], and is hence omitted here. This proof is over. □

Proof of Lemma 3. We use the mathematical introduction to prove the inequality. When $n = 3$, $x^3 \geq \left(x - \frac{1}{2}\right)(x^2+1)$ holds (see Lemma 2 in Ref. [13]). Assume $x^{k-1} \geq \left(\frac{k-2}{2(k-3)}x^{k-3} - \frac{1}{2(k-3)}\right)(x^2+1), k \geq 4$. Then, $x^k \geq \left(\frac{k-2}{2(k-3)}x^{k-2} - \frac{x}{2(k-3)}\right)(x^2+1)$. We only need prove that

$$\frac{k-2}{2(k-3)}x^{k-2} - \frac{x}{2(k-3)} \geq \frac{k-1}{2(k-2)}x^{k-2} - \frac{1}{2(k-2)}.$$

That is to say, we need to verify $\frac{x^{k-2}}{k-2} - x + \frac{k-3}{k-2} \geq 0$. When $x > 1$, it obviously holds. If $0 < x < 1$, let $f(x) = \frac{x^{k-2}}{k-2} - x + \frac{k-3}{k-2}$, $f'(x) = x^{k-3} - 1 \leq 0$. Consequently, $f(x)$ is monotonically decreasing on $[0,1]$. Note that $x = 1$ is the minimum and

$$f(1) = \frac{1}{k-2} - 1 + \frac{k-3}{k-2} = \frac{1-k+2}{k-2} + \frac{k-3}{k-2} = 0;$$

as a result, $f(x) \geq 0, 0 < x < 1$. Therefore, $x^k \geq \left(\frac{k-1}{2(k-2)}x^{k-2} - \frac{1}{2(k-2)}\right)(x^2+1)$. □

Proof of Theorem 1. The criteria (A_1) and (A_2) of Lemma 2 need to be validated. The criterion (A_1) apparently holds [23,30–32]. Moreover, we verify the criterion (A_2):

$$\begin{aligned}\mathfrak{R}_0^S(p) =& \frac{\beta(1-\theta_1)\Lambda\delta}{\left(\mu + \frac{\sigma_{11}^2}{2} + 2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right)\left(\gamma+\sigma+\mu+\frac{\sigma_{31}^2}{2}\right)\left(\delta+\mu+\frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right)} \\ &+ \frac{\beta(1-\theta_1)\theta_2\Lambda}{\left(\mu + \frac{\sigma_{11}^2}{2} + 2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right)\left(\delta+\mu+\frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right)},\end{aligned}$$

where $\mathfrak{R}_0^S(p)$ is a decreasing function of $p \in (0,1)$. Evidently, $\lim_{p\to 0^+}\mathfrak{R}_0^S(p) = \mathfrak{R}_0^S$. In view of model (3), according to Itô's formula, we obtain:

$$\begin{aligned}\mathcal{L}(-(C_1+C_2)\ln S) =& -\frac{(C_1+C_2)\Lambda}{S} + (C_1+C_2)\beta\left(1-\theta_1\right)\left[I+\theta_2 E\right] + (C_1+C_2)\mu \\ &+ (C_1+C_2)\left(\frac{\sigma_{11}^2}{2} + \sigma_{11}\sigma_{12}S + \frac{\sigma_{12}^2}{2}S^2\right), \\ \mathcal{L}(-\ln E) =& -\frac{\beta\left(1-\theta_1\right)SI}{E} - \beta\left(1-\theta_1\right)S\theta_2 + \left(\mu+\delta\right) + \left(\frac{\sigma_{11}^2}{2} + \sigma_{21}\sigma_{22}E\right. \\ &\left.+ \frac{\sigma_{22}^2}{2}E^2\right), \\ \mathcal{L}(-C_3\ln I) =& -\frac{C_3\delta E}{I} + C_3\left(r+\sigma+\mu\right) + C_3\left(\frac{\sigma_{31}^2}{2} + \sigma_{31}\sigma_{32}I + \frac{\sigma_{32}^2}{2}I^2\right),\end{aligned}$$

where C_1, C_2 and C_3 are positive constants which will be determined later. Define

$$V_1 = (C_1 + C_2) \sum_{i=1}^{2} \frac{a_i(S + b_i)^p}{p}, \quad V_2 = \left(d_1 S + \frac{d_2(E + d_3)^p}{p} \right),$$
$$\mathcal{V}_1 = -(C_1 + C_2) \ln S + V_1(S), \quad \mathcal{V}_2 = -\ln E + V_2,$$
$$\mathcal{V}_3 = -C_3 \ln I + \frac{C_3\theta_1(\sigma_{31} + \sigma_{32} I)^p}{p} + \frac{C_3\sigma_{31}\sigma_{32}}{(\gamma + \sigma + \mu)} I,$$

where $a_1, a_2, b_1, b_2, C_1, C_2, C_3, d_1, d_2$ and d_3 are positive constants to be found later. Applying Itô's formula to V_1 and combining the inequality of Lemma 3, we derive:

$$\begin{aligned}
LV_1 =& (C_1 + C_2) \sum_{i=1}^{2} \Big[a_i(S + b_i)^{p-1}(\Lambda - \beta(1 - \theta_1)S(I + \theta_2 E) - \mu S) \\
& - \frac{a_i(1 - p)}{2(S + b_i)^{2-p}} \Big(\sigma_{11} S + \sigma_{12}\Big)^2 S^2 \Big] \\
\leq & (C_1 + C_2) \Big[\sum_{i=1}^{2} \frac{a_i\Lambda}{b_i^{1-p}} - \frac{a_1(1 - p) b_1^{p-2} \sigma_{12}^2 S^4}{2(\frac{S}{b_1} + 1)^{2-p}} - \frac{a_2(1 - p) b_2^{p-2} \sigma_{11}\sigma_{12} S^3}{(\frac{S}{b_2} + 1)^{2-p}} \Big] \\
\leq & (C_1 + C_2) \Big[\sum_{i=1}^{2} \frac{a_i\Lambda}{b_i^{1-p}} - \frac{a_1(1 - p) b_1^{p+2} \sigma_{12}^2 (\frac{S}{b_1})^4}{2(\frac{S}{b_1} + 1)^2} - \frac{a_2(1 - p) b_2^{p+1} \sigma_{11}\sigma_{12} (\frac{S}{b_2})^3}{(\frac{S}{b_2} + 1)^2} \Big] \\
\leq & (C_1 + C_2) \Big[\sum_{i=1}^{2} \frac{a_i\Lambda}{b_i^{1-p}} - \frac{a_1(1 - p) b_1^{p+2} \sigma_{12}^2 (\frac{S}{b_1})^4}{4((\frac{S}{b_1})^2 + 1)} - \frac{a_2(1 - p) b_2^{p+1} \sigma_{11}\sigma_{12} (\frac{S}{b_2})^3}{2((\frac{S}{b_2})^2 + 1)} \Big] \\
\leq & (C_1 + C_2) \Big[\sum_{i=1}^{2} \frac{a_i\Lambda}{b_i^{1-p}} - \frac{a_1(1 - p) b_1^{p+2} \sigma_{12}^2}{4} \Big(\frac{3}{4} \Big(\frac{S}{b_1} \Big)^2 - \frac{1}{4} \Big) - \frac{a_2(1 - p) b_2^{p+1} \sigma_{11}\sigma_{12}}{2} \Big(\frac{S}{b_2} \\
& - \frac{1}{2} \Big) \Big] \\
=& (C_1 + C_2) \Big(\frac{a_1\Lambda}{b_1^{1-p}} + \frac{a_1(1 - p) b_1^{p+2} \sigma_{12}^2}{16} \Big) + (C_1 + C_2) \Big(\frac{a_2 b}{b_2^{1-p}} + \frac{a_2(1 - p) b_2^{p+1} \sigma_{11}\sigma_{12}}{4} \Big) \\
& - \frac{3(C_1 + C_2) a_1 (1 - p) b_1^p \sigma_{12}^2 S^2}{16} - \frac{(C_1 + C_2) a_2 (1 - p) b_2^p \sigma_{11}\sigma_{12}}{2} S.
\end{aligned}$$

Set

$$a_1 = \frac{8}{3(1 - p) b_1^p}, a_2 = \frac{2}{(1 - p) b_2^p}, b_1 = 2\sqrt[3]{\frac{\Lambda}{(1 - p)\sigma_{11}^2}}, b_2 = 2\sqrt{\frac{\Lambda}{(1 - p)\sigma_{11}\sigma_{12}}},$$

then, we have:

$$\mathcal{L}V_1 \leq 2(C_1 + C_2)\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1 - p}} + 2(C_1 + C_2)\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1 - p)^2}} - (C_1 + C_2)\sigma_{11}\sigma_{12} S - \frac{(C_1 + C_2)\sigma_{12}^2}{2} S^2.$$

Thus, it follows that

$$\begin{aligned}\mathcal{L}\mathcal{V}_1 \leq & -\frac{(C_1+C_2)\Lambda}{S} + (C_1+C_2)\beta(1-\theta_1)(I+\theta_2 E) + (C_1+C_2)\mu + C_1\Big(\frac{\sigma_{11}^2}{2} + \sigma_{11}\sigma_{12}S \\ & + \frac{\sigma_{12}^2}{2}S^2\Big) + 2(C_1+C_2)\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2(C_1+C_2)\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}} - (C_1+C_2)\sigma_{11}\sigma_{12}S \\ & - (C_1+C_2)\frac{\sigma_{12}^2}{2}S^2 \\ \leq & -\frac{(C_1+C_2)\Lambda}{S} + (C_1+C_2)\mu + \frac{(C_1+C_2)\sigma_{11}^2}{2} + 2(C_1+C_2)\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} \\ & + 2(C_1+C_2)\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}} + (C_1+C_2)\beta\Big(1-\theta_1\Big)\Big[I+\theta_2 E\Big]. \end{aligned} \tag{A4}$$

Subsequently, in the light of applying Itô's formula to $V_2(S,E)$, we obtain:

$$\begin{aligned}\mathcal{L}\mathcal{V}_2 =& d_1(\Lambda - \beta(1-\theta_1)S(I+\theta_2E) - \mu S) + d_2(E+d_3)^{p-1}[\beta(1-\theta_1)S(I+\theta_2E) \\ & -(\delta+\mu)E] - \frac{d_2(1-p)}{2(E+d_3)^{2-p}}(\sigma_{21}+\sigma_{22}E)^2E^2 \\ \leq & d_1\Lambda + \Big(d_2d_3^{p-1} - d_1\Big)\Big[\beta(1-\theta_1)S(I+\theta_2E)\Big] - \frac{d_2(1-p)d_3^{p-2}}{2\Big(\frac{E}{d_3}+1\Big)^{2-p}}\sigma_{22}^2E^4 \\ \leq & d_1\Lambda + \Big(d_2d_3^{p-1} - d_1\Big)\Big[\beta(1-\theta_1)S(I+\theta_2E)\Big] - \frac{d_2(1-p)d_3^{p-2}}{2\Big(\frac{E}{d_3}+1\Big)^{2}}\sigma_{22}^2E^4 \\ \leq & d_1\Lambda + \Big(d_2d_3^{p-1} - d_1\Big)\Big[\beta(1-\theta_1)S(I+\theta_2E)\Big] - \frac{d_2(1-p)d_3^{p+2}\sigma_{22}^2}{4\Big(\Big(\frac{E}{d_3}\Big)^2+1\Big)} \\ & \times \left(\frac{E}{d_3}\right)^4 \\ \leq & d_1\Lambda + \Big(d_2d_3^{p-1} - d_1\Big)\Big[\beta(1-\theta_1)S(I+\theta_2E)\Big] - \frac{d_2(1-p)d_3^{p+2}\sigma_{22}^2}{4} \\ & \times \left[\frac{3}{4}\left(\frac{E}{d_3}\right)^2 - \frac{1}{4}\right] \\ =& d_1\Lambda + \Big(d_2d_3^{p-1} - d_1\Big)\Big[\beta(1-\theta_1)S(I+\theta_2E)\Big] - \frac{3d_2(1-p)d_3^{p}\sigma_{22}^2}{16}E^2 \\ & + \frac{d_2(1-p)d_3^{p+2}\sigma_{22}^2}{16}. \end{aligned}$$

Choose $d_1 = d_2d_3^{p-1}, d_2 = \dfrac{8}{3(1-p)d_3^p}, d_3 = 2\sqrt[3]{\dfrac{\Lambda}{(1-p)\sigma_{22}^2}}$; then, we have:

$$LV_2(S,E) \leq 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} - \frac{\sigma_{22}^2}{2}E^2.$$

As a result,

$$\begin{aligned}\mathcal{LV}_2 \leq &-\frac{\beta(1-\theta_1)SI}{E} - \beta(1-\theta_1)\theta_2 S + (\delta+\mu) + \frac{\sigma_{21}^2}{2} + \sigma_{21}\sigma_{22}E \\ &+ \frac{\sigma_{22}^2}{2}E^2 + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} - \frac{\sigma_{22}^2}{2}E^2 \\ =&-\frac{\beta(1-\theta_1)SI}{E} + \left(\delta+\mu+\frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right) + \sigma_{21}\sigma_{22}E. \end{aligned} \tag{A5}$$

Similarly,

$$\begin{aligned}\mathcal{L}V_3 =& -\frac{C_3\delta E}{I} + C_3(\gamma+\sigma+\mu) + \frac{C_3\sigma_{31}^2}{2} + C_3\sigma_{31}\sigma_{32}I + \frac{C_3\sigma_{32}^2}{2} \\ &+ C_3\theta_1\sigma_{32}(\sigma_{31}+\sigma_{32}I)^{p-1}(\delta E - (\gamma+\sigma+\mu)I) \\ &- \frac{C_3\theta_1\sigma_{32}^2(1-p)}{2}(\sigma_{31}+\sigma_{32}I)^p I^2 + \frac{C_3\sigma_{31}\sigma_{32}}{(\gamma+\sigma+\mu)}(\delta E - (\gamma+\sigma+\mu)) \\ \leq& -\frac{C_3\delta E}{I} + C_3(\gamma+\sigma+\mu) + \frac{C_3\sigma_{31}^2}{2} + \frac{C_3\sigma_{31}\sigma_{32}\delta E}{\gamma+\sigma+\mu} \\ &+ \frac{C_3\sigma_{32}^2}{2} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{C_3\theta_1\sigma_{32}^2(1-p)\sigma_{31}^p}{2}I^2. \end{aligned}$$

Choose $\theta_1 = \dfrac{1}{(1-p)\sigma_{31}^p}$; then:

$$\mathcal{LV}_3(I) \leq -\frac{C_3\delta E}{I} + C_3(\gamma+\sigma+\mu) + \frac{C_3\sigma_{31}^2}{2} + \left(\frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta\right)E. \tag{A6}$$

Define

$$\mathcal{V}_4 = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3.$$

$$\begin{aligned}\mathcal{LV}_4 \leq& -\frac{\beta(1-\theta_1)S(I+\theta_2 E)}{E} + \left(\delta+\mu+\frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right) - (C_1+C_2)\frac{\Lambda}{S} \\ &+ (C_1+C_2)\beta(1-\theta_1)(I+\theta_2 E) + \sigma_{21}\sigma_{22}E + \Big(\sigma_{21}\sigma_{22} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} \\ &+ C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta\Big)E + (C_1+C_2)\left(\mu + \frac{\sigma_{11}^2}{2} + 2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right) \\ &- \frac{C_3\delta E}{I} + C_3\left(\gamma+\sigma+\mu+\frac{\sigma_{31}^2}{2}\right) \\ \leq& -\left(\frac{\beta(1-\theta_1)SI}{E} + \frac{C_1\Lambda}{S} + \frac{C_3\delta E}{I}\right) - \left(\frac{\beta(1-\theta_1)\theta_2 SE}{E} + \frac{C_2\Lambda}{S}\right) + \Bigg(\delta+\mu \\ &+ \frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\Bigg) + (C_1+C_2)\left(\mu + \frac{\sigma_{11}^2}{2} + 2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right) \\ &+ C_3\left(\gamma+\sigma+\mu+\frac{\sigma_{31}^2}{2}\right) + (C_1+C_2)\beta(1-\theta_1)(I+\theta_2 E) + \sigma_{21}\sigma_{22}E + \Big(\sigma_{21}\sigma_{22} \\ &+ \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta\Big)E \end{aligned}$$

$$
\begin{aligned}
&\leq -3\sqrt[3]{\beta(1-\theta_1)\Lambda\delta C_1 C_3} - 2\sqrt{\beta(1-\theta_1)\theta_2\Lambda C_2} + (C_1+C_2)\Bigg(\mu + \frac{\sigma_{11}^2}{2} \\
&\quad + 2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\Bigg) + C_3\left(\gamma+\sigma+\mu+\frac{\sigma_{31}^2}{2}\right) + \Bigg(\delta+\mu+\frac{\sigma_{21}^2}{2} \\
&\quad + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\Bigg) + (C_1+C_2)\beta(1-\theta_1)(I+\theta_2 E) + \Bigg(\sigma_{21}\sigma_{22} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} \\
&\quad + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta\Bigg)E \\
&= -\frac{\beta(1-\theta_1)\Lambda\delta}{\left(\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right)\left(\gamma+\sigma+\mu+\frac{\sigma_{31}^2}{2}\right)} \\
&\quad - \frac{\beta(1-\theta_1)\theta_2\Lambda}{\left(\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right)} + \left(\delta+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right) \\
&\quad + (C_1+C_2)\beta(1-\theta_1)(I+\theta_2 E) + \Bigg(\sigma_{21}\sigma_{22} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} \\
&\quad + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta\Bigg)E \\
&= -\left(\delta+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right)(\mathfrak{R}_0^S - 1) + (C_1+C_2)\beta(1-\theta_1)(I+\theta_2 E) \\
&\quad + \left(\sigma_{21}\sigma_{22} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta\right)E,
\end{aligned}
\tag{A7}
$$

where

$$
\begin{aligned}
C_1 &= \frac{\Lambda\delta\beta\left(1-\theta_1\right)}{\left(\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right)^2\left(\gamma+\sigma+\mu+\frac{\sigma_{31}^2}{2}\right)}, \\
C_3 &= \frac{\Lambda\delta\beta\left(1-\theta_1\right)}{\left(\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right)\left(\gamma+\sigma+\mu+\frac{\sigma_{31}^2}{2}\right)^2}, \\
C_2 &= \frac{\Lambda\theta_2\beta\left(1-\theta_1\right)}{\left(\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\Lambda^2\sigma_{12}^2}{(1-p)^2}}\right)^2}.
\end{aligned}
$$

Define

$$\mathcal{V}_5 = \frac{\left(C_1+C_2\right)\beta\left(1-\theta_1\right)}{\gamma+\sigma+\mu} I.$$

Define

$$\mathcal{V}_6 = \mathcal{V}_4 + \mathcal{V}_5.$$

In terms of (A7), applying Itô's formula to $\mathcal{V}_6$, we derive:

$$\mathcal{L}\mathcal{V}_6 = -\left(\delta + \mu + \frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right)\left(\mathfrak{R}_0^S(p) - 1\right)$$
$$+ \Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E. \quad \text{(A8)}$$

Define

$$\mathcal{V}_7 = \frac{(\sigma_{11}+\sigma_{12}S)^p}{p} + \frac{(\sigma_{21}+\sigma_{22}E)^p}{p} + \frac{(\sigma_{31}+\sigma_{32}I)^p}{p}.$$

Analogously, we have

$$\begin{aligned}\mathcal{L}\mathcal{V}_7 =&\sigma_{12}(\sigma_{11}+\sigma_{12}S)^{p-1}(\Lambda - \beta(1-\theta_1)S(I+\theta_2E) - \mu S) - \frac{\sigma_{12}^2}{2}(1-p)(\sigma_{11}+\sigma_{12}S)^pS^2\\ &+ \sigma_{22}(\sigma_{21}+\sigma_{22}E)^{p-1}(\beta(1-\theta_1)S(I+\theta_2E) - (\delta+\mu)E) - \frac{\sigma_{22}^2}{2}(1-p)(\sigma_{21}+\sigma_{22}E)^P\\ &\times E^2 + \sigma_{32}(\sigma_{31}+\sigma_{32}I)^{p-1}(\delta E - (\gamma+\sigma+\mu)I) - \frac{\sigma_{32}^2}{2}(1-p)(\sigma_{31}+\sigma_{32}I)^PI^2\\ \le&\sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2E) - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2}\\ &+ \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2}.\end{aligned}$$

Then, we define a C^2-function $\widetilde{\mathcal{V}} : \mathbb{R}_+^3 \to \overline{\mathbb{R}}_+$, as follows:

$$\widetilde{\mathcal{V}} = M\mathcal{V}_6 - \ln S - \ln I + \mathcal{V}_7,$$

where $M > 0$ satisfies $-M\left(\delta + \mu + \frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right)\left(\mathfrak{R}_0^S(p) - 1\right) + C \le -2$ and C is presented at the back. In addition,

$$\liminf_{k\to\infty,(S,E,I)\in\mathbb{R}_+^3\setminus U_k} \widetilde{\mathcal{V}} = +\infty,$$

where $U_k = \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right)$ and $k > 1$ is a sufficiently large integer. We can therefore construct a suitable non-negative C^2-function

$$\mathcal{V}(S,E,I) = \widetilde{\mathcal{V}} - \widetilde{\mathcal{V}}(S_0,E_0,I_0),$$

in which (S_0, E_0, I_0) denotes the minimum. From (A8), we have

$$\begin{aligned}\mathcal{L}\mathcal{V} \le& -\widetilde{M}(\mathfrak{R}_0^S(p) - 1) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{11}\sigma_{22}\delta}{\gamma+\sigma+\mu}\\ &+ C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E - \frac{\Lambda}{S} - \frac{\delta E}{I} + 3\mu + \delta + \gamma + \sigma + \beta(1-\theta_1)[I(t) + \theta_2E(t)]\\ &+ \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2E) - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2}\\ &+ \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2},\end{aligned} \quad \text{(A9)}$$

where $\widetilde{M} = M\left(\delta + \mu + \frac{\sigma_{21}^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}}\right)$.

Define $U_\epsilon = \left\{(S,E,I) \in \mathbb{R}^3_+ : \epsilon \leq S \leq \frac{1}{\epsilon}, \epsilon \leq E \leq \frac{1}{\epsilon}, \epsilon^2 \leq I \leq \frac{1}{\epsilon^2}\right\}$, where $0 < \epsilon < 1$ is a sufficiently small number. In the set $\mathbb{R}^3_+ \backslash U_\epsilon$, we can choose an ϵ thatis sufficiently small such that the following conditions hold:

$$-\frac{\Lambda}{\epsilon} + J \leq -1, \tag{A10}$$

$$-\frac{\sigma_{12}^{p+2}(1-p)}{4\epsilon^{(p+2)}} + J \leq -1, \tag{A11}$$

$$M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)\epsilon \leq 1, \tag{A12}$$

$$-\frac{\delta}{\epsilon} + J \leq -1, \tag{A13}$$

$$-\frac{\sigma_{22}^{p+2}(1-p)}{4\epsilon^{2(p+2)}} + J \leq -1, \tag{A14}$$

$$-\frac{\sigma_{32}^{p+2}(1-p)}{4\epsilon^{2(p+2)}} + J \leq -1, \tag{A15}$$

where

$$\begin{aligned} J = \sup_{(S,E,I)\in\mathbb{R}^3_+} &\Big\{M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{11}\sigma_{22}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta \\ &+ \sigma_{21}\sigma_{22}\Big)E + 3\mu + \delta + \gamma + \sigma + \beta(1-\theta_1)(I+\theta_2 E) + \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2} \\ &+ \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) - \frac{1-p}{4}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2}\Big\} < \infty. \end{aligned}$$

For convenience, we can divide $\mathbb{R}^3_+ \backslash U_\epsilon$ into the following six domains:

$$\begin{aligned} U_1 &= \Big\{(S,E,I)\in\mathbb{R}^3_+ : S \leq \epsilon\Big\}, U_2 = \Big\{(S,E,I)\in\mathbb{R}^3_+ : S \geq \frac{1}{\epsilon}\Big\} \\ U_3 &= \Big\{(S,E,I)\in\mathbb{R}^3_+ : E \leq \epsilon\Big\}, U_4 = \Big\{(S,E,I)\in\mathbb{R}^3_+ : E > \epsilon, I \leq \epsilon^2\Big\} \\ U_5 &= \Big\{(S,E,I)\in\mathbb{R}^3_+ : E \geq \frac{1}{\epsilon}\Big\}, U_6 = \Big\{(S,E,I)\in\mathbb{R}^3_+ : I \geq \frac{1}{\epsilon^2}\Big\}. \end{aligned}$$

It is easy to see that $\mathbb{R}^3_+ \backslash U_\epsilon = U^c_\epsilon = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6$. Next, we will prove that $\mathcal{V}(S,E,I) \leq -1$ for any $(S,E,I) \in U^c_\epsilon$, which is equivalent to proving it on the above six domains, respectively.

Situation 1. When $(S,E,I) \in U_1$, from (A10), we obtain:

$$
\begin{aligned}
\mathcal{LV} \leq & -\frac{\Lambda}{S} - \widetilde{M}\Big(\mathfrak{R}_0^S(p) - 1\Big) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} \\
& + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E - \frac{\delta E}{I} + 3\mu + \delta + \gamma + \sigma + \beta(1-\theta_1)(I+\theta_2 E) + \sigma_{12}\sigma_{11}^{p-1}\Lambda \\
& - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\delta E \\
& - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2} \\
\leq & -\frac{\Lambda}{S} + J \\
\leq & -\frac{\Lambda}{\epsilon} + J \\
\leq & -1.
\end{aligned} \tag{A16}
$$

Situation 2. When $(S, E, I) \in U_2$, from (A11), it follows that:

$$
\begin{aligned}
\mathcal{LV} \leq & -\frac{1-p}{4}\sigma_{11}^{p+2}S^{p+2} - \widetilde{M}\Big(\mathfrak{R}_0^S(p) - 1\Big) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} \\
& + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E - \frac{\Lambda}{S} - \frac{\delta E}{I} + 3\mu + \delta + \sigma + \gamma \\
& + \beta(1-\theta_1)(I+\theta_2 E) + \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2} - \frac{1-p}{4}\sigma_{22}^{p+2}E^{p+2} \\
& + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) + \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2} \\
\leq & -\frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2} + J \\
\leq & -\frac{\sigma_{12}^{p+2}(1-p)}{4\epsilon^{p+2}} + J \\
\leq & -1.
\end{aligned} \tag{A17}
$$

Situation 3. When $(S, E, I) \in U_3$, from (A12), we derive:

$$
\begin{aligned}
\mathcal{LV} \leq & -\widetilde{M}\Big(\mathfrak{R}_0^5(p) - 1\Big) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} \\
& + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E - \frac{\Lambda}{S} - \frac{\delta E}{I} + 3\mu + \delta + \sigma + \gamma + \beta(1-\theta_1)(I+\theta_2 E) \\
& + \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} \\
& + \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2} \\
\leq & -\widetilde{M}\Big(\mathfrak{R}_0^5(p) - 1\Big) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} \\
& + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E + C \\
\leq & -\widetilde{M}\Big(\mathfrak{R}_0^5(p) - 1\Big) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} \\
& + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)\epsilon + C \\
\leq & -2 + 1 \\
= & -1,
\end{aligned} \tag{A18}
$$

where

$$C = \sup_{(S,E,I)\in\mathbb{R}^3_+} \Big\{ 3\mu + \delta + \sigma + \gamma + \beta(1-\theta_1)(I+\theta_2 E) + \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2} \Big\}.$$

Situation 4. When $(S,E,I) \in U_4$, from (A13), we obtain:

$$\begin{aligned}
\mathcal{LV} \le & -\frac{\delta E}{I} - M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta \\
& + \sigma_{21}\sigma_{22}\Big)E - \frac{\Lambda}{S} + 3\mu + \delta + \sigma + \gamma + \beta(1-\theta_1)(I+\theta_2 E) + \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} \\
& + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2} \\
\le & -\frac{\delta E}{I} + J \\
\le & -\frac{\delta\epsilon}{\epsilon^2} + J \\
\le & -\frac{\delta}{\epsilon} + J \\
\le & -1.
\end{aligned} \tag{A19}$$

Situation 5. When $(S,E,I) \in U_5$, from (A14), we derive:

$$\begin{aligned}
\mathcal{LV} \le & -\frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} - \widetilde{M}\Big(\mathfrak{R}_0^S(p) - 1\Big) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} \\
& + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E - \frac{\Lambda}{S} - \frac{\delta E}{I} + 3\mu + \delta + \sigma + \gamma) \\
& + \beta(1-\theta_1)(I+\theta_2 E + \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) \\
& + \sigma_{32}\sigma_{31}^{p-1}\delta E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2} \\
\le & -\frac{1-p}{4}\sigma_{22}^{p+2}E^{p+2} + J \\
\le & -\frac{\sigma_{22}^{p+2}(1-p)}{4\epsilon^{p+2}} + J \\
\le & -1.
\end{aligned} \tag{A20}$$

Situation 6. When $(S,E,I) \in U_6$, from (A15), we obtain:

$$\begin{aligned}
\mathcal{LV} \le & -\frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2} - \widetilde{M}\Big(\mathfrak{R}_0^S(p) - 1\Big) + M\Big(\frac{(C_1+C_2)\beta(1-\theta_1)(\theta_2(\gamma+\sigma+\mu)+\delta)}{\gamma+\sigma+\mu} \\
& + \frac{C_3\sigma_{31}\sigma_{32}\delta}{\gamma+\sigma+\mu} + C_3\theta_1\sigma_{32}\sigma_{31}^{p-1}\delta + \sigma_{21}\sigma_{22}\Big)E - \frac{\Lambda}{S} - \frac{\delta E}{I} + 3\mu + \delta + \sigma + \gamma \\
& + \beta(1-\theta_1)(I+\theta_2 E) + \frac{1}{2}(\sigma_{11}+\sigma_{12}S)^2 + \frac{1}{2}(\sigma_{31}+\sigma_{32}I)^2 + \sigma_{12}\sigma_{11}^{p-1}\Lambda - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} \\
& + \sigma_{22}\sigma_{21}^{p-1}\beta(1-\theta_1)S(I+\theta_2 E) - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\delta E \\
\le & -\frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2} + J \\
\le & -\frac{\sigma_{32}^{p+2}(1-p)}{4\epsilon^{2(p+2)}} + J \\
\le & -1.
\end{aligned} \tag{A21}$$

The proof is over. □

Proof of Theorem 2. Set
$$P(t) = \rho_1 E + \rho_2 I,$$
where ρ_1 and ρ_2 are positive constants to be determined later. Then,
$$d(\ln P) = \mathcal{L}(\ln P) + \frac{1}{P}[\rho_1(\sigma_{21} + \sigma_{22}E)EdB_2(t) + \rho_2(\sigma_{31} + \sigma_{32}I)IdB_3(t)], \tag{A22}$$
where
$$\begin{aligned}
\mathcal{L}(\ln P) =& \frac{1}{P}\Big[\rho_1(\beta(1-\theta_1)SI + \beta(1-\theta_1)\theta_2 SE - (\delta+\mu)E) + \rho_2(\delta E - (\gamma+\sigma+\mu)I)\Big] \\
&- \frac{\rho_1^2(\sigma_{21}E + \sigma_{22}E^2)^2}{2P^2} - \frac{\rho_2^2(\sigma_{31}I + \sigma_{32}I^2)^2}{2P^2} \\
=& \frac{1}{P}\Big[(\rho_1\beta(1-\theta_1)(I+\theta_2 E))S + (\rho_2\delta - \rho_1(\delta+\mu))E - \rho_2(\gamma+\sigma+\mu)I\Big] \\
&- \frac{\rho_1^2(\sigma_{21}E + \sigma_{22}E^2)^2}{2P^2} - \frac{\rho_2^2(\sigma_{31}I + \sigma_{32}I^2)^2}{2P^2}.
\end{aligned}$$
Next, let $\rho_i (i = 1, 2)$ satisfy the following equations:
$$\begin{cases} \rho_1(\delta+\mu) - \rho_2\delta = \dfrac{\Lambda\beta(1-\theta_1)\theta_2}{\mu}, \\ \rho_2(\gamma+\sigma+\mu) = \dfrac{\Lambda\beta(1-\theta_1)}{\mu}. \end{cases}$$
By direct calculation, we derive
$$\rho_2 = \frac{\Lambda\beta(1-\theta_1)}{\mu(\gamma+\sigma+\mu)}, \rho_1 = \frac{\Lambda\beta(1-\theta_1)(\delta + \theta_2(\gamma+\sigma+\mu))}{\mu(\gamma+\sigma+\mu)(\delta+\mu)} = \mathfrak{R}_0.$$
As a result, we can obtain that
$$\begin{aligned}
&\rho_1\beta(1-\theta_1)(I+\theta_2 E)S - (\rho_1(\delta+\mu) - \rho_2\delta)E - \rho_2(\gamma+\sigma+\mu)I \\
=& \rho_1\beta(1-\theta_1)(I+\theta_2 E)\Big(S - \frac{\Lambda}{\mu}\Big) + \frac{\rho_1\Lambda}{\mu}(\beta(1-\theta_1)(I+\theta_2 E)) - (\rho_1(\delta+\mu) - \rho_2\delta)E \\
&- \rho_2(\gamma+\sigma+\mu)I \\
=& \rho_1\beta(1-\theta_1)(I+\theta_2 E)\Big(S - \frac{\Lambda}{\mu}\Big) + \frac{\rho_1\Lambda}{\mu}(\beta(1-\theta_1) - \rho_2(\gamma+\sigma+\mu))I + \frac{\rho_1\Lambda}{\mu}(\beta(1-\theta_1)\theta_2 \\
&- \rho_1(\delta+\mu) + \rho_2\delta)E \\
=& \mathfrak{R}_0\beta(1-\theta_1)(I+\theta_2 E)\Big(S - \frac{\Lambda}{\mu}\Big) + \frac{\Lambda}{\mu}(\beta(1-\theta_1)(I+\theta_2 E))(\mathfrak{R}_0 - 1) \\
\le& \mathfrak{R}_0\beta(1-\theta_1)(I+\theta_2 E)\Big(\widetilde{A} - \frac{\Lambda}{\mu}\Big) \le \mathfrak{R}_0(\beta(1-\theta_1)(I+\theta_2 E))\Big|\widetilde{A} - \frac{\Lambda}{\mu}\Big|.
\end{aligned}$$
Then, one can show that
$$\begin{aligned}
\mathcal{L}(\ln P) =& \frac{1}{P}\Big[\mathfrak{R}_0\beta(1-\theta_1)(I+\theta_2 E)]\Big|\widetilde{A} - \frac{\Lambda}{\mu}\Big|\Big] - \frac{\rho_1^2(\sigma_{21}E + \sigma_{22}E^2)^2}{2P^2} - \frac{\rho_2^2(\sigma_{31}I + \sigma_{32}I^2)^2}{2P^2} \\
&- \frac{\rho_1^2(\sigma_{21}E + \sigma_{22}E^2)^2}{2P^2} - \frac{\rho_2^2(\sigma_{31}I + \sigma_{32}I^2)^2}{2P^2} \\
\le& \mathfrak{R}_0\rho_0\Big|\widetilde{A} - \frac{\Lambda}{\mu}\Big| - \frac{\rho_1^2(\sigma_{21}E + \sigma_{22}E^2)^2}{2P^2} - \frac{\rho_2^2(\sigma_{31}I + \sigma_{32}I^2)^2}{2P^2}, a.s.
\end{aligned}$$

Then,

$$\frac{\ln P(t)}{t} \leq \frac{\ln P(0)}{t} + \frac{\Re_0\rho_0}{t}\int_0^t \left|\widetilde{A}(\tau) - \frac{\Lambda}{\mu}\right| d\tau + \frac{1}{t}\Big(\widetilde{M}_1(t) - \widetilde{N}_1(t)\Big) + \frac{1}{t}\Big(\widetilde{M}_2(t) - \widetilde{N}_2(t)\Big), a.s., \tag{A23}$$

where

$$\widetilde{M}_1(t) = \int_0^t \frac{\rho_1\big(\sigma_{21}E(\tau) + \sigma_{22}E^2(\tau)\big)}{P(\tau)} dB_2(\tau), \widetilde{N}_1(t) = \int_0^t \frac{\rho_1^2\big(\sigma_{21}E(\tau) + \sigma_{22}E^2(\tau)\big)^2}{2P^2(\tau)} d\tau,$$
$$\widetilde{M}_2(t) = \int_0^t \frac{\rho_2\big(\sigma_{31}I(\tau) + \sigma_{32}I^2(\tau)\big)}{P(\tau)} dB_3(\tau), \widetilde{N}_2(t) = \int_0^t \frac{\rho_2^2\big(\sigma_{31}I(\tau) + \sigma_{32}I^2(\tau)\big)^2}{2P^2(\tau)} d\tau.$$

Furthermore, on the basis of the exponential martingales inequality [33], for any positive constants T, ς and v, we have

$$\mathbb{P}\left\{\sup_{0\leq t\leq T}\left[\widetilde{M}_i(t) - \frac{\zeta}{2}\Big\langle \widetilde{M}_i, \widetilde{M}_i\Big\rangle(t)\right] > v\right\} \leq e^{-\zeta v}.$$

Set $T = \varrho, \varsigma = \varepsilon, v = \dfrac{2\ln\varrho}{\varepsilon}$, where ϱ is a positive integer. Then,

$$\mathbb{P}\left\{\sup_{0\leq t\leq \varrho}\left[\widetilde{M}_i(t) - \varepsilon\widetilde{N}_i(t)\right] > \frac{2\ln\varrho}{\varepsilon}\right\} \leq \frac{1}{\varrho^2}.$$

In view of the Borel–Cantelli lemma [33], $\forall\ \omega_t \in \Omega$, $\exists n_t(\omega_t) \in N$, when $t \in (\varrho - 1, \varrho]$, $\varrho \geq n_t(\omega_t)$, a.s., we have:

$$\widetilde{M}_i(t) \leq \varepsilon\widetilde{N}_i(t) + \frac{2\ln\varrho}{\varepsilon},$$

which implies

$$\begin{aligned}
\frac{1}{t}\sum_{i=1}^{2}\Big(\widetilde{M}_i(t) - \widetilde{N}_i(t)\Big) &\leq -\frac{1-\varepsilon}{t}\sum_{i=1}^{2} N_i(t) + \frac{4\ln\varrho}{\varepsilon t} \\
&\leq -\frac{1-\varepsilon}{t}\int_0^t \frac{\rho_1^2\sigma_{22}^2E^2(u) + \rho_2^2\sigma_{32}^2I^2(u)}{2\Big(\rho_1E(u) + \rho_2I(u)\Big)^2} d\tau + \frac{4\ln\varrho}{\varepsilon(\varrho-1)} \\
&\leq -\frac{1-\varepsilon}{t}\int_0^t \frac{\rho_1^2\sigma_{22}^2E^2(u) + \rho_2^2\sigma_{32}^2I^2(u)}{4\Big(\rho_1^2E^2(u) + \rho_2^2I(u)\Big)^2} d\tau + \frac{4\ln\varrho}{\varepsilon(\varrho-1)} \\
&\leq -\frac{(1-\varepsilon)(\sigma_{22}^2 \wedge \sigma_{32}^2)}{4} + \frac{4\ln\varrho}{\varepsilon(\varrho-1)}.
\end{aligned} \tag{A24}$$

Additionally,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left|\widetilde{A}(\tau) - \frac{\Lambda}{\mu}\right| d\tau = \int_0^\infty \left|x - \frac{\Lambda}{\mu}\right| \pi(x)dx, \ \ a.s.$$

By combining (A23) and (A24), we obtain:

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\ln P(t)}{t} &\leq \lim_{t\to\infty}\frac{\Re_0\rho_0}{t}\int_0^t \left|\widetilde{A}(\tau) - \frac{\Lambda}{\mu}\right| d\tau + \lim_{n\to\infty}\frac{4\ln\varrho}{\varepsilon(\varrho-1)} - \frac{(1-\varepsilon)(\sigma_{22}^2 \wedge \sigma_{32}^2)}{4} \\
&= \Re_0\rho_0\int_0^\infty \left|x - \frac{\Lambda}{\mu}\right| \pi(x)dx - \frac{(1-\varepsilon)(\sigma_{22}^2 \wedge \sigma_{32}^2)}{4}, a.s.
\end{aligned}$$

When $\varepsilon \to 0^+$, one can draw the conclusion that

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln P(t)}{t} &\leq \Re_0\rho_0 \int_0^{\infty} \left| x - \frac{\Lambda}{\mu} \right| \pi(x)dx - \frac{\sigma_{22}^2 \wedge \sigma_{32}^2}{4} \\ &= \Re_0^E < 0, \ \ a.s.\end{aligned}$$

Then, $\lim_{t\to\infty} P(t) = 0$, a.s., and

$$\lim_{t\to\infty} E(t) = 0, \quad \lim_{t\to\infty} I(t) = 0, \ \ a.s.$$

The proof is over. □

Proof of Theorem 3. According to Roozen [34], the density function $\varphi(y) = \varphi(y_1, y_2, y_3)$ of the quasi-stationary distribution of model (7) around the quasi-endemic equilibrium E^* satisfies the following Fokker–Plank equation:

$$\begin{aligned}&-\sum_{i=1}^{3} \frac{\sigma_{i1}^2}{2} \frac{\partial^2}{\partial y_i^2}\varphi + \frac{\partial}{\partial y_1}[(-a_{11}y_1 - a_{12}y_2 - a_{13}y_3)\varphi] + \frac{\partial}{\partial y_2}[(a_{21}y_1 - a_{22}y_2 + a_{23}y_3)\varphi] \\ &+ \frac{\partial}{\partial y_3}[(a_{32}y_2 - a_{33}y_3)\varphi] = 0,\end{aligned}$$

where

$$\varphi(y) = c\exp\Big\{ -\frac{1}{2}(Y - Y^*)Q(Y - Y^*)^T \Big\}, \tag{A25}$$

where $Y^* = (0,0,0)$, and Q is a real symmetric matrix which satisfies $QG^2Q + A^TQ + QA = 0$. If Q is positive definite, let $Q^{-1} = \Sigma$; then,

$$G^2 + A\Sigma + \Sigma A^T = 0. \tag{A26}$$

From [35], Equation (A26) satisfies

$$G_i^2 + A\Sigma_i + \Sigma_i A^T = 0, (i = 1,2,3),$$

where $G_1 = diag(\sigma_{11}, 0, 0), G_2 = diag(0, \sigma_{21}, 0), G_3 = diag(0, 0, \sigma_{31}), \Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3, G = G_1 + G_2 + G_3$. And, $\varphi_A(\lambda) = \lambda^3 + r_1\lambda^2 + r_2\lambda + r_3$, where

$$\begin{aligned}&r_1 = a_{11} + a_{22} + a_{32} > 0, r_2 = a_{11}a_{32} + a_{11}a_{22} + a_{12}a_{21} > 0, r_3 = a_{13}a_{21}a_{32} + a_{32}a_{12}a_{21} > 0, \\ &r_1r_2 - r_3 = a_{11}^2a_{32} + a_{11}^2a_{22} + a_{11}a_{12}a_{21} + a_{11}a_{22}a_{32} + a_{11}a_{22}^2 + a_{12}a_{21}a_{22} + a_{11}a_{32}^2 + a_{11}a_{22}a_{32} \\ &\qquad - a_{13}a_{21}a_{32} > 0.\end{aligned}$$

From Lemma 2.6 in Ref. [16], we obtain that Σ of Equation (A26) is positive definite.

Step 1. Notice that

$$G_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0, \tag{A27}$$

where $G_1 = diag(\sigma_{11}, 0, 0)$. Let $B_1 = M_1AM_1^{-1}$, where the standard transform matrix

$$M_1 = \begin{pmatrix} a_{21}a_{32} & (-a_{33} - a_{22})a_{32} & a_{33}^2 + a_{23}a_{32} \\ 0 & a_{32} & -a_{33} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$\begin{aligned}&(M_1J_1)G_1^2(M_1J_1)^T + (M_1J_1)A(M_1J_1)^{-1})(M_1J_1)\Sigma_1(M_1J_1)^T \\ &+ (M_1J_1)\Sigma_1(M_1J_1)^T((M_1J_1)A(M_1J_1)^{-1})^T = 0,\end{aligned}$$

namely,

$$G_0^2 + B_1\Sigma_0 + \Sigma_0 B_1^T = 0,$$

where $\Sigma_0 = \frac{1}{\rho_1^2}(M_1J_1)\Sigma_1(M_1J_1)^T$, $\rho_1 = a_{21}a_{32}\sigma_{11}$. By Lemma 4.1, we have that Σ_0 is positive definite and

$$\Sigma_0 = \begin{pmatrix} \frac{r_2}{2(r_1r_2-r_3)} & 0 & -\frac{1}{2(r_1r_2-r_3)} \\ 0 & \frac{1}{2(r_1r_2-r_3)} & 0 \\ -\frac{1}{2(r_1r_2-r_3)} & 0 & \frac{r_1}{2r_3(r_1r_2-r_3)} \end{pmatrix}.$$

Consequently,

$$\Sigma_1 = \rho_1^2(M_1J_1)^{-1}\Sigma_0[(M_1J_1)^{-1}]^T. \tag{A28}$$

Step 2. Notice that

$$G_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0, \tag{A29}$$

where $G_2 = diag(0, \sigma_{21}, 0)$. Similarly, let $A_2 = J_2AJ_2^T$, where $J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$; then,

$$A_2 = \begin{pmatrix} -a_{22} & a_{23} & a_{21} \\ a_{32} & -a_{33} & 0 \\ -a_{12} & -a_{13} & -a_{11} \end{pmatrix}.$$

Let $M_2 = J_3A_2J_3^{-1}$, where the standard transform matrix

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{a_{12}}{a_{32}} & 1 \end{pmatrix}.$$

Then,

$$M_2 = \begin{pmatrix} -a_{22} & a_{23} - \frac{a_{12}a_{21}}{a_{32}} & a_{21} \\ a_{32} & -a_{33} & 0 \\ 0 & \Delta_1 & -a_{11} \end{pmatrix},$$

where $\Delta_1 = -\frac{a_{12}a_{33}}{a_{32}} - a_{13} + \frac{a_{12}a_{11}}{a_{32}}$. Let $B_2 = J_4M_2J_4^{-1}$, where

$$J_4 = \begin{pmatrix} a_{32}\Delta_1 & (a_{11}-a_{33})\Delta_1 & a_{11}^2 \\ 0 & \Delta_1 & a_{11} \\ 0 & 0 & 1 \end{pmatrix}.$$

And $B_2 = B_1$. In addition, (A29) can be expressed by:

$$G_0^2 + B_2\Big(\frac{1}{\rho_2^2}(J_4J_3J_2)\Sigma_2(J_4J_3J_2)^T\Big) + \Big(\frac{1}{\rho_2^2}(J_4J_3J_2)\Sigma_2(J_4J_3J_2)^T\Big)B_2^T = 0,$$

where $\Sigma_0 = \Big(\frac{1}{\rho_2^2}(J_4J_3J_2)\Sigma_2(J_4J_3J_2)^T\Big)$, $\rho_2 = a_{32}\Delta_1\sigma_{21}$. As discussed in Step 1, Σ_0 is positive definite. Consequently,

$$\Sigma_2 = \rho_2^2(J_4J_3J_2)^{-1}\Sigma_0[(J_4J_3J_2)^{-1}]^T. \tag{A30}$$

Step 3. Notice that

$$G_3^2 + A\Sigma_3 + \Sigma_3 A^T = 0, \tag{A31}$$

where $G_3 = diag(0,0,\sigma_{31})$. Firstly, let $A_3 = J_3 A J_3^T$, where $J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$; thus,

$$A_3 = \begin{pmatrix} -a_{33} & 0 & a_{32} \\ -a_{13} & -a_{11} & -a_{12} \\ a_{23} & a_{21} & -a_{22} \end{pmatrix}.$$

Then, performing transformation $B_3 = P_3 A_3 P_3^{-1}$, where $P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{a_{23}}{a_{13}} & 1 \end{pmatrix}$:

$$B_3 = \begin{pmatrix} -a_{33} & -\frac{a_{32}a_{23}}{a_{13}} & a_{32} \\ -a_{13} & -a_{11} + \frac{a_{12}a_{23}}{a_{13}} & -a_{12} \\ 0 & \Delta_2 & -a_{22} - \frac{a_{23}}{a_{13}} a_{12} \end{pmatrix},$$

where $\Delta_2 = a_{21} - \frac{a_{23}}{a_{13}} a_{11} + \frac{a_{22}a_{23}}{a_{13}} + \frac{a_{23}^2}{a_{13}^2} a_{12}$.

Case 1. If $\Delta_2 \neq 0$, on the basis of the method in Step 1 or 2, we set $C_3 = M_3 B_3 M_3^{-1}$, where the standard transform matrix is

$$M_3 = \begin{pmatrix} -a_{13}\Delta_2 & \left(- a_{11} + \frac{a_{12}a_{23}}{a_{13}} - a_{22} - \frac{a_{23}}{a_{13}} a_{12} \right)\Delta_2 & \left(- a_{22} - \frac{a_{23}}{a_{13}} a_{12} \right)^2 - a_{12}\Delta_2 \\ 0 & \Delta_2 & -a_{22} - \frac{a_{23}}{a_{13}} a_{12} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we easily have $C_3 = B_1$. Concurrently, (A31) can also be transformed into

$$G_0^2 + C_3\Sigma_0 + \Sigma_0 C_3^T = 0,$$

where $\Sigma_0 = \frac{1}{\rho_3^2}(M_3P_3J_3)\Sigma_3(M_3P_3J_3)^T, \rho_3 = -a_{13}\Delta_2\sigma_{31}$, Therefore, we obtain:

$$\Sigma_3 = \rho_3^2 (M_3P_3J_3)^{-1}\Sigma_0 \left[(M_3P_3J_3)^{-1} \right]^T. \tag{A32}$$

Case 2. If $\Delta_2 = 0$, in terms with the method as that in Step 1 or 2, we set $C_{3\varpi} = M_{3\varpi} B_3 M_{3\varpi}^{-1}$, where the standard transform matrix

$$M_{3\varpi} = \begin{pmatrix} -a_{13} & -a_{11} - \frac{a_{12}a_{23}}{a_{13}} & -a_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$C_{3\varpi} = \begin{pmatrix} -b_1 & -b_2 & -b_3 \\ 1 & 0 & 0 \\ 0 & 0 & -a_{22} - \frac{a_{23}a_{12}}{a_{13}} \end{pmatrix}.$$

At the same time,

$$\begin{aligned} \varphi_A(\lambda) =& \lambda^3 + r_1\lambda^2 + r_2\lambda + r_3 = \varphi_{C_{3\varpi}} = \lambda^3 + \left[b_1 + \left(a_{22} + \frac{a_{23}}{a_{13}} a_{12} \right) \right] \lambda^2 \\ &+ \left[b_2 + b_1 \left(a_{22} + \frac{a_{23}}{a_{13}} a_{12} \right) \right] \lambda + b_2 \left(a_{22} + \frac{a_{23}}{a_{13}} a_{12} \right). \end{aligned}$$

Consequently,

$$b_1 = r_1 - \left(a_{22} + \frac{a_{23}}{a_{13}} a_{12} \right), b_2 = r_2 - b_1 \left(a_{22} + \frac{a_{23}}{a_{13}} a_{12} \right).$$

Analogously, we transform (A31) into the following equation:

$$(M_{3\varpi}P_3J_3)G_3^2(M_{3\varpi}P_3J_3)^T + \left((M_{3\varpi}P_3J_3)A(M_{3\varpi}P_3J_3)^{-1}\right)(M_{3\varpi}P_3J_3)\Sigma_3(M_{3\varpi}P_3J_3)^T$$
$$+ (M_{3\varpi}P_3J_3)\Sigma_3(M_{3\varpi}P_3J_3)^T\left((M_{3\varpi}P_3J_3)A(M_{3\varpi}P_3J_3)^{-1}\right)^T = 0,$$

i.e.,

$$G_0^2 + C_{3\varpi}\theta_0 + \theta_0 C_{3\varpi}^T = 0,$$

where $\theta_0 = \frac{1}{\rho_{3\varpi}^2}(M_{3\varpi}P_3J_3)\Sigma_3(M_{3\varpi}P_3J_3)^T$, $\rho_{3\varpi} = -a_{13}\sigma_{31}$. From Lemma 4.2, we derive that θ_0 is semi-positive definite, and its form is as follows:

$$\theta_0 = \begin{pmatrix} \frac{1}{2b_1} & 0 & 0 \\ 0 & \frac{1}{2b_1b_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\Sigma_3 = \rho_{3\varpi}^2(M_{3\varpi}P_3J_3)^{-1}\theta_0\left[(M_{3\varpi}P_3J_3)^{-1}\right]^T. \tag{A33}$$

The proof is over. □

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