fractal and fractional

Editorial

Advances in Boundary Value Problems for Fractional Differential Equations

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Fractional-order differential and integral operators and fractional differential equations have extensive applications in the mathematical modelling of real-world phenomena which occur in scientific and engineering disciplines such as physics, chemistry, biophysics, biology, medical sciences, financial economics, ecology, bioengineering, control theory, signal and image processing, aerodynamics, transport dynamics, thermodynamics, viscoelasticity, hydrology, statistical mechanics, electromagnetics, astrophysics, cosmology, and rheology. Fractional differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer-order differential equations. The Special Issue "Advances in Boundary Value Problems for Fractional Differential Equations" covers aspects of the recent developments in the theory and applications of fractional differential equations, inclusions, inequalities, and systems of fractional differential equations with Riemann–Liouville derivatives, Caputo derivatives, or other generalized fractional derivatives subject to various boundary conditions. In the papers published in this Special Issue, the authors study the existence, uniqueness, multiplicity, and nonexistence of classical or mild solutions, the approximation of solutions, and the approximate controllability of mild solutions for diverse models. I will present these papers in the following, grouped according to their subject.

1. Equations and Systems of Equations with Sequential Fractional Derivatives

In paper [1], the authors investigate the differential equation

$$D^{\sigma_n} \mathrm{z}(t) = \mathcal{A}\mathrm{z}(t) + \mathrm{f}(t), \quad t \in (0, T], \tag{1}$$

with the initial conditions

$$D^{\sigma_k} \mathrm{z}(0) = \mathrm{z}_k, \quad k = 0, 1, \ldots, n-1, \tag{2}$$

where the operator $\mathcal{A} : D_{\mathcal{A}} \subset \mathcal{Z} \to \mathcal{Z}$ is linear and closed with its domain $D_{\mathcal{A}}$ (a dense set), $\mathcal{Z}$ is a Banach space, $\mathrm{f} : [0, T] \to \mathcal{Z}$ is a given function, and D^{σ_k}, $k = 0, 1, \ldots, n$ are the Dzhrbashyan–Nersesyan fractional derivatives. For the set of numbers $\{\alpha_k\}_0^n$, with $\alpha_k \in (0, 1]$, $k = 0, 1, \ldots, n$, they introduced the numbers $\sigma_k = \sum_{j=0}^{k} \alpha_j - 1$, $k = 0, 1, \ldots, n$, with the condition $\sigma_n > 0$. The fractional derivatives D^{σ_k}, $k = 0, 1, \ldots, n$ are given by $D^{\sigma_0}\mathrm{z}(t) = D_t^{\alpha_0 - 1}\mathrm{z}(t)$, $D^{\sigma_k}\mathrm{z}(t) = D_t^{\alpha_k - 1} D_t^{\alpha_{k-1}} D_t^{\alpha_{k-2}} \ldots D_t^{\alpha_0}\mathrm{z}(t)$, for $k = 1, 2, \ldots, n$, where D_t^{β} is the Riemann–Liouville integral for $\beta \leq 0$ and the Riemann–Liouville derivative for $\beta > 0$. The Dzhrbashyan–Nersesyan fractional derivative D^{σ_n} is a generalization of the Riemann–Liouville and Caputo fractional derivatives. The authors prove firstly the existence and uniqueness of the k-resolving families of operators (for $k = 0, \ldots, n-1$) for the homogeneous equation $D^{\sigma_n}z(t) = \mathcal{A}z(t)$, and then they give a criterion for the existence and uniqueness of analytic k-resolving families, namely $\mathcal{A}$ belongs to a class of operators denoted by $\mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$. Different properties of the resolving families are also studied, and a perturbation theorem for operators from $\mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$ is presented. Then, the authors prove the existence and uniqueness of a solution for problem (1),(2), where f is continuous in the graph norm of $\mathcal{A}$ or it is a Hölderian function. As an application,

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they show the existence of a unique solution for an initial boundary value problem to a fractional linearized model of viscoelastic Oldroyd fluid dynamics.

Paper [2] deals with a nonlinear coupled system of sequential fractional differential equations

$$\begin{cases} ({}^cD^{q+1} + {}^cD^q)\mathrm{x}(t) = \mathrm{f}(t, \mathrm{x}(t), \mathrm{y}(t)), \ \ t \in [0,1], \\ ({}^cD^{p+1} + {}^cD^p)\mathrm{y}(t) = \mathrm{g}(t, \mathrm{x}(t), \mathrm{y}(t)), \ \ t \in [0,1], \end{cases} \tag{3}$$

supplemented with the coupled multipoint and Riemann–Stieltjes integral boundary conditions

$$\begin{cases} \mathrm{x}(0) = 0, \ \mathrm{x}'(0) = 0, \ \mathrm{x}'(1) = 0, \\ \mathrm{x}(1) = k\int_0^\rho \mathrm{y}(s)\, d\mathcal{A}(s) + \sum_{i=1}^{n-2} \alpha_i \mathrm{y}(\sigma_i) + k_1 \int_\nu^1 \mathrm{y}(s)\, d\mathcal{A}(s), \\ \mathrm{y}(0) = 0, \ \mathrm{y}'(0) = 0, \ \mathrm{y}'(1) = 0, \\ \mathrm{y}(1) = h\int_0^\rho \mathrm{x}(s)\, d\mathcal{A}(s) + \sum_{i=1}^{n-2} \beta_i \mathrm{x}(\sigma_i) + h_1 \int_\nu^1 \mathrm{x}(s)\, d\mathcal{A}(s), \end{cases} \tag{4}$$

where $p, q \in (2,3]$, ${}^cD^\kappa$ denotes the Caputo fractional derivative of order $\kappa \in \{q, p\}$, $0 < \rho < \sigma_i < \nu < 1$, $\mathrm{f}, \mathrm{g} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $k, h, k_1, h_1, \alpha_i, \beta_i \in \mathbb{R}$, for $i = 1, 2, \ldots, n-2$, and $\mathcal{A}$ is a function of bounded variation. The word sequential is used in the sense that the operator ${}^cD^{q+1} + {}^cD^q$ can be written as the composition of operators ${}^cD^q$ and $D + I$, where D is the usual differential operator and I is the identity operator. Under some assumptions of the data of the problem, the authors prove the existence and uniqueness of solutions for problem (3),(4) by applying the Leray–Schauder alternative and the Banach contraction mapping principle.

2. Resonance Problems for Caputo Fractional Differential Equations

Paper [3] is concerned with the nonlinear boundary value problem for a fractional differential equation of variable order at resonance

$$\begin{cases} {}^cD_{0+}^{\mathrm{u}(t)}\mathrm{x}(t) = \mathrm{g}(t, \mathrm{x}(t)), \ \ t \in [0,T], \\ \mathrm{x}(0) = \mathrm{x}(T), \end{cases} \tag{5}$$

where ${}^cD_{0+}^{\mathrm{u}(t)}$ is the Caputo derivative of variable order $\mathrm{u}(t)$ with $\mathrm{u} : [0,T] \to (0,1]$ and $\mathrm{g} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. This problem is at resonance, that is, the corresponding linear homogeneous boundary value problem has non-trivial solutions. The authors transform firstly problem (5) to an equivalent standard boundary value problem at resonance with a fractional derivative of constant order by using some generalized intervals and piece-wise constant functions. Then, by applying Mawhin's continuation theorem, they demonstrate the existence of at least one solution to (5).

In paper [4], the authors study the fractional differential equation in space $\mathbb{R}^n$

$${}^cD_{0+}^{\alpha}\mathrm{u}(t) = \mathrm{f}(t, \mathrm{u}(t), {}^cD_{0+}^{\alpha-1}\mathrm{u}), \ \ t \in (0,1), \tag{6}$$

subject to the boundary conditions

$$\mathrm{u}(0) = \mathcal{B}\mathrm{u}(\xi), \ \ \mathrm{u}(1) = \mathcal{C}\mathrm{u}(\eta), \tag{7}$$

where ${}^cD_{0+}^{k}$ denotes the Caputo fractional derivative of order $k \in \{\alpha, \alpha-1\}$, $\xi, \eta \in (0,1)$, $\alpha \in (1,2]$, $\mathrm{f} : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ satisfies Carathéodory conditions, and $\mathcal{B}, \mathcal{C}$ are n-order nonzero square matrices. They prove the existence of solutions of problem (6),(7) by using Mawhin coincidence degree theory.

3. Approximations of Solutions for Caputo Fractional Differential Equations

Paper [5] is devoted to the Caputo fractional differential equation with variable coefficients

$$D_x^{\lambda}\mathrm{u}(x)+c_1(x)\mathrm{u}'(x)+c_0(x)\mathrm{u}(x)=\mathrm{g}(x),\ \ 0<x<1, \tag{8}$$

with the boundary conditions

$$p_0\mathrm{u}(0)-q_0\mathrm{u}'(0)=b_0,\ \ p_1\mathrm{u}(1)+q_1\mathrm{u}'(1)=b_1, \tag{9}$$

where $\lambda\in(1,2]$, D_x^{λ} is the Caputo fractional derivative of order λ, c_1, c_0, and g are continuous functions, $p_0,p_1,q_0,q_1\geq 0$, and $p_0p_1+p_0q_1+q_0p_1\neq 0$. By using the shifted Chebyshev polynomials of the first kind and the collocation method, the authors present approximate solutions to problem (8),(9).

4. Systems of Fractional Differential Equations with *p*-Laplacian Operators

In paper [6], the authors investigate the system of fractional differential equations with r_1-Laplacian and r_2-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma_1}\left(\varphi_{r_1}\left(D_{0+}^{\delta_1}\mathrm{u}(t)\right)\right)=\mathrm{f}\left(t,\mathrm{u}(t),\mathrm{v}(t),I_{0+}^{\sigma_1}\mathrm{u}(t),I_{0+}^{\sigma_2}\mathrm{v}(t)\right),\ \ t\in(0,1),\\ D_{0+}^{\gamma_2}\left(\varphi_{r_2}\left(D_{0+}^{\delta_2}\mathrm{v}(t)\right)\right)=\mathrm{g}\left(t,\mathrm{u}(t),\mathrm{v}(t),I_{0+}^{\varsigma_1}\mathrm{u}(t),I_{0+}^{\varsigma_2}\mathrm{v}(t)\right),\ \ t\in(0,1), \end{cases} \tag{10}$$

supplemented with the uncoupled nonlocal boundary conditions

$$\begin{cases} \mathrm{u}^{(i)}(0)=0,\ \ i=0,\ldots,p-2,\ \ D_{0+}^{\delta_1}\mathrm{u}(0)=0,\\ \varphi_{r_1}(D_{0+}^{\delta_1}\mathrm{u}(1))=\displaystyle\int_0^1\varphi_{r_1}(D_{0+}^{\delta_1}\mathrm{u}(\tau))\,d\mathcal{H}_0(\tau),\ \ D_{0+}^{\alpha_0}\mathrm{u}(1)=\sum_{k=1}^{n}\int_0^1 D_{0+}^{\alpha_k}\mathrm{u}(\tau)\,d\mathcal{H}_k(\tau),\\ \mathrm{v}^{(j)}(0)=0,\ \ j=0,\ldots,q-2,\ \ D_{0+}^{\delta_2}\mathrm{v}(0)=0,\\ \varphi_{r_2}(D_{0+}^{\delta_2}\mathrm{v}(1))=\displaystyle\int_0^1\varphi_{r_2}(D_{0+}^{\delta_2}\mathrm{v}(\tau))\,d\mathcal{K}_0(\tau),\ \ D_{0+}^{\beta_0}\mathrm{v}(1)=\sum_{k=1}^{m}\int_0^1 D_{0+}^{\beta_k}\mathrm{v}(\tau)\,d\mathcal{K}_k(\tau), \end{cases} \tag{11}$$

where $\gamma_1,\gamma_2\in(1,2]$, $p,q\in\mathbb{N}$, $p,q\geq 3$, $\delta_1\in(p-1,p]$, $\delta_2\in(q-1,q]$, $n,m\in\mathbb{N}$, $\sigma_1,\varsigma_1,\sigma_2,\varsigma_2>0$, $\alpha_i\in\mathbb{R}$, $i=0,\ldots,n$, $0\leq\alpha_1<\alpha_2<\ldots<\alpha_n\leq\alpha_0<\delta_1-1$, $\alpha_0\geq 1$, $\beta_j\in\mathbb{R}$, $j=0,\ldots,m$, $0\leq\beta_1<\beta_2<\ldots<\beta_m\leq\beta_0<\delta_2-1$, $\beta_0\geq 1$, $\varphi_{r_k}(\tau)=|\tau|^{r_k-2}\tau$, $r_k>1$, $k=1,2$, the functions $\mathrm{f},\mathrm{g}:(0,1)\times\mathbb{R}_+^4\to\mathbb{R}_+$ are continuous, singular at $t=0$ and/or $t=1$, ($\mathbb{R}_+=[0,\infty)$), $I_{0+}^{\varkappa}$ is the Riemann–Liouville fractional integral of order $\varkappa$ (for $\varkappa=\sigma_1,\varsigma_1,\sigma_2,\varsigma_2$), $D_{0+}^{\varkappa}$ is the Riemann–Liouville fractional derivative of order $\varkappa$ (for $\varkappa=\gamma_1,\gamma_2,\delta_1,\delta_2,\alpha_0,\ldots,\alpha_n,\beta_0,\ldots,\beta_m$), and the integrals from the boundary conditions (11) are Riemann–Stieltjes integrals with $\mathcal{H}_i:[0,1]\to\mathbb{R}$, $i=0,\ldots,n$ and $\mathcal{K}_j:[0,1]\to\mathbb{R}$, $j=0,\ldots,m$ functions of bounded variation. By using the Guo–Krasnoselskii fixed point theorem of cone expansion and norm-type compression, they prove the existence and multiplicity of positive solutions for problem (10),(11).

Paper [7] is focused on the system of fractional differential equations (10) subject to the nonlocal coupled boundary conditions

$$\begin{cases} \mathrm{u}^{(i)}(0)=0,\ \ i=0,\ldots,p-2,\ \ D_{0+}^{\delta_1}\mathrm{u}(0)=0,\\ \varphi_{r_1}(D_{0+}^{\delta_1}\mathrm{u}(1))=\displaystyle\int_0^1\varphi_{r_1}(D_{0+}^{\delta_1}\mathrm{u}(\tau))\,d\mathcal{H}_0(\tau),\ \ D_{0+}^{\alpha_0}\mathrm{u}(1)=\sum_{i=1}^{n}\int_0^1 D_{0+}^{\alpha_i}\mathrm{v}(\tau)\,d\mathcal{H}_i(\tau),\\ \mathrm{v}^{(j)}(0)=0,\ \ j=0,\ldots,q-2,\ \ D_{0+}^{\delta_2}\mathrm{v}(0)=0,\\ \varphi_{r_2}(D_{0+}^{\delta_2}\mathrm{v}(1))=\displaystyle\int_0^1\varphi_{r_2}(D_{0+}^{\delta_2}\mathrm{v}(\tau))\,d\mathcal{K}_0(\tau),\ \ D_{0+}^{\beta_0}\mathrm{v}(1)=\sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}\mathrm{u}(\tau)\,d\mathcal{K}_j(\tau), \end{cases} \tag{12}$$

where $\alpha_i\in\mathbb{R}$, $i=0,\ldots,n$, $0\leq\alpha_1<\alpha_2<\ldots<\alpha_n\leq\beta_0<\delta_2-1$, $\beta_0\geq 1$, $\beta_j\in\mathbb{R}$, $j=0,\ldots,m$, $0\leq\beta_1<\beta_2<\ldots<\beta_m\leq\alpha_0<\delta_1-1$, $\alpha_0\geq 1$. The authors present existence and multiplicity results for the positive solutions of problem (10),(12) by applying the Guo–Krasnoselskii fixed point theorem.

Paper [8] deals with a system of fractional differential equations with ϱ_1-Laplacian and ϱ_2-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}u(t))) + a(t)f(v(t)) = 0, \ \ t \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}v(t))) + b(t)g(u(t)) = 0, \ \ t \in (0,1), \end{cases} \tag{13}$$

with the coupled nonlocal boundary conditions

$$\begin{cases} u^{(i)}(0) = 0, \ i = 0,\dots,p-2; \ \ D_{0+}^{\delta_1}u(0) = 0, \ \ D_{0+}^{\alpha_0}u(1) = \displaystyle\sum_{i=1}^{n}\int_0^1 D_{0+}^{\alpha_i}v(\tau)\,d\mathcal{H}_i(\tau) + c_0, \\ v^{(j)}(0) = 0, \ j = 0,\dots,q-2; \ \ D_{0+}^{\delta_2}v(0) = 0, \ \ D_{0+}^{\beta_0}v(1) = \displaystyle\sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}u(\tau)\,d\mathcal{K}_j(\tau) + d_0, \end{cases} \tag{14}$$

where $\gamma_1, \gamma_2 \in (0,1]$, $p, q \in \mathbb{N}$, $p, q \geq 3$, $\delta_1 \in (p-1, p]$, $\delta_2 \in (q-1, q]$, $n, m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ for all $i = 0,1,\dots,n$, $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\beta_j \in \mathbb{R}$ for all $j = 0,1,\dots,m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, the functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ and $a, b : [0,1] \to \mathbb{R}_+$ are continuous, c_0 and d_0 are positive parameters, $\varrho_1, \varrho_2 > 1$, $\varphi_{\varrho_i}(\zeta) = |\zeta|^{\varrho_i - 2}\zeta$, $i = 1,2$, the functions $\mathcal{H}_j$, $j = 1,\dots,n$ and $\mathcal{K}_i$, $i = 1,\dots,m$ have bounded variation, and D_{0+}^{κ} denotes the Riemann–Liouville derivative of order κ (for $\kappa = \gamma_1, \gamma_2, \delta_1, \delta_2, \alpha_i$ for $i = 0,1,\dots,n$, β_j for $j = 0,1,\dots,m$). The authors give sufficient conditions for the functions f and g, and intervals for the parameters c_0 and d_0 such that problem (13),(14) have at least one positive solution or they have no positive solutions. They apply the Schauder fixed point theorem in the proof of the main existence result.

In paper [9], the authors study a system of nonlinear Fredholm fractional integro-differential equations with p-Laplacian operator

$$\begin{cases} {}_tD_T^{\gamma_j}(\mathrm{k}_j(t)\phi_p({}_0^cD_t^{\gamma_j}\mathrm{z}_j(t))) + \mathrm{l}_j(t)\phi_p(\mathrm{z}_j(t)) \\ \quad = \lambda \mathrm{f}_{\mathrm{z}_j}(t, \mathrm{z}_1(t), \dots, \mathrm{z}_m(t)) + \displaystyle\int_0^T \mathrm{g}_j(t,s)\phi_p(\mathrm{z}_j(s))\,ds, \ \ t \in [0,T], \ \ j = 1,2,\dots,m, \\ \mathrm{z}_j(t) = \displaystyle\int_0^T \mathrm{g}_j(t,s)\phi_p(\mathrm{z}_j(s))\,ds, \ \ t \in [0,T], \ \ j = 1,2,\dots,m, \end{cases} \tag{15}$$

supplemented with the Sturm–Liouville boundary conditions

$$\begin{cases} c_j\mathrm{k}_j(0)\phi_p(\mathrm{z}_j(0)) - c'_j\,{}_tD_T^{\gamma_j-1}(\mathrm{k}_j(0)\phi_p({}_0^cD_t^{\gamma_j}\mathrm{z}_j(0))) = 0, \ \ j = 1,2,\dots,m, \\ d_j\mathrm{k}_j(T)\phi_p(\mathrm{z}_j(T)) + d'_j\,{}_tD_T^{\gamma_j-1}(\mathrm{k}_j(T)\phi_p({}_0^cD_t^{\gamma_j}\mathrm{z}_j(T))) = 0, \ \ j = 1,2,\dots,m, \end{cases} \tag{16}$$

where λ is a positive parameter, $\mathrm{k}_i, \mathrm{l}_i \in L^\infty[0,T]$ with $\operatorname{ess\,inf}_{[0,T]}\mathrm{k}_i(t) > 0$ and $\operatorname{ess\,inf}_{[0,T]}$ $\mathrm{l}_i(t) \geq 0$, c_i, d_i, c'_i, d'_i, $i = 1,2,\dots,m$, are positive constants, $p \in (1,\infty)$, $\phi_p(s) = |s|^{p-2}s$, $(s \neq 0)$, $\phi_p(0) = 0$, the functions $\mathrm{f} : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ and $\mathrm{g}_i : [0,T] \times [0,T] \to \mathbb{R}$, $i = 1,\dots,m$ satisfy some conditions, and ${}_0^cD_t^{\gamma_j}$ and ${}_tD_T^{\gamma_j}$ denote the left Caputo fractional derivative and the right Riemann–Liouville fractional derivative of order γ_j, respectively. By using the critical point theory, they prove the existence of infinitely many solutions of problem (15),(16).

5. Approximate Controllability for Fractional Differential Equations in Banach Spaces

Paper [10] is concerned with the fractional evolution equation of Sobolev type in the Hilbert space X, with a control and a nonlocal condition

$$\begin{cases} {}^LD_t^{\alpha}(\mathcal{E}\mathrm{x}(t)) = \mathcal{A}\mathrm{x}(t) + \mathrm{f}(t, \mathrm{x}(t)) + \mathcal{B}\mathrm{u}(t), \ \ t \in (0,b], \\ I_t^{1-\alpha}(\mathcal{E}\mathrm{x}(t))|_{t=0} + \mathrm{g}(\mathrm{x}) = x_0, \end{cases} \tag{17}$$

where $\alpha \in (0,1)$, $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ and $\mathcal{E} : D(\mathcal{E}) \subset X \to X$ are linear operators, $\mathcal{B} : U \to X$ is a linear bounded operator, U is another Hilbert space, the control function $\mathrm{u} \in L^p([0,b],U)$ for $p\alpha > 1$, $x_0 \in X$, the functions f and g satisfy some assumptions, $I_t^{1-\alpha}$ is the Riemann–Liouville fractional integral operator of order $1-\alpha$, and ${}^L D_t^\alpha$ denotes the Riemann–Liouville fractional derivative of order α. By using the Schauder fixed point theorem and operator semigroup theory, the authors prove firstly the existence of mild solutions for problem (17) without the compactness of the operator semigroup. Then, they show that if the corresponding linear problem is approximately controllable on $[0,b]$, then problem (17) is also approximately controllable on $[0,b]$. An example with an initial boundary value problem for a partial differential equation with Riemann–Liouville fractional derivatives is finally presented.

Paper [11] is devoted to the fractional differential evolution equation in the Banach space X with a finite delay and a control

$$ {}^c\mathcal{D}^\beta \mathrm{x}(t) = \mathcal{A}\mathrm{x}(t) + \mathrm{f}(t,\mathrm{x}_t) + \mathcal{B}\mathrm{u}(t), \ \ t \in [0,a], \tag{18} $$

subject to the initial date

$$ \mathrm{x}(t) = \phi(t), \ \ t \in [-b,0], \tag{19} $$

or to the nonlocal condition with a parameter

$$ \mathrm{x}(t) + \lambda \mathrm{g}_t(\mathrm{x}) = \phi(t), \ \ t \in [-b,0], \tag{20} $$

where $\mathcal{A} : \mathcal{D} \subset X \to X$ is a closed linear unbounded operator on X, where its domain $\mathcal{D}$ is a dense set; u is the control function; $\mathcal{B} : L^2([0,a];U) \to L^2([0,a];\mathcal{D})$ is a linear bounded operator, where U is another Banach space; $\phi \in L^1([-b,0];X)$, x_t denotes the history of the state function defined by $\mathrm{x}_t(\theta) = \{\mathrm{x}(t+\theta)$, if $t+\theta \geq 0$; $\phi(t+\theta)$, if $t+\theta \leq 0\}$ for $\theta \in [-b,0]$; λ is a parameter; $\mathrm{g}_t : C([-b,a];X) \to X$ is a given function satisfying some assumptions; and ${}^c\mathcal{D}^\beta$ is the Caputo fractional derivative of order β, with $\beta \in (1/2,1]$. Under the assumption that $\mathcal{A}$ is the infinitesimal generator of a differentiable resolvent operator, the authors prove the existence and uniqueness of mild solutions for problems (18),(19) and (18),(20) by utilizing the Banach contraction mapping principle. Then, based on the iterative method, they give sufficient conditions for the approximate controllability of (18),(19) and (18),(20). As an application, an example of a Caputo fractional partial differential equation with delay in the space $X = L^2([0,\pi])$ is finally addressed.

6. Fractional Differential Inclusions and Inequalities

In paper [12], the authors investigate the neutral impulsive semi-linear fractional differential inclusion with delay and initial date

$$ \begin{cases} {}^c\mathcal{D}^\alpha_{0,t}[\mathrm{x}(t) - \mathrm{h}(t,\varkappa(t)\mathrm{x})] \in \mathcal{A}\mathrm{x}(t) + \mathcal{F}(t,\varkappa(t)\mathrm{x}), \ \ \text{a.e. } t \in [0,b] \setminus \{t_1,\ldots,t_m\}, \\ I_i\mathrm{x}(t_i^-) = \mathrm{x}(t_i^-) - \mathrm{x}(t_i^+), \ \ i = 1,\ldots,m, \\ \mathrm{x}(t) = \psi(t), \ \ t \in [-r,0], \end{cases} \tag{21} $$

where $\alpha \in (0,1)$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $r > 0$, the operator $\mathcal{A}$ is the infinitesimal generator of the non-compact semigroup $\mathcal{T} = \{Y(t), \ t \geq 0\}$ on the Banach space E, and $\mathcal{F} : [0,b] \times \Theta \to 2^E \setminus \{\phi\}$ is a multifunction. Here, $\mathrm{h} : [0,b] \times \Theta \to E$, $I_i : E \to E$, $i = 1,\ldots,m$, $\psi \in \Theta$, and for every $t \in [0,b]$, the function $\varkappa(t) : \mathcal{H} \to \Theta$ is defined by $(\varkappa(t)\mathrm{x})(\theta) = \mathrm{x}(t+\theta)$ for $\theta \in [-r,0]$. ${}^c\mathcal{D}^\alpha_{0,t}$ denotes the Caputo fractional derivative of order α and the spaces Θ and $\mathcal{H}$ are defined in the paper. They show that the set of mild solutions to problem (21) is nonempty, compact, and an R_δ-set in a complete metric space H.

Paper [13] is focused on the Hilfer fractional neutral integro-differential inclusion with initial date

$$\begin{cases} D_{0+}^{k,\epsilon}[\mathrm{y}(t)-\mathcal{N}(t,\mathrm{y}(t))] \in \mathcal{A}y(t)+\mathcal{G}\left(t,\mathrm{y}(t),\int_0^t \mathrm{e}(t,s,\mathrm{y}(s))\,ds\right), \ \ t\in(0,d], \\ I_{0+}^{(1-k)(1-\epsilon)}\mathrm{y}(0)=y_0, \end{cases} \tag{22}$$

where $D_{0+}^{k,\epsilon}$ denotes the Hilfer fractional derivative of order k and type ϵ, with $k\in(0,1)$ and $\epsilon\in[0,1]$, $I_{0+}^{(1-k)(1-\epsilon)}$ is the Riemann–Liouville fractional integral of order $(1-k)(1-\epsilon)$, and $\mathcal{A}$ is an almost sectorial operator of the analytic semigroup $\{\mathcal{T}(t),\ t\geq 0\}$ on the Banach space Y. Here, $\mathcal{G}:[0,d]\times Y\times Y\to 2^Y\setminus\{\phi\}$ is a nonempty, bounded, closed, convex multivalued map and $\mathcal{N}:[0,d]\times Y\to Y$ and $\mathrm{e}:[0,d]\times[0,d]\times Y\to Y$ are appropriate functions. By using the Martelli fixed point theorem, the authors prove the existence of mild solutions to problem (22).

In paper [14], the authors study the damped wave inequality

$$\frac{\partial^2 \mathrm{u}}{\partial t^2}-\frac{\partial^2 \mathrm{u}}{\partial x^2}+\frac{\partial \mathrm{u}}{\partial t}\geq x^\sigma|\mathrm{u}|^p, \ \ t>0, \ \ x\in(0,L), \tag{23}$$

subject to initial boundary conditions

$$\begin{cases} (\mathrm{u}(t,0),\mathrm{u}(t,L))=(\mathrm{f}(t),\mathrm{g}(t)), \ \ t>0, \\ \left(\mathrm{u}(0,x),\dfrac{\partial \mathrm{u}}{\partial t}(0,x)\right)=(u_0(x),u_1(x)), \ \ x\in(0,L), \end{cases} \tag{24}$$

where $L>0$, $\sigma\in\mathbb{R}$, $p>1$, $\mathrm{f}\in L^1_{loc}([0,\infty))$, $\mathrm{g}(t)=C_g t^\gamma$ with $C_g\geq 0$ and $\gamma>-1$, and u_0, $u_1\in L^1([0,L])$. They also investigate the time-fractional damped wave inequality

$$\frac{\partial^\alpha \mathrm{u}}{\partial t^\alpha}-\frac{\partial^2 \mathrm{u}}{\partial x^2}+\frac{\partial^\beta \mathrm{u}}{\partial t^\beta}\geq x^\sigma|\mathrm{u}|^p, \ \ t>0, \ \ x\in(0,L), \tag{25}$$

supplemented with the initial boundary conditions in (24), where $\alpha\in(1,2)$, $\beta\in(0,1)$, and $\frac{\partial^\kappa}{\partial t^\kappa}$ is the time Caputo fractional derivative of order κ, for $\kappa\in\{\alpha,\beta\}$. By using the test function method, the authors give sufficient conditions depending on the above data under which problems (23),(24) and (23),(25) admit no global weak solutions.

7. Fractional q-Difference Equations and Systems

Paper [15] deals with the fractional q-difference equation in a Banach space E, with nonlinear integral conditions

$$\begin{cases} ({}^c D_q^\alpha \mathrm{y})(t)=\mathrm{f}(t,\mathrm{y}(t)), \ \ \text{a.e. } t\in[0,T], \\ \mathrm{y}(0)-\mathrm{y}'(0)=\int_0^T \mathrm{g}(s,\mathrm{y}(s))\,ds, \\ \mathrm{y}(T)+\mathrm{y}'(T)=\int_0^T \mathrm{h}(s,\mathrm{y}(s))\,ds, \end{cases} \tag{26}$$

where $T>0$, $q\in(0,1)$, ${}^c D_q^\alpha$ denotes the Caputo fractional q-derivative of order α, with $\alpha\in(1,2]$, and $\mathrm{f},\mathrm{g},\mathrm{h}:[0,T]\times E\to E$ are given functions satisfying some assumptions. By using the measures of noncompactness technique and the Mönch fixed point theorem, the authors prove the existence of solutions to problem (26).

Paper [16] is concerned with the system of nonlinear fractional q-difference equations

$$\begin{cases} (D_q^\alpha \mathrm{u})(t)+\mathrm{P}(t,\mathrm{u}(t),\mathrm{v}(t),I_q^{\omega_1}\mathrm{u}(t),I_q^{\delta_1}\mathrm{v}(t))=0, \ \ t\in(0,1), \\ (D_q^\beta \mathrm{v})(t)+\mathrm{Q}(t,\mathrm{u}(t),\mathrm{v}(t),I_q^{\omega_2}\mathrm{u}(t),I_q^{\delta_2}\mathrm{v}(t))=0, \ \ t\in(0,1), \end{cases} \tag{27}$$

subject to the coupled nonlocal boundary conditions

$$\begin{cases} \mathrm{u}(0) = D_q\mathrm{u}(0) = \ldots = D_q^{n-2}\mathrm{u}(0) = 0, \;\; D_q^{\zeta_0}\mathrm{u}(1) = \int_0^1 D_q^{\zeta}\mathrm{v}(t)\, d_q\mathcal{H}(t), \\ \mathrm{v}(0) = D_q\mathrm{v}(0) = \ldots = D_q^{m-2}\mathrm{v}(0) = 0, \;\; D_q^{\xi_0}\mathrm{v}(1) = \int_0^1 D_q^{\xi}\mathrm{u}(t)\, d_q\mathcal{K}(t), \end{cases} \tag{28}$$

where $q \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geq 2$, $\omega_1, \delta_1, \omega_2, \delta_2 > 0$, $\zeta \in [0,\beta-1)$, $\xi \in [0,\alpha-1)$, $\zeta_0 \in [0,\alpha-1)$, $\xi_0 \in [0,\beta-1)$. Here, D_q^{κ} denotes the Riemann–Liouville q-derivative of order κ for $\kappa \in \{\alpha, \beta, \zeta_0, \zeta, \xi_0, \xi\}$, I_q^k is the Riemann–Liouville q-integral of order k for $k \in \{\omega_1, \delta_1, \omega_2, \delta_2\}$, P and Q are nonlinear functions, and the integrals from conditions (28) are Riemann–Stieltjes integrals with $\mathcal{H}$, $\mathcal{K}$ functions of bounded variation. By applying varied fixed point theorems, the authors obtain existence and uniqueness results for the solutions of problem (27),(28).

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