fractal and fractional

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Combined Liouville–Caputo Fractional Differential Equation

McSylvester Ejighikeme Omaba *, Hamdan Al Sulaimani, Soh Edwin Mukiawa, Cyril Dennis Enyi and Tijani Abdul-Aziz Apalara

Department of Mathematics, College of Science, University of Hafr Al Batin, P.O. Box 1803, Hafr Al Batin 31991, Saudi Arabia
* Correspondence: mcomaba@uhb.edu.sa

Abstract: This paper studies a fractional differential equation combined with a Liouville–Caputo fractional differential operator, namely, ${}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}Q(t) = \lambda\vartheta(t, Q(t))$, $t \in [c, d]$, $\beta, \gamma \in (0, 1]$, $\eta \in [0, 1]$, where $Q(c) = q_c$ is a bounded and non-negative initial value. The function $\vartheta : [c, d] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in the second variable, $\lambda > 0$ is a constant and the operator ${}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}$ is a convex combination of the left and the right Liouville–Caputo fractional derivatives. We study the well-posedness using the fixed-point theorem, estimate the growth bounds of the solution and examine the asymptotic behaviours of the solutions. Our findings are illustrated with some analytical and numerical examples. Furthermore, we investigate the effect of noise on the growth behaviour of the solution to the combined Liouville–Caputo fractional differential equation.

Keywords: well-posedness; growth estimate; asymptotic behaviours; combined Liouville–Caputo fractional derivative; numerical simulations; stochastic models; second moment bound

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Citation: Omaba, M.E.; Al Sulaimani, H.; Mukiawa, S.E.; Enyi, C.D.; Apalara, T.A.-A. Combined Liouville–Caputo Fractional Differential Equation. *Fractal Fract.* **2023**, *7*, 366. https://doi.org/10.3390/fractalfract7050366

Academic Editors: Hari Mohan Srivastava, Riccardo Caponetto and Sameerah Jamal

Received: 28 January 2023
Revised: 4 March 2023
Accepted: 29 March 2023
Published: 28 April 2023

1. Introduction

Fractional-order derivatives are known to give more accurate and realistic mathematical models when compared to the classical-order models. The Liouville–Caputo fractional derivative is unarguably a useful operator that mostly models nonlocal behaviours by fractional DEs [1]. The Liouville–Caputo fractional derivative and others alike have found applications in different science and engineering fields and have been used to model many real-life problems. For example, they are used in the mathematical modelling of a human brain tissue (HBT) constitutive model in the framework of anisotropic hyperelasticity [2]; modelling the growth of many economical processes, specifically memory effect on the economic growth model, that is, in the application of economic growth models with memory effect [3] in physics and the environment; studying the chaotic behaviour(s) of dynamical systems; and developing the fractional-order models of neurons [4] and porous media, among others. See also [5,6] for other fractional models. It is worthy of note that all of the above fractional differential equations and more studied in the literature make use of one-sided (left- or right-sided) fractional derivative operators.

The new operator is a convex combination of left and right Liouville–Caputo fractional operators. There is little that one can find in the literature regarding this combined fractional derivative operators. Importantly, this new combined fractional operator is more general than other fractional derivatives [7]. The new operators were studied and defined by Malinowska and Torres [7] as follows:

$$ {}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma} = \eta\, {}_{c}^{LC}\mathcal{D}_{t}^{\beta} + (1-\eta)\, {}_{t}^{LC}\mathcal{D}_{d}^{\gamma}, \ \beta, \gamma \in (0, 1] \,\&\, \eta \in [0, 1], \tag{1} $$

where

$$ {}^{LC}\mathcal{D}_{0}^{\beta,\gamma}Q(t) = {}_{t}^{LC}\mathcal{D}_{d}^{\gamma}Q(t), $$

and

$$ {}^{LC}\mathcal{D}_{1}^{\beta,\gamma}Q(t) = {}_{c}^{LC}\mathcal{D}_{t}^{\beta}Q(t). $$

Another advantage of this new fractional derivative ${}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}$ is that it can describe variational problems in a broad perspective [7]. By drawing inspirations from the diamond-alpha derivative on a time scale, which is a linear combination of the delta and nabla derivatives [8–10], the model (1) was birthed. Malinowska and Torres [8] showed that the approximation of exact derivatives by the diamond-alpha derivative was better that those of the delta and nabla derivatives.

Therefore, this research considered a special operator by studying the combined Liouville–Caputo fractional differential equation as follows:

$$ {}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}Q(t) = \lambda\vartheta(t, Q(t)), \quad \beta, \gamma \in (0,1] \,\&\, \eta \in [0,1], \tag{2}$$

where $Q(c) = q_c$ is the initial condition, $\vartheta : [c,d] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz in the second variable, $\lambda > 0$ and ${}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}$ is known as the combined Liouville–Caputo fractional derivative operator. The operator ${}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}$ is a convex combination of left and right Liouville–Caputo fractional derivatives. Thus, our contribution and aim in this research paper was to examine the qualitative properties, the well-posedness, estimates of the growth bounds, and the asymptotic behaviour of the solution to a class of combined fractional differential equations.

The organization of the paper is as follows. Section 2 contains the preliminary concepts and definitions needed in the article; Section 3 contains the main results—existence, uniqueness, upper growth estimate, asymptotic behaviour of solution to the combined L–C fractional differential equation. Numerical and analytical illustration of our results are given in Section 4. In Section 5, we consider the stochastic (non-deterministic) case of our equation. Summary of the paper is given in Section 6.

2. Preliminary Concepts

Here, we give some definitions and basic materials. Readers can refer to [5] for further materials on fractional calculus.

Definition 1 ([7])**.** *For $\beta, \gamma \in (0,1)$ and $0 \le \eta \le 1$, the combined Liouville–Caputo fractional derivative operator ${}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}$ is a convex combination of left and right Liouville–Caputo fractional derivatives, defined by*

$$ {}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma} = \eta\, {}^{LC}_{\ c}\mathcal{D}_{t}^{\beta} + (1-\eta)\, {}^{LC}_{\ t}\mathcal{D}_{d}^{\gamma}.$$

Definition 2 ([11])**.** *Let $0 < c < d$ and $f : [c,d] \to \mathbb{R}$ be an integrable function. The left-sided Katugampola fractional integral of order $\beta > 0$ and parameter $\sigma > 0$ is given by*

$$\mathcal{I}_{c^+}^{\beta,\sigma}u(t) = \frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_c^t s^{\sigma-1}(t^{\sigma} - s^{\sigma})^{\sigma-1}u(s)ds,$$

provided the integral converges.

For $\sigma = 1$, we define the fractional integral as follows.

Definition 3 ([11])**.** *Let $u : [c,d] \to \mathbb{R}$ be an integrable function where $c, d > 0$ with $c < d$. Then,*

$${}_c\mathcal{I}_t^{\beta}u(t) = \frac{1}{\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}u(s)ds, \tag{3}$$

is the left-sided Riemann–Liouville fractional integral of u of order $0 < \beta < 1$, provided the integral converges.

Definition 4 ([5])**.** *Let $u : [c,d] \to \mathbb{R}$, then*

$${}_c\mathcal{D}_t^{\beta}u(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_c^t (t-s)^{-\beta}u(s)ds,$$

is the Riemann–Liouville fractional derivative of u of order $0 < \beta < 1$ *provided the integral converges.*

Definition 5 ([11])**.** *Let* $c, d > 0$ *with* $c < d$, $\sigma > 0$, $\beta \in \mathbb{R}^+$ *and* $m \in \mathbb{N}$ *satisfying* $m - 1 < \beta < m$. *Let* $u : [c, d] \to \mathbb{R}$ *be a* C^m*-function. Then,*

$$ {}^{C}\mathcal{D}^{\beta,\sigma}_{c^+} u(t) = \mathcal{I}^{\beta,\sigma}_{c^+} \left(t^{1-\sigma} \frac{d}{dt} \right)^m u(t) = \frac{\sigma^{1-m+\beta}}{\Gamma(m-\beta)} \int_c^t s^{\sigma-1} (t^\sigma - s^\sigma)^{\beta-1} \left(t^{1-\sigma} \frac{d}{ds} \right)^m u(s) ds $$

is the left-sided Caputo–Katugampola fractional derivative of u of order β *and parameter* σ.

For $\sigma = 1$ and $m = 1$, we define the Liouville–Caputo fractional derivative as follows.

Definition 6. *Given* $c, d > 0$ *with* $c < d$, $\beta \in \mathbb{R}^+$ *so that* $0 < \beta < 1$, *and a* C^1*-function* $u : [c, d] \to \mathbb{R}$. *The left-sided Liouville–Caputo fractional derivative of order* β *is given by*

$$ {}^{LC}_{c}\mathcal{D}^{\beta}_{t} u(t) = \frac{1}{\Gamma(1-\beta)} \int_c^t (t-s)^{-\beta} u'(s) ds, $$

provided that the integral converges. Similarly, the right-sided Liouville–Caputo fractional derivative of order γ *is given by*

$$ {}^{LC}_{t}\mathcal{D}^{\gamma}_{d} u(t) = \frac{1}{\Gamma(1-\beta)} \int_t^d (s-t)^{-\gamma} u'(s) ds, $$

provided that the integral converges.

Lemma 1 ([12])**.** *The Liouville–Caputo derivative is connected to the Riemann–Liouville derivatives by*

$$ {}^{LC}_{c}\mathcal{D}^{\beta}_{t} u(t) = {}_{c}\mathcal{D}^{\beta}_{t} u(t) - \sum_{k=0}^{m-1} \frac{(t-c)^{k-\beta}}{\Gamma(k-\beta+1)} u^{(k)}(c), \; m - 1 < \beta \leq m. $$

In particular, when $m = 1$, *one obtains*

$$ {}^{LC}_{c}\mathcal{D}^{\beta}_{t} u(t) = {}_{c}\mathcal{D}^{\beta}_{t} u(t) - \frac{(t-c)^{-\beta}}{\Gamma(1-\beta)} u(c). $$

Suppose the function $u : [c, d] \to \mathbb{R}$ *is defined as* $u(t) = t^a$. *Then,*

$$ {}_{c}\mathcal{D}^{\beta}_{t} u(t) = \frac{\Gamma(a+1)}{\Gamma(a-\beta+1)} t^{a-\beta}. $$

Theorem 1 ([11])**.** *Suppose* $u \in C^m[c, d]$, *then for* $m - 1 < \beta \leq m$, $0 < \sigma \leq 1$

$$ \mathcal{I}^{\beta,\sigma}_{c^+} {}^{C}\mathcal{D}^{\beta,\sigma}_{c^+} u(t) = u(t) - \sum_{k=0}^{m-1} \frac{\sigma^{-k}}{k!} (t^\sigma - c^\sigma)^k u_{(k)}(c). $$

Note that if $u \in C^1[c, d]$ *and* $\sigma = 1$, *then*

$$ \mathcal{I}^{\beta}_{c^+} \, {}^{LC}\mathcal{D}^{\beta}_{c^+} u(t) = u(t) - u(c). $$

Thus,

Corollary 1. *Suppose* $u \in C^1[c, d]$, $0 < \beta \leq 1$ *and* $\sigma = 1$, *we have*

$$ {}_{c}\mathcal{I}^{\beta}_{t} \, {}^{LC}_{c}\mathcal{D}^{\beta}_{t} u(t) = u(t) - u(c). \tag{4} $$

The next result is a formula for the generalized fractional integration by parts:

Theorem 2 ([11]). *Suppose $u \in C[c,d]$ and $v \in C^m[c,d]$. Then, for $m-1<\beta \le m$, $0<\sigma\le 1$,*

$$\begin{aligned}\int_c^d u(t)^C\mathcal{D}_{c^+}^{\beta,\sigma}v(t)dt &= \int_c^d t^{\sigma-1}v(t)\mathcal{D}_{d^-}^{\beta,\sigma}(t^{1-\sigma})u(t)dt \\ &+ \left[\sum_{j=0}^{m-1}\left(-t^{1-\sigma}\frac{d}{dt}\right)^j \mathcal{I}_{d^-}^{m-\beta,\sigma}(t^{1-\beta}u(t))v_{(m-j-1)}(t)\right]_{t=c}^{t=d}.\end{aligned}$$

Particularly for $m=1$ and $\sigma=1$, we have

$$\int_c^d u(t)^{LC}\mathcal{D}_{c^+}^{\beta}v(t)dt = \int_c^d v(t)\mathcal{D}_{d^-}^{\beta}u(t)dt.$$

Therefore,

Corollary 2. *Suppose $u \in C[c,d]$ and $v \in C^1[c,d]$, then for $0<\gamma\le 1$ and $\sigma=1$,*

$$\int_c^d u(t)\,{}_t^{LC}\mathcal{D}_d^{\gamma}v(t)dt = \int_c^d v(t)\,{}_c\mathcal{D}_t^{\gamma}u(t)dt. \tag{5}$$

Formulation of the Solution

Here, we apply the properties or relationship between the fractional integral and fractional differential operators in Equation (4) to make sense of the solution to problem (2).

Lemma 2. *Let $\eta \in (0,1]$. Then, the solution to fractional differential Equation (2) is defined as*

$$Q(t) = q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)}\int_c^t (t-s)^{\beta-\gamma-1}Q(s)ds + \frac{\lambda}{\eta\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}\vartheta(s,Q(s))ds.$$

Proof. The application of the integral operator ${}_c\mathcal{I}_t^{\beta}$ to Equation (2) on both sides gives

$${}_c\mathcal{I}_t^{\beta}\left[\,{}^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}Q(t)\right] = \lambda\,{}_c\mathcal{I}_t^{\beta}[\vartheta(t,Q(t))].$$

That is,

$${}_c\mathcal{I}_t^{\beta}\left[\eta\,{}_c^{LC}\mathcal{D}_t^{\beta}Q(t) + (1-\eta)\,{}_t^{LC}\mathcal{D}_d^{\gamma}Q(t)\right] = \lambda\,{}_c\mathcal{I}_t^{\beta}[\vartheta(t,Q(t))].$$

By the linearity of the operator ${}_c\mathcal{I}_t^{\beta}$, we have

$$\eta\,{}_c\mathcal{I}_t^{\beta}\,{}_c^{LC}\mathcal{D}_t^{\beta}Q(t) + (1-\eta)\,{}_c\mathcal{I}_t^{\beta}\,{}_t^{LC}\mathcal{D}_d^{\gamma}Q(t) = \lambda\,{}_c\mathcal{I}_t^{\beta}[\vartheta(t,Q(t))].$$

From Equation (4) in Corollary 1 and Equation (3), we get

$$\eta[Q(t)-q_c] + \frac{1-\eta}{\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}\,{}_s^{LC}\mathcal{D}_d^{\gamma}Q(s)ds = \frac{\lambda}{\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}\vartheta(s,Q(s))ds.$$

By Equation (5) in Corollary 2, we obtain

$$\eta[Q(t)-q_c] + \frac{1-\eta}{\Gamma(\beta)}\int_c^t Q(s)\,{}_c\mathcal{D}_s^{\gamma}[(t-s)^{\beta-1}]ds = \frac{\lambda}{\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}\vartheta(s,Q(s))ds. \tag{6}$$

The derivative ${}_c\mathcal{D}_s^{\gamma}\left((t-s)^{\beta-1}\right)$ is evaluated as,

$${}_c\mathcal{D}_s^{\gamma}\left((t-s)^{\beta-1}\right) = -\frac{\Gamma(\beta-1+1)}{\Gamma(\beta-1-\gamma+1)}(t-s)^{\beta-\gamma-1} = -\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}(t-s)^{\beta-\gamma-1}.$$

From Equation (6), one gets

$$\eta[Q(t) - q_c] - \frac{1-\eta}{\Gamma(\beta-\gamma)} \int_c^t (t-s)^{\beta-\gamma-1} Q(s) ds = \frac{\lambda}{\Gamma(\beta)} \int_c^t (t-s)^{\beta-1} \vartheta(s, Q(s)) ds.$$

Thus, for $\eta \in (0,1]$, one obtains

$$Q(t) = q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)} \int_c^t (t-s)^{\beta-\gamma-1} Q(s) ds + \frac{\lambda}{\eta\Gamma(\beta)} \int_c^t (t-s)^{\beta-1} \vartheta(s, Q(s)) ds.$$

□

The following sup-norm on Q defined by

$$\|Q\| := \sup_{c \le t \le d} |Q(t)|,$$

is useful in the next section.

3. Main Results

The following global Lipschitz condition on the function $\vartheta(., Q)$ with respect to the second variable is important for establishing our main result.

Condition 1. *Suppose* $0 < \text{Lip}_\vartheta < \infty$, $t \in [c, d]$ *and* $\forall$ Q, $R \in \mathbb{R}$, *we assume*

$$|\vartheta(t, Q) - \vartheta(t, R)| \le \text{Lip}_\vartheta |Q - R|. \tag{7}$$

For convenience, we shall set $\vartheta(t, 0) = 0$ *in our computations.*

3.1. Well-Posedness

Here, we apply the Banach's fixed-point theorem to prove the existence and uniqueness of the solution to our problem (2). We begin by defining the operator

$$\mathcal{A}Q(t) = q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)} \int_c^t (t-s)^{\beta-\gamma-1} Q(s) ds + \frac{\lambda}{\eta\Gamma(\beta)} \int_c^t (t-s)^{\beta-1} \vartheta(s, Q(s)) ds, \tag{8}$$

and show that $\mathcal{A}$ has a unique fixed point which gives the solution to problem (2).

Lemma 3. *Suppose* Q *is a solution to problem* (2) *and Condition 1 holds. Then, for* $0 < \gamma < \beta < 1$ *and* $\eta \in (0,1]$, *we have*

$$\|\mathcal{A}Q\| \le q_c + c_1 \|Q\|, \tag{9}$$

with positive constant

$$c_1 := \left[\frac{1-\eta}{\eta\Gamma(\beta-\gamma+1)} (d-c)^{\beta-\gamma} + \frac{\lambda \text{Lip}_\vartheta}{\eta\Gamma(\beta+1)} (d-c)^\beta \right] < \infty.$$

Proof. Taking the absolute value of Equation (8) leads to

$$|\mathcal{A}Q(t)| \le q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)} \int_c^t (t-s)^{\beta-\gamma-1} |Q(s)| ds + \frac{\lambda}{\eta\Gamma(\beta)} \int_c^t (t-s)^{\beta-1} |\vartheta(s, Q(s))| ds.$$

Applying Equation (7) of Condition 1, we have

$$\begin{aligned}|\mathcal{A}Q(t)| &\leq q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)}\int_c^t (t-s)^{\beta-\gamma-1}|Q(s)|ds + \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}|Q(s)|ds \\ &\leq q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)}\|Q\|\int_c^t (t-s)^{\beta-\gamma-1}ds + \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta)}\|Q\|\int_c^t (t-s)^{\beta-1}ds \\ &= q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)}\|Q\|\frac{(t-c)^{\beta-\gamma}}{\beta-\gamma} + \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta)}\|Q\|\frac{(t-c)^{\beta}}{\beta} \\ &= q_c + \left[\frac{1-\eta}{\eta\Gamma(\beta-\gamma+1)}(t-c)^{\beta-\gamma} + \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta+1)}(t-c)^{\beta}\right]\|Q\|.\end{aligned}$$

Next, we take the supremum over $t \in [c,d]$ to get

$$\|\mathcal{A}Q\| \leq q_c + \left[\frac{1-\eta}{\eta\Gamma(\beta-\gamma+1)}(d-c)^{\beta-\gamma} + \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta+1)}(d-c)^{\beta}\right]\|Q\|,$$

and the result is obtained. □

Lemma 4. *Let Condition 1 hold. Suppose Q_1 and Q_2 are two solutions of (2); for $0<\gamma<\beta<1$ and $\eta \in (0,1]$, we have*

$$\|\mathcal{A}Q_1 - \mathcal{A}Q_2\| \leq c_1\|Q_1 - Q_2\|. \tag{10}$$

Proof. Proceeding with similar steps as in the proof of Lemma 3, we arrive at the desired result. □

Theorem 3. *Suppose Condition 1 holds, $0<\gamma<\beta<1$ and $\eta \in (0,1]$. There exists a constant $0<c_1<1$ such that Equation (2) possesses a unique solution.*

Proof. Applying the Banach fixed point theorem gives $Q(t) = \mathcal{A}Q(t)$. It follows from Equation (9) in Lemma 3 that

$$\|Q\| = \|\mathcal{A}Q\| \leq q_c + c_1\|Q\|.$$

This gives $\|Q\|[1-c_1] \leq q_c$ and therefore, $\|Q\| < \infty$ whenever $c_1 < 1$.

Next, suppose for a contradiction that $Q_1 \neq Q_2$ are solutions of Equation (2). Then, from Equation (10) of Lemma 4, we have

$$\|Q_1 - Q_2\| = \|\mathcal{A}Q_1 - \mathcal{A}Q_2\| \leq c_1\|Q_1 - Q_2\|.$$

Therefore, $\|Q_1 - Q_2\|[1-c_1] \leq 0$. However, $1-c_1>0$, therefore $\|Q_1 - Q_2\| < 0$. This is a contradiction, hence $\|Q_1 - Q_2\| = 0$. By the contraction principle, the existence and uniqueness result follows. □

3.2. Upper Growth Bound

Agarwal et al. [13] presented the following retarded Gronwall type inequality:

$$u(t) \leq f(t) + \sum_{i=1}^{n}\int_{r_i(t_0)}^{r_i(t)} h_i(t,s)w_i(u(s))ds, \;\; t_0 \leq t < t_1. \tag{11}$$

Theorem 4 (Theorem 2.1 of [13])**.** *Assume the hypotheses of (Theorem 2.1 of [13]) hold and $u(t)$ is a non-negative continuous function on $[t_0,t_1)$ satisfying (11). Then,*

$$u(t) \leq \mathrm{X}_n^{-1}\left[\mathrm{X}_n(q_n(t)) + \int_{r_n(t_0)}^{r_n(t)} \max_{t_0\leq\tau\leq t} h_n(\tau,s)ds\right], \;\; t_0 \leq t \leq T_1,$$

where $q_n(t)$ *is determined recursively by*

$$q_1(t) := f(t_0) + \int_{t_0}^{t} |f'(s)|ds,$$

$$q_{i+1} := X_i^{-1}\left[X_i(q_i(t)) + \int_{r_i(t_0)}^{r_i(t)} \max_{t_0 \le \tau \le t} h_i(\tau, s)ds\right],\ i = 1, ..., n-1,$$

and $X_i(\tau, \tau_i) := \int_{\tau_i}^{\tau} \frac{d\omega}{w_i(\omega)}$.

Remark 1. *For the case* $n = 2$, *if*

$$u(t) \le f(t) + \int_{r_1(t_0)}^{r_1(t)} h_1(t, s)w_1(u(s))ds + \int_{r_2(t_0)}^{r_2(t)} h_2(t, s)w_2(u(s))ds,$$

then

$$u(t) \le X_2^{-1}\left[X_2(q_2(t)) + \int_{r_2(t_0)}^{r_2(t)} \max_{t_0 \le \tau \le t} h_2(\tau, s)ds\right],$$

with $q_2(t) = X_1^{-1}\left[X_1(r_1(t)) + \int_{r_1(t_0)}^{r_1(t)} \max_{t_0 \le \tau \le t} h_1(\tau, s)ds\right]$.
Here, we take $w_i(u(s)) = u(s)$, $r_i(t_0) = t_0 = c$, *and* $r_i(t) = t$ *for* $i = 1, 2$.

Consequently, we present the upper growth bound estimate for the solution.

Theorem 5. *Assume Condition 1 holds. Then,* $\forall\ t \in [c, d]$, $0 < c < d$, *and* $c_2, c_3 > 0$, *we have*

$$|Q(t)| \le \frac{q_c}{\exp\left(c_2(c-t)^{\beta-\gamma} + c_3(c-t)^{\beta}\right)},$$

with $c_2 = \frac{1-\eta}{\eta\Gamma(\beta-\gamma+1)}$, $c_3 = \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta+1)}$, *for* $\eta \in (0, 1]$, $0 < \gamma < \beta < 1$.

Proof. We already obtained from Lemma 3 that

$$|Q(t)| \le q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)} \int_c^t (t-s)^{\beta-\gamma-1}|Q(s)|ds + \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta)} \int_c^t (t-s)^{\beta-1}|Q(s)|ds.$$

Let $g(t) := |Q(t)|$, $t \in [c, d]$; it follows that

$$g(t) \le\ q_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)} \int_c^t (t-s)^{\beta-\gamma-1}g(s)ds + \frac{\lambda \mathrm{Lip}_\vartheta}{\eta\Gamma(\beta)} \int_c^t (t-s)^{\beta-1}g(s)ds. \quad (12)$$

Applying Theorem 4 to (12), we take $w_i(\omega) = \omega$ for $i = 1, 2$ and it follows that

$$X_2(\tau, \tau_2) = \int_{\tau_2}^{\tau} \frac{d\omega}{\omega} = \ln\left(\frac{\tau}{\tau_2}\right).$$

If one takes $\tau_2 = 1$ for convenience, then $X_2(\tau) = \ln\tau$ with the inverse $X_2^{-1}(\tau) = e^{\tau}$. Similarly, $X_1(\tau) = \ln\tau$ has the inverse $X_1^{-1}(\tau) = e^{\tau}$. Moreover, $f(t) = q_c$ and $f'(t) = 0$, hence $q_1(t) = q_c$. Now, we define the non-negative functions h_1, $h_2 : [c, d] \times [c, d] \to \mathbb{R}_+$ as follows:

$$h_1(\zeta, s) := \frac{1-\eta}{\eta\Gamma(\beta-\gamma)}(\zeta-s)^{\beta-\gamma-1},$$

and

$$h_2(\zeta, s) := \frac{\lambda \text{Lip}_\vartheta}{\eta \Gamma(\beta)} (\zeta - s)^{\beta - 1}.$$

Let $c \leq s < \zeta$; since $\beta - \gamma - 1 < 0$, then h_1 is decreasing and continuous, thus

$$\max_{c \leq \tau \leq t} h_1(\zeta, s) = \frac{1 - \eta}{\eta \Gamma(\beta - \gamma)} (c - s)^{\beta - \gamma - 1},$$

and it follows that

$$\begin{aligned} q_2(t) &= \exp\left[\ln(q_c) + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)} \int_c^t (c-s)^{\beta-\gamma-1} ds\right] \\ &= \exp\left[\ln(q_c) - \frac{1-\eta}{\eta\Gamma(\beta-\gamma)} \frac{(c-t)^{\beta-\gamma}}{\beta-\gamma}\right] \\ &= \exp\left[\ln(q_c) - \frac{1-\eta}{\eta\Gamma(\beta-\gamma+1)} (c-t)^{\beta-\gamma}\right]. \end{aligned}$$

In addition, for $c \leq s < \zeta$, and for all $\beta < 1$, h_2 is decreasing and continuous, and

$$\max_{c \leq \zeta \leq t} h_2(\zeta, s) = \frac{\lambda \text{Lip}_\vartheta}{\eta \Gamma(\beta)} (c - s)^{\beta - 1}.$$

Thus,

$$\begin{aligned} g(t) &\leq \exp\left[\ln(q_2(t)) + \frac{\lambda \text{Lip}_\vartheta}{\eta\Gamma(\beta)} \int_c^t (c-s)^{\beta-1} ds\right] \\ &= \exp\left[\ln(q_c) - \frac{1-\eta}{\eta\Gamma(\beta-\gamma+1)} (c-t)^{\beta-\gamma} - \frac{\lambda \text{Lip}_\vartheta}{\eta\Gamma(\beta)} \frac{(c-t)^\beta}{\beta}\right] \\ &= q_c \exp\left[-\frac{1-\eta}{\eta\Gamma(\beta-\gamma+1)} (c-t)^{\beta-\gamma} - \frac{\lambda \text{Lip}_\vartheta}{\eta\Gamma(\beta+1)} (c-t)^\beta\right], \end{aligned}$$

and this completes the proof. □

3.3. Asymptotic Behaviours

Here, we show that the solution exhibits some asymptotic properties. By the growth bound in Theorem 5, we have

$$|Q(t)| \leq q_c \exp\left[-\frac{1}{\eta}\left(\frac{1-\eta}{\Gamma(\beta-\gamma+1)} (c-t)^{\beta-\gamma} + \frac{\lambda \text{Lip}_\vartheta}{\Gamma(\beta+1)} (c-t)^\beta\right)\right].$$

Now, taking the limit as $\eta \to 0$, we obtain

$$\lim_{\eta \to 0} |Q(t)| = 0.$$

Next, taking limit as $t \to c^+$, we get

$$\lim_{t \to c^+} |Q(t)| \leq q_c.$$

4. Examples

The example below illustrates Theorem 3 as follows: For $\beta = \frac{9}{10}$, $\gamma = \frac{1}{10}$, $\eta = \frac{1}{2}$, and the nonlinear Lipschitz continuous function $\vartheta : [0.01, 0.05] \times \mathbb{R} \to \mathbb{R}$ defined by

$\vartheta(t, Q(t)) = \sin(Q(t))$ having a Lipschitz constant $\text{Lip}_\vartheta = 1$, the Combined Liouville–Caputo fractional differential equation

$$\begin{cases} {}^{LC}\mathcal{D}_{\frac{1}{2}}^{\frac{9}{10},\frac{1}{10}} Q(t) = \lambda \sin(Q(t)), & 0.01 < t \leq 0.05, \\ Q(0.01) = q_c, \end{cases}$$

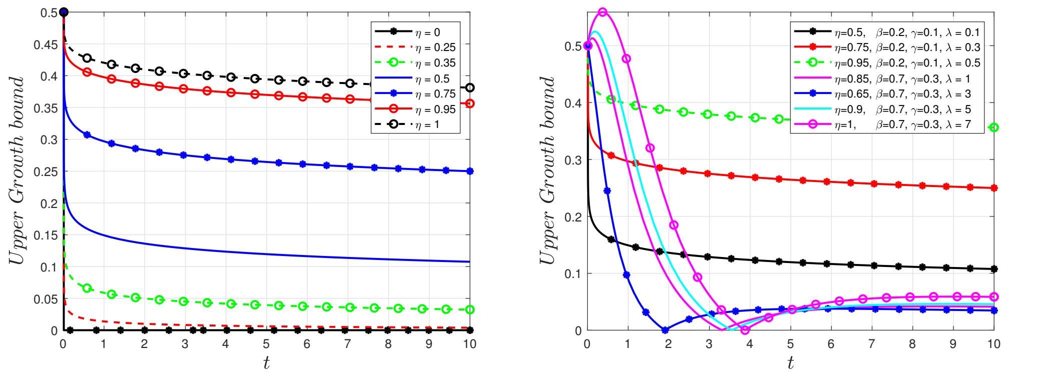
has a unique solution provided by

$$c_1 := \frac{(1-\frac{1}{2})}{\frac{1}{2}\eta(\frac{9}{10}-\frac{1}{10}+1)}(0.04)^{0.8} + \frac{\lambda}{\frac{1}{2}\eta(\frac{9}{10}+1)}(0.04)^{0.9} < 1,$$

if and only if $c_1 = 0.081756 + \frac{0.0551892\lambda}{0.480883} < 1$. That is, for all λ such that $0 < \lambda < 8.00099$.

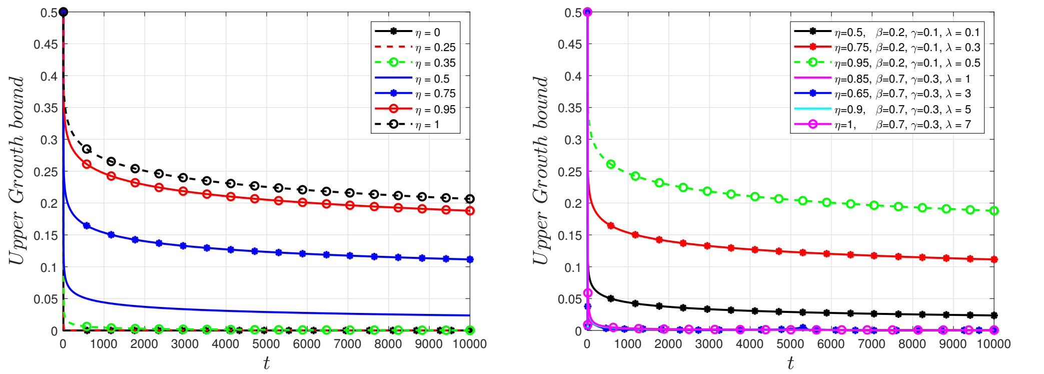
Numerical Comparisons

We present numerical simulations and different plots of the upper growth bound functions $q_c \exp\left[-\frac{1}{\eta}\left(\frac{1-\eta}{\Gamma(\beta-\gamma+1)}(c-t)^{\beta-\gamma} + \frac{\lambda \text{Lip}_\vartheta}{\eta(\beta+1)}(c-t)^\beta\right)\right]$ and compare their behaviours for various values of parameters β, γ, η and λ over different time intervals. See Figures 1–6 below.



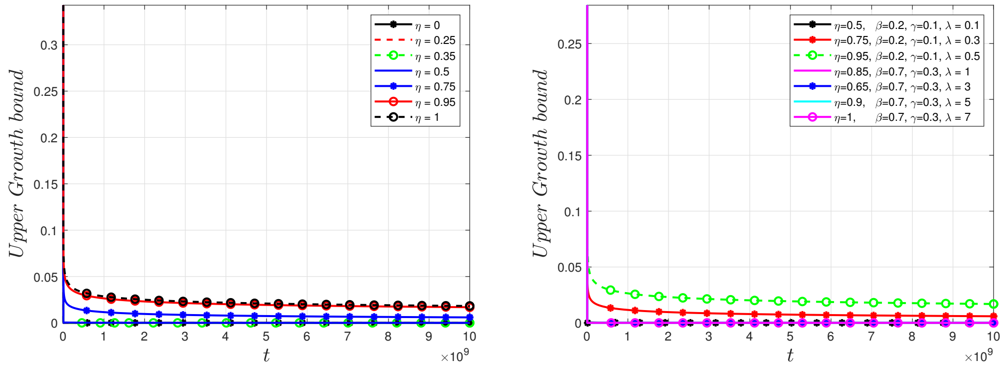
(**a**) (**b**)

Figure 1. Behaviour of the upper growth bound for different η and λ values: (**a**) $0.01 \leq t \leq 10$, $\beta = 0.2,\ \gamma = 0.1,\ \lambda = 1$; (**b**) $0.01 \leq t \leq 10$.



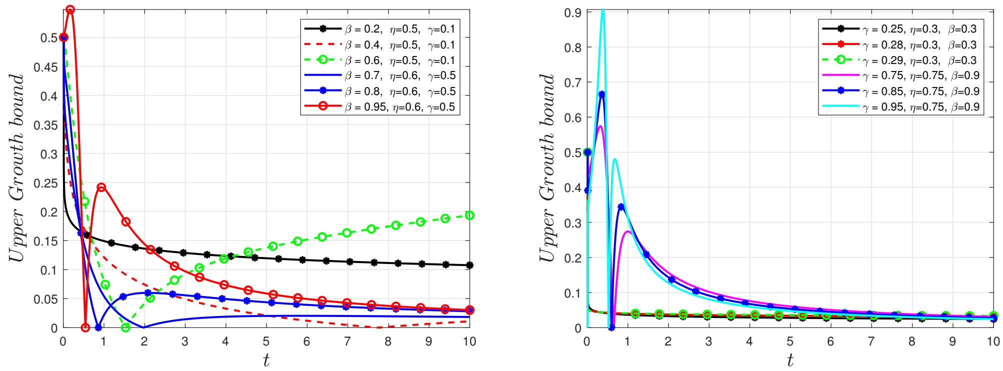
(**a**) (**b**)

Figure 2. Behaviour of the upper growth bound for different η and λ values: (**a**) $0.01 \leq t \leq 10^4$, $\beta = 0.2,\ \gamma = 0.1,\ \lambda = 1$; (**b**) $0.01 \leq t \leq 10^4$.



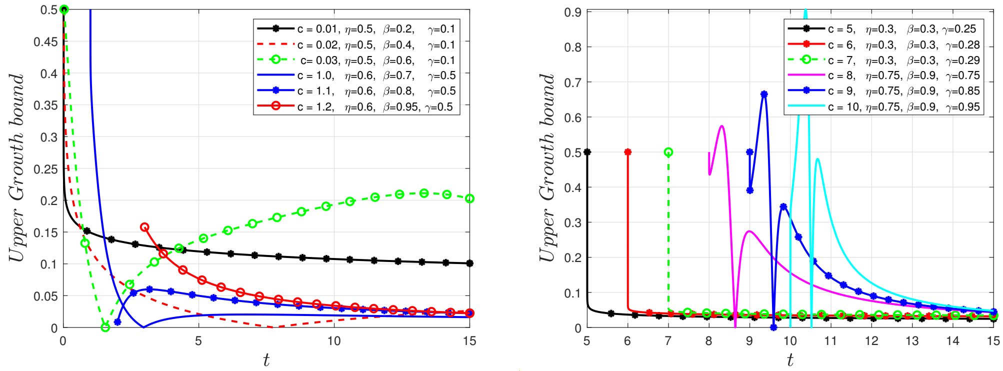
(**a**) (**b**)

Figure 3. Behaviour of the upper growth bound for different η and λ values: (**a**) $0.01 \leq t \leq 10^{10}$, $\beta = 0.2,\ \gamma = 0.1,\ \lambda = 1$; (**b**) $0.01 \leq t \leq 10^{10}$.



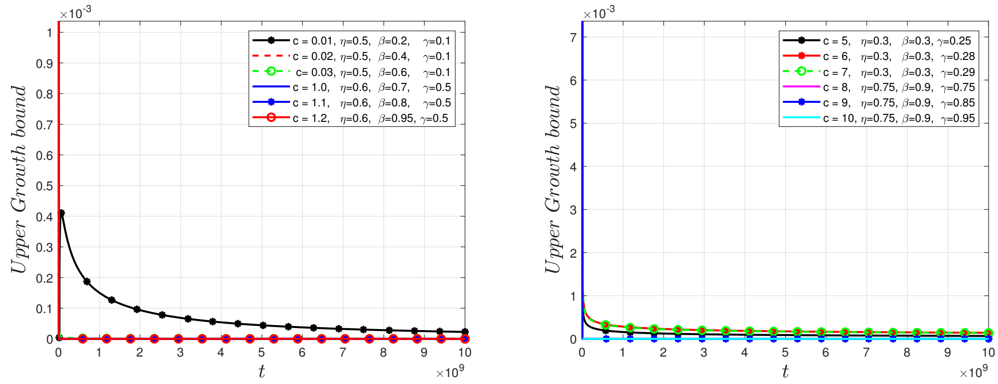
(**a**) (**b**)

Figure 4. Behaviour of the upper growth bound for different η, β and γ values: (**a**) $0.01 \leq t \leq 10,\ \lambda = 1$; (**b**) $0.01 \leq t \leq 10,\ \lambda = 1$.



(**a**) (**b**)

Figure 5. Behaviour of the upper growth bound for different initial points $t = c$: (**a**) $c \leq t \leq 15,\ \lambda = 1$; (**b**) $c \leq t \leq 15,\ \lambda = 1$.



(**a**) (**b**)

Figure 6. Behaviour of the upper growth bound for different initial points $t = c$: (**a**) $c \leq t \leq 10^{10}$, $\lambda = 1$; (**b**) $c \leq t \leq 10^{10}$, $\lambda = 1$.

5. Stochastic Combined Fractional Differential Equation

In this section, we study the effect of an external force (a noise term) on the growth behaviour of Equation (2). Thus, we perturb the combined L–C fractional differential equation with a multiplicative noise term $\dot{w}(t)$ known as the generalized derivative of a Wiener process $w(t)$ (a Gaussian white noise process) and consider the following stochastic combined L–C fractional differential equation:

$$^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}\Phi(t) = \lambda\vartheta(t,\Phi(t))\dot{w}(t),\ \ \beta,\gamma \in (0,1]\ \&\ \eta \in [0,1], \tag{13}$$

where $\Phi(c) = \rho_c$ is the initial condition, $\vartheta : [c,d] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz in the second variable, $^{LC}\mathcal{D}_{\eta}^{\beta,\gamma}$ is known as the combined Liouville–Caputo fractional derivative operator, $\dot{w}(t)$ is the noise term and $\lambda > 0$ denotes the level of the noise term. For recent work on stochastic fractional differential equations, see [14,15] and their references.

Following the formulation of solution in Lemma 2, the solution of Equation (13) is given by

$$\Phi(t) = \rho_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)}\int_c^t (t-s)^{\beta-\gamma-1}\Phi(s)ds + \frac{\lambda}{\eta\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}\vartheta(s,\Phi(s))dw(s). \tag{14}$$

5.1. Well-Posedness of the Solution to Equation (13)

We define the norm of the solution (14) above by

$$\|\Phi\|_2^2 := \sup_{c\leq t\leq d} \mathrm{E}|\Phi(t)|^2$$

and define the operator as follows:

$$\mathcal{L}\Phi(t) = \rho_c + \frac{1-\eta}{\eta\Gamma(\beta-\gamma)}\int_c^t (t-s)^{\beta-\gamma-1}\Phi(s)ds + \frac{\lambda}{\eta\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}\vartheta(s,\Phi(s))dw(s). \tag{15}$$

We show that the fixed point of the operator $\mathcal{L}$ gives the solution to Equation (13).

Lemma 5. *Suppose Φ is a solution to problem (13) and Condition 1 holds. Then, for $0 < \gamma < \beta < 1$ and $\eta \in (0,1]$, we have*

$$\|\mathcal{L}\Phi\|_2^2 \leq 3\rho_c^2 + c_4\|\Phi\|_2^2,$$

with positive constant

$$c_4 := \left[\frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)[2(\beta-\gamma)-1]}(d-c)^{2(\beta-\gamma)} + \frac{3\lambda^2\mathrm{Lip}_{\vartheta}^2}{\eta^2\Gamma^2(\beta)[2\beta-1]}(d-c)^{2\beta-1}\right] < \infty.$$

Proof. Take the second moment of Equation (15) to get

$$\mathrm{E}|\mathcal{L}\Phi(t)|^2 \leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\mathrm{E}\left|\int_c^t (t-s)^{\beta-\gamma-1}\Phi(s)ds\right|^2 + \frac{3\lambda^2}{\eta^2\Gamma^2(\beta)}\mathrm{E}\left|\int_c^t (t-s)^{\beta-1}\vartheta(s,\Phi(s))dw(s)\right|^2.$$

Applying Holder's inequality on the first integral and Itô isometry on the second integral, we obtain

$$\mathrm{E}|\mathcal{L}\Phi(t)|^2 \leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\mathrm{E}\left[\left(\int_c^t (t-s)^{2(\beta-\gamma-1)}ds\right)^{\frac{1}{2}}\left(\int_c^t |\Phi(s)|^2 ds\right)^{\frac{1}{2}}\right]^2 + \frac{3\lambda^2}{\eta^2\Gamma^2(\beta)}\int_c^t (t-s)^{2(\beta-1)}\mathrm{E}|\vartheta(s,\Phi(s))|^2 ds.$$

Next, we apply Equation (7) of Lipschitz Condition 1 to arrive at

$$\begin{aligned}
\mathrm{E}|\mathcal{L}\Phi(t)|^2 &\leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\int_c^t (t-s)^{2(\beta-\gamma-1)}ds\int_c^t \mathrm{E}|\Phi(s)|^2 ds \\
&\quad + \frac{3\lambda^2\mathrm{Lip}_\vartheta}{\eta^2\Gamma^2(\beta)}\int_c^t (t-s)^{2(\beta-1)}\mathrm{E}|\Phi(s)|^2 ds \\
&\leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\int_c^t (t-s)^{2(\beta-\gamma-1)}ds\|\Phi\|_2^2\int_c^t 1ds \\
&\quad + \frac{3\lambda^2\mathrm{Lip}_\vartheta}{\eta^2\Gamma^2(\beta)}\|\Phi\|_2^2\int_c^t (t-s)^{2(\beta-1)}ds \\
&= 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\frac{(t-c)^{2(\beta-\gamma)-1}}{2(\beta-\gamma)-1}\|\Phi\|_2^2(t-c) + \frac{3\lambda^2\mathrm{Lip}_\vartheta}{\eta^2\Gamma^2(\beta)}\|\Phi\|_2^2\frac{(t-c)^{2\beta-1}}{2\beta-1} \\
&= 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\frac{(t-c)^{2(\beta-\gamma)}}{2(\beta-\gamma)}\|\Phi\|_2^2 + \frac{3\lambda^2\mathrm{Lip}_\vartheta}{\eta^2\Gamma^2(\beta)}\|\Phi\|_2^2\frac{(t-c)^{2\beta-1}}{2\beta-1}.
\end{aligned}$$

For $0 < \gamma < \beta < 1$ such that $\beta > \frac{1}{2}$ and $\beta - \gamma > \frac{1}{2}$, we take the supremum over $t \in [c,d]$ to arrive at

$$\|\mathcal{L}\Phi\|_2^2 \leq 3\rho_c^2 + \left[\frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\frac{(d-c)^{2(\beta-\gamma)}}{2(\beta-\gamma)} + \frac{3\lambda^2\mathrm{Lip}_\vartheta}{\eta^2\Gamma^2(\beta)}\frac{(d-c)^{2\beta-1}}{2\beta-1}\right]\|\Phi\|_2^2,$$

and the result follows. □

Similarly, we obtain the following result:

Lemma 6. *Let Condition 1 hold. Suppose Φ and Θ are two solutions of (13); for $0 < \gamma < \beta < 1$ and $\eta \in (0,1]$, we have*

$$\|\mathcal{L}\Phi - \mathcal{L}\Theta\|_2^2 \leq c_4\|\Phi - \Theta\|_2^2.$$

Applying Lemmas 5 and 6, we obtain the following existence and uniqueness result:

Theorem 6. *Suppose Condition 1 holds, $0 < \gamma < \beta < 1$ and $\eta \in (0,1]$. There exists a constant $0 < c_4 < 1$ such that Equation (13) possesses a unique solution.*

5.2. Growth Moment Bound

Here, we state and prove the second moment growth estimate for the solution to Equation (13):

Theorem 7. *Assume Condition 1 holds. Then,* $\forall\, t \in [c,d]$*,* $0 < c < d$ *and* $c_5, c_6 > 0$*, we have*

$$\mathrm{E}|\Phi(t)|^2 \leq \frac{3\rho_c^2}{\exp\left(c_5(c-t)^{2(\beta-\gamma)-1} + c_6(c-t)^{2\beta-1}\right)},$$

with $c_5 = \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)\left(2(\beta-\gamma)-1\right)}$*,* $c_6 = \frac{3\lambda^2\mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)(2\beta-1)}$*, for* $\eta \in (0,1]$*,* $0 < \gamma < \beta < 1$*.*

Proof. Following the proof of Lemma 5, we obtain that

$$\mathrm{E}|\Phi(t)|^2 \leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\mathrm{E}\left| \int_c^t (t-s)^{\beta-\gamma-1}\Phi(s)ds \right|^2 + \frac{3\lambda^2}{\eta^2\Gamma^2(\beta)}\mathrm{E}\left| \int_c^t (t-s)^{\beta-1}\vartheta(s,\Phi(s))dw(s) \right|^2.$$

According to Holder's inequality and Itô isometry on the first and second integrals, respectively, together with Equation (7) of Condition 1, one arrives at

$$\begin{aligned}
\mathrm{E}|\Phi(t)|^2 &\leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}\mathrm{E}\left[\left(\int_c^t (t-s)^{2(\beta-\gamma-1)}|\Phi(s)|^2 ds\right)^{1/2}\left(\int_c^t 1ds\right)^{1/2}\right]^2 \\
&\quad + \frac{3\lambda^2\mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}\int_c^t (t-s)^{2(\beta-1)}\mathrm{E}|\Phi(s)|^2 ds \\
&= 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}(t-c)\int_c^t (t-s)^{2(\beta-\gamma-1)}\mathrm{E}|\Phi(s)|^2 ds \\
&\quad + \frac{3\lambda^2\mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}\int_c^t (t-s)^{2(\beta-1)}\mathrm{E}|\Phi(s)|^2 ds \\
&\leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}(d-c)\int_c^t (t-s)^{2(\beta-\gamma-1)}\mathrm{E}|\Phi(s)|^2 ds \\
&\quad + \frac{3\lambda^2\mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}\int_c^t (t-s)^{2(\beta-1)}\mathrm{E}|\Phi(s)|^2 ds.
\end{aligned}$$

The last line follows since $t \in [c,d]$. Let $g(t) := \mathrm{E}|\Phi(t)|^2$, $t \in [c,d]$; it follows that

$$g(t) \leq 3\rho_c^2 + \frac{3(1-\eta)^2}{\eta^2\Gamma^2(\beta-\gamma)}(d-c)\int_c^t (t-s)^{2(\beta-\gamma-1)}g(s)ds + \frac{3\lambda^2\mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}\int_c^t (t-s)^{2(\beta-1)}g(s)ds. \quad (16)$$

Next, we apply the retarded Gronwall-type inequality of Theorem 4 on Equation (16) and follow the proof of Theorem 5. Consequently, the non-negative functions h_1, h_2 : $[c,d] \times [c,d] \to \mathbb{R}_+$ are defined as follows:

$$h_1(\zeta,s) := \frac{3(1-\eta)^2(d-c)}{\eta^2\Gamma^2(\beta-\gamma)}(\zeta-s)^{2(\beta-\gamma-1)},$$

and

$$h_2(\zeta,s) := \frac{3\lambda^2\mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}(\zeta-s)^{2(\beta-1)}.$$

For $c \leq s < \zeta$ and $\beta-\gamma-1 < 0$, it follows that h_1 is continuous and decreasing and one obtains

$$\max_{c\leq\tau\leq t} h_1(\zeta,s) = \frac{3(1-\eta)^2(d-c)}{\eta^2\Gamma^2(\beta-\gamma)}(c-s)^{2(\beta-\gamma-1)}.$$

This gives (for $\beta - \gamma > \frac{1}{2}$),

$$\begin{aligned} q_2(t) &= \exp\left[\ln(3\rho_c^2) + \frac{3(1-\eta)^2(d-c)}{\eta^2\Gamma^2(\beta-\gamma)}\int_c^t (c-s)^{2(\beta-\gamma-1)}ds\right] \\ &= \exp\left[\ln(3\rho_c^2) - \frac{3(1-\eta)^2(d-c)}{\eta^2\Gamma^2(\beta-\gamma)}\frac{(c-t)^{2(\beta-\gamma)-1}}{2(\beta-\gamma)-1}\right]. \end{aligned}$$

Similarly, for h_2, we have that, given $c \le s < \zeta$ and $\beta < 1$, h_2 is decreasing, continuous and

$$\max_{c\le\zeta\le t} h_2(\zeta, s) = \frac{3\lambda^2 \mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}(c-s)^{2(\beta-1)}.$$

Then, it follows that (for $\beta > \frac{1}{2}$ and $\beta - \gamma > \frac{1}{2}$)

$$\begin{aligned} g(t) &\le \exp\left[\ln(q_2(t)) + \frac{3\lambda^2 \mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}\int_c^t (c-s)^{2(\beta-1)}ds\right] \\ &= \exp\left[\ln(3\rho_c^2) - \frac{3(1-\eta)^2(d-c)}{\eta^2\Gamma^2(\beta-\gamma)}\frac{(c-t)^{2(\beta-\gamma)-1}}{2(\beta-\gamma)-1} - \frac{3\lambda^2 \mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}\frac{(c-t)^{2\beta-1}}{2\beta-1}\right] \\ &= 3\rho_c^2 \exp\left[-\frac{3(1-\eta)^2(d-c)}{\eta^2\Gamma^2(\beta-\gamma)}\frac{(c-t)^{2(\beta-\gamma)-1}}{2(\beta-\gamma)-1} - \frac{3\lambda^2 \mathrm{Lip}_\vartheta^2}{\eta^2\Gamma^2(\beta)}\frac{(c-t)^{2\beta-1}}{2\beta-1}\right], \end{aligned}$$

and this proves the result. □

Remark 2. *The mild solution in Equation* (14) *also satisfies the same asymptotic properties of Section 3.3. Since*

$$\mathrm{E}|\Phi(t)|^2 \le 3\rho_c^2 \exp\left[-\frac{3}{\eta^2}\left(\frac{(1-\eta)^2(d-c)}{\Gamma^2(\beta-\gamma)}\frac{(c-t)^{2(\beta-\gamma)-1}}{2(\beta-\gamma)-1} + \frac{\lambda^2 \mathrm{Lip}_\vartheta^2}{\Gamma^2(\beta)}\frac{(c-t)^{2\beta-1}}{2\beta-1}\right)\right],$$

then, taking limits as $\eta \to 0$ and $t \to c^+$, we obtain the following: $\lim_{\eta\to 0} \mathrm{E}|\Phi(t)|^2 = 0$ and $\lim_{t\to c^+} \mathrm{E}|\Phi(t)|^2 \le 3\rho_c^2$.

6. Conclusions

The combined Liouville–Caputo fractional derivative operator plays a major role in describing a more general class of variational problems. We used this new operator to investigate the behaviour of the solution to a class of combined Liouville–Caputo fractional differential equations. The existence and uniqueness of the solution was established through Banach's fixed-point theorem. The solution's growth bound and asymptotic behaviours were also established. Moreover, in Figures 1–6, we presented a numerical simulation of the behaviour of the combined Liouville–Caputo fractional differential equations via the upper growth bound, with respect to different initial points $t = c$, different values β and γ of the fractional orders, different convex combination parameters η, as well as different values of λ. In general, the solution function Q of problem (2) decayed to zero; however, the speed of decay was determined by the values of the parameters η, β, γ and λ. We further investigated the effects of a noise term on the growth of the solution to the combined fractional differential equation and observed that the presence or introduction of the noise term does not affect the growth behaviour of the solution to the combined fractional differential equation. For future work, we will study the dependence of the solution(s) on the initial condition and will also estimate the lower bounds, etc.

Author Contributions: Conceptualization, M.E.O.; methodology, M.E.O., H.A.S., S.E.M., C.D.E. and T.A.-A.A.; software, C.D.E.; validation, M.E.O., H.A.S., S.E.M., C.D.E. and T.A.-A.A.; software, C.D.E.; formal analysis, M.E.O., H.A.S., S.E.M., C.D.E. and T.A.-A.A.; investigation, M.E.O., H.A.S., S.E.M., C.D.E. and T.A.-A.A.; writing—original draft preparation, M.E.O.; writing—review and editing, M.E.O., H.A.S., S.E.M., C.D.E. and T.A.-A.A.; project administration, M.E.O.; funding acquisition, M.E.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the University of Hafr Al Batin, Institutional Financial Program under project number IFP-A-2022-2-1-09.

Data Availability Statement: Not applicable.

Acknowledgments: The authors gratefully acknowledge the technical and financial support from the Agency for Research and Innovation, Ministry of Education and University of Hafr Al Batin, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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