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Multiplicity of Positive Solutions to Hadamard-Type Fractional Relativistic Oscillator Equation with *p*-Laplacian Operator

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Abstract: The purpose of this paper is to investigate the initial value problem of Hadamard-type fractional relativistic oscillator equation with p-Laplacian operator. By overcoming the perturbation of singularity to fractional relativistic oscillator equation, the multiplicity of positive solutions to the problem were proved via the methods of reducing and topological degree in cone, which extend and enrich some previous results.

Keywords: fractional relativistic oscillator equation; initial value problem; fixed point; multiplicity

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1. Introduction

In this paper, the following initial value problem of the Hadamard-type fractional relativistic oscillator equation was considered:

$$\begin{cases} {}^{H}D^{\gamma}_{1^+}\left(\varphi_p\left(\frac{{}^{H}D^{\alpha}_{1^+}u(t)}{\sqrt{1-|{}^{H}D^{\alpha}_{1^+}u(t)|^2}}\right)\right) = f(t,u(t),{}^{H}D^{\alpha}_{1^+}u(t)),\ t\in(1,T), \\ u(1) = {}^{H}D^{\alpha}_{1^+}u(1) = 0, \end{cases} \tag{1}$$

where ${}^{H}D^{\gamma}_{1^+}$ and ${}^{H}D^{\alpha}_{1^+}$ stand for the Hadamard-type fractional derivatives with orders γ, $\alpha \in (0,1)$, $1 < \alpha+\gamma < 2$, for $s \in \mathbb{R}$, $\varphi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, $\varphi_p(0) = 0$, φ_q is the inverse of φ_p, $\frac{1}{p}+\frac{1}{q} = 1$, $1 < p,\ q < +\infty$, $f(t,u,v) \in C([1,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. Let ϕ represent the relativistic acceleration operator, given by

$$\phi(v) = \frac{v}{\sqrt{1-|v|^2}},\ v \in \mathcal{B}(1),$$

$\mathcal{B}(\delta)$ means the open ball of center 0 and radius δ.

With the continuous development and improvement of the basic theory of fractional calculus (see [1–7]), scholars are increasingly realizing the importance of fractional differential equations in many fields. Recently, Hadamard-type fractional differential equation that is an important component of the fractional differential equations and originates from mechanical problems have been highly concerned and investigated. Based on some fixed point theorems, Ahmad, Ntouyas [8] considered the existence and uniqueness of solutions for a class of nonlocal initial value problem to Hadamard-type fractional differential equation as follows.

$$\begin{cases} {}^{H}D^{\alpha}_{1^+}u(t) = f(t,u(t)),\ t\in(1,T), \\ u(1) + \sum\limits_{i=1}^{m}\zeta_i u(t_i) = 0, \end{cases} \tag{2}$$

where $0 < \alpha \leq 1$, f meets L^1-Carathéodory condition, $\zeta_i \in \mathbb{R}$ satisfying $1+\sum\limits_{i=1}^{m}\zeta_i \neq 0$, $1 \leq t_1 \leq t_2 \leq \cdots \leq t_m < T$. Moreover, in [9], they also studied the existence of solutions for an initial value problem to Hadamard-type fractional hybrid differential equations by using a fixed point theorem due to Dhage as follows.

$$\begin{cases} {}^H D_{1^+}^{\alpha}\left(\frac{u(t)}{f(t,u(t))}\right) = g(t,u(t)),\ t \in [1,T], \\ {}^H I_{1^+}^{1-\alpha} u(1) = \eta, \end{cases} \tag{3}$$

where $0 < \alpha \leq 1$, $\eta \in \mathbb{R}$, ${}^H I_{1^+}^{1-\alpha}$ is Hadamard-type fractional integral of order $1-\alpha$, $f \in C([1,T]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g \in C([1,T]\times\mathbb{R},\mathbb{R})$. Subsequently, Jiang, O'Regan, Xu and Cui [10] dealt with the existence and uniqueness of solutions for a class of three-point boundary value problem to Hadamard-type fractional differential equations with p-Laplacian operator via the fixed point index theory and a fixed point theorem as follows.

$$\begin{cases} {}^H D_{1^+}^{\alpha} \varphi_p({}^H D_{1^+}^{\beta} u(t)) = f(t,u(t)),\ t \in (1,e), \\ {}^H D_{1^+}^{\alpha} u(1) = {}^H D_{1^+}^{\beta} u(e) = 0, \\ u(1) = u'(1) = 0, u(e) = au(\xi), \end{cases} \tag{4}$$

where $1 < \alpha \leq 2$, $2 < \beta \leq 3$, $1 < \xi < e$, $a > 0$ $f \in C([1,e]\times[0,+\infty),[0,+\infty))$. For more meaningful research papers on this issue, please refer to [11–17] and references therein.

On the other hand, it well known that the relativistic oscillator equation comes from classical theory of relativity (see [18–20]), which is a singular equation. Due to its strong physical background, it has been studied by many scholars from perspective of qualitative theory of differential equations. For examples, Bereanu, Jebelean and Mawhin [21] was concerned with the existence and multiplicity of radial solutions for the Neumann boundary problem by critical point theory in Minkowski space as follows.

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = g(|x|,u),\ \text{on}\ \mathcal{A}, \\ \frac{\partial u}{\partial \nu} = 0,\ \text{on}\ \partial\mathcal{A}, \end{cases} \tag{5}$$

where $0 \leq R_1 < R_2$, $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$, $g : [R_1,R_2]\times\mathbb{R} \to \mathbb{R}$ is continuous. Subsequently, based on a geometric method in critical point theory, Jebelean, Mawhin and Şerban [22] studied the multiplicity of periodic solutions to the N-dimensional relativistic pendulum equation with periodic nonlinearity as follows.

$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1-|u'(t)|^2}}\right)' = \nabla F(t,u(t)),\ a.e.\ t \in [0,T], \\ u(0) = u(T),\ u'(0) = u'(T), \end{cases} \tag{6}$$

where $F(t,u)$ satisfies Carathéodory condition, $F(t,0) = 0$, and there exists function $a \in L^1([0,T],[0,+\infty))$ such that, for all $u \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, $|\nabla F(t,u)| \leq a(t)$. For more meaningful research papers on this topic, please refer to [23–27] and references therein.

Motivated by the works mentioned above. A meaningful question naturally appears in the brain. Can we investigate the multiplicity of solutions for a initial value problem to Hadamard-type fractional relativistic oscillator equation with p-Laplacian operator? In this paper, a positive response will be provided. Let us emphasize some features of this article. Firstly, we overcome the perturbation of singularity to fractional relativistic oscillator equation and obtain the estimations of inequality and prior bound. Secondly, the equivalent truncation equation is constructed under the influence of p-Laplacian operator and relativistic acceleration operator. Thirdly, as far as we know, there are few paper investigating the multiplicity of solutions for initial value problems to Hadamard-type fractional relativistic oscillator equation with p-Laplacian operator. Furthermore, if $p = 2$, the operator ${}^H D_{1^+}^{\gamma}(\varphi_p(\phi({}^H D_{1^+}^{\alpha} u)))$ reduces to ${}^H D_{1^+}^{\gamma}(\phi({}^H D_{1^+}^{\alpha} u))$, so the model we considered are more general and complex.

The rest of this paper is organized as follows. To begin with, the basic space, the definitions and properties of Hadamard-type fractional integrals and derivatives, some necessary lemmas are given in Section 2. Moreover, based on the methods of reducing and topological degree in cone, multiplicity of solutions to the problem (1) are obtained in Section 3.

2. Preliminaries

Letting $\mathbb{R}_+ = [0, +\infty)$ and $C := C([1, T], \mathbb{R})$ with the norm $\|u\|_\infty = \max_{t\in[1,T]} |u(t)|$, define

$$C^{H,\alpha} = \{u : [1, T] \to \mathbb{R} \mid u \in C \text{ and} {}^H D^\alpha_{1^+} u \in C\},$$

whose norm is $\|u\| = \max\{\|u\|_\infty, \|{}^H D^\alpha_{1^+} u\|_\infty\}$, which yields that it is a Banach space. Let P be a cone of $C^{H,\alpha}$ that $P = \{x \in C^{H,\alpha} | \ u(t) \geq 0\}$, where the partial ordering $\leq$ with respect to P by $x \leq y$ if only if $y - x \in P$. Define $P_c = \{u \in P | \|u\| \leq c\}$, where the constant c is positive. Next, the basic definitions and properties of the Hadamard-type fractional integral and derivative are shown.

Definition 1. ([1, 2]). *The Hadamard-type fractional integral of order $\alpha > 0$ of a function $u : [1, T] \to \mathbb{R}$, is defined by*

$${}^H I^\alpha_{1^+} u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (ln\frac{t}{s})^{\alpha-1} u(s) \frac{ds}{s},$$

provided the right-side exists.

Definition 2. ([1, 2]). *The Hadamard-type fractional derivative of order $\alpha > 0$ of a function $u : [1, T] \to \mathbb{R}$, is defined by*

$${}^H D^\alpha_{1^+} u(t) = \frac{1}{\Gamma(n-\alpha)} (t\frac{d}{dt})^n \int_1^t (ln\frac{t}{s})^{n-\alpha-1} u(s) \frac{ds}{s},$$

provided the right-side exists, where $n = [\alpha] + 1$.

Lemma 1. ([1]). *Letting $\alpha > 0$, $n = [\alpha] + 1$, the equality ${}^H D^\alpha_{1^+} u(t) = 0$ is valid if and only if*

$$u(t) = \sum_{i=1}^n c_i (lnt)^{\alpha-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Lemma 2. ([1]). *Letting $\alpha > 0$, $1 \leq \beta \leq +\infty$, then for $u \in L^\beta(1, T)$*

$${}^H D^\alpha_{1^+} {}^H I^\alpha_{1^+} u = u.$$

Moreover, by Lemma 1, if ${}^H D^\alpha_{1^+} u \in L^\beta(1, T)$, then the following formula holds.

$${}^H I^\alpha_{1^+} {}^H D^\alpha_{1^+} u(t) = u(t) + \sum_{i=1}^n c_i (lnt)^{\alpha-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n - 1 < \alpha < n$.

Lemma 3 ([28])**.** *Let P be a cone of Banach space E and $\Psi : P_c \to P_c$ be completely continuous map. There exists a nonnegative continuous concave functional θ such that $\theta(u) \leq \|u\|$ for $u \in P$ and numbers $0 < a < b < d \leq c$ satisfying the following conditions.*

(i) $\{u \in P(\theta, b, d) | \theta(u) > b\} \neq \emptyset$ *and* $\theta(\Psi u) > b$ *for* $u \in P(\theta, b, d)$, *where* $P(\theta, b, d) = \{u \in P | \theta(u) \geq b \text{ and } \|u\| \leq d\}$.

(ii) $\|\Psi u\| < a$ *for* $u \in P_a$.

(iii) $\theta(\Psi u) > b$ *for* $u \in P(\theta, b, c)$ *with* $\|\Psi u\| > d$.

Then Ψ *has at least three fixed points* u_1, u_2, u_3 *in* P_c.

In order to obtain a priori bound, we need the following assumption.

H1. *There exist fuctions* $\mu_i \in C([1,T], \mathbb{R}_+)$, $i = 1,2,3$ *such that for* $(t,u,v) \in [1,T] \times \mathbb{R}^2$,

$$|f(t,u,v)| \le \mu_1(t) + \mu_2(t)|u|^{\nu} + \mu_3(t)|u|^{\lambda},\ \nu \in (0, \frac{1}{q-1}], \lambda \in (0,1],$$

where $\frac{2^{2q-2}(lnT)^{\gamma q-\gamma+\alpha}\|\mu_2\|_\infty^{q-1}}{\Gamma(\alpha+1)(\Gamma(\gamma+1))^{q-1}} < 1$.

- A priori bound

If $u(t)$ is a solution of (1), taking the Hadamard-type fractional calculus ${}^HI_{1^+}^{\gamma}$ on both sides of the equation, it follows

$${}^HD_{1^+}^{\alpha}u(t) = \phi^{-1}(\varphi_q({}^HI_{1^+}^{\gamma}f(t,u(t), {}^HD_{1^+}^{\alpha}u(t)) + c_1(\ln t)^{\gamma-1})),$$

which together with ${}^HD_{1^+}^{\alpha}u(1) = 0$ yield that $c_1 = 0$ and

$$u(t) = {}^HI_{1^+}^{\alpha}(\phi^{-1}(\varphi_q({}^HI_{1^+}^{\gamma}f(t,u(t), {}^HD_{1^+}^{\alpha}u(t))))) + c_2(\ln t)^{\alpha-1}.$$

By $u(1) = 0$, we have $c_2 = 0$ and

$$u(t) = {}^HI_{1^+}^{\alpha}(\phi^{-1}(\varphi_q({}^HI_{1^+}^{\gamma}f(t,u(t), {}^HD_{1^+}^{\alpha}u(t))))).$$

From (H1) and the basic inequality $(a+b)^k \le 2^k(a^k+b^k)$ for $a,b,k > 0$, one has

$$\begin{aligned}
|{}^HI_{1^+}^{\gamma}f(t,u(t), {}^HD_{1^+}^{\alpha}u(t))|^{q-1} &\le |\frac{1}{\Gamma(\gamma)}\int_1^t(\ln\frac{t}{s})^{\gamma-1}d(\ln s)(\|\mu_1\|_\infty + \|\mu_2\|_\infty\|u\|_\infty^{\nu} \\
&\quad +\|\mu_3\|_\infty\|{}^HD_{1^+}^{\alpha}u\|_\infty^{\lambda})|^{q-1} \\
&\le \frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1}\|u\|_\infty^{\nu(q-1)} \\
&\quad +2^{q-1}\|\mu_3\|_\infty^{q-1}\|{}^HD_{1^+}^{\alpha}u\|_\infty^{\lambda(q-1)}).
\end{aligned}$$

Since $|{}^HD_{1^+}^{\alpha}u(t)| < 1$, it follows that

$$\begin{aligned}
|u(t)| &\le {}^HI_{1^+}^{\alpha}(\phi^{-1}(\frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1}\|u\|_\infty^{\nu(q-1)}))) \\
&= {}^HI_{1^+}^{\alpha}(\frac{\frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1}\|u\|_\infty^{\nu(q-1)})}{\sqrt{1 + \frac{2^{2q-2}(\ln T)^{2\gamma q-2\gamma}}{(\Gamma(\gamma+1))^{2q-2}}(\|\mu_1\|_\infty^{q-1} + +2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1}\|u\|_\infty^{\nu(q-1)})^2}}) \\
&\le {}^HI_{1^+}^{\alpha}(\frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1}\|u\|_\infty^{\nu(q-1)})) \\
&\le \frac{2^{q-1}(\ln T)^{\gamma q-\gamma+\alpha}}{\Gamma(\alpha+1)(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1}\|u\|_\infty^{\nu(q-1)}),
\end{aligned}$$

which together with $\frac{2^{2q-2}(\ln T)^{\gamma q-\gamma+\alpha}\|\mu_2\|_\infty^{q-1}}{\Gamma(\alpha+1)(\Gamma(\gamma+1))^{q-1}} < 1$ yield that there exists a positive constant $l > 1$ such that $|u(t)| < l$. Therefore, the solutions of (1) must be located in $\mathcal{B}(l)$.

Let $\Upsilon := \phi^{-1}(\bar{\mathcal{B}}(\eta)) \subset \mathcal{B}(1)$, where

$$\eta = \frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1} l^{\nu(q-1)}).$$

Moreover, choose $\zeta \in (0,1)$ such that

$$\frac{\zeta}{\sqrt{1-\zeta^2}} \geq \eta,\ \Upsilon \subset \bar{\mathcal{B}}(\zeta). \tag{7}$$

Define $\psi : \mathbb{R} \to \mathbb{R}$:

$$\psi(x) = \begin{cases} \frac{x}{\sqrt{1-|x|^2}}, & |x| \leq \zeta, \\ \frac{x}{\sqrt{1-|\zeta|^2}}, & |x| \geq \zeta. \end{cases}$$

Let $\overline{\Delta} = \{u \in C \mid \|u\|_\infty \leq l\}$. Following [22], the following lemma can be established.

Lemma 4. *A function $u \in C^{H,\alpha} \cap \overline{\Delta}$ is a solution of problem (1) if and only if it is a solution of the following problem:*

$$\begin{cases} {}^H D_{1^+}^\gamma(\varphi_p(\psi({}^H D_{1^+}^\alpha u(t))) = f(t, u(t), {}^H D_{1^+}^\alpha u(t)),\ t \in (1,T), \\ u(1) = {}^H D_{1^+}^\alpha u(1) = 0. \end{cases} \tag{8}$$

Proof. Assuming that $u \in C^{H,\alpha} \cap \overline{\Delta}$ is a solution of problem (1), from ${}^H D_{1^+}^\alpha u(1) = 0$, one has

$$\phi({}^H D_{1^+}^\alpha u(t)) = \varphi_q({}^H I_{1^+}^\gamma f(t, u(t), {}^H D_{1^+}^\alpha u(t))),$$

which together with $|{}^H D_{1^+}^\alpha u(t)| < 1$ implies that

$$|\phi({}^H D_{1^+}^\alpha u(t))| \leq \frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1} l^{\nu(q-1)}) = \eta.$$

Hence, for any $t \in [1,T]$, ${}^H D_{1^+}^\alpha u(t) \in \Upsilon$ and $|{}^H D_{1^+}^\alpha u(t)| \leq \zeta$. It follows that $\phi({}^H D_{1^+}^\alpha u(t)) = \psi({}^H D_{1^+}^\alpha u(t))$.

Assuming that $u \in C^{H,\alpha} \cap \overline{\Delta}$ is a solution of problem (8), we just need to prove that $|{}^H D_{1^+}^\alpha u(t)| \leq \zeta$ for any $t \in [1,T]$. If not, we can find that there exists a $t_0 \in [1,T]$ such that $|{}^H D_{1^+}^\alpha u(t_0)| > \zeta$. From

$${}^H I_{1^+}^\gamma({}^H D_{1^+}^\gamma(\varphi_p(\psi({}^H D_{1^+}^\alpha u(t)))) = \varphi_p(\psi({}^H D_{1^+}^\alpha u(t))) - c_3(\ln t)^{\gamma-1}, \tag{9}$$

and ${}^H D_{1^+}^\alpha u(1) = 0$, we have $c_3 = 0$ and

$$\eta \geq |\varphi_q({}^H I_{1^+}^\gamma({}^H D_{1^+}^\gamma(\varphi_p(\psi({}^H D_{1^+}^\alpha u(t)))))\ |_{t=t_0}| = |\psi({}^H D_{1^+}^\alpha u(t_0))| > \frac{\zeta}{\sqrt{1-|\zeta|^2}}, \tag{10}$$

which contradicts the definition of η. Hence, the proof is complete. □

3. Main Results

Define the operator $\Psi : C^{H,\alpha} \to C^{H,\alpha}$ by

$$\Psi u(t) = {}^H I_{1^+}^\alpha(\psi^{-1}(\varphi_q({}^H I_{1^+}^\gamma f(t, u(t), {}^H D_{1^+}^\alpha u(t))))), \tag{11}$$

where

$$\psi^{-1}(x) = \begin{cases} \frac{x}{\sqrt{1+|x|^2}}, & |x| \leq \frac{\zeta}{\sqrt{1-\zeta^2}}, \\ x\sqrt{1-|\zeta|^2}, & |x| \geq \frac{\zeta}{\sqrt{1-\zeta^2}}. \end{cases}$$

Set

$$\Xi = \{u \in C^{H,\alpha} \cap \overline{\Delta} \mid u = \Psi u \text{ and } |^{H}D^{\alpha}_{1^+}u| \le \zeta\}.$$

Thus, a function $u \in \Xi$ is a solution of the problem (1). Define

$$\overline{U} = \{u \in C^{H,\alpha} \mid \|u\| \le r\},$$

where $r = \max\{\zeta, \frac{\zeta(\ln T)^\alpha}{\Gamma(\alpha+1)}\}$. Clearly, $\overline{U}$ is a nonempty convex closed set. If $(\ln T)^\alpha \le \Gamma(\alpha+1)$, we have $\overline{U} \subset C^{H,\alpha} \cap \overline{\Delta}$.

Lemma 5. *If the condition (H1) is satisfied and* $\frac{\zeta(\ln T)^\alpha}{\Gamma(\alpha+1)} < 1$, $\Psi : \overline{U} \to \overline{U}$ *is completely continuous.*

Proof. Following the continuity of f, one has Ψ is continuous. From (H1), for $(t,u) \in [1,T] \times \overline{\Omega}$, we can obtain that

$$\begin{aligned} |^{H}I^{\gamma}_{1^+}f(t,u(t),{}^{H}D^{\alpha}_{1^+}u(t))|^{q-1} &\le \frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|^{q-1}_\infty + +2^{q-1}\|\mu_3\|^{q-1}_\infty \\ &\quad +2^{q-1}\|\mu_2\|^{q-1}_\infty r^{\nu(q-1)}) \\ &\le \eta. \end{aligned}$$

By $\frac{\zeta}{\sqrt{1-\zeta^2}} \ge \eta$, one has

$$|^{H}D^{\alpha}_{1^+}(\Psi u(t))| \le \frac{\eta}{\sqrt{1+\eta^2}} \le \zeta \le r$$

and

$$|\Psi u(t)| \le \frac{\eta(\ln T)^\alpha}{\Gamma(\alpha+1)\sqrt{1+\eta^2}} \le \frac{\zeta(\ln T)^\alpha}{\Gamma(\alpha+1)} \le r.$$

Thus, it follows $\|\Psi u\| \le r$ which means that $\Psi(\overline{U})$ is uniformly bounded in $C^{H,\alpha}$. Moreover, $\forall t_1,\ t_2 \in [1,T]$, assuming that $t_1 \le t_2$, for any $u \in \overline{\Omega}$, we have

$$\begin{aligned} &|\Psi u(t_2) - \Psi u(t_1)| \\ &= |\frac{1}{\Gamma(\alpha)}\int_1^{t_1}((\ln\frac{t_2}{s})^{\alpha-1} - (\ln\frac{t_1}{s})^{\alpha-1})\frac{\psi^{-1}(\varphi_q({}^{H}I^{\gamma}_{1^+}f(s,u(s),{}^{H}D^{\alpha}_{1^+}u(s))))}{s}ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\ln\frac{t_2}{s})^{\alpha-1}\frac{\psi^{-1}(\varphi_q({}^{H}I^{\gamma}_{1^+}f(s,u(s),{}^{H}D^{\alpha}_{1^+}u(s))))}{s}ds| \\ &\le \frac{r}{\Gamma(\alpha)}(\int_1^{t_1}((\ln t_1 - \ln s)^{\alpha-1} - (\ln t_2 - \ln s)^{\alpha-1})d(\ln s) + \int_{t_1}^{t_2}(\ln t_2 - \ln s)^{\alpha-1}d(\ln s)) \\ &= \frac{r}{\Gamma(\alpha+1)}(2(\ln t_2 - \ln t_1)^\alpha + (\ln t_1)^\alpha - (\ln t_2)^\alpha) \to 0 \text{ uniformly as } t_1 \to t_2. \end{aligned}$$

Thus, $\Psi(\overline{U})$ is equicontinuous and uniformly bounded in C, which implies $\Psi(\overline{U})$ is relatively compact in C. Furthemore, we have

$$\begin{aligned}
&\left|\varphi_p(\psi({}^H D_{1^+}^{\alpha}(\Psi u(t_2)))) - \varphi_p(\psi({}^H D_{1^+}^{\alpha}(\Psi u(t_1))))\right| \\
&= |\frac{1}{\Gamma(\gamma)}\int_1^{t_1}((\ln\frac{t_2}{s})^{\gamma-1} - (\ln\frac{t_1}{s})^{\gamma-1})\frac{f(s,u(s),{}^H D_{1^+}^{\alpha}(s))}{s}ds \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}(\ln\frac{t_2}{s})^{\gamma-1}\frac{f(s,u(s),{}^H D_{1^+}^{\alpha}u(s))}{s}ds| \\
&\le \frac{\|\mu_1\|_\infty + \|\mu_3\|_\infty + \|\mu_2\|_\infty r^\nu}{\Gamma(\gamma+1)}(2(\ln t_2 - \ln t_1)^\gamma + (\ln t_1)^\gamma - (\ln t_2)^\gamma) \to 0 \\
&\quad \text{uniformly as } t_1 \to t_2.
\end{aligned}$$

Hence, $\varphi_p(\psi({}^H D_{1^+}^{\alpha}(\Psi(\overline{U}))))$ is equicontinuous and uniformly bounded in C, which yields $\varphi_p(\psi({}^H D_{1^+}^{\alpha}(\Psi(\overline{U}))))$ is relatively compact in C. Since ψ is monotonous and uniformly continuous on $[-r,r]$ and φ_p is monotonous and uniformly continuous on $[\psi(-r),\psi(r)]$, it follows that ${}^H D_{1^+}^{\alpha}(\Psi(\overline{U}))$ is relatively compact in C. Hence, $\Psi(\overline{U})$ is relatively compact in $C^{H,\alpha}$. Therefore, $\Psi : \overline{U} \to \overline{U}$ is completely continuous. □

Define a nonnegative continuous functional θ on P by

$$\theta(\Psi u) = \rho \min_{t\in[1+\tau,T]} \Psi u(t),$$

where the constant $\rho > \max\{2, \frac{\Gamma(\alpha+\gamma(q-1)+1)(\ln T)^{\gamma(q-1)}}{\Gamma(\gamma(q-1)+1)(\ln(1+\tau))^{\alpha+\gamma(q-1)}}\}$, $T^{2-\frac{1}{\alpha}} - 1 < \tau < T - 1$. Since, for any $u, v \in C^{H,\alpha}$ and $s \in [0,1]$, we have

$$\begin{aligned}
\theta(s\Psi u + (1-s)\Psi v) &= \rho \min_{t\in[1+\tau,T]}\{s\Psi_1 u(t) + (1-s)\Psi v(t)\} \\
&\ge s(\rho \min_{t\in[1+\tau,T]} \Psi u(t) + (1-s)\rho \min_{t\in[1+\tau,T]} \Psi v(t)) \\
&= s\theta(\Psi u) + (1-s)\theta(\Psi v),
\end{aligned}$$

which implies that the functional θ is concave. Choose $0 < a < b < d \le c \le \zeta < 1$ satisfying

$$\eta_1 < \frac{a}{\sqrt{1-a^2}},$$
$$\eta_2 \le \frac{d}{\sqrt{1-d^2}},$$
$$\eta_3 \le \frac{c}{\sqrt{1-c^2}},$$

where

$$\eta_1 = \frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1} a^{\nu(q-1)}),$$

$$\eta_2 = \frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1} d^{\nu(q-1)}),$$

$$\eta_3 = \frac{2^{q-1}(\ln T)^{\gamma q-\gamma}}{(\Gamma(\gamma+1))^{q-1}}(\|\mu_1\|_\infty^{q-1} + 2^{q-1}\|\mu_3\|_\infty^{q-1} + 2^{q-1}\|\mu_2\|_\infty^{q-1} c^{\nu(q-1)}).$$

For investigating the positive solution of the initial value problem in the cone, the following assumptions are naturally needed.

H2. *$f(t,u,v) : [1,T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous.*
H3. *$f(t,u,v) > \varphi_p(\Lambda b)$ for $(t,u,v) \in [1,T] \times [0,d] \times \mathbb{R}$,*

where

$$\Lambda = \frac{(\Gamma(1+\gamma))^{q-1}\Gamma(\alpha+\gamma(q-1)+1)}{\sqrt{\rho^2(\ln(1+\tau))^{2\alpha+2\gamma(q-1)}(\Gamma(\gamma(q-1)+1))^2-(\ln T)^{2\gamma(q-1)}(\Gamma(\alpha+\gamma(q-1)+1))^2}}.$$

Lemma 6. *If the condition (H1) and (H2) are satisfied, $\Psi : P_c \to P_c$ is completely continuous, provided that $(lnT)^\alpha \leq \Gamma(\alpha+1)$.*

Proof. Firstly, from (H1) and (H2), it follows that $\Psi u \geq 0$ for any $u \in P_c$ and Ψ is continuous. Moreover, since $\eta_3 \leq \frac{c}{\sqrt{1-c^2}}$, we can obtain that for any $u \in P_c$,

$$|{}^H D^\alpha_{1^+}(\Psi u(t))| \leq \frac{\eta_3}{\sqrt{1+\eta_3^3}} \leq c \leq r$$

and

$$|\Psi u(t)| \leq \frac{c(\ln T)^\alpha}{\Gamma(1+\alpha)} \leq c \leq r.$$

Hnece, $\Psi : P_c \to P_c$ is uniformly bounded. By the same way to Lemma 5, it follows Ψ is equicontinuous on P_c. Based on the Arzelà-Ascoli theorem, we can obtain that $\Psi : P_c \to P_c$ is completely continuous. □

Theorem 1. *If the conditions (H1), (H2) and (H3) are satisfied, for $d = c$, there exist at least three fixed points u_1, u_2, u_3 in P_c satisfying $\Psi u = u$, provided that $(lnT)^\alpha = \Gamma(\alpha+1)$.*

Proof. In view of (H1), by the same way to Lemma 5, it follows that $\|\Psi u\| < a$ for $u \in P_a$, which yields that the condition (ii) of Lemma 3 is satisfied. Letting

$$u_*(t) = \frac{b+d}{2(\ln T)^\alpha}(\ln t)^\alpha,$$

it follows $\|u_*\|_\infty = \frac{b+d}{2} \leq d$. Moreover, since $0 < \alpha < 1$, we obtain

$$\begin{aligned}
{}^H D^\alpha_{1^+} u_*(t) &= \frac{1}{\Gamma(1-\alpha)} t \frac{d}{dt}\left(\int_1^t (\ln\frac{t}{s})^{-\alpha} u_*(s)\frac{ds}{s}\right) \\
&= \frac{b+d}{2(\ln T)^\alpha \Gamma(1-\alpha)} t \frac{d}{dt}\left(\int_1^t (\ln t - \ln s)^{-\alpha}(\ln t)^\alpha d(\ln s)\right) \\
&\overset{\varsigma=\frac{\ln s}{\ln t}}{=} \frac{b+d}{2(\ln T)^\alpha \Gamma(1-\alpha)} t \frac{d}{dt}\left(\ln t \int_0^1 (1-\varsigma)^{-\alpha}\varsigma^\alpha d\varsigma\right) \\
&= \frac{(b+d)\Gamma(1+\alpha)}{2(\ln T)^\alpha} = \frac{b+d}{2},
\end{aligned}$$

which yields that $\|{}^H D^\alpha_{1^+} u_*\|_\infty = \frac{b+d}{2} \leq d$. Hence, $\|u_*\| \leq d$. Since $T^{2^{-\frac{1}{\alpha}}} - 1 < \tau < T - 1$, we have

$$\theta(u_*) = \frac{\rho(b+d)}{2(\ln T)^\alpha}(\ln(1+\tau))^\alpha > \frac{b+d}{2} > b.$$

So,

$$\{u \in P(\theta, b, d) | \theta(u) > b\} \neq \emptyset.$$

If $u \in P(\theta, b, d)$, for $t \in [1, T]$, one has $0 \leq u(t) \leq d$. Thus, by (H2) and (H3), we have

$$\begin{aligned}
\theta(\Psi u) &= \rho \min_{t\in[1+\tau,T]} \Psi u(t) \\
&= \frac{\rho}{\Gamma(\alpha)} \min_{t\in[1+\tau,T]} \int_1^t (\ln\frac{t}{s})^{\alpha-1}\psi^{-1}(\varphi_q({}^H I_{1^+}^{\gamma} f(s,u(s),{}^H D_{1^+}^{\alpha} u(s))))\frac{ds}{s} \\
&> \frac{\rho}{\Gamma(\alpha)} \min_{t\in[1+\tau,T]} \int_1^t (\ln\frac{t}{s})^{\alpha-1}\psi^{-1}(\frac{\Lambda b(\ln s)^{\gamma(q-1)}}{(\Gamma(1+\gamma))^{q-1}})\frac{ds}{s} \\
&\geq \frac{\rho\Lambda b}{(\Gamma(1+\gamma))^{q-1}\Gamma(\alpha)\sqrt{1+\frac{\Lambda^2 b^2(\ln T)^{2\gamma(q-1)}}{(\Gamma(1+\gamma))^{2q-2}}}} \min_{t\in[1+\tau,T]} \int_1^t (\ln\frac{t}{s})^{\alpha-1}(\ln s)^{\gamma(q-1)}\frac{ds}{s} \\
&\geq \frac{\rho\Lambda b(\ln(1+\tau))^{\alpha+\gamma(q-1)}\Gamma(\gamma(q-1)+1)}{(\Gamma(1+\gamma))^{q-1}\Gamma(\alpha+\gamma(q-1)+1)\sqrt{1+\frac{\Lambda^2(\ln T)^{2\gamma(q-1)}}{(\Gamma(1+\gamma))^{2q-2}}}} = b,
\end{aligned}$$

which means that the condition (i) of Lemma 3 holds. If $d = c$, the condition (i) implies (iii) in Lemma 3. Then there exist at least three fixed points $u_1, u_2, u_3 \in P_c$ satisfying $\Psi u = u$. $\square$

Remark 1. *Based on Theorem 1, we can obtain that the fixed points $u_1, u_2, u_3 \in P_c \subset \Xi$, which implies that the problem (1) has at least three solutions in P_c.*

Next, an example is given to verify our main conclusions.
An Example. Consider the following problem:

$$\begin{cases} {}^H D_{1^+}^{\frac{2}{3}}(\varphi_{\frac{4}{3}}(\frac{{}^H D_{1^+}^{\frac{1}{2}} u(t)}{\sqrt{1-|{}^H D_{1^+}^{\frac{1}{2}} u(t)|^2}})) = f(t,u(t),{}^H D_{1^+}^{\frac{1}{2}} u(t)),\ t\in(1,T), \\ u(1) = {}^H D_{1^+}^{\frac{1}{2}} u(1) = 0, \end{cases} \tag{12}$$

where $\gamma = \frac{2}{3}$, $\alpha = \frac{1}{2}$, $p = \frac{4}{3}$, $q = 4$, $T = e^{(\Gamma(\frac{3}{2}))^2}$, $v = \frac{1}{3}$, $\lambda = 1$,

$$f(t,u,v) = 1 + \frac{t}{32T}|u|^{\frac{1}{3}} + \frac{\sqrt[3]{2}}{8}|v|.$$

It is obvious that $(\ln T)^{\frac{1}{2}} = \Gamma(\frac{3}{2})$ and (H2) is satisfied. Letting $\mu_1(t) = 1$, $\mu_2(t) = \frac{t}{32T}$, $\mu_3(t) = \frac{\sqrt[3]{2}}{8}$, one has

$$\frac{2^6(\Gamma(\frac{3}{2}))^5(\frac{1}{32})^3}{\Gamma(\frac{3}{2})(\Gamma(\frac{5}{3}))^3} < 1,$$

where implies that (H1) holds. Choosing

$$a = \frac{199}{200} < b = \frac{299}{300} < d = \frac{399}{400} = c < \zeta = \frac{999}{1000},$$

we have $\eta_1 \approx 7.166076 < \frac{\frac{199}{200}}{\sqrt{1-(\frac{199}{200})^2}} \approx 10.081053$, $\eta_2 = \eta_3 \approx 7.166079 < \frac{\frac{399}{400}}{\sqrt{1-(\frac{399}{400})^2}} \approx 199.779691$. By calculation, setting $\rho = 5$, $\tau = 1$, we have

$$\Lambda = \frac{(\Gamma(\frac{5}{3}))^3\Gamma(\frac{7}{2})}{\sqrt{25(\ln 2)^5(\Gamma(3))^2 - (\Gamma(\frac{3}{2}))^8(\Gamma(\frac{7}{2}))^2}} \approx 0.832090,$$

and for $(t.u, v) \in [1, T] \times [0, \frac{399}{400}] \times \mathbb{R}$,

$$f(t, u, v) > |\Lambda b|^3 \approx 0.570375,$$

which means that (H3) is satisfied. By Theorem 1, the problem (12) has at least three solutions in P_c.

4. Conclusions

The Hadamard-type fractional differential equation is an important component of the fractional differential equations and originates from mechanical problems have been thoroughly investigated. However, there are few papers in the literature which have investigated the multiplicity of solutions for initial value problems to Hadamard-type fractional relativistic oscillator equation with the p-Laplacian operator. To begin with, we overcome the perturbation of singularity to the fractional relativistic oscillator equation and obtain the estimations of inequality and prior bound. Moreover, the equivalent truncation equation is constructed under the influence of the p-Laplacian operator and relativistic acceleration operator. Furthermore, if $p = 2$, the operator ${}^H D^{\gamma}_{1^+}(\varphi_p(\phi({}^H D^{\alpha}_{1^+} u)))$ reduces to ${}^H D^{\gamma}_{1^+}(\phi({}^H D^{\alpha}_{1^+} u))$, so the model we considered is more general and complex. Based on the methods of reducing and topological degree in cone, the multiplicity of positive solutions to the problem were proved, which extend and enrich some previous results. For boundary value problems of the Hadamard-type fractional relativistic oscillator equation, the difficulty lies in correcting singularity, which is the focus of future research on such problems.

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