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Fractional-Order Accumulative Generation with Discrete Convolution Transformation

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Abstract: A new fractional accumulation technique based on discrete sequence convolution transform was developed. The accumulation system, whose unit impulse response is the accumulation convolution sequence, was constructed; then, the order was extended to fractional orders. The fractional accumulative convolution grey forecasting model $GM^{r*}(1,1)$ was established on the sequence convolution. From the viewpoint of sequence convolution, we can better understand the mechanism of accumulative generation. Real cases were used to verify the validity and effectiveness of the fractional accumulative convolution method.

Keywords: fractional accumulation; discrete sequence convolution transform; unit impulse; grey forecasting

1. Introduction

Due to the finite cognitive competence of human beings, the information obtained from the investigated system is always incomplete and inaccurate. The incompleteness and inaccuracy of information are absolute, while completeness and accuracy are relative [1]. Prof. Julong Deng proposed the grey system theory to solve undetermined problems, in which the given information is lacking and the data for modelling are few.

In contrast to traditional statistical models, such as the Bayesian approach [2], maximum likelihood estimation [3], modern artificial neural network [4], and machine learning methods [5], which usually need many samples, the grey model requires less data modelling [1]. It makes full use of the limited data from a small sample and excavates more useful information from the data. In the early stage, Deng pointed out that only four data points were sufficient for $GM(1,1)$ modelling [6]. Yao et al. put forward the mathematical proof indicating that small samples have greater accuracy than large samples [7]. Matrix perturbation theory was introduced to analyse the grey model, suitable for modelling small samples, by Wu, and Xu et al. [8,9]. Wang applied the grey power model to simulate small oscillating samples [10]. Talafuse modelled a small sample on discrete reliability growth with the grey model and achieved more accurate predictions than the traditional parametric and non-parametric methods [11]. Ma proposed $KGM(1,n)$ combined with kernel learning to model the non-linearity of multi-input and single-output sequences and proved it to be more stable and efficient than the machine learning models, such as least squares support vector machines [12]. The grey model has been applied with high efficiency in modelling small data samples in science and engineering technologies [13], energy resources [14], traffic control [15], health care [16], economics [17], management [18], ecology [19], agriculture [20], etc., and has achieved significant economic and social benefits [21–24].

The grey model employs a kind of sequence operator called accumulative generation operator to act on the system behaviour sequence to provide intermediate data for the grey model [25], distinguishing itself from other forecasting models that directly model the original data series. The acting of the accumulative generation operator on the original data sequence removes random fluctuation in the data, and the acted-upon sequence

approximates to the quasi-exponential law. Many systems in economics, ecosystems, etc., are general energy systems, and energy transfer follows the power law, which is the foundation of grey modelling [26]. Prof. Sifeng Liu proposed the concept of the grey sequence operator [27] and then considered the accumulative generation procedure as the action of the sequence operator called the accumulative generation operator (AGO) [26]. The AGO can make the data sequence follow the power law, but the over-acting of the AGO can also destroy the power law [21].

In nature, there are many systems with non-integral-order derivative effects; the fractional-order derivative is a better way to describe this kind of system behaviour than ordinary integral-order models. With the concept of "in between", the fractional-order derivative is the general extension of the normal integral-order derivative, applied in many disciplines, such as fractional system analysis and control [28–30], energy resource modelling and prediction [31,32], air pollution and environment protection [33,34], etc.

A higher-order operator is a natural concept indicating the repeated action of the operator on the data sequence, i.e., the AGO acting m times on the data sequence leads to the m-th-order AGO, where m is a positive integer [22]. Wu et al. considered the accumulative generation operator a square matrix; then, the action of AGOs of different orders was the same as the power of the square matrix [35]. From this viewpoint, the explicit expression of the positive integral m-th-order AGO was derived; then, it could be very quickly extended to fractional orders, i.e., the fractional-order accumulative operator (FAO), establishing the fractional-order grey forecasting model (FGM) [35–38]. Due to great improvement in forecasting precision, FAO and FGM have attracted a lot of attention and have become hot topics in recent years. Xiao, and Mao et al. used matrix analysis to explore the modelling mechanism and theoretical significance of fractional accumulation grey models [39,40]. Soon, the order was extended to arbitrary real numbers by Meng, and Zeng et al. [41–43]. Recently, Wu et al. unified the expression of AGOs; then, they extended the order numbers to the widest field, i.e., the complex number order [44].

Although the AGO has been discussed for a long time and has been widely used in many fields, little attention has been paid to its essence and mechanism. Wei, and Xie et al. introduced the integral matching method to explain that the AGO is a discrete form of the integral of the continuous function approximated by a piecewise constant in each subinterval, and due to the approximated discrete form of the integral being cumulative summation, the operator gained a new name, cumulative sum operator [45–47]. Chen used convolution transformation to improve the accumulative generation procedure and pointed out that convolution transformation could enhance the smoothness of the data sequence, which could be used in grey modelling [48]. Lin et al. quantitatively studied the mechanism and power of the AGO by applying spectrum analysis in the frequency domain [49–52].

The paper proposes fractional accumulation with the discrete convolution transform of finite sequences and aims to interpret its physical meaning from the perspective of signal processing. The main contributions are listed as follows:

(1) The concept of discrete convolution transform is introduced in accumulative generation. In fact, the unit impulse response of the accumulative generation system is found and is named accumulative generation convolution sequence. This is the discretization of the AGO in the time domain.

(2) By extending the concept in (1) to the fractional-order accumulative generation convolution sequence, the unit impulse response of the fractional-order accumulative generation system is obtained. In fact, the discrete form of the fractional accumulation operator (FAO) in the time domain is explicitly represented, which makes the physical meaning of the FAO self-evident.

(3) The fractional-order accumulative generation convolution transform and its inversion are mutually inverse. They do not impose any extra error on data transformation. The inversion of the fractional-order accumulative generation convolution sequence can

be calculated directly by assigning the minus fractional order, without demanding a round number order (compare with [8] (p. 1780)).

(4)   According to model fitting error, the fractional accumulation grey model can dynamically adjust the order to model and predict the system behaviour data better.

Convolution transformation is a powerful tool in digital signal process [53]. In this framework, a new viewpoint to understand the mechanism of AGO emerges. An AGO can be understood as a linear time invariant system. Convolving an input sequence with the unit impulse responsesequence yields an accumulated output sequence [54].

The remainder of the paper is organized as follows. The integral-order accumulative convolution sequence is introduced and extended to arbitrary real numbers in Section 2. The fractional accumulative convolution grey model is discussed in Section 3. Some real cases are used to demonstrate the validity of the fractional accumulative convolution transform in Section 4. Finally, Section 5 discusses the conclusions.

2. The Accumulative Generation with Discrete Convolution Transform

In this section, we start from the classical definition of the AGO, then introduce the finite sequence convolution and construct the accumulative convolution sequence to fulfil the accumulative generation procedure and the extend the accumulative convolution sequenceto the integral- and fractional-order.

Definition 1. *The accumulative generation operator (AGO) for a sequence $\{x^{(0)}[n]\}_{n=0}^{N-1}$ is as follows [25]*

$$\text{AGO}\left(x^{(0)}[n]\right) = x^{(0)}[0] + \cdots + x^{(0)}[n] = x^{(1)}[n]. \tag{1}$$

Here, $\{x^{(0)}[n]\}_{n=0}^{N-1}$ is a sequence with length N with the integral-index n varying from 0 to $N-1$ and $x^{(0)}[n]$ is one datum in the sequence. The superscript represents the order of accumulation, e.g., $x^{(0)}$ is the zeroth-order accumulated sequence, i.e., the original sequence; while $x^{(1)}$ represents the first-order accumulated sequence.

Applying the AGO m times leads to the integral-order accumulated sequence [21,26]

$$\text{AGO}\left(x^{(m-1)}[n]\right) = x^{(m)}[n], \tag{2}$$

where $m \in \mathbb{N}$ (set of natural numbers).

Definition 2. *Given two sequences with length N, $\{x[n]\}_{n=0}^{N-1}$ and $\{y[n]\}_{n=0}^{N-1}$, then their discrete convolution is [54]*

$$x[n] * y[n] = \sum_{k=0}^{n} x[k] \cdot y[n-k]. \tag{3}$$

By Definition 2, the discrete convolution operation is commutative.

Definition 3. *Set $\{\delta[n]\}_{n=0}^{N-1}$ to be the unit impulse sequence on the non-negative part of the time axis, i.e., [55]*

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n = 1, 2, \cdots, N-1. \end{cases} \tag{4}$$

By Definitions 2 and 3, the invariance of the unit impulse sequence in the convolution operation is obtained, i.e., $x[n] * \delta[n]$ does not change anything and is identically equal to $x[n]$. Therefore, in the convolution operation, $\{\delta[n]\}_{n=0}^{N-1}$ is an identity element. Moreover, for a positive integer m,

$$x[n] * \delta[n-m] = x[n-m]. \tag{5}$$

Definition 4. *Set $\{x[n]\}_{n=0}^{N-1}$ and $\{y[n]\}_{n=0}^{N-1}$ to be two sequences. If their convolution yields to the unit impulse sequence, then they are mutually inverse, denoted by $x[n] = y^{-1*}[n]$.*

Based on the above definitions, we arrived at Theorem (1) to represent the AGO in the form of finite sequence convolution. Since the classical accumulative generation operator AGO($\cdot$) is denoted in uppercase, the accumulative convolution sequence is denoted by ago[n] in lowercase for distinction.

Theorem 1. *For a sequence $\{x^{(0)}[n]\}_{n=0}^{N-1}$, denote the first-order accumulated sequence obtained by the AGO in (1) as $\{x^{(1)}[n]\}_{n=0}^{N-1}$. Then it can be regenerated by convolution*

$$x^{(0)}[n] * \mathrm{ago}[n] = x^{(1)}[n].\tag{6}$$

The $\{\mathrm{ago}[n]\}_{n=0}^{N-1}$ is called the accumulative convolution sequence, and is represented by

$$\mathrm{ago}[n] = \delta[0] + \delta[1] + \cdots + \delta[n] = 1.\tag{7}$$

Proof. From Definition 2 and Equation (5),

$$
\begin{aligned}
&x^{(0)}[n] * (\delta[0] + \cdots + \delta[n])\\
=&x^{(0)}[n] * (\delta[n-n] + \cdots + \delta[n-0])\\
=&x^{(0)}[0] + \cdots + x^{(0)}[n] = x^{(1)}[n].
\end{aligned}
$$

which concludes the result. □

Remark 1. *The* AGO($\cdot$) *acts on a sequence, denoted with a pair of round brackets, i.e.,* AGO($x^{(0)}[n]$)*, which yields to the $\{x^{(1)}[n]\}_{n=0}^{N-1}$; while the accumulative convolution sequence $\{\mathrm{ago}[n]\}_{n=0}^{N-1}$ is a data sequence by itself, in which a pair of square brackets indicates that $ago[n]$ is a datum. Convolution transforming the original sequence by the accumulative convolution sequence also generates the first-order accumulated sequence, i.e., $x^{(0)}[n] * ago[n] = x^{(1)}[n]$, which plays the same role of the AGO. To avoid confusion, the same name but in lower case "ago" denotes the accumulative convolution sequence. Simply,* AGO($\cdot$) *is an operator; while* ago[n] *is a datum from an accumulative convolution sequence. They are fundamentally different.*

Definition 5. *The convolution power is defined by a sequence $\{x[n]\}_{n=0}^{N-1}$ convolving with itself for k times, where k is a positive integer,*

$$\underbrace{x[n] * \cdots * x[n]}_{k} = x^{k*}[n].\tag{8}$$

For completeness of Definition 5, the zeroth convolution power is set to be the identity element, i.e.,

$$x^{0*}[n] = \delta[n].\tag{9}$$

By (5),

$$\delta^{k*}[n-1] = \delta[n-k].\tag{10}$$

Comparing (7) and (10), the accumulative convolution sequence $\{\mathrm{ago}[n]\}_{n=0}^{N-1}$ can be represented by summation of the first n terms of the convolution power series

$$\mathrm{ago}[n] = \sum_{k=0}^{n} \delta^{k*}[n-1].\tag{11}$$

Convolving $\delta[n] - \delta[n-1]$ with (11) yields the unit impulse sequence, i.e.,

$$\mathrm{ago}[n] * (\delta[n] - \delta[n-1]) = \delta[n].\tag{12}$$

According to Definition 4, $\delta[n] - \delta[n-1]$ is the inverse accumulative convolution sequence of $\mathrm{ago}[n]$. It is denoted by $\mathrm{ago}^{-1*}[n] = \delta[n] - \delta[n-1]$.

Based on (12), the first-order accumulative convolution sequence $\{\mathrm{ago}[n]\}_{n=0}^{N-1}$ is extended to the integral-order, i.e., $\{\mathrm{ago}^{m*}[n]\}_{n=0}^{N-1}$, with $m \in \mathbb{Z}^+$ (set of positive integers),

$$\mathrm{ago}^{m*}[n] * \mathrm{ago}^{-m*}[n] = \delta[n]. \tag{13}$$

This leads to the definition of the positive integral-order accumulative convolution sequence. Using Newton's generalized binomial theorem and the unit impulse sequence in Definition 3,

$$\mathrm{ago}^{m*}[n] = (\delta[n] - \delta[n-1])^{-m*} = \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} -m \\ k \end{pmatrix} \delta^{k*}[n-1], \tag{14}$$

where $\begin{pmatrix} -m \\ k \end{pmatrix}$ is the extended combination number, given by

$$\begin{pmatrix} -m \\ k \end{pmatrix} = \frac{(-m)(-m-1)\cdots(-m-k+1)}{k!}. \tag{15}$$

According to (4) and (10), the m-th power of the accumulative convolution sequence yields

$$\mathrm{ago}^{m*}[n] = \sum_{k=0}^{n} \begin{pmatrix} k+m-1 \\ k \end{pmatrix} \delta[n-k], \tag{16}$$

For $n = 0$, (16) is always equal to 1, i.e., $\mathrm{ago}^{m*}[0] = 1$. For $n = 1, 2\ldots$, by (4), the unit impulse is equal to zero when $k \neq n$, thus

$$\mathrm{ago}^{m*}[n] = \sum_{k=0}^{n} \frac{(m+k-1)\cdots m}{k!} \delta[n-k] = \prod_{i=0}^{n-1} \frac{m+i}{1+i}.$$

Therefore, the positive-integral m-th power of the accumulative convolution sequence is

$$\mathrm{ago}^{m*}[n] = \begin{cases} 1, & n = 0, \\ \prod\limits_{i=0}^{n-1} \frac{m+i}{1+i}, & n \geq 1. \end{cases} \tag{17}$$

For the negative-integral $-m$-th-order accumulative convolution sequence,

$$\mathrm{ago}^{-m*}[n] = (\delta[n] - \delta[n-1])^{m*} = \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} \delta^{k*}[n-1]. \tag{18}$$

For $n = 0$, (18) is also equal to 1, i.e., $\mathrm{ago}^{-m*}[0] = 1$. For $n \geq 1$,

$$\mathrm{ago}^{-m*}[n] = \sum_{k=0}^{n} (-1)^k \frac{m\cdots(m-k+1)}{k!} \delta[n-k] = \sum_{k=0}^{m} \prod_{i=0}^{k-1} \frac{-m+i}{1+i} \delta[n-k].$$

If $m \geq n$, k can reach n and $\delta[n-k] = \delta[0] = 1$ if and only if $k = n$. Thus,

$$\mathrm{ago}^{-m*}[n] = \prod_{i=0}^{n-1} \frac{-m+i}{1+i}.$$

Otherwise, if $m < n$, k is also less than n, which causes the unit impulse to equal zero. Thus

$$\mathrm{ago}^{-m*}[n] = \prod_{i=0}^{n-1} \frac{-m+i}{1+i} = 0.$$

Therefore, the negative-integral $-m$-th power of the accumulative convolution sequence is

$$\mathrm{ago}^{-m*}[n] = \begin{cases} 1, & n = 0, \\ \prod\limits_{i=0}^{n-1} \frac{-m+i}{1+i}, & n \geq 1. \end{cases} \tag{19}$$

According to Equations (17) and (19), the following theorem unifies the integral m-th-order accumulative convolution sequence.

Theorem 2. *Let $\{\mathrm{ago}[n]\}_{n=0}^{N-1}$ be the accumulative convolution sequence, then the integral m-th-order accumulative convolution sequence with $m \in \mathbb{Z}$ (set of integers) is unified, i.e.*

$$\mathrm{ago}^{m*}[n] = \begin{cases} 1, & n = 0, \\ \prod\limits_{i=0}^{n-1} \frac{m+i}{1+i}, & n \geq 1. \end{cases} \tag{20}$$

Next, the order of accumulative convolution sequence is extended to fractional orders. Based on (13), for a positive real number $r \in \mathbb{R}^+$ (set of positive real numbers),

$$\mathrm{ago}^{r*}[n] * \mathrm{ago}^{-r*}[n] = \delta[n]. \tag{21}$$

Thus, the r-th power of the accumulative convolution sequence is

$$\mathrm{ago}^{r*}[n] = \left(\delta[n] - \delta[n-1]\right)^{-r*}.$$

Using Newton's generalized binomial theorem and the definition of $\delta[n]$ in (4),

$$\mathrm{ago}^{r*}[n] = \sum_{k=0}^{\infty} (-1)^k \frac{(-r)\cdots(-r-k+1)}{k!} \delta^{k*}[n-1] = \sum_{k=0}^{n} \binom{k+r-1}{k} \delta[n-k]. \tag{22}$$

Similarly, (22) is always equal to 1 when $n = 0$, i.e., $\mathrm{ago}^{r*}[0] = 1$. For $n = 1, 2, \cdots$,

$$\mathrm{ago}^{r*}[n] = \sum_{k=0}^{n} \frac{(r+k-1)\cdots r}{k!} \delta[n-k] = \prod_{i=0}^{n-1} \frac{r+i}{1+i}.$$

Therefore, the positive real r-th power of the accumulative convolution sequence is

$$\mathrm{ago}^{r*}[n] = \begin{cases} 1, & n = 0, \\ \prod\limits_{i=0}^{n-1} \frac{r+i}{1+i}, & n \geq 1. \end{cases} \tag{23}$$

For the negative real $-r$-th-order accumulative convolution sequence,

$$\mathrm{ago}^{-r*}[n] = \left(\delta[n] - \delta[n-1]\right)^{r*}.$$

Using Newton's generalized binomial theorem again,

$$\mathrm{ago}^{-r*}[n] = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \delta^{k*}[n-1] = \sum_{k=0}^{\infty} (-1)^k \frac{r\cdots(r-k+1)}{k!} \delta[n-k].$$

By the definition of unit impulse in (4),

$$\mathrm{ago}^{-r*}[n] = \sum_{k=0}^{n} \binom{k-r-1}{k} \delta[n-k]. \tag{24}$$

Thus, (24) is always equal to 1 for $n = 0$, i.e., $\text{ago}^{-r*}[0] = 1$. For $n = 1, 2, \cdots$,

$$\text{ago}^{-r*}[n] = \sum_{k=0}^{n} \frac{(-r+k-1)\cdots(-r)\cdots r}{k!} \delta[n-k] = \prod_{i=0}^{n-1} \frac{-r+i}{1+i}.$$

Therefore, (24) becomes

$$\text{ago}^{-r*}[n] = \begin{cases} 1, & n = 0, \\ \prod_{i=0}^{n-1} \frac{-r+i}{1+i}, & n \geq 1. \end{cases} \tag{25}$$

According to Equations (23) and (25), a unified expression of the real number r-th-order accumulative convolution sequence is obtained in the following theorem.

Theorem 3. *Let $\{\text{ago}[n]\}_{n=0}^{N-1}$ be the accumulative convolution sequence, then the real r-th-order accumulative convolution sequence with $r \in \mathbb{R}$ (set of real numbers) is unified, i.e.,*

$$\text{ago}^{r*}[n] = \begin{cases} 1, & n = 0, \\ \prod_{i=0}^{n-1} \frac{r+i}{1+i}, & n \geq 1. \end{cases} \tag{26}$$

3. Grey Forecasting Model with Fractional Accumulative Convolution

The fractional accumulative convolution is introduced into $\text{GM}(1,1)$ in this section, leading to a new grey model.

For a given sequence $\{x^{(0)}[n]\}_{n=0}^{N-1}$ and the accumulation order r, according to Definition 2 and the Theorem 3, the accumulated sequence can be represented by the convolution transform with the r-th-order accumulative convolution sequence $\{\text{ago}^{r*}[n]\}_{n=0}^{N-1}$.

Definition 6. *Given an original sequence $\{x^{(0)}[n]\}_{n=0}^{N-1}$, and a real number r-th-order accumulative convolution sequence $\{\text{ago}^{r*}[n]\}_{n=0}^{N-1}$ defined in (26), their discrete convolution yields the r-th-order accumulated sequence $\{x^{(r)}[n]\}_{n=0}^{N-1}$, i.e.,*

$$x^{(0)}[n] * \text{ago}^{r*}[n] = x^{(r)}[n]. \tag{27}$$

Suppose $\{x^{(r)}[n]\}_{n=0}^{N-1}$ be the equidistant sampling sequence from a differentiable function $x^{(r)}(t)$, satisfying the first-order ordinary differential equation

$$\frac{dx^{(r)}}{dt} = c_1 x^{(r)}(t) + c_2. \tag{28}$$

For a given sequence $\{x^{(r)}[n]\}_{n=0}^{N-1}$, the coefficients c_1 and c_2 in (28) are determined. Both sides of (28) are integrated on subinterval $[n-1, n]$,

$$x^{(r)}[n] - x^{(r)}[n-1] = \int_{n-1}^{n} \left(c_1 x^{(r)}(t) + c_2 \right) dt. \tag{29}$$

By the trapezoidal rule, a discrete form of (28) is obtained,

$$x^{(r)}[n] - x^{(r)}[n-1] = 0.5c_1 \left(x^{(r)}[n-1] + x^{(r)}[n] \right) + c_2. \tag{30}$$

Remark 2. *Equation (30) is the so-called fractional r-th-order accumulative convolution grey model, denoting $\text{GM}^{r*}(1,1)$ with $r \in \mathbb{R}$. The superscript $r*$ represents the accumulative convolution sequence with order r. When $r = 1$, it turns out to be the $\text{GM}(1,1)$ model .*

For $n = 1, \cdots, N - 1$, (30) leads to a linear system

$$Ac = d, \tag{31}$$

where

$$A = \begin{pmatrix} 0.5(x^{(r)}[0] + x^{(r)}[1]) & 1 \\ 0.5(x^{(r)}[1] + x^{(r)}[2]) & 1 \\ \vdots & \vdots \\ 0.5(x^{(r)}[N-2] + x^{(r)}[N-1]) & 1 \end{pmatrix}, d = \begin{pmatrix} x^{(r)}[1] - x^{(r)}[0] \\ x^{(r)}[2] - x^{(r)}[1] \\ \vdots \\ x^{(r)}[N-1] - x^{(r)}[N-2] \end{pmatrix}, c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Linear system (31) contains $N - 1$ equations, but there are only two unknowns, c_1 and c_2. It is an over-determined system when $N - 1 > 2$. Therefore, the solution of (31) in the sense of least squares is equivalent to the solution of the normal equation

$$A^T Ac = A^T d, \tag{32}$$

and the unknown coefficients can be estimated by

$$\hat{c} = \left(A^T A\right)^{-1} A^T d. \tag{33}$$

The ODE (28) can be solved by

$$x(t) = \left(x(0) + \frac{c_2}{c_1}\right) e^{c_1 t} - \frac{c_2}{c_1}. \tag{34}$$

Then the time response of $GM^{r*}(1,1)$ can be achieved by substituting coefficients obtained from (33) into (34)

$$\hat{x}^{(r)}[n] = \left(x^{(r)}[0] + \frac{\hat{c}_2}{\hat{c}_1}\right) e^{\hat{c}_1 n} - \frac{\hat{c}_2}{\hat{c}_1}. \tag{35}$$

The timeresponse sequence from (35) can be convolved with the corresponding inverse $-r$-th-order accumulative convolution sequence $\{ago^{-r*}[n]\}_{n=0}^{N-1}$ to obtain the estimated data sequence, i.e.,

$$ago^{(-r*)}[n] * \hat{x}^{(r)}[n] = \hat{x}^{(0)}[n]. \tag{36}$$

The mean absolute percentage error (MAPE) is employed here to indicate errors between the estimated data $\hat{x}^{(0)}[n]$ and original data $x^{(0)}[n]$, i.e.,

$$\text{MAPE} = \frac{1}{N-1} \sum_{n=1}^{N-1} \left| \frac{x^{(0)}[n] - \hat{x}^{(0)}[n]}{x^{(0)}[n]} \right| \times 100\%. \tag{37}$$

Finally, the modelling procedure for the fractional-order accumulative convolution $GM^{r*}(1,1)$ model is described in Algorithm 1, and depends on Algorithm 2 for finite sequence convolution based on Definition 2 and Algorithm 3 for the fractional accumulative convolution sequence based on Theorem 3.

Algorithm 1 Fractional-order accumulative convolution GM$^{r*}(1,1)$ model

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1: Input: sequence $\{x^{(0)}[n]\}_{n=0}^{N-1}$, and order r.
2: Get fractional-order accumulative convolution sequence $\{\mathrm{ago}^{r*}[n]\}_{n=0}^{N-1} \leftarrow \mathrm{ago}(N, r)$;
3: Get $\{x^{(r)}[n]\}_{n=0}^{N-1} \leftarrow \mathrm{conv}(\{\mathrm{ago}^{r*}[n]\}_{n=0}^{N-1}, \{x^{(0)}[n]\}_{n=0}^{N-1})$;
4: Get matrix A and load d of GM$(1,1)$ difference equations by (31);
5: Get coefficients by (33);
6: Get time response sequence by (35);
7: Get inverse accumulative convolution sequence $\{\mathrm{ago}^{-r*}[n]\}_{n=0}^{N-1} \leftarrow \mathrm{ago}(N, -r)$;
8: Recover data, $\{\hat{x}^{(0)}[n]\}_{n=0}^{N-1} \leftarrow \mathrm{conv}(\{\mathrm{ago}^{-r*}[n]\}_{n=0}^{N-1}, \{\hat{x}^{(r)}[n]\}_{n=0}^{N-1})$;
9: Get MAPE by (37).
10: Output: sequence $\{\hat{x}^{(0)}[n]\}_{n=0}^{N-1}$, and MAPE.

Algorithm 2 Finite sequence convolution

1: function $\{z[n]\}_{n=0}^{N-1} = \mathrm{conv}(\{x[n]\}_{n=0}^{N-1}, \{y[n]\}_{n=0}^{N-1})$
2: for $n = 0 : N - 1$
3:     $z[n] \leftarrow 0$;
4:     for $k = 0 : n$
5:         $z[n] \leftarrow x[k] \cdot y[n-k] + z[n]$;
6:     end
7: end

Algorithm 3 Fractional-order accumulative convolution sequence

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1: function $\{\mathrm{ago}^{r*}[n]\}_{n=0}^{N-1} = \mathrm{ago}(N, r)$
2: for $n = 0 : N - 1$
3:     $\mathrm{ago}^{r*}[n] \leftarrow 1$;
4:     for $i = 0 : n - 1$
5:         $\mathrm{ago}^{r*}[n] \leftarrow \dfrac{r+i}{1+i} \cdot \mathrm{ago}^{r*}[n]$;
6:     end
7: end

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4. Cases Study

Case 1 (Positive fractional accumulative convolution sequence). *Set $N = 8$, generate fractional-order accumulative convolution sequences $\{\mathrm{ago}^{r*}[n]\}_{n=0}^{N-1}$ with different orders $r = \frac{1}{2}$, 1, $\sqrt{2}$, $\sqrt{3}$, 2, e, 3, π. The fractional r-order accumulative convolution sequences are generated by Algorithm 3, and displayed in Table 1 and Figure 1.*

Table 1. Accumulative convolution sequences with fractional r-th order.

n r	$\dfrac{1}{2}$	1	$\sqrt{2}$	$\sqrt{3}$	2	e	3	π
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	0.500	1.000	1.414	1.732	2.000	2.718	3.000	3.142
2	0.375	1.000	1.707	2.366	3.000	5.054	6.000	6.506
3	0.313	1.000	1.943	2.943	4.000	7.948	10.000	11.150
4	0.273	1.000	2.144	3.482	5.000	11.363	15.000	17.119
5	0.246	1.000	2.322	3.992	6.000	15.267	21.000	24.452
6	0.226	1.000	2.482	4.479	7.000	19.640	28.000	33.179
7	0.209	1.000	2.629	4.947	8.000	24.461	36.000	43.330
8	0.196	1.000	2.765	5.400	9.000	29.714	45.000	54.930
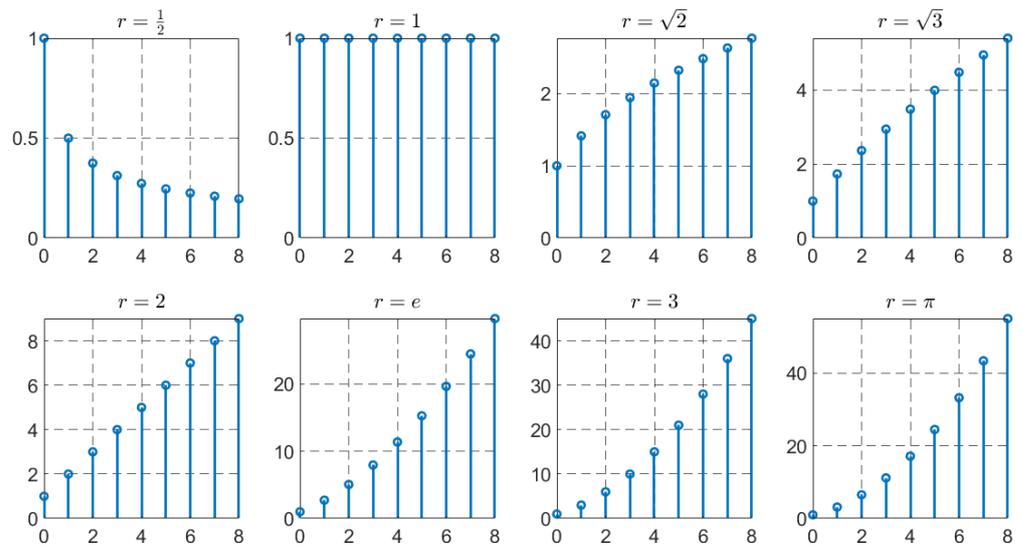
Figure 1. Accumulative convolution sequences with fractional *r*-th order.

Case 2 (Inverse fractional accumulative convolution sequence). *Set $N = 8$, generate the inverse $-r$-th-order accumulative convolution sequence $\{ago^{r*}[n]\}_{n=0}^{N-1}$ with different orders $r = -\frac{1}{2}$, $-1, -\sqrt{2}, -\sqrt{3}, -2, -e, -3, -\pi$. The inverse fractional $-r$-th-order accumulative convolution sequences are generated by Algorithm 3, and displayed in Table 2 and Figure 2.*

Table 2. Accumulative convolution sequences with inverse fractional $-r$-th order.

n \ r	$-\dfrac{1}{2}$	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2	$-e$	-3	$-\pi$
0	1.0000	1	1.0000	1.0000	1	1.0000	1	1.0000
1	−0.5000	−1	−1.4142	−1.7321	−2	−2.7183	−3	−3.1416
2	−0.1250	0	0.2929	0.6340	1	2.3354	3	3.3640
3	−0.0625	0	0.0572	0.0566	0	−0.5592	−1	−1.2801
4	−0.0391	0	0.0227	0.0179	0	−0.0394	0	0.0453
5	−0.0273	0	0.0117	0.0081	0	−0.0101	0	0.0078
6	−0.0205	0	0.0070	0.0044	0	−0.0038	0	0.0024
7	−0.0161	0	0.0046	0.0027	0	−0.0018	0	0.0010
8	−0.0131	0	0.0032	0.0018	0	−0.0010	0	0.0005
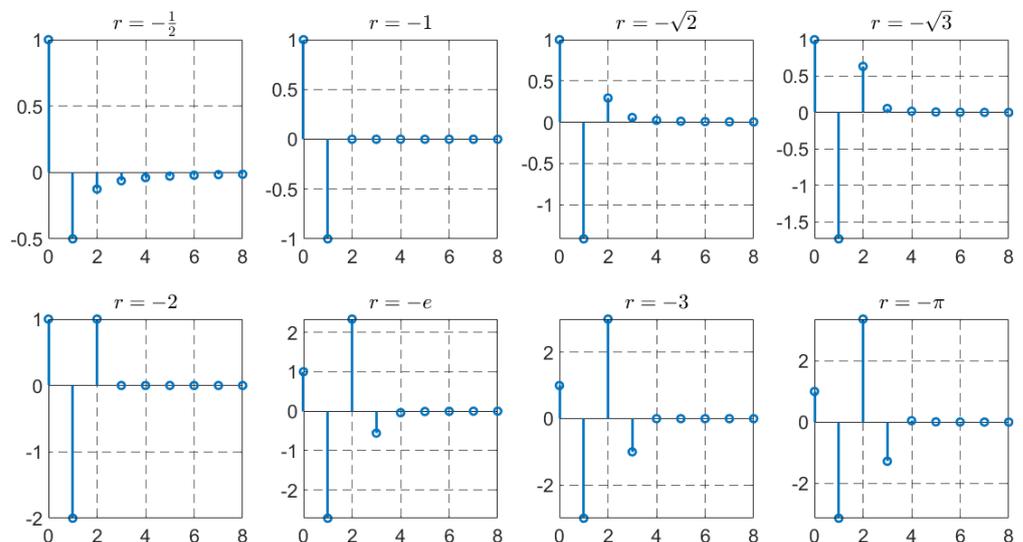
Figure 2. Accumulative convolution sequences with inverse fractional $-r$-th order.

Case 3 (Mutually inverse relationships). *Calculate the convolution of the r-th-order convolution sequences in Case 1 and the inverse −r-th-order convolution sequences in Case 2 to verify their mutually inverse relationship (21),*

$$\mathrm{ago}^{r*}[n] * \mathrm{ago}^{-r*}[n] = \delta[n],$$

with different orders $r = \frac{1}{2}, 1, \sqrt{2}, \sqrt{3}, 2, e, 3, \pi$. The convolution operations follow Algorithm 2 and the results are displayed in Figure 3.
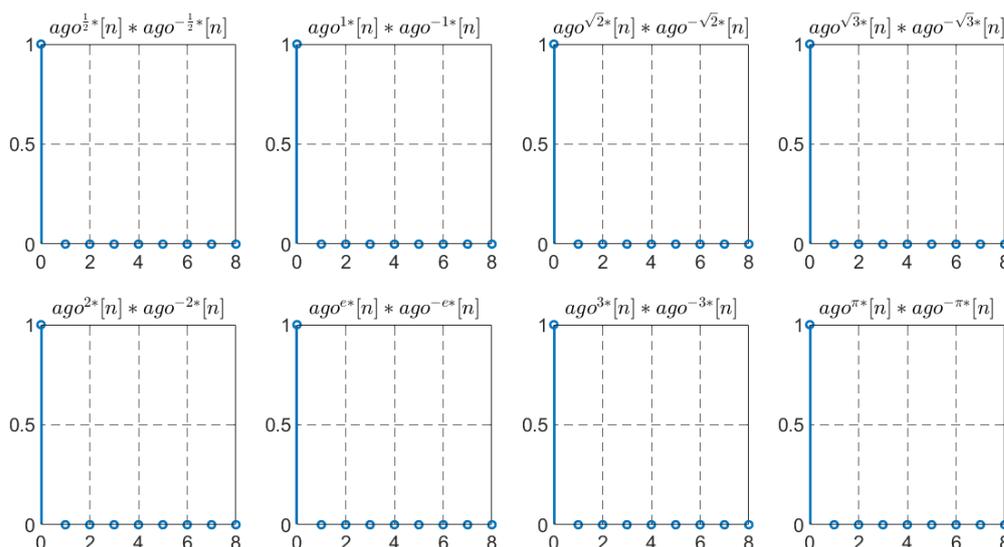
Figure 3. Convolution results of $\mathrm{ago}^{r*}[n] * \mathrm{ago}^{-r*}[n]$ with different $r-$th order. The first and third $*$ on the superscript represent the exponentiated convolution, and the second $*$ represents the convolution operation.

Case 4 (Lorenz system [56]). *Consider a Lorenz map*

$$L(x) = \begin{cases} 1 - 2|x|^2, & x \le 0, \\ -1 + 2x^2, & x > 0. \end{cases} \tag{38}$$

Using (38) as the iterative function, and starting from a random initial point $x_0 = -0.65$, with 19 iterations, where 16 points checks the model's coefficient estimation and 3 points check its predictive power. Figure 4 is a cobweb-diagram of the iteration procedure, and the obtained sequence is displayed in Figure 5, illustrating the chaotic characteristic of the sequence obtained from the Lorenz iteration.

The Lorenz system was introduced by a meteorologist, Edward Lorenz, in 1963, describing the unpredictable chaotic motion of convection flow in the atmosphere surrounding the Earth. The Lorenz system, with unstable topological structure, is sensitive to the initial value condition. A small difference at the beginning leads to great unpredictable diversity in the following, which is called "chaos". Another system family with chaotic features is the one-dimensional map. Guckenheimer and Williams studied the geometric and topological structure of the Lorenz attractor which is a one-dimensional self-map. The chaotic features can also be found in grey models. Tan re-established the context value of the grey model leading to the logistic equation, which mapped the properties of chaos [57].Wang studied the unbiased grey model with chaotic characteristics [58] which improved upon Deng's model. Zhang applied the grey model to simulate and predict data sequences from a Lorenz chaotic system [56], achieving a predictive precision of over 90%. To make the results comparable with Zhang's work, the same Lorenz map configuration was used is set as the one in [56].

Due to the chaotic characteristics of the Lorenz iteration, two data transforms are employed before grey modeling. Firstly, the absolute values of the Lorenz sequence are taken, denoted by $|x[n]|$ in Figure 6. Secondly, the sequence of absolute values by convolution transform are accumulated, shown in Figure 7. Then the transformed data is denoted as the input sequence $x^{(0)}[n]$ to the grey model.
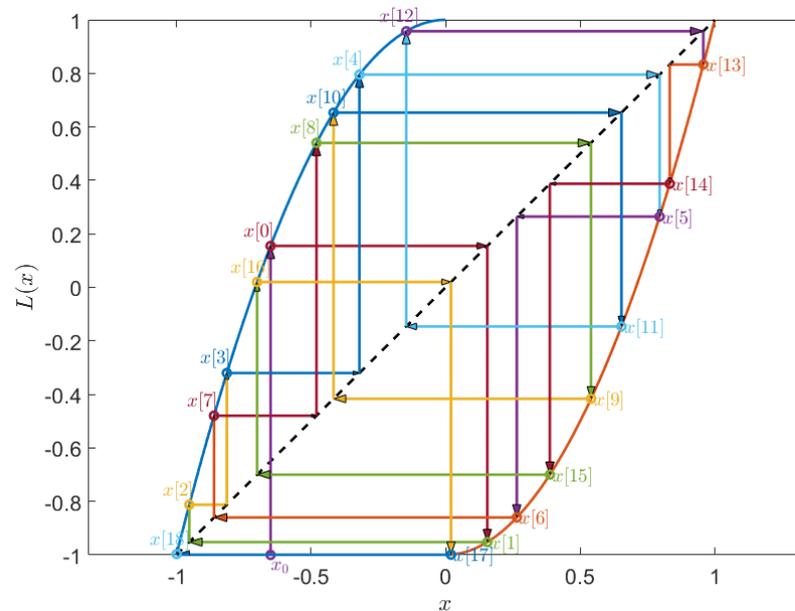
Figure 4. Cobweb-diagram of 19 Lorenz iterations with the initial condition $x_0 = -0.65$. One colour represents one step of the iteration process.
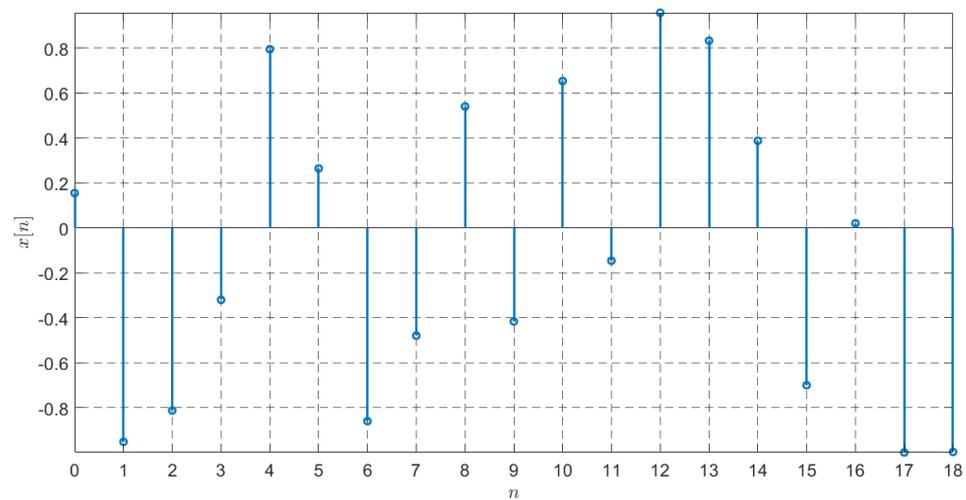
Figure 5. The sequence obtained from the Lorenz iterations.
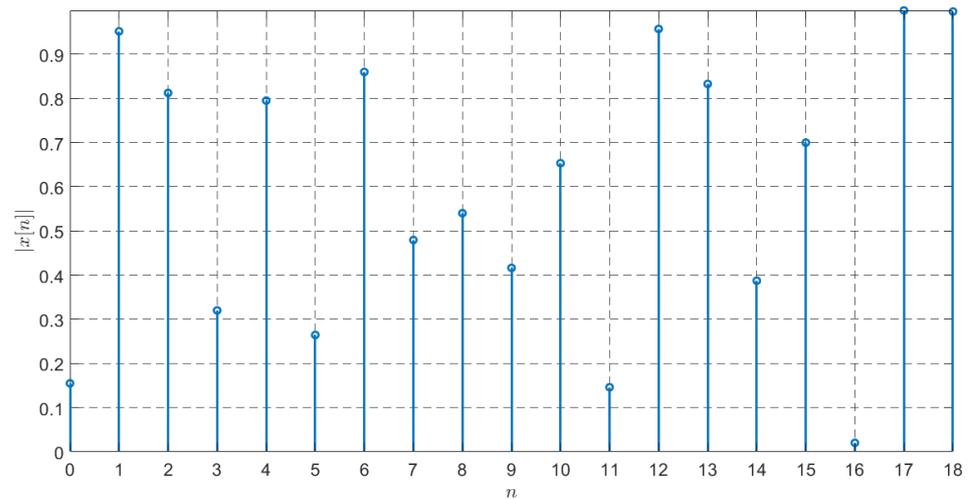
Figure 6. Data transformation—absolute values of the Lorenz iteration sequence.
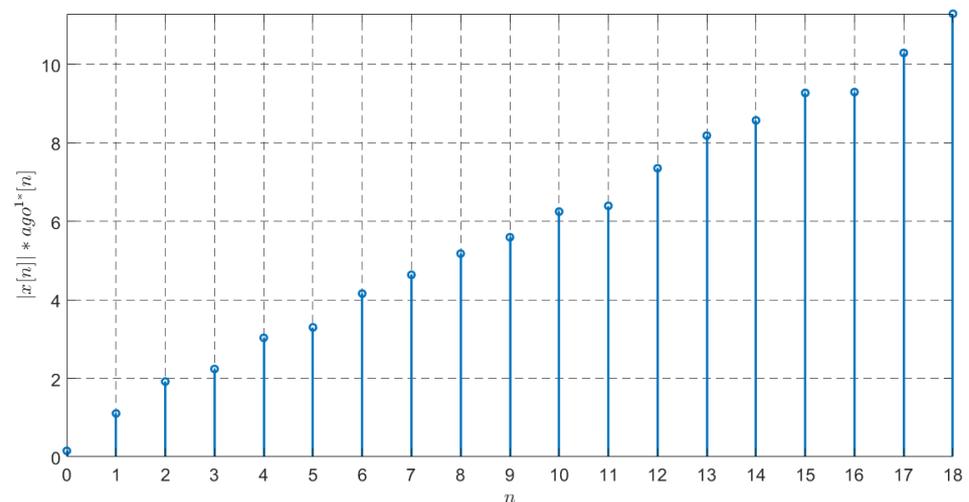
Figure 7. Data transformation—first-order accumulative absolute values by convolution. The first $*$ represents the convolution operation, and the second $*$ on the superscript represents the exponentiated convolution.

Following Algorithm 1, three different $GM^{r*}(1,1)$ models are established with $r = 1$, 0.5 and 0.362. The fitted sequences of the different models are compared with the input sequence in Table 3 and Figure 8. For $r = 1$, it turns out to be the $GM(1,1)$ model, and its fitted values are the same as those in Zhang's paper (Table 6 on p.1008 in [56]). In fact, the in-sample simulation precision was 86% (MAPE in-sample $= 13.9748\%$). This is close to 90% but a little below. However, the out-of-sample predictive precision declined to 78% (MAPE out-of-sample $= 22.0589\%$). The reason Zhang claimed the precision of the $GM(1,1)$ model exceeded 90% is that he measured the error at a single datum (Table 6 on p.1008 in [56]), which is the local error when he predicted the Lorenz chaotic system; while MAPE belongs to the global error which objectively measures the modelling precision over the whole data sequence. The new fractional-order accumulative convolution grey models are compared with Zhang's model in Table 3. It can be seen that both grey models with fractional accumulative convolution transform MAPEs are lower than the $GM(1,1)$ model. Obviously, from Figure 8, two fractional accumulative convolution grey models fitted original data better than ordinary grey model. The MAPE curve with respect to r in Figure 9 demonstrates that $r = 0.362$ is the best fractional-order since the $GM^{0.362*}(1,1)$ achieves the lowest MAPE in-sample, a precision over 97% (MAPE in-sample $= 2.5166\%$), and gives the best prediction out-of-sample, over 96% (MAPE out-of-sample $= 3.5414\%$).
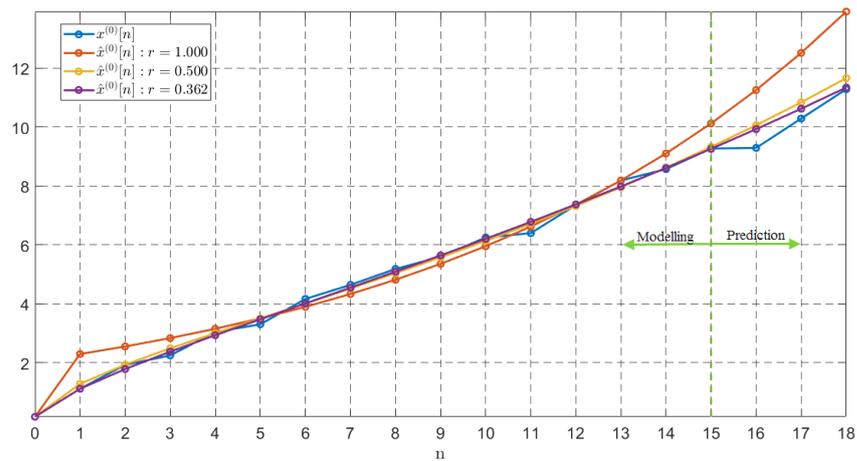
Figure 8. Comparison of fitted results from different orders.
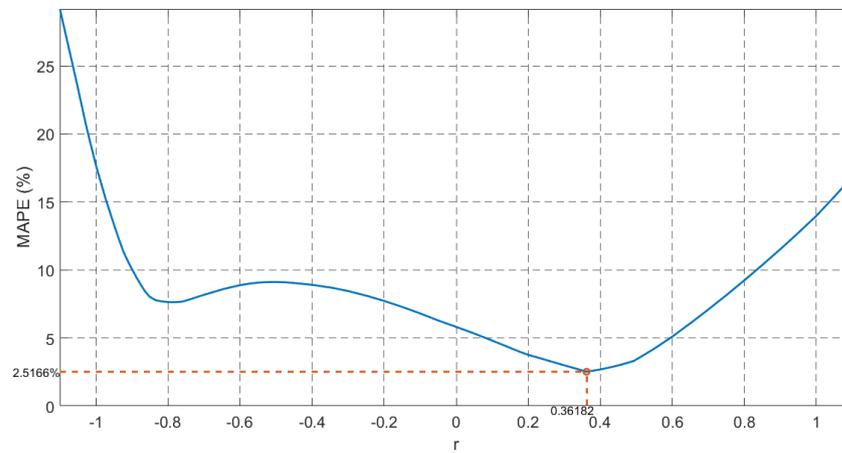
Figure 9. MAPE curve with respect to the accumulative convolution order *r*.

Table 3. Comparison of the fitted results from different order GM^{r*}(1,1) models.

n	$x^{(0)}[n]$	GM(1,1)	GM$^{0.5*}$(1,1)	GM$^{0.36*}$(1,1)
0	0.1550	0.1550	0.1550	0.1550
1	1.1069	2.2874	1.2752	1.1070
2	1.9194	2.5438	1.9276	1.7767
3	2.2394	2.8290	2.4809	2.3684
4	3.0346	3.1461	2.9973	2.9264
5	3.2991	3.4988	3.5005	3.4690
6	4.1591	3.8910	4.0029	4.0062
7	4.6386	4.3272	4.5122	4.5444
8	5.1788	4.8122	5.0339	5.0878
9	5.5951	5.3517	5.5721	5.6399
10	6.2485	5.9516	6.1304	6.2032
11	6.3946	6.6187	6.7118	6.7801
12	7.3519	7.3607	7.3193	7.3724
13	8.1848	8.1858	7.9556	7.9820
14	8.5722	9.1034	8.6234	8.6105
15	9.2721	10.1239	9.3253	9.2596
MAPE	in-sample	13.9748%	3.4191%	2.5166%
16	9.2923	11.2587	10.0642	9.9308
17	10.2915	12.5208	10.8427	10.6256
18	11.2882	13.9244	11.6639	11.3454
MAPE	out-of-sample	22.0589%	5.6635%	3.5414%

Case 5. *Consider time sequences of the electrocardiogram (ECG), which can be downloaded from the MIT-BIH Database [59]. Four ECG sequences: "hr.207", "hr.237", "hr.11839" and "hr.7257" are studied here. Sequences "hr.207" and "hr.237" contain 950 measured points of the transient heart rate (unit: beats per minute); while sequences "hr.11839" and "hr.7257" contain 1800 points. The time spacing between the measure points is 0.5 s.*

Figure 10 shows the fitted results from the $GM^{r*}(1,1)$ model compared with the original ECG sequence "hr.207". Figure 11 represents the absolute percentage error (APE) of the fitting at each sampling point. Furthermore, Figure 12 is the mean absolute percentage error (MAPE) of the fitting in each sample block, containing 10 sampling points. Similarly, Figures 13–15 display the ECG sequence for "hr.237". Figures 16–18 display the ECG sequence for "hr.11839". Figures 19–21 display the ECG sequence for "hr.7257".
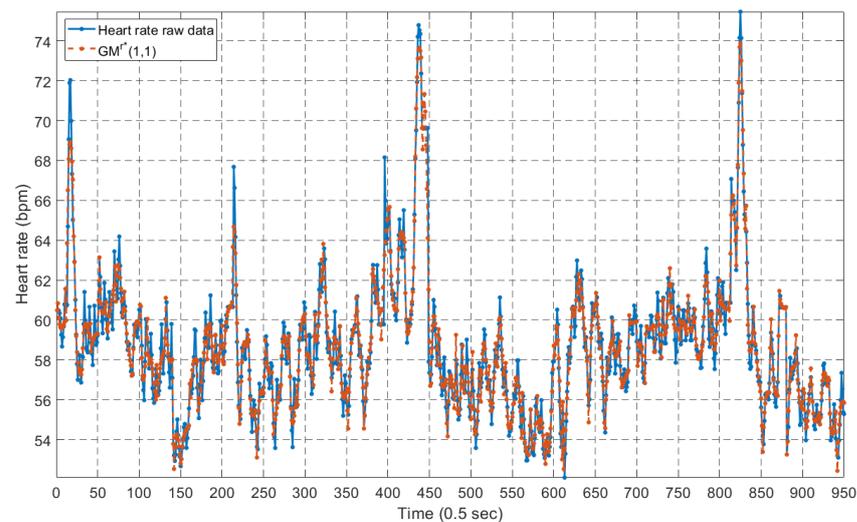
Figure 10. Comparison of the original sequence "hr.207" and the fitted results from the $GM^{r*}(1,1)$ model.
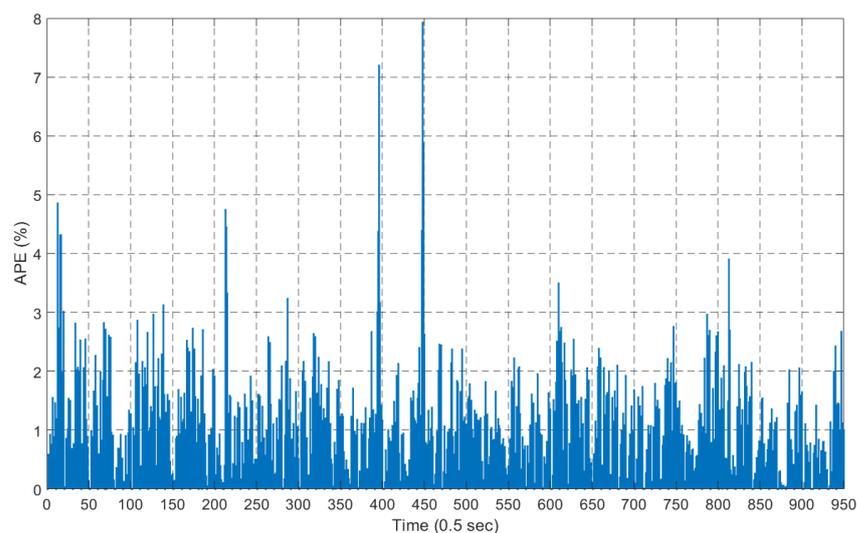
Figure 11. APE of the original sequence "hr.207" using the $GM^{r*}(1,1)$ model.
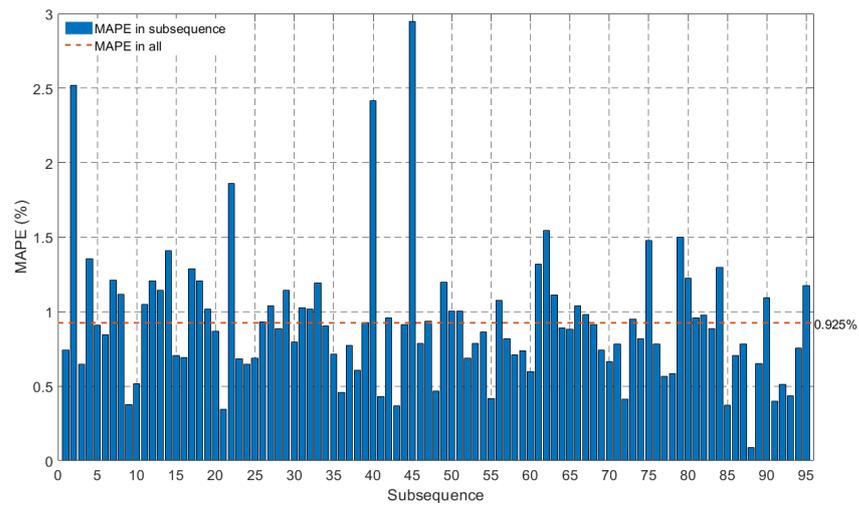
Figure 12. MAPE of the original sequence "hr.207" using the $GM^{r*}(1,1)$ model.
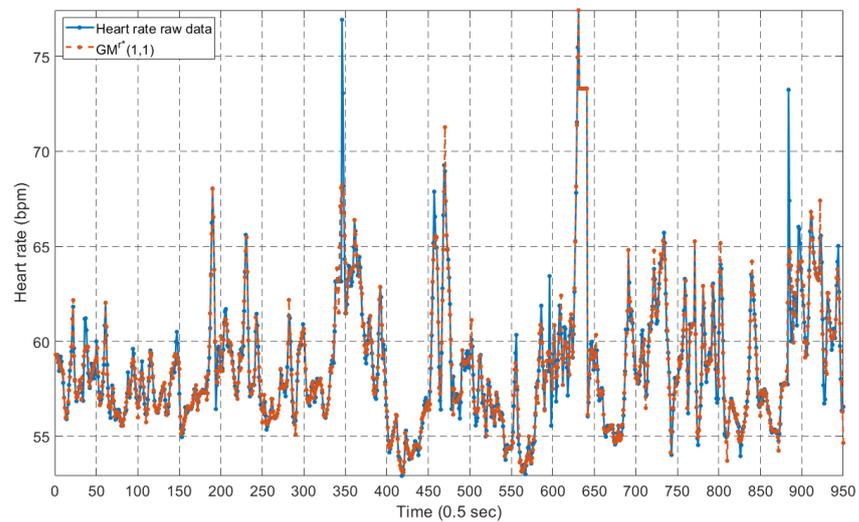
Figure 13. Comparison of the original sequence "hr.237" and the fitted results from the $GM^{r*}(1,1)$ model.
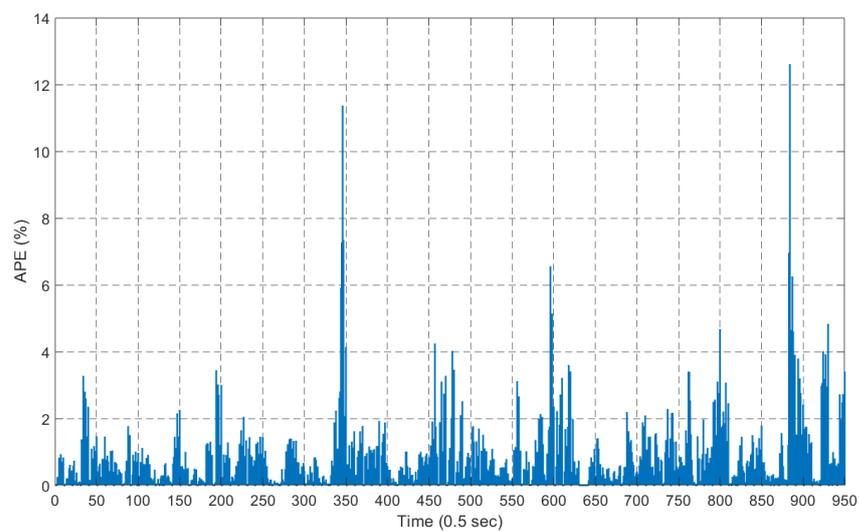
Figure 14. APE of the original sequence "hr.237" using the $GM^{r*}(1,1)$ model.
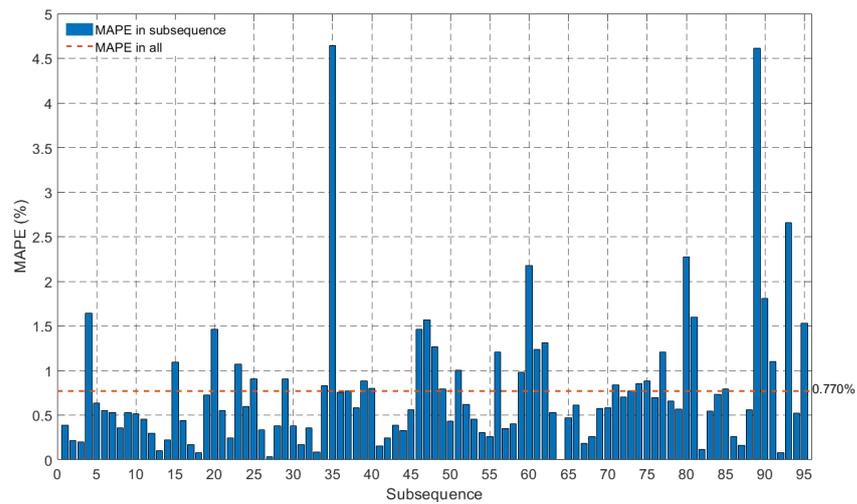
Figure 15. APE of the original sequence "hr.237" using the $GM^{r*}(1,1)$ model.
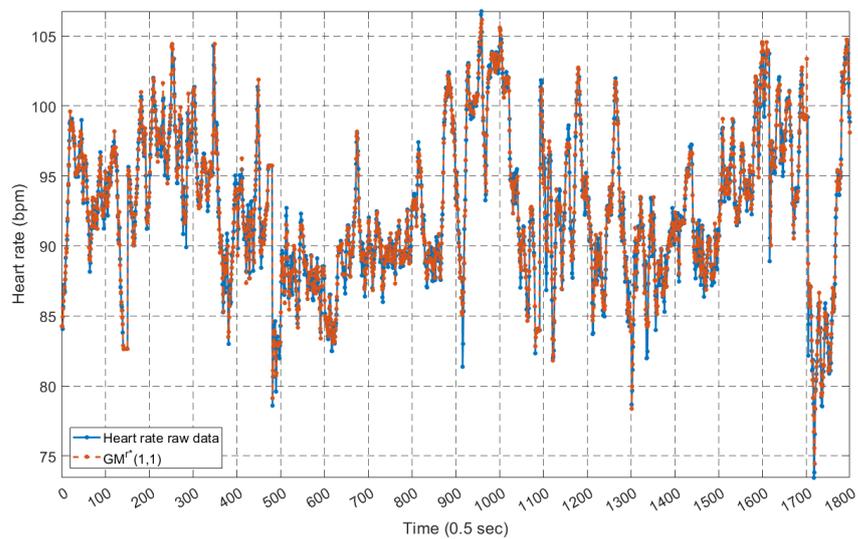
Figure 16. Comparison of the original sequence "hr.11839" and the fitted results from the $GM^{r*}(1,1)$ model.
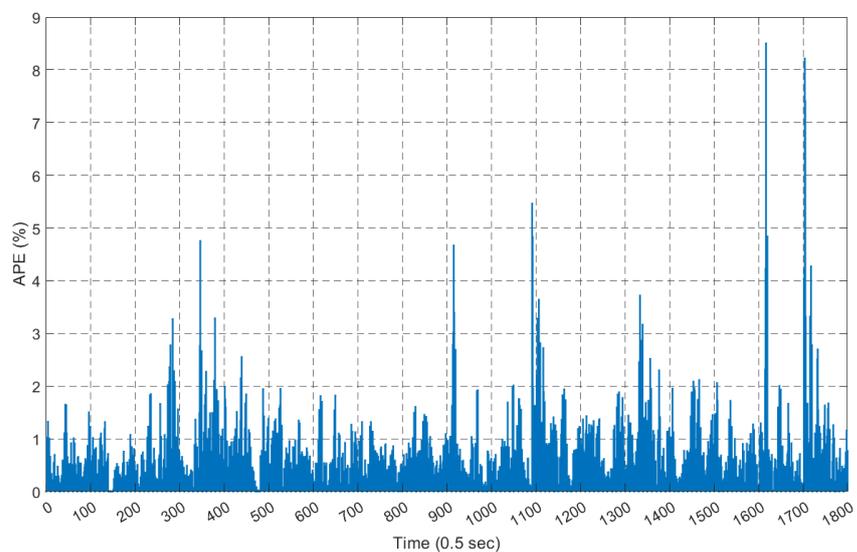
Figure 17. APE of the original sequence "hr.11839" using the $GM^{r*}(1,1)$ model.
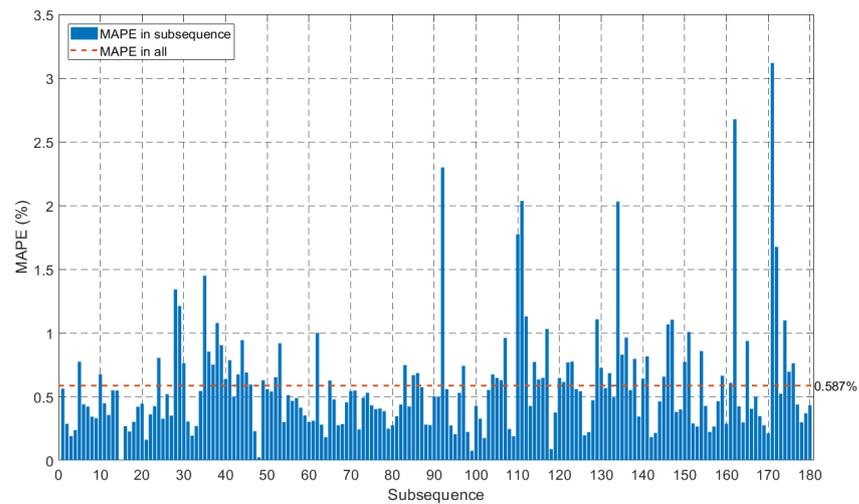
Figure 18. MAPE of the original sequence "hr.11839" using the $GM^{r*}(1,1)$ model.
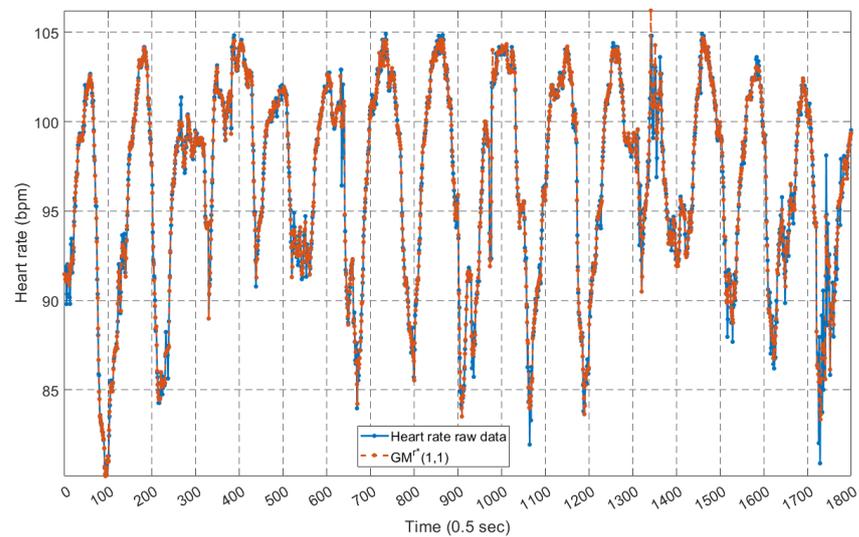
Figure 19. Comparison of the original sequence "hr.7257" and the fitted results from the $GM^{r*}(1,1)$ model.
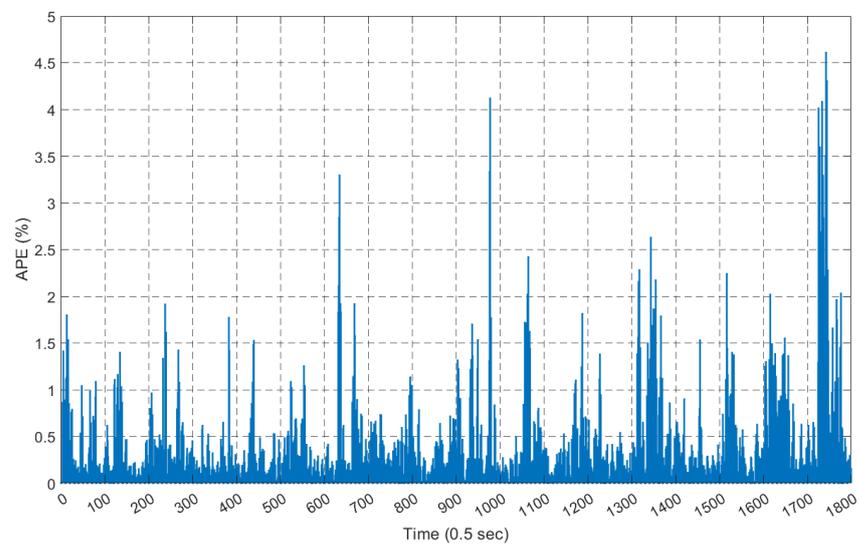
Figure 20. APE of original sequence "hr.7257" by $GM^{r*}(1,1)$ model.
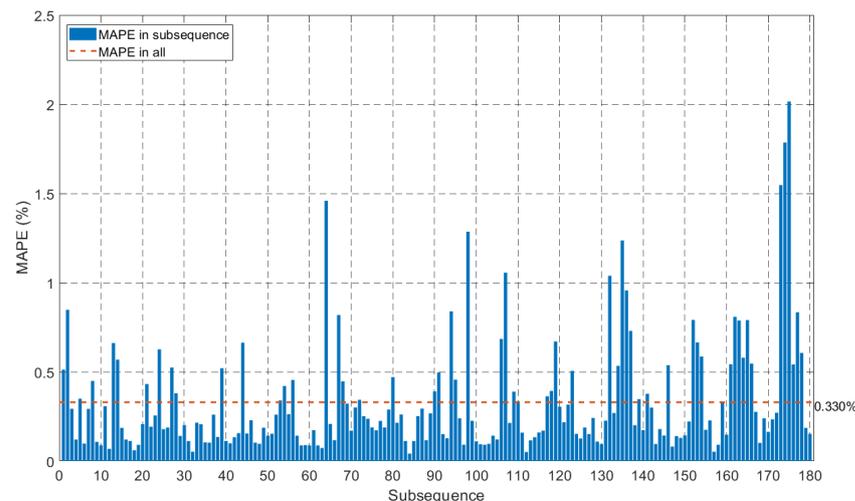
Figure 21. MAPE of the original sequence "hr.7257" using the $GM^{r*}(1,1)$ model.

5. Conclusions

The accumulation generation and its inversion are essential procedures in grey modelling. They are implemented here in a new way—sequence convolution. The fractional-order accumulative convolution sequence is constructed to fulfil the fractional-order accumulative generation procedure. Based on accumulative convolution, the fractional accumulative convolution grey model is established. The main results of this paper are as follows.

(i)   The accumulative generation convolution sequence is constructed by the unit impulse sequence in Theorem 1. The classical accumulative generation procedure can be fulfilled by the sequence convolution, i.e., $\text{ago}[n] * x^{(0)}[n]$ yields to accumulated sequence, and its inverse, $\text{ago}^{-1*}[n] * x^{(1)}[n]$ returns to the original sequence. They are mutually inverse in the sense of convolution operation, i.e., $\text{ago}[n] * \text{ago}^{-1*}[n] = \delta[n]$.

(ii)   The fractional-order accumulative convolution sequence is constructed in Theorem 3. With the convolution transform, $\text{ago}^{r*}[n] * x^{(0)}[n]$ yields to the fractional accumulated sequence $x^{(r)}[n]$, and with the convolution transform of its inverse sequence, $\text{ago}^{-r*}[n] * x^{(r)}[n]$ recovers the data. Furthermore, the two fractional-order accumulative convolution sequences are mutually inverse in the convolution operation, $\text{ago}^{r*}[n] * \text{ago}^{-r*}[n] = \delta[n]$.

(iii)   Under the fractional-order accumulative convolution transform, the new $GM^{r*}(1,1)$ is established in Algorithm 1. The cases above verify and demonstrate the validity and effectiveness of the new model.

Accumulative convolution transform and its inversion are new concepts for data transformation. The success in the grey forecasting $GM^{r*}(1,1)$ model is one application of the fractional-order accumulative convolution transformation. It could be found in more extensive applications in other disciplines. Some possible research directions in the future are listed as follows.

(i)   Since the unit impulse response of the fractional-order accumulative system has been obtained, the powerful tools in digital signal process could be used to analysis the more impressive properties of the fractional-order accumulative system.

(ii)   Fractional-order accumulative convolution transform is the discrete form of integration, and can be applied in fractional calculus and fractional differential equations.

Institutional Review Board Statement: The study did not require ethical approval.

Data Availability Statement: Publicly available datasets were analysed in this study. This data can be found at: http://ecg.mit.edu/time-series/index.html (accessed on 5 May 2023).

Conflicts of Interest: The authors declare no conflicts of interest.

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