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Boundary Value Problem for Impulsive Delay Fractional Differential Equations with Several Generalized Proportional Caputo Fractional Derivatives

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Abstract: A scalar nonlinear impulsive differential equation with a delay and generalized proportional Caputo fractional derivatives (IDGFDE) is investigated. The linear boundary value problem (BVP) for the given fractional differential equation is set up. The explicit form of the unique solution of BVP in the special linear case is obtained. This formula is a generalization of the explicit solution of the case without any delay as well as the case of Caputo fractional derivatives. Furthermore, this integral form of the solution is used to define a special proportional fractional integral operator applied to the determination of a mild solution of the studied BVP for IDGFDE. The relation between the defined mild solution and the solution of the BVP for the IDGFDE is discussed. The existence and uniqueness results for BVP for IDGFDE are proven. The obtained results in this paper are a generalization of several known results.

Keywords: generalized proportional Caputo fractional derivatives; delays; impulses; boundary value problem; integral presentation of the solution; existence; uniqueness

MSC: 34A34; 34K45; 34A08

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1. Introduction

Fractional differential equations play an important role in the modeling and development of many fields in science and engineering. In the literature, several different types of fractional derivatives are defined and applied (see, for example, Refs. [1,2] for Hilfer operators, Refs. [3–5] for derivatives depending on another function, and Refs. [6,7] for derivatives involving arbitrary kernels). In the last decade, the generalized proportional fractional derivative is defined [8,9] and studied (see, for example, for stability properties [10–12] and for stochastic fractional differential equations [13]).

In this paper, the boundary value problem (BVP) for nonlinear impulsive fractional differential equations with a delay and generalized proportional Caputo-type derivatives (GPCFD) is investigated. Initially, the linear case of a scalar impulsive differential equation with several GPCFDs is considered. The explicit proportional fractional integral form of the solution is obtained. As partial cases of the new formula, the BVP for a linear impulsive differential equation with one GPCFD with several Caputo fractional derivatives and one Caputo fractional derivative is considered and the integral form of the exact solutions is presented. These forms are generalizations of the solutions for the initial value problem of a linear fractional equation with and without impulses. Some of the partial cases coincide with the known results in the literature. The new integral form of the exact solution is applied to define a proportional fractional integral operator and the mild solution of the studied nonlinear problem. Existence and uniqueness results are provided. We use

Banach's fixed point theorem and the defined proportional fractional integral operator to prove the existence of a mild solution.

Note that the BVP for a couple of multi-term delay-generalized proportional Caputo fractional differential equations is studied in [14]. In this paper, impulses are involved in the differential equation. These impulses totally change the differential equation as well as the behavior of the solution. It is connected with the memory property of the fractional derivatives. We consider the case when the lower limit of the GPCFD is changed at any impulsive time (see [14]).

2. Basic Notes on Fractional Calculus

For a better understanding of the main results, we will provide the basic definitions and some known in the literature results which will be used in the proofs.

Let $\tau < T \leq \infty$ be given numbers and $y : [\tau, T] \to \mathbb{R}$.

The generalized proportional fractional integral (GPFI) is defined by (see [8,9])

$$({}_\tau\mathcal{I}^{\alpha,\rho}y)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-\sigma)}(t-\sigma)^{\alpha-1}y(\sigma)\,d\sigma, \quad t \in (\tau, T], \ \alpha \geq 0, \ \rho \in (0,1], \tag{1}$$

and the generalized Caputo proportional fractional derivative (GPCFD) is defined by

$$\begin{aligned}({}_\tau^C\mathcal{D}^{\alpha,\rho}y)(t) &= \frac{1-\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-\sigma)^{-\alpha}y(\sigma)\,d\sigma \\ &+ \frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-\sigma)}(t-\sigma)^{-\alpha}y'(\sigma)\,d\sigma \\ &\text{for } \ t \in (\tau, T], \ \alpha \in (0,1), \ \rho \in (0,1].\end{aligned} \tag{2}$$

Remark 1. *The GPFI and the GPCFD are generalizations of the fractional integral ${}_\tau I^\alpha y(t)$ and the Caputo fractional derivative ${}_\tau^C D^\alpha y(t)$, respectively, ($\rho = 1$ in (1)) (see, for example, [15]).*

We will provide some known results from the literature about GPFI and GPCFD which will be used later.

Lemma 1 (Proposition 3.7 [8])**.** *For $\rho \in (0,1]$, $\alpha, \gamma > 0$ we have*

$$\Big({}_\tau\mathcal{I}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}\cdot}(.\,-\tau)^{\gamma-1}\Big)(t) = \frac{\Gamma(\gamma)}{\rho^\alpha\Gamma(\gamma+\alpha)}e^{\frac{\rho-1}{\rho}t}(t-\tau)^{\alpha+\gamma-1}, \quad t > \tau.$$

Corollary 1. *Let $\rho \in (0,1]$, $\alpha > 0$. Then $\Big({}_\tau\mathcal{I}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}\cdot}\Big)(t) = \frac{1}{\rho^\alpha\Gamma(1+\alpha)}e^{\frac{\rho-1}{\rho}t}(t-\tau)^\alpha, \ \ t > \tau,$ holds.*

We will use the following partial case of proposition 5.2 [8].

Lemma 2. *For $\rho \in (0,1]$, $\alpha \in (0,1)$, $\gamma > 1$ we have*

$$\Big({}_\tau^C\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}\cdot}(.\,-\tau)^{\gamma-1}\Big)(t) = \frac{\rho^\alpha\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}e^{\frac{\rho-1}{\rho}t}(t-\tau)^{\gamma-1-\alpha}, \quad t > \tau,$$

and

$$\Big({}_\tau^C\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}\cdot}\Big)(t) = 0, \quad t > \tau.$$

Lemma 3 (Theorem 3.8 [8])**.** *Let $\rho \in (0,1]$, $\beta, \gamma > 0$ and $\nu \in C([\tau, T], \mathbb{R})$. Then*

$$\Big({}_\tau\mathcal{I}^{\beta,\rho}({}_\tau\mathcal{I}^{\gamma,\rho}\nu)\Big)(t) = ({}_\tau\mathcal{I}^{\beta+\gamma,\rho}\nu)(t), \ \ t \in (\tau, T].$$

Furthermore, consider the following partial case of Theorem 5.3 [8] for $\alpha \in (0,1)$.

Lemma 4. *For $\rho \in (0,1]$, $\alpha \in (0,1)$ and $\nu \in C^1[\tau, T]$, we have*

$$\left({}_\tau\mathcal{I}^{\alpha,\rho}\left({}_\tau^C\mathcal{D}^{\alpha,\rho}\nu\right)\right)(t) = \nu(t) - \nu(\tau)e^{\frac{\rho-1}{\rho}(t-\tau)},\ \ t \in (\tau, T].$$

Corollary 2 ([8])**.** *Let $\alpha \in (0,1)$, $\rho \in (0,1]$ and $\nu \in C[\tau, T]$. Then $({}_\tau^C\mathcal{D}^{\alpha,\rho}({}_\tau\mathcal{I}^{\alpha,\rho}\nu))(t) = \nu(t)$ for $t \in (\tau, T]$.*

3. Impulsive Linear Differential Equations with Caputo-Type Fractional Derivatives

Let the points $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$ be given, and $T < \infty$.

Let $\alpha \in (0,1),\ \rho \in (0,1]$. Consider the classes of functions

$$\begin{aligned}
PC[0,T] &= \Big\{y \in C(\cup_{k=0}^{p}(t_k, t_{k+1}], \mathbb{R}) :\ \lim_{t\to t_k,\ t>t_k} y(t) = y(t_k+0) < \infty, \\
&\qquad \lim_{t\to t_k, t<t_k} y(t) = y(t_k)\Big\}; \\
PC^1[0,T] &= \Big\{y \in PC[0,T] :\ y \in C^1((t_k, t_{k+1}], \mathbb{R}),\ k = 0,1,2,\ldots,p,\Big\}; \\
PC^{\alpha,\rho}[0,T] &= \Big\{y \in PC^1[0,T] :\ \ ({}_{t_k}\mathcal{D}^{\alpha,\rho}y)(t)\ \text{ exists for } t \in (t_k, t_{k+1}],\ k = 0,1,\ldots,p\Big\}; \\
PI^{\alpha,\rho}[0,T] &= \Big\{y \in PC[0,T] :\ \ ({}_{t_k}\mathcal{I}^{\alpha,\rho}y)(t)\ \text{ exists for } t \in (t_k, t_{k+1}],\ k = 0,1,\ldots,p\Big\}.
\end{aligned}$$

3.1. Generalized Proportional Caputo Fractional Derivatives

Let the sequence of numbers $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_N > 0$ be given.

Consider the linear impulsive fractional differential equation with several generalized proportional Caputo fractional derivatives (LIGPCDE)

$$\begin{aligned}
&\sum_{i=1}^{N} C_i({}_{t_k}^C\mathcal{D}^{\alpha_i,\rho}z)(t) = F(t),\ \text{for }\ t \in (t_k, t_{k+1}],\ k = 0,1,2,\ldots,p, \\
&z(t_k+0) = B_k z(t_k),\ k = 1,2,3,\ldots,p,
\end{aligned} \tag{3}$$

with the boundary condition

$$z(0) + \mu z(T) = \Psi, \tag{4}$$

where $\rho \in (0,1]$, $B_k \in \mathbb{R}, (k = 1,2,\ldots,p)$, $C_i \in \mathbb{R}, (i = 2,\ldots N),\ C_1 = 1$, $\mu \in \mathbb{R}$, $F : [0,T] \to \mathbb{R}$, $\Psi \in \mathbb{R}$ and

$$1 + \mu e^{\frac{\rho-1}{\rho}T} \prod_{m=0}^{p} B_m \sum_{j=1}^{N} \frac{C_j}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}(t_{m+1}-t_m)^{\alpha_1-\alpha_j} \neq 0. \tag{5}$$

Theorem 1. *Let the function $F \in PI^{\alpha_1,\rho}[0,T]$ and inequality* (5) *be fulfilled. Then the BVP for the LIGPCDE* (3) (4) *has a unique solution satisfying the integral presentation*

$$\begin{aligned}
z(t) = &\frac{\Psi M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu M_p(T)\prod_{m=0}^{p-1} M_m(t_{m+1})} + \frac{\mu M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu M_p(T)\prod_{m=0}^{p-1} M_m(t_{m+1})} \\
&\times \sum_{j=2}^{N} C_j\Big[M_p(T)\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1})\right)({}_{t_{p-m}}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t_{p-m+1}) \\
&+ ({}_{t_p}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(T)\Big]
\end{aligned} \tag{6}$$

$$
\begin{aligned}
&-\frac{\mu M_k(t)\prod_{m=0}^{k-1}M_m(t_{m+1})}{1+\mu M_p(T)\prod_{m=0}^{p-1}M_m(t_{m+1})}M_p(T)\\
&\quad\times\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1}M_{p-l}(t_{p-l+1})\right)({}_{t_{p-m}}\mathcal{I}^{\alpha_1,\rho}F)(t_{p-m+1})\\
&-\frac{\mu M_k(t)\prod_{m=0}^{k-1}M_m(t_{m+1})}{1+\mu M_p(T)\prod_{m=0}^{p-1}M_m(t_{m+1})}({}_{t_p}\mathcal{I}^{\alpha_1,\rho}F)(T)\\
&-\sum_{j=2}^{N}C_j\Bigg[M_k(t)\sum_{m=1}^{k}\left(\prod_{l=1}^{m-1}M_{k-l}(t_{k-l+1})\right)({}_{t_{k-m}}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t_{k-m+1})\\
&\quad+({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t)\Bigg]\\
&+M_k(t)\sum_{m=1}^{k}\left(\prod_{l=1}^{m-1}M_{k-l}(t_{k-l+1})\right)({}_{t_{k-m}}\mathcal{I}^{\alpha_1,\rho}F)(t_{k-m+1})\\
&+({}_{t_k}\mathcal{I}^{\alpha_1,\rho}F)(t)\ \textit{for}\quad t\in(t_k,t_{k+1}],\ k=0,1,2,\dots,,
\end{aligned}
$$

where

$$
A_k(t)=\sum_{j=1}^{N}\frac{C_j}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}(t-t_k)^{\alpha_1-\alpha_j},\ k=0,1,2,\dots,p,
$$

$$
M_k(t)=B_kA_k(t)e^{\frac{\rho-1}{\rho}(t-t_k)},\ k=0,1,2,\dots,p,\quad B_0=1.
$$

Proof. Let $t\in(t_k,t_{k+1}]$, $k=0,1,2,\dots,p$. Since $\alpha_1>\alpha_j, j=2,3,\dots,N$, we take a GPFI $({}_{t_k}\mathcal{I}^{\alpha_1,\rho}z)(t)$ from both sides of (3), use Lemma 1 with $\tau=t_k$, $\alpha=\alpha_1-\alpha_j$, Lemma 3, Lemma 4 and get for any $j=2,3,\dots,N$ the equalities

$$
\begin{aligned}
&\left({}_{t_k}\mathcal{I}^{\alpha_1,\rho}\left({}^{C}_{t_k}\mathcal{D}^{\alpha_j,\rho}z\right)\right)(t)=\left({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_j,\rho}\left({}^{C}_{t_k}\mathcal{D}^{\alpha_j,\rho}z\right)\right)\right)(t)\\
&=\left({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}\left(z(t)-z(t_k+0)e^{\frac{\rho-1}{\rho}(t-t_k)}\right)\right)\\
&=({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t)-z(t_k+0)(\,{}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}e^{\frac{\rho-1}{\rho}(t-t_k)})\\
&=({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t)-z(t_k+0)\frac{e^{\frac{\rho-1}{\rho}(t-t_k)}}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}(t-t_k)^{\alpha_1-\alpha_j}.
\end{aligned}
\tag{7}
$$

From (3) on $(t_k,t_{k+1}]$, Lemma 4, Corollary 1 with $\tau=t_k,\alpha=\alpha_1-\alpha_j$ and Equation (7) we get

$$
\begin{aligned}
z(t)&=z(t_k+0)e^{\frac{\rho-1}{\rho}(t-t_k)}-\sum_{j=2}^{N}C_j\Big(({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t)\\
&\quad-z(t_k+0)({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}e^{\frac{\rho-1}{\rho}(t-t_k)}\Big)+({}_{t_k}\mathcal{I}^{\alpha_1,\rho}F)(t)\\
&=z(t_k+0)e^{\frac{\rho-1}{\rho}(t-t_k)}A_k(t)-\sum_{j=2}^{N}C_j({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t)\\
&\quad+({}_{t_k}\mathcal{I}^{\alpha_1,\rho}F)(t),\quad t\in(t_k,t_{k+1}],\ k=0,1,2,\dots,p.
\end{aligned}
\tag{8}
$$

From (8) with $k=1$ we obtain

$$
z(t_1)=z(0)M_0(t_1)-\sum_{j=2}^{N}C_j({}_{t_0}\mathcal{I}^{\alpha_1-\alpha_j,\rho}z)(t_1)+({}_{t_0}\mathcal{I}^{\alpha_1,\rho}F)(t_1).
\tag{9}
$$

Apply Equation (9) to (8) with $k = 1$, use the impulsive condition for $k = 1$ in (3) to get

$$\begin{aligned} z(t) &= M_1(t)z(t_1) - \sum_{j=2}^{N} C_j ({}_{t_1}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t) + ({}_{t_1}\mathcal{I}^{\alpha_1,\rho} F)(t) \\ &= M_0(t_1)M_1(t)z(0) \\ &\quad - \sum_{j=2}^{N} C_j \big[M_1(t)({}_{t_0}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t_1) + ({}_{t_1}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t) \big] \\ &\quad + M_1(t)({}_{t_0}\mathcal{I}^{\alpha_1,\rho} F)(t_1) + ({}_{t_1}\mathcal{I}^{\alpha_1,\rho} F)(t), \ t \in (t_1, t_2]. \end{aligned} \tag{10}$$

From (8) with $k = 2$, (10) for $t = t_2$ and the impulsive condition for $k = 2$ in (3) we obtain

$$\begin{aligned} z(t) &= M_2(t)z(t_2) - \sum_{j=2}^{N} C_j ({}_{t_2}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t) + ({}_{t_2}\mathcal{I}^{\alpha_1,\rho} F)(t) \\ &= M_0(t_1)M_1(t_2)M_2(t)z(0) \\ &\quad - \sum_{j=2}^{N} C_j \Big[M_2(t)M_1(t_2)({}_{t_0}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t_1) + M_2(t)({}_{t_1}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t_2) \\ &\qquad + ({}_{t_2}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t) \Big] \\ &\quad + M_1(t_2)M_2(t)({}_{t_0}\mathcal{I}^{\alpha_1,\rho} F)(t_1) + M_2(t)({}_{t_1}\mathcal{I}^{\alpha_1,\rho} F)(t_2) \\ &\quad + ({}_{t_2}\mathcal{I}^{\alpha_1,\rho} F)(t), \quad t \in (t_2, t_3]. \end{aligned} \tag{11}$$

By induction with respect to the intervals between two consecutive impulses, we obtain

$$\begin{aligned} z(t) &= z(0)M_k(t) \prod_{m=0}^{k-1} M_m(t_{m+1}) \\ &\quad - \sum_{j=2}^{N} C_j \Big\{ \sum_{m=1}^{k} \Big(\big(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1}) \big) M_k(t) ({}_{t_{k-m}}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t_{k-m+1}) \Big) \\ &\qquad + ({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t) \Big\} \\ &\quad + \sum_{m=1}^{k} \Bigg(\Bigg(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1}) \Bigg) M_k(t)({}_{t_{k-m}}\mathcal{I}^{\alpha_1,\rho} F)(t_{k-m+1}) \Bigg) \\ &\quad + ({}_{t_k}\mathcal{I}^{\alpha_1,\rho} F)(t), \quad t \in (t_k, t_{k+1}], \ k = 0, 1, 2, \dots, p. \end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned} z(T) &= z(t_{p+1}) = z(0)M_p(T) \prod_{m=0}^{p-1} M_m(t_{m+1}) \\ &\quad - \sum_{j=2}^{N} C_j \Big[\sum_{m=1}^{p} \Big(\Big(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \Big) M_p(T)({}_{t_{p-m}}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t_{p-m+1}) \Big) \\ &\qquad + ({}_{t_p}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(T) \Big] \\ &\quad + \sum_{m=1}^{p} \Big(\Big(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \Big) M_p(T)({}_{t_{p-m}}\mathcal{I}^{\alpha_1,\rho} F)(t_{p-m+1}) \Big) \\ &\quad + ({}_{t_p}\mathcal{I}^{\alpha_1,\rho} F)(T). \end{aligned} \tag{13}$$

From Equation (13) and the boundary condition (3) we get

$$
\begin{aligned}
z(0) = & \frac{\Psi}{1+\mu M_p(T)\prod_{m=0}^{p-1} M_m(t_{m+1})} + \frac{\mu}{1+\mu M_p(T)\prod_{m=0}^{p-1} M_m(t_{m+1})} \\
& \times \sum_{j=2}^{N} C_j \Big[\sum_{m=1}^{p} \bigg(\Big(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \Big) M_p(T) ({}_{t_{p-m}}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t_{p-m+1}) \bigg) \\
& + ({}_{t_p}\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(T) \Big] \\
& - \frac{\mu}{1+\mu M_p(T)\prod_{m=0}^{p-1} M_m(t_{m+1})} \sum_{m=1}^{p} \bigg(\Big(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \Big) \\
& \times M_p(T) ({}_{t_{p-m}}\mathcal{I}^{\alpha_1,\rho} F)(t_{p-m+1}) \bigg) \\
& - \frac{\mu}{1+\mu M_p(T)\prod_{m=0}^{p-1} M_m(t_{m+1})} ({}_{t_p}\mathcal{I}^{\alpha_1,\rho} F)(T).
\end{aligned} \tag{14}
$$

Substitute equality (14) in (12) and obtain (7). □

Case 3.1.1. *Generalized proportional Caputo fractional derivatives and no impulses.*

Let us consider the partial case of (3) without impulses, i.e., $t_0 = t_1 = t_2 = \cdots = t_p$. Thus, consider the BVP for the scalar linear differential equation with generalized proportional Caputo fractional derivatives (LGPCDE)

$$
\sum_{i=1}^{N} C_i ({}_0^C\mathcal{D}^{\alpha_i,\rho} z)(t) = F(t), \text{ for } t \in (0,T], \tag{15}
$$

with the boundary condition (4). Then as a partial case of Equation (7) it follows the solution of BVP for LGPCDE (15), (4) is

$$
\begin{aligned}
z(t) = & \frac{\Psi M(t)}{1+\mu M(T)} + \frac{\mu M(t)}{1+\mu M(T)} \sum_{j=2}^{N} C_j ({}_0\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(T) - \frac{\mu M(t)}{1+\mu M(T)} ({}_0\mathcal{I}^{\alpha_1,\rho} F)(T) \\
& - \sum_{j=2}^{N} C_j ({}_0\mathcal{I}^{\alpha_1-\alpha_j,\rho} z)(t) + ({}_0\mathcal{I}^{\alpha_1,\rho} F)(t), \quad t \in (0,T],
\end{aligned} \tag{16}
$$

where $M(t) = e^{\frac{\rho-1}{\rho}t} \sum_{j=1}^{N} \frac{C_j}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)} t^{\alpha_1-\alpha_j}$, with

$$
1 + \mu e^{\frac{\rho-1}{\rho}T} \sum_{j=1}^{N} \frac{C_j}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)} T^{\alpha_1-\alpha_j} \neq 0.
$$

(Compare with [14] with $\gamma = 1, \beta = 0, \Psi(\xi) \equiv const$).

3.2. Caputo Fractional Derivatives

In this section, we will use the fractional integral and the fractional derivative defined in Remark 1.

Let $\alpha \in (0,1)$. Define the classes of functions

$$
\begin{aligned}
PCC^{\alpha}[0,T] &= \Big\{ \nu \in PC^1[0,T]) : \ ({}_{t_k}D^{\alpha}\nu)(t) \text{ exists for } t \in (t_k, t_{k+1}],\ k = 0,1,\dots,p \Big\}, \\
PCI^{\alpha}[0,T] &= \Big\{ \nu \in PC[0,T] : \ ({}_{t_k}I^{\alpha}\nu)(t) \text{ exists for } t \in (t_k, t_{k+1}],\ k = 0,1,\dots,p \Big\}.
\end{aligned}
$$

We will study the following linear impulsive fractional differential equation with several Caputo fractional derivatives (LICDE)

$$\begin{aligned}
&\sum_{i=1}^{N} C_i \ {}_{t_k}^{C}D^{\alpha_i,\rho} z(t) = F(t), \text{ for } t \in (t_k, t_{k+1}], \ k = 0,1,2,\dots,p, \\
&z(t_k + 0) = B_k z(t_k), \ k = 1,2,3,\dots,p,
\end{aligned} \tag{17}$$

where ${}_{t_k}^{C}D^{\alpha_i,\rho}u(t)$ is the Caputo fractional derivative (see Remark 1), $B_k \in \mathbb{R}, (k = 1,2,\dots,p)$, $C_i \in \mathbb{R}, (i = 2,\dots N), \ C_1 = 1, \mu \in \mathbb{R}, F : [0,T] \to \mathbb{R}$ and

$$1 + \mu \prod_{m=0}^{p} B_m \sum_{j=1}^{N} \frac{C_j}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)} (t_{m+1} - t_m)^{\alpha_1-\alpha_j} \neq 0. \tag{18}$$

Theorem 2. *Let $F \in PCI^{\alpha_1}[0,T]$ and inequality (18) holds. Then the BVP for the LICDE (17) (4) has a unique solution*

$$\begin{aligned}
z(t) = {}& \frac{\Psi M_k(t) \prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu M_p(T) \prod_{m=0}^{p-1} M_m(t_{m+1})} + \frac{\mu M_k(t) \prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu M_p(T) \prod_{m=0}^{p-1} M_m(t_{m+1})} \\
& \times \sum_{j=2}^{N} C_j \Bigg[M_p(T) \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) {}_{t_{p-m}}I^{\alpha_1-\alpha_j} z(t_{p-m+1}) \\
& \quad + {}_{t_p}I^{\alpha_1-\alpha_j} z(T) \Bigg] \\
& - \frac{\mu M_k(t) \prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu M_p(T) \prod_{m=0}^{p-1} M_m(t_{m+1})} M_p(T) \\
& \quad \times \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) {}_{t_{p-m}}I^{\alpha_1} F(t_{p-m+1}) \\
& - \frac{\mu M_k(t) \prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu M_p(T) \prod_{m=0}^{p-1} M_m(t_{m+1})} {}_{t_p}I^{\alpha_1} F(T) \\
& - \sum_{j=2}^{N} C_j \Bigg[M_k(t) \sum_{m=1}^{k} \left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1}) \right) {}_{t_{k-m}}I^{\alpha_1-\alpha_j} z(t_{k-m+1}) \\
& \quad + {}_{t_k}I^{\alpha_1-\alpha_j} z(t) \Bigg] \\
& + M_k(t) \sum_{m=1}^{k} \left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1}) \right) {}_{t_{k-m}}I^{\alpha_1} F(t_{k-m+1}) \\
& + {}_{t_k}I^{\alpha_1} F(t), \quad t \in (t_k, t_{k+1}], \ k = 0,1,2,\dots,p,
\end{aligned} \tag{19}$$

where $M_k(t) = B_k \sum_{j=1}^{N} \frac{C_j}{\Gamma(1+\alpha_1-\alpha_j)} (t - t_k)^{\alpha_1-\alpha_j}, \ k = 0,1,2,\dots,p, \quad B_0 = 1.$

Case 3.2.1. *Several Caputo fractional derivatives and no impulses*

Consider the partial case of (17) without impulses, i.e., $t_0 = t_1 = t_2 = \cdots = t_p$. Thus, consider the scalar linear differential equation with Caputo fractional derivatives (LCDE)

$$\sum_{i=1}^{N} C_i \ {}_{0}^{C}D^{\alpha_i,\rho} z(t) = F(t) \quad \text{for } t \in (0,T], \tag{20}$$

where $C_i \in \mathbb{R}, (i = 2,\dots N), \ C_1 = 1, \mu \in \mathbb{R}, F : [0,T] \to \mathbb{R}$.

As a partial case of Equation (19) the solution of BVP for LCDE (20) (4) is

$$\begin{aligned} z(t) &= \frac{\Psi M(t)}{1+\mu M(T)} + \frac{\mu M(t)}{1+\mu M(T)} \sum_{j=2}^{N} C_j \ {}_0I^{\alpha_1-\alpha_j} z(T) \\ &- \frac{\mu M(t)}{1+\mu M(T)} \ {}_0I^{\alpha_1} F(T) - \sum_{j=2}^{N} C_j \ {}_0I^{\alpha_1-\alpha_j} z(t) + {}_0I^{\alpha_1} F(t), \ \ t \in (0,T], \end{aligned} \tag{21}$$

where $M(t) = \sum_{j=1}^{N} \frac{C_j}{\Gamma(1+\alpha_1-\alpha_j)} t^{\alpha_1-\alpha_j}$ with $1 + \mu \sum_{j=1}^{N} \frac{C_j}{\Gamma(1+\alpha_1-\alpha_j)} T^{\alpha_1-\alpha_j} \neq 0$.

Case 3.2.2. *One Caputo fractional derivative and no impulses*

Consider the partial case of Equation (20) without impulses, i.e., $t_0 = t_1 = t_2 = \cdots = t_p$ and $C_k = 0,\ k = 2,3,\ldots,N$. We will study the scalar linear Caputo fractional differential equation

$$ {}_0^C D^{\alpha} z(t) = F(t) \quad \text{for} \ \ t \in (0,T], \tag{22}$$

where $\alpha \in (0,1)$, $F : [0,T] \to \mathbb{R}$

As a partial case of Equation (21), the solution of the BVP (22) (4) is

$$ z(t) = \frac{\Psi}{1+\mu} - \frac{\mu}{1+\mu} \frac{1}{\Gamma(\alpha)} \int_0^T \frac{F(s)}{(T-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{F(s)}{(t-s)^{1-\alpha}} ds, \qquad t \in (0,T], \tag{23}$$

where $1 + \mu \neq 0$.

Case 3.2.3. *One Caputo fractional derivative and no impulses—Initial value problem*

Consider the scalar linear Caputo fractional differential Equation (22) with boundary condition (4) with $\mu = 0$. Then the solution of (22) with the initial condition $z(0) = \Psi$ is given by

$$ z(t) = \Psi + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{F(s)}{(t-s)^{1-\alpha}} ds. $$

(see, for example, [15]).

3.3. One GPCFD and Impulses

Consider the following impulsive linear generalized proportional Caputo fractional differential equation (IGPFDE)

$$\begin{aligned} &({}_{t_k}^C\mathcal{D}^{\alpha,\rho} z)(t) = F(t), \ \text{for} \ \ t \in (t_k, t_{k+1}], \ k = 0,1,2,\ldots,p, \\ &z(t_k + 0) = B_k z(t_k), \ k = 1,2,3,\ldots,p, \end{aligned} \tag{24}$$

where $\alpha \in (0,1),\ \rho \in (0,1]$, $B_k \in \mathbb{R}$, $(k = 1,2,\ldots,p)$, $\mu \in \mathbb{R}$, $F : [0,T] \to \mathbb{R}$ and

$$ 1 + \mu e^{\frac{\rho-1}{\rho}(T-t_0)} \prod_{m=0}^{p} B_m \neq 0. \tag{25}$$

As a partial case of Theorem 1 and Formula (7) with $C_i = 0,\ i = 2,3,\ldots,N$, we obtain

Theorem 3. *Let $F \in PI^{\alpha,\rho}[0,T]$ and inequality (25) be satisfied. Then the BVP for the IGPFDE (24) (4) has an unique solution given by*

$$\begin{aligned} z(t) &= \frac{\Psi e^{\frac{\rho-1}{\rho}(t-t_0)} \prod_{m=0}^{k} B_m}{1 + \mu e^{\frac{\rho-1}{\rho}(T-t_0)} \prod_{m=0}^{p} B_m} + \frac{\mu e^{\frac{\rho-1}{\rho}(t-t_0)} \prod_{m=0}^{k} B_m}{1 + \mu e^{\frac{\rho-1}{\rho}(T-t_0)} \prod_{m=0}^{p} B_m} ({}_{t_p}\mathcal{I}^{\alpha,\rho} F)(T) \\ &- \frac{\mu e^{\frac{\rho-1}{\rho}(t-t_0)} \prod_{m=0}^{k} B_m}{1 + \mu e^{\frac{\rho-1}{\rho}(T-t_0)} \prod_{m=0}^{p} B_m} \end{aligned} \tag{26}$$

$$\times \sum_{m=1}^{p}\left(\left(\prod_{l=0}^{m-1} B_{p-l}\right) e^{\frac{\rho-1}{\rho}(T-t_{p-m+1})}({}_{t_{p-m}}\mathcal{I}^{\alpha,\rho}F)(t_{p-m+1})\right)$$
$$+\sum_{m=1}^{k}\left(\left(\prod_{l=0}^{m-1} B_{k-l}\right) e^{\frac{\rho-1}{\rho}(t-t_{k-m+1})}({}_{t_{k-m}}\mathcal{I}^{\alpha,\rho}F)(t_{k-m+1})\right)$$
$$+({}_{t_k}\mathcal{I}^{\alpha,\rho}F)(t), \quad t\in(t_k,t_{k+1}],\ k=0,1,2,\dots,p.$$

Case 3.3.1. *GPCFD and no impulse*

We will study the partial case of Equation (24) without impulses, i.e., $t_0 = t_1 = t_2 = \cdots = t_p$. Thus, consider the scalar linear generalized proportional Caputo fractional differential equation (GPFDE)

$$({}_0^C\mathcal{D}^{\alpha_i,\rho}z)(t) = F(t), \quad t\in(0,T], \tag{27}$$

where $\mu\in\mathbb{R}$, $F:[0,T]\to\mathbb{R}$.

As a partial case of Equation (27) the solution of BVP for GPFDE (27), (4) is given by

$$\begin{aligned} z(t) =& \frac{\Psi e^{\frac{\rho-1}{\rho}t}}{1+\mu e^{\frac{\rho-1}{\rho}T}} - \frac{\mu e^{\frac{\rho-1}{\rho}t}}{1+\mu e^{\frac{\rho-1}{\rho}T}}\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^T e^{\frac{\rho-1}{\rho}(T-s)}\frac{F(s)}{(T-s)^{1-\alpha}}ds \\ &+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{F(s)}{(t-s)^{1-\alpha}}ds, \quad t\in(0,T], \end{aligned} \tag{28}$$

where $1+\mu e^{\frac{\rho-1}{\rho}T}\neq 0$.

Case 3.3.2. *GPCFD and no impulse—Initial value problem*

Consider GPFDE (27) with the boundary condition (4) with $\mu = 0$. Then as a partial case of (28) the GPFDE (27) with the initial condition $z(0) = \Psi$ has a solution

$$z(t) = \Psi e^{\frac{\rho-1}{\rho}t} + \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{F(s)}{(t-s)^{1-\alpha}}ds,$$

(see, Example 5.7 [8] with $\lambda = 0$).

4. Nonlinear Impulsive Delay Differential Equations with Several GPCFDE

Let the numbers $1>\alpha_1>\alpha_2>\cdots>\alpha_N>0$ and the points $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=T$ be given.

Consider the nonlinear impulsive delay differential equation with several GPCDE (NIGPDE)

$$\begin{aligned} &\sum_{i=1}^{N} C_i({}_{t_k}^C\mathcal{D}^{\alpha_i,\rho}x)(t) = f(t,x(t),x(\lambda t)), \text{ for } t\in(t_k,t_{k+1}],\ k=0,1,\dots,p, \\ &x(t_k+0) = B_k x(t_k), \quad k=1,2,\dots,p, \end{aligned} \tag{29}$$

with the boundary value condition (4), where $\lambda\in(0,1)$, $C_i, i = 2,\dots N$, $C_1 = 1$, $B_i\in\mathbb{R}$, $i=1,2,3,\dots,p$, $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$.

4.1. Mild Solution of the BVP for NIGPDE

Based on the integral form (7) of the solution of BVP for LIGPCDE (3) (4) we define the fractional integral operator $\Omega : \cup_{k=2}^{N} PI^{\alpha_1-\alpha_k,\rho}[0,T] \to PC[0,T]$ by the equality

$$
\begin{aligned}
(\Omega x)(t) = {} & \frac{\Psi M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p} M_m(t_{m+1})} \\
& + \frac{\mu M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p} M_m(t_{m+1})} \sum_{j=2}^{N} C_j \Bigg[M_p(T) \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) \\
& \quad \times \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \int_{t_{p-m}}^{t_{p-m+1}} e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)} \frac{x(s)}{(t_{p-m+1}-s)^{1-\alpha_1+\alpha_j}} ds \\
& + \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \int_{t_p}^{T} e^{\frac{\rho-1}{\rho}(T-s)} \frac{x(s)}{(T-s)^{1-\alpha_1+\alpha_j}} ds \Bigg] \\
& - \frac{\mu M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p} M_m(t_{m+1})} M_p(T) \\
& \quad \times \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_{t_{p-m}}^{t_{p-m+1}} e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)} \frac{f(s,x(s),x(\lambda s))}{(t_{p-m+1}-s)^{1-\alpha_1}} ds \\
& - \frac{\mu M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p} M_m(t_{m+1})} \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_{t_p}^{T} e^{\frac{\rho-1}{\rho}(T-s)} \frac{f(s,x(s),x(\lambda s))}{(T-s)^{1-\alpha_1}} ds \\
& - \sum_{j=2}^{N} C_j \Bigg[M_k(t) \sum_{m=1}^{k} \left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1}) \right) \\
& \quad \times \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \int_{t_{k-m}}^{t_{k-m+1}} e^{\frac{\rho-1}{\rho}(t_{k-m+1}-s)} \frac{x(s)}{(t_{k-m+1}-s)^{1-\alpha_1+\alpha_j}} ds \\
& + \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \int_{t_k}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{x(s)}{(t-s)^{1-\alpha_1+\alpha_j}} ds \Bigg] \\
& + M_k(t) \sum_{m=1}^{k} \left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1}) \right) \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_{t_{k-m}}^{t_{k-m+1}} e^{\frac{\rho-1}{\rho}(t_{k-m+1}-s)} \frac{f(s,x(s),x(\lambda s))}{(t_{k-m+1}-s)^{1-\alpha_1}} ds \\
& + \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_{t_k}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{f(s,x(s),x(\lambda s))}{(t-s)^{1-\alpha_1}} ds, \\
& t \in (t_k, t_{k+1}],\ k = 0,1,2,\dots,p,
\end{aligned} \tag{30}
$$

where

$$M_0(t) = e^{\frac{\rho-1}{\rho}t} \sum_{j=1}^{N} \frac{C_j}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)} t^{\alpha_1-\alpha_j}, \tag{31}$$

and

$$M_k(t) = B_k e^{\frac{\rho-1}{\rho}(t-t_k)} \sum_{j=1}^{N} \frac{C_j}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)} (t-t_k)^{\alpha_1-\alpha_j},\ k = 1,2,\dots,p. \tag{32}$$

Definition 1. *The function $x \in PI^{\alpha_1-\alpha_k,\rho}[0,T], k = 2,3,\dots,N$, is a mild solution of the BVP for NIGPDE (29) (4) if it is a fixed point of the operator Ω, defined by (30).*

Theorem 4. *Let the inequality (5) hold and $x(t),\ t \in [0,T]$, be a mild solution of the BVP for NIGPDE (29) (4) such that $x \in PC^{\alpha_k,\rho}[0,T]$ for $k = 1,2,\dots,N$. Then the function $x(t)$ is a solution to the same problem.*

Proof. From Equation (30) applying $M_0(0) = 1$ and the following two equalities

$$
\begin{aligned}
x(T) &= \frac{\Psi \prod_{m=0}^{p} M_m(t_{m+1})}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} \\
&+ \left(\frac{\mu \prod_{m=0}^{p} M_m(t_{m+1})}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} - 1 \right) \sum_{j=2}^{N} C_j M_p(T) \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) \\
&\quad \times \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \int_{t_{p-m}}^{t_{p-m+1}} e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)} \frac{x(s)}{(t_{p-m+1}-s)^{1-\alpha_1+\alpha_j}} ds \\
&+ \left(\frac{\mu \prod_{m=0}^{p} M_m(t_{m+1})}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} - 1 \right) \sum_{j=2}^{N} C_j \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \\
&\quad \times \int_{t_p}^{T} e^{\frac{\rho-1}{\rho}(T-s)} \frac{x(s)}{(T-s)^{1-\alpha_1+\alpha_j}} ds \\
&- \left(\frac{\mu \prod_{m=0}^{p} M_m(t_{m+1})}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} - 1 \right) M_p(T) \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \\
&\quad \times \int_{t_{p-m}}^{t_{p-m+1}} e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)} \frac{f(s,x(s),x(\lambda s))}{(t_{p-m+1}-s)^{1-\alpha_1}} ds \\
&- \left(\frac{\mu \prod_{m=0}^{p} M_m(t_{m+1})}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} - 1 \right) \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_{t_p}^{T} e^{\frac{\rho-1}{\rho}(T-s)} \frac{f(s,x(s),x(\lambda s))}{(T-s)^{1-\alpha_1}} ds,
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
x(0) &= \frac{\Psi}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} \\
&+ \frac{\mu}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} \sum_{j=2}^{N} C_j M_p(T) \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) \\
&\quad \times \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \int_{t_{p-m}}^{t_{p-m+1}} e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)} \frac{x(s)}{(t_{p-m+1}-s)^{1-\alpha_1+\alpha_j}} ds \\
&+ \frac{\mu}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} \sum_{j=2}^{N} C_j \frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)} \int_{t_p}^{T} e^{\frac{\rho-1}{\rho}(T-s)} \frac{x(s)}{(T-s)^{1-\alpha_1+\alpha_j}} ds \\
&- \frac{\mu}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} M_p(T) \sum_{m=1}^{p} \left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1}) \right) \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \\
&\quad \times \int_{t_{p-m}}^{t_{p-m+1}} e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)} \frac{f(s,x(s),x(\lambda s))}{(t_{p-m+1}-s)^{1-\alpha_1}} ds \\
&- \frac{\mu}{1+\mu \prod_{m=0}^{p} M_m(t_{m+1})} \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_{t_p}^{T} e^{\frac{\rho-1}{\rho}(T-s)} \frac{f(s,x(s),x(\lambda s))}{(T-s)^{1-\alpha_1}} ds,
\end{aligned}
\tag{34}
$$

we obtain

$$
\begin{aligned}
x(0)+\mu x(T) &= \frac{\Psi}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}+\mu\frac{\Psi\prod_{m=0}^{p}M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})} \\
&+\frac{\mu}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}\sum_{j=2}^{N}C_jM_p(T)\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1}M_{p-l}(t_{p-l+1})\right) \\
&\quad\times\frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)}\int_{t_{p-m}}^{t_{p-m+1}}e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)}\frac{x(s)}{(t_{p-m+1}-s)^{1-\alpha_1+\alpha_j}}ds \\
&+\mu\left(\frac{\mu\prod_{m=0}^{p}M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}-1\right)\sum_{j=2}^{N}C_jM_p(T)\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1}M_{p-l}(t_{p-l+1})\right) \\
&\quad\times\frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)}\int_{t_{p-m}}^{t_{p-m+1}}e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)}\frac{x(s)}{(t_{p-m+1}-s)^{1-\alpha_1+\alpha_j}}ds \\
&+\frac{\mu}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}\sum_{j=2}^{N}C_j\frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)}\int_{t_p}^{T}e^{\frac{\rho-1}{\rho}(T-s)}\frac{x(s)}{(T-s)^{1-\alpha_1+\alpha_j}}ds \\
&+\mu\left(\frac{\mu\prod_{m=0}^{p}M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}-1\right)\sum_{j=2}^{N}C_j\frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(\alpha_1-\alpha_j)}\int_{t_p}^{T}e^{\frac{\rho-1}{\rho}(T-s)}\frac{x(s)}{(T-s)^{1-\alpha_1+\alpha_j}}ds \\
&-\frac{\mu}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}M_p(T) \\
&\quad\times\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1}M_{p-l}(t_{p-l+1})\right)\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_{t_{p-m}}^{t_{p-m+1}}e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)}\frac{f(s,x(s),x(\lambda s))}{(t_{p-m+1}-s)^{1-\alpha_1}}ds \\
&-\mu\left(\frac{\mu\prod_{m=0}^{p}M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}-1\right)M_p(T) \\
&\quad\times\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1}M_{p-l}(t_{p-l+1})\right)\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_{t_{p-m}}^{t_{p-m+1}}e^{\frac{\rho-1}{\rho}(t_{p-m+1}-s)}\frac{f(s,x(s),x(\lambda s))}{(t_{p-m+1}-s)^{1-\alpha_1}}ds \\
&-\frac{\mu}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_{t_p}^{T}e^{\frac{\rho-1}{\rho}(T-s)}\frac{f(s,x(s),x(\lambda s))}{(T-s)^{1-\alpha_1}}ds \\
&-\mu\left(\frac{\mu\prod_{m=0}^{p}M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p}M_m(t_{m+1})}-1\right)\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_{t_p}^{T}e^{\frac{\rho-1}{\rho}(T-s)}\frac{f(s,x(s),x(\lambda s))}{(T-s)^{1-\alpha_1}}ds \\
&=\Psi.
\end{aligned}
\tag{35}
$$

Equalities (35) proves the boundary condition (4) holds for $x(t)$.

Define $w_j(t)=({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x)(t)$, $j=2,3,\dots,N$, $t\in(t_k,t_{k_1}]$, $k=1,2,\dots,p$. According to Corollary 2, the equalities

$$\left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x\right)\right)(t)=\left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}w_j\right)(t)=\left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}\left({}_{t_k}^{C}\mathcal{D}^{\alpha_j,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_j,\rho}w_j\right)\right)\right)(t)$$

hold. According to Lemma 2 we get

$$\left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x\right)\right)(t)=\left({}_{0}^{C}\mathcal{D}^{\alpha_j,\rho}\left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_1,\rho}x\right)\right)\right)(t)=\left({}_{t_k}^{C}\mathcal{D}^{\alpha_j,\rho}x\right)(t). \tag{36}$$

According to Lemma 1 with $\alpha=\alpha_1-\alpha_j$, we have the equality

$$\frac{1}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}e^{\frac{\rho-1}{\rho}(t-t_k)}(t-t_k)^{\alpha_1-\alpha_j}={}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}e^{\frac{\rho-1}{\rho}t}$$

and in view of Lemma 3 with $\alpha = \alpha_j, y = x$ we get

$$x(t) - x(t_k + 0)e^{\frac{\rho-1}{\rho}(t-t_k)} = \left({}_{t_k}\mathcal{I}^{\alpha_j,\rho}\left({}_{t_k}^{C}\mathcal{D}^{\alpha_j,\rho}x\right)\right)(t).$$

Then using (30) for $t_k + 0 \in (t_k, t_{k+1}])$, $M_k(t_k + 0) = B_k$, $k = 1,2,\ldots,p$, we obtain $x(t_k + 0) = B_k x(t_k)$, i.e., the mild solution is satisfying the impulsive condition in (29).

Thus, equality $x(t) = (\Omega x)(t)$ for the mild solution, with operator Ω defined by (30). Thus,

$$\begin{aligned} x(t) = M_k(t)\Xi_k + \sum_{j=2}^{N} C_j({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x)(t) + ({}_{t_k}\mathcal{I}^{\alpha_1,\rho}f(.,x(.),x(\lambda\ .))(t) \\ t \in (t_k, t_{k+1}],\ k = 0,1,2,\ldots,p. \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Xi_k = &\frac{\Psi\prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu\prod_{m=0}^{p} M_m(t_{m+1})} \\ &+ \frac{\mu\prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu\prod_{m=0}^{p} M_m(t_{m+1})}\sum_{j=2}^{N} C_j\Bigg[M_p(T)\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1})\right) \\ &\quad\times ({}_{t_{p-m}}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x)(t_{p-m+1}) + ({}_{t_p}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x)(T)\Bigg] \\ &- \frac{\mu\prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu\prod_{m=0}^{p} M_m(t_{m+1})}M_p(T)\sum_{m=1}^{p}\left(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1})\right) \\ &\quad\times ({}_{t_{p-m}}\mathcal{I}^{\alpha_1,\rho}f(.,x(.),x(\lambda_1.)))(t_{p-m+1}) \\ &- \frac{\mu\prod_{m=0}^{k-1} M_m(t_{m+1})}{1 + \mu\prod_{m=0}^{p} M_m(t_{m+1})}({}_{t_p}\mathcal{I}^{\alpha_1,\rho}f(.,x(.),x(\lambda\ .))(T) \\ &- \sum_{j=2}^{N} C_j\sum_{m=1}^{k}\left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1})\right)({}_{t_{k-m}}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x)(t_{k-m+1}) \\ &+ M_k(t)\sum_{m=1}^{k}\left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1})\right)({}_{t_{k-m}}\mathcal{I}^{\alpha_1,\rho}f(.,x(.),x(\lambda\ .))(t_{k-m+1}). \end{aligned} \tag{38}$$

We take the GPCFD ${}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}$ of both sides of (37) apply Equation (36), Lemma 3, Corollary 2, the equality

$$\left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}e^{\frac{\rho-1}{\rho}(.\ -t_k)}\right)\right)(t) = \left({}_{t_k}^{C}\mathcal{D}^{\alpha_j,\rho}e^{\frac{\rho-1}{\rho}(,\ -t_k)}\right)(t) = 0$$

and obtain

$$\begin{aligned} \left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}x\right)(t) &= \sum_{j=2}^{N} C_j\left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_1-\alpha_j,\rho}x\right)\right)(t) + \left({}_{t_k}^{C}\mathcal{D}^{\alpha_1,\rho}\left({}_{t_k}\mathcal{I}^{\alpha_1,\rho}f(.,x(.),x(\lambda\ .))\right)\right)(t) \\ &= \sum_{j=2}^{N} C_j\left({}_{t_k}^{C}\mathcal{D}^{\alpha_j,\rho}x\right)(t) + f(t,x(t),x(\lambda t)),\ \ t \in (t_k, t_{k+1}],\ k = 0,1,\ldots,p. \end{aligned} \tag{39}$$

It proves the claim. □

Theorem 5. *Let the inequality* (5) *hold, $x(t)$ be a solution of the BVP for NIGPDE* (29) (4) *and $F \in PI^{\alpha_1,\rho}[0,T]$, where $F(t) = f(t,x(t),x(\lambda t))$. Then the function $x(t)$ is a mild solution of the same problem.*

The proof of Theorem 5 is similar to the one of Theorem 1 and we omit it.

We will obtain some sufficient conditions for the existence of the mild solutions of (29) (4). The proofs are based on fixed point theorems and the application of the operator Ω.

Theorem 6. *Let:*

1. *The inequality* (5) *holds.*
2. *There exist constants* $L, M > 0:$

$$|f(t,x_1,z_1) - f(t,x_2,z_2)| \le L|x_1 - x_2| + M|z_1 - z_2|, \ \ t \in [0,T], \ x_i, z_i \in \mathbb{R}, \ i = 1,2.$$

3. *For* $k = 0,1,2,\dots,p$, *the inequality*

$$\Big(\sum_{j=2}^{N} |C_j| \frac{\xi^{\alpha_1-\alpha_j}}{\Gamma(1+\alpha_1-\alpha_j)} + (L+M)\frac{\xi^{\alpha_1}}{\Gamma(1+\alpha_1)}\Big)\Phi_k < 1, \tag{40}$$

holds where

$$\xi_k = \frac{t_{k+1}-t_k}{\rho}, \ k = 0,1,2,\dots,p, \qquad \xi = \max_{k=0,1,2,\dots,p} \xi_k,$$

$$\sigma_m = \sum_{j=1}^{N} \frac{|C_j|}{\xi_m^{\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}, \ m = 0,1,2,\dots,p,$$

$$\beta_m = \sum_{j=1}^{N} \frac{C_j}{\xi_m^{\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}, \ m = 0,1,2,\dots,p,$$

$$\eta = \Big|1 + \mu \prod_{m=0}^{p} B_m \xi_m^{\alpha_1} e^{(\rho-1)\xi_m}\beta_m\Big|,$$

$$\Phi_k = \frac{|\mu| \prod_{m=0}^{k} |B_m|\xi_m^{\alpha_1}\sigma_m}{\eta} \sum_{m=0}^{p}\Bigg(\prod_{l=0}^{m-1} |B_{p-l}|\xi_{p-l}^{\alpha_1}\sigma_{p-l}\Bigg) + \sum_{m=0}^{k}\Bigg(\prod_{l=0}^{m-1} |B_{k-l}|\xi_{k-l}^{\alpha_1}\sigma_{k-l}\Bigg).$$

Then the BVP for NIGPDE (29) (4) *has a unique mild solution.*

Proof. We use the norm $||x|| = \sup_{t\in[0,T]} |x(t)|$ for any $x \in PC[0,T]$.

Then $|M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})| \le \prod_{m=0}^{k} |B_m\xi_m^{\alpha_1}| \sum_{j=1}^{N} \frac{|C_j|}{\xi_m^{\alpha_j}\Gamma(1+\alpha_1-\alpha_j)} = \prod_{m=0}^{k} |B_m|\xi_m^{\alpha_1}\sigma_m$, where $M_k(t), \ k = 0,1,2,\dots,p$ are defined by (31) and (32).

Consider the integral fractional operator Ω defined by (30). Let $x, y \in PC([0,T])$. Then $\int_{t_k}^{t} \frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^{1-\alpha}} ds \le \frac{(t-t_k)^\alpha}{\alpha}$. Thus, we obtain for $t \in (t_k, t_{k+1}], \ k = 0,1,2,\dots,p,$

$$\begin{aligned} |\Omega x(t) - \Omega y(t)| \le & \Bigg\{ \frac{\mu M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p} M_m(t_{m+1})} \sum_{j=2}^{N} C_j \Bigg[M_p(T) \\ & \times \sum_{m=1}^{p}\Bigg(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1})\Bigg)\frac{(t_{p-m+1}-t_{p-m})^{\alpha_1-\alpha_j}}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)} \\ & + \frac{(T-t_p)^{\alpha-1-\alpha_j)}}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}\Bigg] \\ & + (L+M)\frac{\mu M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p} M_m(t_{m+1})} M_p(T) \\ & \times \sum_{m=1}^{p}\Bigg(\prod_{l=1}^{m-1} M_{p-l}(t_{p-l+1})\Bigg)\frac{(t_{p-m+1}-t_{p-m})^{\alpha_1}}{\rho^{\alpha_1}\Gamma(1+\alpha_1)} \end{aligned} \tag{41}$$

$$
\begin{aligned}
&+ (L+M)\frac{\mu M_k(t)\prod_{m=0}^{k-1} M_m(t_{m+1})}{1+\mu\prod_{m=0}^{p} M_m(t_{m+1})}\,\frac{(T-t_p)^{\alpha_1}}{\rho^{\alpha_1}\Gamma(1+\alpha_1)}\\
&+ \sum_{j=2}^{N} C_j\Bigg[M_k(t)\sum_{m=1}^{k}\left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1})\right)\frac{(t_{k-m+1}-t_{k-m})^{\alpha_1-\alpha_j}}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}\\
&\quad+\frac{(t-t_k)^{\alpha_1-\alpha_j}}{\rho^{\alpha_1-\alpha_j}\Gamma(1+\alpha_1-\alpha_j)}\Bigg]\\
&+ (L+M)M_k(t)\sum_{m=1}^{k}\left(\prod_{l=1}^{m-1} M_{k-l}(t_{k-l+1})\right)\frac{(t_{k-m+1}-t_{k-m})^{\alpha_1}}{\rho^{\alpha_1}\Gamma(1+\alpha_1)}\\
&\quad+ (L+M)\frac{(t-t_k)^{\alpha_1}}{\rho^{\alpha_1}\Gamma(1+\alpha_1)}\Bigg\}||x-y||
\end{aligned}
$$

or

$$
\begin{aligned}
|\Omega x(t)-\Omega y(t)| \le &\Bigg\{\frac{|\mu|\prod_{m=0}^{k}|B_m|\xi_m^{\alpha_1}\sigma_m}{\eta}\sum_{j=2}^{N}|C_j|\Bigg[\sum_{m=1}^{p}\left(\prod_{l=0}^{m-l}|B_{p-l}|\xi_{p-l}^{\alpha_1}\sigma_{p-l}\right)\\
&\times\frac{\xi_{p-m}^{\alpha_1-\alpha_j}}{\Gamma(1+\alpha_1-\alpha_j)}+\frac{\xi_{p}^{\alpha_1-\alpha_j}}{\Gamma(1+\alpha_1-\alpha_j)}\Bigg]\\
&+\sum_{j=2}^{N}|C_j|\Bigg[\sum_{m=1}^{k}\left(\prod_{l=0}^{m-1}|B_{k-l}|\xi_{k-l}^{\alpha_1}\sigma_{k-l}\right)\frac{\xi_{k-m}^{\alpha_1-\alpha_j}}{\Gamma(1+\alpha_1-\alpha_j)}+\frac{\xi_{k}^{\alpha_1-\alpha_j}}{\Gamma(1+\alpha_1-\alpha_j)}\Bigg]\\
&+(L+M)\Bigg[\frac{|\mu|\prod_{m=0}^{k}|B_m|\xi_m^{\alpha_1}\sigma_m}{\eta}\sum_{m=1}^{p}\left(\prod_{l=0}^{m-1}|B_{p-l}|\xi_{p-l}^{\alpha_1}\sigma_{p-l}\right)\\
&+\sum_{m=1}^{k}\left(\prod_{l=0}^{m-1}|B_{k-l}|\xi_{k-l}^{\alpha_1}\sigma_{k-l}\right)+\frac{|\mu|\prod_{m=0}^{k}|B_m|\xi_m^{\alpha_1}\sigma_m}{\eta}+1\Bigg]\frac{\xi^{\alpha_1}}{\Gamma(1+\alpha_1)}\Bigg\}||x-y||\\
\le &\Bigg\{\sum_{j=2}^{N}|C_j|\frac{\xi^{\alpha_1-\alpha_j}}{\Gamma(1+\alpha_1-\alpha_j)}\Bigg[\frac{|\mu|\prod_{m=0}^{k}|B_m|\xi_m^{\alpha_1}\sigma_m}{\eta}\sum_{m=0}^{p}\left(\prod_{l=0}^{m-l}|B_{p-l}|\xi_{p-l}^{\alpha_1}\sigma_{p-l}\right)\\
&+\sum_{m=0}^{k}\left(\prod_{l=0}^{m-1}|B_{k-l}|\xi_{k-l}^{\alpha_1}\sigma_{k-l}\right)\Bigg]\\
&+(L+M)\Bigg[\frac{|\mu|\prod_{m=0}^{k}|B_m|\xi_m^{\alpha_1}\sigma_m}{\eta}\sum_{m=0}^{p}\left(\prod_{l=0}^{m-1}|B_{p-l}|\xi_{p-l}^{\alpha_1}\sigma_{p-l}\right)\\
&+\sum_{m=0}^{k}\left(\prod_{l=0}^{m-1}|B_{k-l}|\xi_{k-l}^{\alpha_1}\sigma_{k-l}\right)\Bigg]\frac{\xi^{\alpha_1}}{\Gamma(1+\alpha_1)}\Bigg\}||x-y||\\
= &\left(\sum_{j=2}^{N}|C_j|\frac{\xi^{\alpha_1-\alpha_j}}{\Gamma(1+\alpha_1-\alpha_j)}+(L+M)\frac{\xi^{\alpha_1}}{\Gamma(1+\alpha_1)}\right)\Phi_k||x-y||.
\end{aligned}
\tag{42}
$$

Inequality (42) and condition 3 proves the existence of a constant $K\in(0,1)$ such that

$$||\Omega x-\Omega y|| \le K||x_1-y||.$$

Thus, the Banach contraction principle applied to the operator Ω proves that there exists a unique fixed point $x^*\in PC[0,T]$ of Ω, which is a mild solution of (29) and (4). □

4.2. Mild Solution of BVP for the Impulsive Delay Fractional Differential Equation with GPCDE

We consider the nonlinear impulsive delay differential equation with one GPCDE (NIGDE) of the type

$$\begin{aligned} &({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}x)(t) = f(t, x(t), x(\lambda t))), \text{ for } t \in (t_k, t_{k+1}],\ k = 0,1,2,\dots,p \\ &z(t_k + 0) = B_k z(t_k),\ k = 1,2,3,\dots,p, \end{aligned} \tag{43}$$

with the boundary value condition (4), where $\alpha \in (0,1)$, $\rho \in (0,1]$, $\lambda \in (0,1)$, the numbers $B_i \in \mathbb{R}$, $k = 1,2,3,\dots,p$, the function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and

$$1 + \mu e^{\frac{\rho-1}{\rho}T} \prod_{m=0}^{p} B_m \neq 0. \tag{44}$$

Based on the integral presentation (24) we define the integral fractional operator $\mathcal{W} : PC[0,T] \to PC[0,T]$ by

$$\begin{aligned} (\mathcal{W}x)(t) = &\frac{\Psi e^{\frac{\rho-1}{\rho}(t-t_0)} \prod_{m=0}^{k} B_m}{1 + \mu e^{\frac{\rho-1}{\rho}(T-t_0)} \prod_{m=0}^{p} B_m} \\ &+ \frac{\mu e^{\frac{\rho-1}{\rho}(t-t_0)} \prod_{m=0}^{k} B_m}{1 + \mu e^{\frac{\rho-1}{\rho}(T-t_0)} \prod_{m=0}^{p} B_m} ({}_{t_p}\mathcal{I}^{\alpha_1,\rho} f(., x(.), x(\lambda\, .)))(T) \\ &- \frac{\mu e^{\frac{\rho-1}{\rho}(t-t_0)} \prod_{m=0}^{k} B_m}{1 + \mu e^{\frac{\rho-1}{\rho}(T-t_0)} \prod_{m=0}^{p} B_m} \\ &\times \sum_{m=1}^{p} \left(\left(\prod_{l=0}^{m-1} B_{p-l} \right) e^{\frac{\rho-1}{\rho}(T-t_{p-m+1})} ({}_{t_{p-m}}\mathcal{I}^{\alpha_1,\rho} f(., x(.), x(\lambda\, .)))(t_{p-m+1}) \right) \\ &+ \sum_{m=1}^{k} \left(\left(\prod_{l=0}^{m-1} B_{k-l} \right) e^{\frac{\rho-1}{\rho}(t-t_{k-m+1})} ({}_{t_{k-m}}\mathcal{I}^{\alpha_1,\rho} f(., x(.), x(\lambda\, .)))(t_{k-m+1}) \right) \\ &+ ({}_{t_k}\mathcal{I}^{\alpha_1,\rho} f(., x(.), x(\lambda\, .)))(t), \qquad t \in (t_k, t_{k+1}],\ k = 0,1,2,\dots,p. \end{aligned} \tag{45}$$

Definition 2. *The function $x(t)$ is a mild solution of the BVP for NIGPDE (43) (4) if it is a fixed point of the integral fractional operator $\mathcal{W}$, defined by (45).*

Theorem 7. *Let:*

1. *The inequality (44) holds.*
2. *There exist constants $L, M > 0 :$*

$$|f(t, x_1, z_1) - f(t, x_2, z_2)| \leq L|x_1 - x_2| + M|z_1 - z_2|,\ t \in [0,T], x_i, z_i \in \mathbb{R},\ i = 1,2.$$

3. *For $k = 0,1,2,\dots,p$, the inequality*

$$(L + M)\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}\Phi_k < 1, \tag{46}$$

holds where

$$\begin{aligned} \xi &= \max_{k=0,1,2,\dots,p} \frac{t_{k+1} - t_k}{\rho}, \\ \eta &= \left| 1 + \mu e^{\frac{\rho-1}{\rho}T} \prod_{m=0}^{p} B_m \xi_m^{\alpha_1} \right|, \\ \Phi_k &= \frac{|\mu| \prod_{m=0}^{k} |B_m| \xi_m^{\alpha_1}}{\eta} \sum_{m=0}^{p} \left(\prod_{l=0}^{m-1} |B_{p-l}| \xi_{p-l}^{\alpha_1} \right) + \sum_{m=0}^{k} \left(\prod_{l=0}^{m-1} |B_{k-l}| \xi_{k-l}^{\alpha_1} \right). \end{aligned} \tag{47}$$

Then the BVP for NIGPDE (43) (4) *has a unique mild solution.*

The claim of Theorem 7 follows from Theorem 6.

Remark 2. *From the above results for BVP for NIGPDE, we obtain some results for BVP for impulsive fractional differential equations with Caputo fractional derivatives with lower changeable limits at the impulsive points. The case of Caputo fractional derivative with a fixed lower limit at the initial time is studied in [16].*

5. Conclusions

In this paper, two main goals are reached. First, an integral form of the unique solutions of the BVP for various types of linear impulsive fractional differential equations is obtained. Second, based on these presentations, new types of integral fractional operators are defined. These operators are applied to obtain sufficient conditions for the uniqueness and existence of mild solutions of BVP for several types of nonlinear delay impulsive fractional differential equations.

The obtained integral presentations of the solutions of the BVP for the linear impulsive fractional differential equations could be applied in some approximate methods such as the monotone-iterative technique for constructing successive approximations. The newly defined integral fractional operator could be successfully used to study the Ulam-type stability of BVP for various types of nonlinear fractional differential equations with impulses.

The ideas in this paper are also applicable to studying various kinds of boundary conditions, such as nonlinear ones, nonlocal ones, and integral ones. Furthermore, the case of the fractional order $\alpha > 1$ could be investigated.

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