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Article

Fractional Telegraph Equation with the Caputo Derivative

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Abstract: The Cauchy problem for the telegraph equation $(D_t^\rho)^2 u(t) + 2\alpha D_t^\rho u(t) + Au(t) = f(t)$ $(0 < t \leq T,\, 0 < \rho < 1,\, \alpha > 0)$, with the Caputo derivative is considered. Here, A is a selfadjoint positive operator, acting in a Hilbert space, H; D_t is the Caputo fractional derivative. Conditions are found for the initial functions and the right side of the equation that guarantee both the existence and uniqueness of the solution of the Cauchy problem. It should be emphasized that these conditions turned out to be less restrictive than expected in a well-known paper by R. Cascaval et al. where a similar problem for a homogeneous equation with some restriction on the spectrum of the operator, A, was considered. We also prove stability estimates important for the application.

Keywords: telegraph-type equations; Caputo derivatives; stability inequalities

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1. Introduction

Consider a separable Hilbert space, H, with inner product, $(\cdot,\cdot)$, and norm, $||\cdot||$. Let $A : H \to H$ be an arbitrary selfadjoint unbounded positive operator with a domain of definition, $D(A)$, assuming that A has a complete orthonormal system of eigenvalues, $\{v_k\}$, and a innumerable set of positive eigenfunctions, λ_k. We also assume that the spectrum of the operator, A, has no finite limit points. In particular, the multiplicity of each eigenvalue, λ_k, is finite. Without the loss of generality, we assume that the eigenvalues do not decline as their numbers rise, i.e., $0 < \lambda_1 \leq \lambda_2 \cdots \to +\infty$.

For vector functions (or just functions), $h : \mathbb{R}_+ \to H$, fractional integrals and derivatives are defined similarly with scalar functions, and known formulas and qualities are preserved [1]. Recall that fractional integrals of the order $\sigma < 0$ of the function $h(t)$ defined on $\mathbb{R}_+$ have the form (see, e.g., [2])

$$J_t^\sigma h(t) = \frac{1}{\Gamma(-\sigma)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\sigma+1}} d\xi, \quad t > 0, \tag{1}$$

supplied the right-hand side exists. Here, $\Gamma(\sigma)$ is a Euler's gamma function. Using this definition, one can define the Caputo fractional derivative of order $\rho \in (0,1)$:

$$D_t^\rho h(t) = J_t^{\rho-1} \frac{d}{dt} h(t).$$

Note that if $\rho = 1$, then the fractional derivative coincides with the ordinary classical derivative of the first order: $D_t h(t) = \frac{d}{dt} h(t)$.

Let $C[0,T]$ stand for the set of continuous functions defined by $[0,T]$ with the standard max-norm $||\cdot||_{C[0,T]}$, and let $C(H) = C([0,T];H)$ be a space of continuous H-valued functions $h(t)$ defined by $[0,T]$, and supplied with the norm

$$||h||_{C(H)} = \max_{0\leq t\leq T} ||h(t)||.$$

Let $\rho \in (0,1)$ be a fixed number. Consider the following Cauchy problem:

$$\begin{cases} (D_t^\rho)^2 u(t) + 2\alpha D_t^\rho u(t) + Au(t) = f(t), \quad 0 < t \le T; \\ \lim_{t\to 0} D_t^\rho u(t) = \varphi_0, \\ u(0) = \varphi_1, \end{cases} \tag{2}$$

where $f(t) \in C(H)$ and φ_0, φ_1 are the known elements of H. We call this problem **the forward problem**.

Note that one can study the above equation with the operator $D_t^{2\rho}$ instead of $(D_t^\rho)^2$. However, these two operators are not the same and the corresponding problems are completely different (see [3]). As a simple example, we can take the function:

$$u(t) = t^{\rho-1}, \qquad t > 0.$$

It is easy to see that

$$(D_t^\rho)^2 = D^\rho(D^\rho u) \ne D^{2\rho} u.$$

In this paper, following the paper in [3], we consider the telegraph equation in the form of (2).

Definition 1. *If the function $u(t)$ with the properties $(D_t^\rho)^2 u(t), Au(t) \in C((0,T];H)$ and $u(t)$, $D_t^\rho u(t) \in C(H)$ satisfies Condition (2), then it is called **the strong solution** of the forward problem.*

To formulate the main results of this article for an arbitrary real number, τ, we define the degree of the operator, A, acting in H as

$$A^\tau h = \sum_{k=1}^\infty \lambda_k^\tau h_k v_k, \ h_k = (h, v_k).$$

Naturally, the domain of definition of this operator has the form

$$D(A^\tau) = \{h \in H : \sum_{k=1}^\infty \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

It immediately follows from this definition that $D(A^\tau) \subseteq D(A^\sigma)$ for any $\tau \ge \sigma$.

On the set $D(A^\tau)$, we define the inner product

$$(h,g)_\tau = \sum_{k=1}^\infty \lambda_k^{2\tau} h_k \overline{g}_k = (A^\tau h, A^\tau g)$$

Then, $D(A^\tau)$ becomes a Hilbert space with the norm $||h||_\tau^2 = (h,h)_\tau$.

Theorem 1. *Let $\alpha > 0$, $\varphi_0 \in H$ and $\varphi_1 \in D(A^{\frac{1}{2}})$. Further, let $\epsilon \in (0,1)$ be any fixed number and $f(t) \in C([0,T]; D(A^\epsilon))$. Then, the forward problem has a unique strong solution.*

Furthermore, there is a constant, $C > 0$, such that the following stability estimate holds:

$$||(D_t^\rho)^2 u|| + ||D_t^\rho u|| + ||Au|| \le C\left[t^{-\rho}(||\varphi_0|| + ||\varphi_1||_{\frac{1}{2}}) + \max_{0\le t\le T} ||f(t)||_\epsilon\right], \quad t > 0$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded N-dimensional domain with a sufficiently smooth boundary $\partial\Omega$. Let A_0 stand for the operator in $L_2(\Omega)$: $A_0 h(x) = -\triangle h(x)$, with the domain of definition $D(A_0) = \{h \in C^2(\Omega) \cap C(\overline{\Omega}) : h(x) = 0, \ x \in \partial\Omega\}$, where $\triangle$ is the Laplace operator. Then (see, e.g., [4]), A_0 has a system of orthonormal eigenfunctions, $\{v_k(x)\}$, complete in $L_2(\Omega)$, and a countable set of non-negative eigenvalues, λ_k: $0 < \lambda_1(\Omega) = \lambda_1 \le \lambda_2 \cdots \to +\infty$.

Let A be the operator: $Ah(x) = \sum \lambda_k h_k v_k(x)$ with $D(A) = \{h \in L_2(\Omega) : \sum \lambda_k^2 h_k^2 < \infty\}$. Then, one can easily verify that A is a positive selfadjoint extension in $L_2(\Omega)$ of operator A_0. Hence, one is able to assign Theorem 1 to operator A and, therefore, to the problem:

$$\begin{cases} (D_t^\rho)^2 u(x,t) + 2\alpha D_t^\rho u(x,t) - \triangle u(x,t) = f(x,t),\, x \in \Omega,\, 0 < t \le T; \\ u(x,t) = 0,\, x \in \partial\Omega,\;\, 0 < t \le T; \\ \lim\limits_{t\to 0} D_t^\rho u(x,t) = \varphi_0(x); \\ u(x,0) = \varphi_1(x), \end{cases} \tag{3}$$

where $f(x,t) \in C(L_2(\Omega))$ and $\varphi_0(x), \varphi_1(x)$ are known elements of $L_2(\Omega)$.

For simplicity, one can take $N = 1$ and $\Omega = [0, \pi]$. Then, we have the following problem:

$$\begin{cases} (D_t^\rho)^2 u(x,t) + 2\alpha D_t^\rho u(x,t) - u_{xx}(x,t) = f(x,t), \quad (x,t) \in \Omega; \\ u(0,t) = u(\pi,t) = 0, \quad 0 \le t \le T; \\ \lim\limits_{t\to 0} D_t^\rho u(x,t) = \varphi_0(x), \quad 0 \le x \le \pi; \\ u(x,0) = \varphi_1(x), \quad 0 \le x \le \pi. \end{cases}$$

In this case, we know that the eigenvalues and eigenfunctions have the form: $\lambda_k = k^2$ and $v_k = \sin(kx)$. From this, we see that whenever $\alpha \notin \mathbb{N}$, then $\alpha^2 \ne \lambda_k$ for all k. Otherwise, for some k, we may have the equality, $\alpha^2 = \lambda_k$.

The telegraph equation first appeared in the work of Oliver Heaviside in 1876. When simulating the passage of electrical signals in marine telegraph cables, he obtained the following equation:

$$u_{tt} + au_t + bu - cu_{xx} = 0,$$

where a and b are non-negative constants, and c is a positive constant (see, e.g., [5,6]). Then, specialists came to this equation when modeling various physical processes. A small overview of various applications of the telegraph equation is given in [7]. This is shown in the theory of superconducting electrodynamics, where it illustrates the propagation electromagnetic waves in superconducting media (see, e.g., [8]). In [7], the propagation of digital and analog signals through media, which, in general, are both dissipative and dispersive, is modeled using the telegraph equation. Some applications of the telegraph equation to the theory of random walks are contained in [9]. Another field of application of the telegraph equation is the biological sciences (see, e.g., [5,10,11]).

In recent decades, fractional calculus has attracted the attention of many mathematicians and researchers as non-integer derivative operators have come to play a larger role in describing physical phenomena, modeling more accurately and efficiently than classical derivatives [12–14]. Various forms of the time-fractional telegraph equation was considered by a number of researchers (see, e.g., [15–17]), with the elliptic part of the equation in the form $Au(x,t) = u_{xx}(x,t)$. Thus, in the works of ref. [18] (in the case of $\rho = 1/2$) and ref. [19] (in the case of fractional derivatives of rational order, $\rho = m/n$, with $m < n$), fundamental solutions for problem (2) are constructed. In the work of ref. [20], a fundamental solution of the Cauchy problem in cases $x \in \mathbb{R}$ and $x \in \mathbb{R}_+$ is found using Fourier–Laplace transforms and their inverse transforms. Additionally, for the case of a bounded spatial domain, the solution of the boundary value problem is found in the form of a series using the Sine–Laplace transformation method.

The authors of [21] studied problem (2) in a bounded spatial domain, with operator $A = (-\triangle)^{\beta/2}$, $\beta \in (0, 2]$, and they found the formal analytical solutions under nonhomogeneous Dirichlet and Neumann boundary conditions by using the method of separation of variables. The obtained solutions are expressed as a Fourier series through multivariate Mittag-Leffler-type functions. However, it should be noted that the authors did not study the convergence and differentiability of these series, i.e., it is not shown whether the function represented by these series is really the solution of the problems under study.

A number of specialists have developed efficient and optimally accurate numerical algorithms for solving problem (2) for different operators, A. Reviews of some works in this direction are contained in the papers in [7,22].

The closest to our article is the fundamental work of R. Cascaval et al. [3]. In this paper, problem (2) is considered for a homogeneous equation in the case when the parameter, α^2, is not included in the spectrum of the operator, A. The main goal of this paper is to study the asymptotic behavior of the solution, $u(t)$, of problem (2) for large t. The authors succeeded in proving the existence of a solution, $v(t)$, of the equation $2\alpha D_t^{\rho} v(t) + Av(t) = 0$, for which the asymptotic relation,

$$u(t) = v(t) + o(v(t)), \quad t \to +\infty,$$

is valid.

We note that, in this paper, it was also conjectured that for the existence and uniqueness of a strong solution to problem (2) (recall that, in this paper, a homogeneous equation is considered, and it is assumed that α^2 is not included in the spectrum of the operator, A), the initial functions must be from the following classes: $\varphi_0 \in D(A^{\frac{1}{2}})$ and $\varphi_1 \in D(A)$. However, as Theorem 1 shows, for problem (2) (even in a more general case), to be well posed, it suffices to require much fewer conditions on the initial functions. We also emphasize the importance of the stability estimate obtained in Theorem 1, which was not known even for the homogeneous telegraph equation.

The present paper consists of four sections: Section 2 provides some background and preliminaries for the forward problem. Here, we prove several important lemmas. In Section 3, complete proof of the existence and uniqueness of the solution to problem (1) is provided. Moreover, we present here the stability result for the same problem. The article ends with the Conclusion.

2. Preliminaries

In this part, we recall several data about the Mittag-Leffler functions, differential and integral equations, which we will utilize in the following parts.

For $0 < \rho < 1$ and an arbitrary complex number μ, by $E_{\rho,\mu}(z)$, we denote the Mittag-Leffler function of a complex argument, z, with two parameters:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}. \tag{4}$$

If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_{\rho}(z) = E_{\rho,1}(z)$. Prabhakar (see [23]) introduced the function, $E^{\gamma}_{\rho,\mu}(z)$, of the form

$$E^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\rho k + \mu)} \cdot \frac{z^k}{k!}, \tag{5}$$

where $z \in C$, ρ, μ, and γ are arbitrary positive constants, and $(\gamma)_k$ is the Pochhammer symbol. When $\gamma = 1$, one has $E^{1}_{\rho,\mu}(z) = E_{\rho,\mu}(z)$. We also have [23]

$$E^{2}_{\rho,\mu}(z) = \frac{1}{\rho}\Big(E_{\rho,\mu-1}(z) + (1-\rho+\mu)E_{\rho,\mu}(z)\Big). \tag{6}$$

Obviously, since $E_{\rho,\mu}(z)$ is an analytic function of z, then it is bounded for $|z| \le 1$. We also note the notorious asymptotic assess of the Mittag-Leffler function (see, e.g., [24] (p. 133)):

Lemma 1. *Let μ be an arbitrary complex number. Further, let β be a fixed number, such that $\frac{\pi}{2}\rho < \beta < \pi\rho$ and $\beta \leq |\arg z| \leq \pi$. Then, the following asymptotic estimate holds:*

$$E_{\rho,\mu}(z) = -\frac{z^{-1}}{\Gamma(\rho-\mu)} + O(|z|^{-2}),\ |z| > 1.$$

Corollary 1. *Under the conditions of Lemma 1, one has*

$$|E_{\rho,\mu}(z)| \leq \frac{M}{1+|z|}, \quad |z| \geq 0,$$

where M is a constant, independent of z.

We also use the following estimate for sufficiently large $\lambda > 0$ and $\alpha > 0, \quad 0 < \epsilon < 1$:

$$|t^{\rho-1}E_{\rho,\mu}(-(\alpha-\sqrt{\alpha^2-\lambda})t^\rho)| \leq \frac{t^{\rho-1}M}{1+\sqrt{\lambda}t^\rho} \leq M\lambda^{\epsilon-\frac{1}{2}}t^{2\epsilon\rho-1}, \quad t > 0, \tag{7}$$

which is easy to verify. Indeed, let $(\lambda)^{\frac{1}{2}}t^\rho < 1$, then $t < \lambda^{-\frac{1}{2\rho}}$ and

$$t^{\rho-1} = t^{\rho-2\epsilon\rho}t^{2\epsilon\rho-1} < \lambda^{\epsilon-\frac{1}{2}}t^{2\epsilon\rho-1}.$$

If $(\lambda)^{\frac{1}{2}}t^\rho \geq 1$, then $\lambda^{-\frac{1}{2}} \leq t^\rho$ and

$$\lambda^{-\frac{1}{2}}t^{-1} = \lambda^{\epsilon-\frac{1}{2}}\lambda^{-\epsilon}t^{-1} \leq \lambda^{\epsilon-\frac{1}{2}}t^{2\rho\epsilon-1}.$$

Lemma 2. *If $\rho > 0$ and $\lambda \in C$, then (see [25] (p. 446))*

$$D_t^\rho E_{\rho,1}(\lambda t^\rho) = \lambda E_{\rho,1}(\lambda t^\rho) \quad t > 0. \tag{8}$$

The following lemma is an extension of the result of [3], where the authors considered only a homogeneous equation with an extra condition, $\alpha^2 \neq \lambda$.

Lemma 3. *Let $g(t) \in C[0,T]$ and φ_0, φ_1 be known numbers. Then, the unique solution of the Cauchy problem,*

$$\begin{cases} (D_t^\rho)^2 y(t) + 2\alpha D_t^\rho y(t) + \lambda y(t) = g(t), \quad 0 < t \leq T; \\ \lim\limits_{t\to 0} D_t^\rho y(t) = \varphi_0; \\ y(0) = \varphi_1, \end{cases} \tag{9}$$

has the form

$$y(t) = \begin{cases} y_1(t), & \alpha^2 \neq \lambda; \\ y_2(t), & \alpha^2 = \lambda. \end{cases} \tag{10}$$

Here,

$$\begin{aligned} y_1(t) = {} & \frac{\left(\sqrt{\alpha^2-\lambda}+\alpha\right)\varphi_1}{2\sqrt{\alpha^2-\lambda}} E_{\rho,1}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)t^\rho\right) \\ & + \frac{\left(\sqrt{\alpha^2-\lambda}-\alpha\right)\varphi_1}{2\sqrt{\alpha^2-\lambda}} E_{\rho,1}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)t^\rho\right) \\ & + \frac{1}{2\sqrt{\alpha^2-\lambda}}\left(E_{\rho,1}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)t^\rho\right) - E_{\rho,1}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)t^\rho\right)\right)\varphi_0 \\ & + \frac{1}{2\sqrt{\alpha^2-\lambda}} \int_0^t (t-\tau)^{\rho-1} E_{\rho,\rho}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)(t-\tau)^\rho\right) g(\tau)d\tau \end{aligned}$$

$$-\frac{1}{2\sqrt{\alpha^2-\lambda}}\int_0^t (t-\tau)^{\rho-1}E_{\rho,\rho}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)(t-\tau)^\rho\right)g(\tau)d\tau,$$
$$y_2(t) = t^\rho E^2_{\rho,1+\rho}(-\alpha t^\rho)\varphi_0 + \alpha t^\rho E^2_{\rho,1+\rho}(-\alpha t^\rho)\varphi_1 + E_{\rho,1}(-\alpha t^\rho)\varphi_1$$
$$+\int_0^t (t-\tau)^{2\rho-1}E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho)g(\tau)d\tau.$$

Proof. We utilize the Laplace transform to prove the lemma. Let us be reminded that the Laplace transform of a function, $f(t)$, is defined as (see [26])

$$L[f](s) = \hat{f}(s) = \int_0^\infty e^{-st}f(t)dt.$$

The inverse Laplace transform is defined by

$$L^{-1}[\hat{f}](t) = \frac{1}{2\pi i}\int_C e^{st}\hat{f}(s)ds,$$

where C is a contour parallel to the imaginary axis and to the right of the singularities of $\hat{f}$.

Let us apply the Laplace transform to Equation (9). Then, Equation (9) becomes:

$$s^{2\rho}\hat{y}(s) + 2\alpha s^\rho\hat{y}(s) + \lambda\hat{y}(s) - s^{2\rho-1}y(0) - s^{\rho-1}\lim_{t\to 0}D_t^\rho y(t) - 2\alpha s^{\rho-1}y(0) = \hat{g}(s),$$

it follows from this

$$\hat{y}(s) = \frac{\hat{g}(s) + s^{2\rho-1}y(0) + s^{\rho-1}\lim\limits_{t\to 0}D_t^\rho y(t) + 2\alpha s^{\rho-1}y(0)}{s^{2\rho}+2\alpha s^\rho+\lambda}. \tag{11}$$

Case 1. Let $\alpha^2 \neq \lambda$.

Write $\hat{y}(s) = \hat{y}_0(s) + \hat{y}_1(s)$, where

$$\hat{y}_0(s) = \frac{(s^{2\rho-1}+2\alpha s^{\rho-1})\varphi_0 + s^{\rho-1}\varphi_1}{s^{2\rho}+2\alpha s^\rho+\lambda}, \hat{y}_1(s) = \frac{\hat{g}(s)}{s^{2\rho}+2\alpha s^\rho+\lambda}.$$

Furthermore,

$$y(t) = L^{-1}[\hat{y}_0(s)] + L^{-1}[\hat{y}_1(s)].$$

As in the work in [3], when we apply the inverse Laplace transform, we obtain the following expression:

$$L^{-1}[\hat{y}_0(s)] = \frac{\left(\sqrt{\alpha^2-\lambda}+\alpha\right)\varphi_1+\varphi_0}{2\sqrt{\alpha^2-\lambda}}E_{\rho,1}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)t^\rho\right) \tag{12}$$
$$+\frac{\left(\sqrt{\alpha^2-\lambda}-\alpha\right)\varphi_1-\varphi_0}{2\sqrt{\alpha^2-\lambda}}E_{\rho,1}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)t^\rho\right).$$

For the second term of $y(t)$, one can obtain the inverse by splitting the function, $\hat{y}_1$, into simpler functions:

$$L^{-1}[\hat{y}_1(s)] = L^{-1}\left[\frac{\hat{g}(s)}{s^{2\rho}+2\alpha s^\rho+\lambda}\right] = L^{-1}\left[\frac{1}{s^{2\rho}+2\alpha s^\rho+\lambda}\right] * L^{-1}[\hat{g}(s)]. \tag{13}$$

Using $f * g$, we denote the Laplace convolution of functions defined by

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

The following simple observations show that

$$L^{-1}\left[\frac{1}{s^{2\rho}+2\alpha s^{\rho}+\lambda}\right]=L^{-1}\left[\frac{1}{2\sqrt{\alpha^2-\lambda}}\left(\frac{1}{s^{\rho}+\alpha-\sqrt{\alpha^2-\lambda}}-\frac{1}{s^{\rho}+\alpha+\sqrt{\alpha^2-\lambda}}\right)\right]$$

$$=\frac{1}{2\sqrt{\alpha^2-\lambda}}L^{-1}\left[\frac{1}{s^{\rho}+\alpha-\sqrt{\alpha^2-\lambda}}\right]-\frac{1}{2\sqrt{\alpha^2-\lambda}}L^{-1}\left[\frac{1}{s^{\rho}+\alpha+\sqrt{\alpha^2-\lambda}}\right],$$

or (see [3])

$$\frac{1}{2\sqrt{\alpha^2-\lambda}}L^{-1}\left[\frac{1}{s^{\rho}+\alpha-\sqrt{\alpha^2-\lambda}}\right]=\frac{t^{\rho-1}}{2\sqrt{\alpha^2-\lambda}}E_{\rho,\rho}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)t^{\rho}\right)$$

and

$$\frac{1}{2\sqrt{\alpha^2-\lambda}}L^{-1}\left[\frac{1}{s^{\rho}+\alpha+\sqrt{\alpha^2-\lambda}}\right]=\frac{t^{\rho-1}}{2\sqrt{\alpha^2-\lambda}}E_{\rho,\rho}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)t^{\rho}\right).$$

Plugging this function into (13) and combining it with (12), we have:

$$\begin{aligned}y(t)=&\frac{\left(\sqrt{\alpha^2-\lambda}+\alpha\right)\varphi_1}{2\sqrt{\alpha^2-\lambda}}E_{\rho,1}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)t^{\rho}\right)\\&+\frac{\left(\sqrt{\alpha^2-\lambda}-\alpha\right)\varphi_1}{2\sqrt{\alpha^2-\lambda}}E_{\rho,1}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)t^{\rho}\right)\\&+\frac{1}{2\sqrt{\alpha^2-\lambda}}\left(E_{\rho,1}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)t^{\rho}\right)-E_{\rho,1}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)t^{\rho}\right)\right)\varphi_0\\&+\frac{1}{2\sqrt{\alpha^2-\lambda}}\int_0^t(t-\tau)^{\rho-1}E_{\rho,\rho}\left(\left(-\alpha+\sqrt{\alpha^2-\lambda}\right)(t-\tau)^{\rho}\right)g(\tau)d\tau\\&-\frac{1}{2\sqrt{\alpha^2-\lambda}}\int_0^t(t-\tau)^{\rho-1}E_{\rho,\rho}\left(\left(-\alpha-\sqrt{\alpha^2-\lambda}\right)(t-\tau)^{\rho}\right)g(\tau)d\tau.\end{aligned}$$

Case 2. Let $\alpha^2=\lambda$. In this case, (11) has the following form:

$$\hat{y}(s)=\frac{\hat{g}(s)+s^{2\rho-1}y(0)+s^{\rho-1}\lim_{t\to 0}D_t^{\rho}y(t)+2\alpha s^{\rho-1}y(0)}{(s^{\rho}+\alpha)^2}.$$

Therefore,

$$\hat{y}(s)=\frac{s^{\rho-1}}{s^{\rho}+\alpha}y(0)+\frac{\alpha s^{\rho-1}}{(s^{\rho}+\alpha)^2}y(0)+\frac{s^{\rho-1}}{(s^{\rho}+\alpha)^2}\lim_{t\to 0}D_t^{\rho}y(t)+\frac{1}{(s^{\rho}+\alpha)^2}\hat{g}(s).$$

Passing to the inverse Laplace transform (see [26] (p. 226, E67)):

$$\begin{aligned}y(t)=L^{-1}\left[\frac{s^{\rho-1}}{s^{\rho}+\alpha}y(0)\right]+L^{-1}\left[\frac{\alpha s^{\rho-1}}{(s^{\rho}+\alpha)^2}y(0)\right]+L^{-1}\left[\frac{s^{\rho-1}}{(s^{\rho}+\alpha)^2}\lim_{t\to 0}D_t^{\rho}y(t)\right]\\+L^{-1}\left[\frac{1}{(s^{\rho}+\alpha)^2}\hat{g}(s)\right],\end{aligned}$$

one has

$$\begin{aligned}y(t)=E_{\rho,1}(-\alpha t^{\rho})\varphi_1+\alpha t^{\rho}E^2_{\rho,1+\rho}(-\alpha t^{\rho})\varphi_1+t^{\rho}E^2_{\rho,1+\rho}(-\alpha t^{\rho})\varphi_0\\+\int_0^t(t-\tau)^{2\rho-1}E^2_{\rho,2\rho}(-\alpha(t-\tau)^{\rho})g(\tau)d\tau.\end{aligned}$$

□

Remark 1. *A similar result can also be obtained using the Melling transform (see [27]).*

Lemma 4. *Let $g(t) \in C[0,T]$.Then, the unique solution of the Cauchy problem*

$$\begin{cases} D_t^\rho u(t) + 2\alpha u(t) + \alpha^2 J_t^{-\rho} u(t) = J_t^{-\rho} g(t), \quad 0 < t \leq T; \\ u(0) = 0, \end{cases} \tag{14}$$

where $0 < \rho < 1$ and $\alpha \in \mathbb{C}$, has the form

$$u(t) = \int_0^t (t-\tau)^{2\rho-1} E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(\tau) d\tau.$$

Proof. Allow us to apply the Laplace transform to Equation (14). Then, Equation (14) becomes:

$$s^\rho \hat{u}(s) - s^{\rho-1} u(0) + 2\alpha \hat{u}(s) + \alpha^2 s^{-\rho} \hat{u}(s) = s^{-\rho} \hat{g}(s),$$

It follows from this that

$$\hat{u}(s) = \frac{s^{-\rho}\hat{g}(s)}{s^\rho + 2\alpha + \alpha^2 s^{-\rho}} = \frac{\hat{g}(s)}{(s^\rho + \alpha)^2}.$$

Passing to the inverse Laplace transform, we obtain:

$$u(t) = L^{-1}[\frac{1}{(s^\rho + \alpha)^2}] * L^{-1}[\hat{g}(s)].$$

The first term in the convolution is known (see [26] (p. 226, E67)), and one has

$$u(t) = \int_0^t (t-\tau)^{2\rho-1} E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(\tau) d\tau.$$

□

Lemma 5. *The solution to the Cauchy problem*

$$\begin{cases} D_t^\rho u(t) - \lambda u(t) = f(t), \quad 0 < t \leq T; \\ u(0) = 0, \end{cases} \tag{15}$$

where $0 < \rho < 1$ and $\lambda \in \mathbb{C}$, has the form

$$u(t) = \int_0^t (t-\tau)^{\rho-1} E_{\rho,\rho}(\lambda(t-\tau)^\rho) f(\tau) d\tau.$$

The proof of this lemme for $\lambda \in \mathbb{R}$ can be found in [28] (p. 231). In a complex case, similar ideas will lead us to the same conclusion.

Regarding the operator, $E_{\rho,\mu}(t^\rho A) : H \to H$, defined by the spectral theorem of J. von Neumann:

$$E_{\rho,\mu}(t^\rho A) g = \sum_{k=1}^{\infty} E_{\rho,\mu}(t^\rho \lambda_k) g_k v_k,$$

here and throughout below, by g_k, we will denote the Fourier coefficients of a vector, $g \in H$: $g_k = (g, v_k)$.

Lemma 6. *Let $\alpha > 0$. Then, for any $g(t) \in C(H)$, one has $E_{\rho,\mu}(-St^{\rho})g(t) \in C(H)$ and $SE_{\rho,\mu}(-St^{\rho})g(t) \in C((0,T];H)$. Furthermore, the following values hold:*

$$||E_{\rho,\mu}(-t^{\rho}S)g(t)||_{C(H)} \leq M||g(t)||_{C(H)}, \tag{16}$$

$$||SE_{\rho,\mu}(-t^{\rho}S)g(t)|| \leq C_1 t^{-\rho}||g(t)||_{C(H)}, \quad t > 0. \tag{17}$$

If $g(t) \in D(A^{\frac{1}{2}})$ for all $t \in [0,T]$, then

$$||SE_{\rho,\mu}(-t^{\rho}S)g(t)||_{C(H)} \leq C_2 \max_{0\leq t\leq T} ||g(t)||_{\frac{1}{2}}, \tag{18}$$

$$||AE_{\rho,\mu}(-t^{\rho}S)g(t)|| \leq C_3 t^{-\rho} \max_{0\leq t\leq T} ||g(t)||_{\frac{1}{2}}, \quad t > 0. \tag{19}$$

Here, S has two states: S^- and S^+,

$$S^- = \alpha I - (\alpha^2 I - A)^{\frac{1}{2}}, \quad S^+ = \alpha I + (\alpha^2 I - A)^{\frac{1}{2}}.$$

Proof. By using Parseval's equality, one has

$$||E_{\rho,\mu}(-S^- t^{\rho})g(t)||^2 = \sum_{k=1}^{\infty} \left| E_{\rho,\mu}\left(-\left(\alpha - \sqrt{\alpha^2 - \lambda_k}\right)t^{\rho}\right)g_k(t)\right|^2.$$

According to Corollary 1, we write the following:

$$||E_{\rho,\mu}(-(\alpha - \sqrt{\alpha^2 - \lambda_k})t^{\rho})g(t)||^2 \leq M^2 \sum_{k=1}^{\infty} \left| \frac{g_k(t)}{1 + |\alpha - \sqrt{\alpha^2 - \lambda_k}|t^{\rho}} \right|^2 \leq M^2||g(t)||^2,$$

which concludes the assertion (16). On the other hand,

$$||S^- E_{\rho,\mu}(-t^{\rho}S^-)g(t)||^2 \leq M^2 \sum_{k=1}^{\infty} \frac{|\alpha - \sqrt{\alpha^2 - \lambda_k}|^2 |g_k(t)|^2}{(1 + t^{\rho}|\alpha - \sqrt{\alpha^2 - \lambda_k}|)^2},$$

$$u_{\lambda_k}(t) = \frac{|\alpha - \sqrt{\alpha^2 - \lambda_k}|^2 |g_k(t)|^2}{(1 + t^{\rho}|\alpha - \sqrt{\alpha^2 - \lambda_k}|)^2} \underset{\lambda_k \to \infty}{\sim} v_{\lambda_k}(t) = \frac{\lambda_k |g_k(t)|^2}{(1 + t^{\rho}\sqrt{\lambda_k})^2},$$

$$\sum_{k=1}^{\infty} v_{\lambda_k}(t) = \sum_{k=1}^{\infty} \frac{\lambda_k |g_k(t)|^2}{(1 + t^{\rho}\sqrt{\lambda_k})^2} \leq t^{-2\rho}||g(t)||^2, \quad t > 0.$$

We have used the notation here: $u_{\lambda_k} \underset{\lambda_k \to \infty}{\sim} v_{\lambda_k}$ means $\lim_{\lambda_k \to \infty} \frac{u_{\lambda_k}}{v_{\lambda_k}} = 1$. Therefore,

$$||S^- E_{\rho,\mu}(-t^{\rho}S^-)g(t)||^2 \leq M^2 C t^{-2\rho}||g(t)||^2 = C_1 t^{-2\rho}||g(t)||^2, \quad t > 0.$$

Obviously, if $g(t) \in D(A^{\frac{1}{2}})$ for all $t \in [0,T]$, then

$$||S^- E_{\rho,\mu}(-t^{\rho}S^-)g(t)||_{C(H)} \leq C_2 \max_{0\leq t\leq T} ||g(t)||_{\frac{1}{2}},$$

$$||AE_{\rho,\mu}(-t^{\rho}S^-)g(t)|| \leq C_3 t^{-\rho} \max_{0\leq t\leq T} ||g(t)||_{\frac{1}{2}}, \quad t > 0.$$

A similar estimate is proven in precisely the same way, with the operator, S^-, replaced by the operator, S^+. □

Lemma 7. *Let $\alpha > 0$ and $\lambda_k \neq \alpha^2$ for all k. Then, for any $g(t) \in C(H)$, one has $R^{-1}E_{\rho,\mu}(-St^\rho)g(t)$, $SR^{-1}E_{\rho,\mu}(-St^\rho)g(t) \in C(H)$, and $AR^{-1}E_{\rho,\mu}(-t^\rho S)g(t) \in C((0,T],H)$. Furthermore, the following values hold:*

$$||R^{-1}E_{\rho,\mu}(-t^\rho S)g(t)||_{C(H)} \leq C_4||g(t)||_{C(H)}, \tag{20}$$

$$||SR^{-1}E_{\rho,\mu}(-t^\rho S)g(t)||_{C(H)} \leq C_5||g(t)||_{C(H)}, \tag{21}$$

$$||AR^{-1}E_{\rho,\mu}(-t^\rho S)g(t)|| \leq C_6 t^{-\rho}||g(t)||_{C(H)}, \quad t > 0. \tag{22}$$

Here,

$$R^{-1} = (\alpha^2 I - A)^{-\frac{1}{2}}.$$

Proof. In proving the lemma, we use Parseval's equality and Corollary 1 similarly to the proof of Lemma 6:

$$||R^{-1}E_{\rho,\mu}(-t^\rho S^-)g(t)||^2 \leq M^2 \sum_{k=1}^{\infty} \left| \frac{1}{\sqrt{\alpha^2-\lambda_k}} \frac{g_k(t)}{1+t^\rho|\alpha - \sqrt{\alpha^2-\lambda_k}|} \right|^2,$$

$$u_{\lambda_k}(t) = \frac{|g_k(t)|^2}{|\sqrt{\alpha^2-\lambda_k}|^2(1+t^\rho|\alpha-\sqrt{\alpha^2-\lambda_k}|)^2} \underset{\lambda_k\to\infty}{\sim} v_{\lambda_k}(t) = \frac{|g_k(t)|^2}{\lambda_k(1+t^\rho\sqrt{\lambda_k})^2},$$

$$\sum_{k=1}^{\infty} v_{\lambda_k}(t) = \sum_{k=1}^{\infty} \frac{|g_k(t)|^2}{\lambda_k(1+t^\rho\sqrt{\lambda_k})^2} \leq C^*||g(t)||^2.$$

Therefore,

$$||R^{-1}E_{\rho,\mu}(-t^\rho S^-)g(t)||^2 \leq M^2C^*||g(t)||^2_{C(H)} = C_4||g(t)||^2_{C(H)}.$$

Similarly,

$$||S^-R^{-1}E_{\rho,\mu}(-t^\rho S^-)g(t)||^2 \leq M^2 \sum_{k=1}^{\infty} \left| \frac{|\alpha-\sqrt{\alpha^2-\lambda_k}|}{\sqrt{\alpha^2-\lambda_k}} \frac{g_k(t)}{1+t^\rho|\alpha - \sqrt{\alpha^2-\lambda_k}|} \right|^2,$$

$$u_{\lambda_k}(t) = \frac{|\alpha-\sqrt{\alpha^2-\lambda_k}|^2|g_k(t)|^2}{|\sqrt{\alpha^2-\lambda_k}|^2(1+t^\rho|\alpha-\sqrt{\alpha^2-\lambda_k}|)^2} \underset{\lambda_k\to\infty}{\sim} v_{\lambda_k}(t) = \frac{|g_k(t)|^2}{(1+t^\rho\sqrt{\lambda_k})^2},$$

$$\sum_{k=1}^{\infty} v_{\lambda_k}(t) = \sum_{k=1}^{\infty} \frac{|g_k(t)|^2}{(1+t^\rho\sqrt{\lambda_k})^2} \leq ||g(t)||^2.$$

It remains to prove estimate (22). We consider the case with the operator S^-. We have

$$||AR^{-1}E_{\rho,\mu}(-t^\rho S^-)g(t)||^2 \leq M^2 \sum_{k=1}^{\infty} \left| \frac{1}{\sqrt{\alpha^2-\lambda_k}} \frac{\lambda_k g_k(t)}{1+t^\rho|\alpha - \sqrt{\alpha^2-\lambda_k}|} \right|^2,$$

$$u_{\lambda_k}(t) = \frac{\lambda_k^2|g_k(t)|^2}{|\sqrt{\alpha^2-\lambda_k}|^2(1+t^\rho|\alpha-\sqrt{\alpha^2-\lambda_k}|)^2} \underset{\lambda_k\to\infty}{\sim} v_{\lambda_k}(t) = \frac{\lambda_k^2|g_k(t)|^2}{\lambda_k(1+t^\rho\sqrt{\lambda_k})^2},$$

$$\sum_{k=1}^{\infty} v_{\lambda_k}(t) = \sum_{k=1}^{\infty} \frac{\lambda_k|g_k(t)|^2}{(1+t^\rho\sqrt{\lambda_k})^2} \leq t^{-2\rho}||g(t)||^2, \quad t > 0.$$

Then,

$$||AR^{-1}E_{\rho,\mu}(-t^\rho S^-)g(t)||^2 \leq M^2C^{**}t^{-2\rho}||g(t)||^2_{C(H)} = C_6 t^{-2\rho}||g(t)||^2_{C(H)}, \quad t > 0.$$

Similar estimates are proven in precisely the same way for the operator, S^+. □

Lemma 8. *Let $\alpha > 0$ and $\lambda_k \neq \alpha^2$ for all k. Then, for any $g(t) \in C([0,T]; D(A^\epsilon))$, with $0 < \epsilon < 1$, we have*

$$\left|\left|\int_0^t (t-\tau)^{\rho-1} AR^{-1} E_{\rho,\rho}(-(t-\tau)^\rho S) g(\tau) d\tau\right|\right| \leq C \max_{0\leq t\leq T} ||g(t)||_\epsilon, \tag{23}$$

$$\left|\left|\int_0^t (t-\tau)^{\rho-1} SR^{-1} E_{\rho,\rho}(-(t-\tau)^\rho S) g(\tau) d\tau\right|\right| \leq C \max_{0\leq t\leq T} ||g(t)||_\epsilon, \tag{24}$$

$$\left|\left|\int_0^t (t-\tau)^{\rho-1} R^{-1} E_{\rho,\rho}(-(t-\tau)^\rho S) g(\tau) d\tau\right|\right| \leq C \max_{0\leq t\leq T} ||g(t)||_\epsilon. \tag{25}$$

Proof. Let

$$S_j(t) = \sum_{k=1}^{j} \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-(\alpha - \sqrt{\alpha^2 - \lambda_k})\eta^\rho) g_k(t-\eta) d\eta\right] \frac{\lambda_k}{\sqrt{\alpha^2-\lambda_k}} v_k.$$

We may write

$$||S_j(t)||^2 = \sum_{k=1}^{j} \left|\frac{\lambda_k}{\sqrt{\alpha^2-\lambda_k}}\right|^2 \left|\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-(\alpha - \sqrt{\alpha^2 - \lambda_k})\eta^\rho) g_k(t-\eta) d\eta\right|^2.$$

Apply estimate (7) to a large enough $k \geq j_0$ to obtain

$$||S_j(t)||^2 \leq C \sum_{k=j_0}^{j} \left[\int_0^t \eta^{2\epsilon\rho-1} \frac{\lambda_k}{\sqrt{\lambda_k}} \lambda_k^{\epsilon-\frac{1}{2}} |g_k(t-\eta)| d\eta\right]^2.$$

Minkowski's inequality implies

$$||S_j(t)||^2 \leq C\left[\int_0^t \eta^{2\epsilon\rho-1} \left(\sum_{k=j_0}^{j} \lambda_k^{2\epsilon} |g_k(t-\eta)|^2\right)^{\frac{1}{2}} d\eta\right]^2 \leq C \max_{0\leq t\leq T} ||g(t)||_\epsilon^2.$$

Since

$$\int_0^t (t-\tau)^{\rho-1} AR^{-1} E_{\rho,\rho}(-(t-\tau)^\rho S^-) g(\tau) d\tau = \lim_{j\to\infty} S_j(t),$$

this implies the assertions of Equations (23)–(25) are obtained in the same way as in the proof of (23), combining the fact that $D(A^\epsilon) \subset D(A^{\epsilon-1/2}) \subset D(A^{\epsilon-1})$. □

Lemma 9. *Let $\alpha > 0$ and $g(t) \in C(H)$. Then,*

$$\left|\left|J_t^{-\rho}\left(\int_0^t (t-\tau)^{2\rho-1} E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(\tau) d\tau\right)\right|\right|_{C(H)} \leq \frac{M}{\Gamma(\rho)} \frac{T^{3\rho}}{2\rho^3}(2+\rho)||g(t)||_{C(H)}. \tag{26}$$

Proof. For convenience, let us denote the argument of $J_t^{-\rho}$ by

$$F(t) = \int_0^t (t-\tau)^{2\rho-1} E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(\tau) d\tau.$$

According to (6)

$$F(t) = \frac{1}{\rho}\int_0^t (t-\tau)^{2\rho-1} E_{\rho,2\rho-1}(-\alpha(t-\tau)^\rho) g(\tau) d\tau + \frac{1+\rho}{\rho}\int_0^t (t-\tau)^{2\rho-1} E_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(\tau) d\tau,$$

Then,

$$\max_{0\le t\le T} ||J_t^{-\rho} F(t)|| = \max_{0\le t\le T} || \int_0^t F(\tau)(t-\tau)^{\rho-1} d\tau|| \le \max_{0\le t\le T} \int_0^t ||F(\tau)|||t-\tau|^{\rho-1} d\tau$$

$$\le ||F(t)||_{C(H)} \max_{0\le t\le T} \int_0^t |t-\tau|^{\rho-1} d\tau = ||F(t)||_{C(H)} \max_{0\le t\le T} \frac{t^\rho}{\rho} \le \frac{T^\rho}{\rho} ||F(t)||_{C(H)}.$$

Thus, we need to estimate $\|F(t)\|_{C(H)}$, and this can be performed as follows:

$$||F(t)||_{C(H)} \le \frac{1}{\rho} || \int_0^t (t-\tau)^{2\rho-1} E_{\rho,2\rho-1}(-\alpha(t-\tau)^\rho) g(\tau) d\tau ||_{C(H)}$$

$$+\frac{1+\rho}{\rho} || \int_0^t (t-\tau)^{2\rho-1} E_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(\tau) d\tau ||_{C(H)}$$

$$\le \frac{1}{\rho} ||E_{\rho,2\rho-1}(-\alpha(t-\tau)^\rho) g(t)||_{C(H)} \max_{0\le t\le T} \int_0^t |(t-\tau)|^{2\rho-1} d\tau$$

$$+\frac{1+\rho}{\rho} ||E_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(t)||_{C(H)} \max_{0\le t\le T} \int_0^t |(t-\tau)|^{2\rho} d\tau.$$

Using estimate (16),

$$||F(t)||_{C(H)} \le \frac{MT^{2\rho}}{2\rho^2}(2+\rho)||g(t)||_{C(H)}. \tag{27}$$

□

3. Proof of Theorem on the Forward Problem

In this part, we prove Theorem 1.

Proof. In accordance with the Fourier method, we will seek the solution to this problem in the form

$$u(t) = \sum_{k=1}^{\infty} T_k(t) v_k, \tag{28}$$

where $T_k(t)$ is a solution to the problem

$$\begin{cases} (D_t^\rho)^2 T_k(t) + 2\alpha D_t^\rho T_k(t) + \lambda_k T_k(t) = f_k(t), \\ \lim_{t\to 0} D_t^\rho T_k(t) = \varphi_{0k}, \\ T_k(0) = \varphi_{1k}, \end{cases} \tag{29}$$

Apply Lemma 3 to obtain

$$T_k(t) = \begin{cases} y_{1k}(t), & \alpha^2 \ne \lambda_k; \\ y_2(t), & \alpha^2 = \lambda_k. \end{cases} \tag{30}$$

Therefore, we have two cases:

Case I: $\alpha^2 \ne \lambda_k$ for all $k \in \mathbb{N}$. We have

$$u(t) = \frac{1}{2}\Big[E_{\rho,1}(-S^- t^\rho) + E_{\rho,1}(-S^+ t^\rho)\Big]\varphi_1 + \frac{\alpha}{2} R^{-1} E_{\rho,1}(-S^- t^\rho)\varphi_1 \tag{31}$$

$$-\frac{\alpha}{2} R^{-1} E_{\rho,1}(-S^+ t^\rho)\varphi_1 + \frac{1}{2}\Big[R^{-1} E_{\rho,1}(-S^- t^\rho) - R^{-1} E_{\rho,1}(-S^+ t^\rho)\Big]\varphi_0$$

$$+\frac{1}{2}\int_0^t (t-\tau)^{\rho-1}\Big[R^{-1}E_{\rho,\rho}\big(-S^-(t-\tau)^\rho\big)-R^{-1}E_{\rho,\rho}\big(-S^+(t-\tau)^\rho\big)\Big]f(\tau)d\tau.$$

Case II: $\exists k_0 \in \mathbb{N}$, such that $\alpha^2 = \lambda_{k_0}$.

For simplicity, we assume that there is only one λ_{k_0} of this kind. Then, the solution is

$$u(t) = \frac{1}{2}\Big[\tilde{E}_{\rho,1}(-S^-t^\rho)+\tilde{E}_{\rho,1}(-S^+t^\rho)\Big]\varphi_1 + \frac{\alpha}{2}R^{-1}\tilde{E}_{\rho,1}(-S^-t^\rho)\varphi_1 \tag{32}$$

$$-\frac{\alpha}{2}R^{-1}\tilde{E}_{\rho,1}(-S^+t^\rho)\varphi_1+\frac{1}{2}\Big[R^{-1}\tilde{E}_{\rho,1}(-S^-t^\rho)-R^{-1}\tilde{E}_{\rho,1}(-S^+t^\rho)\Big]\varphi_0+\alpha t^\rho E^2_{\rho,1+\rho}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}$$

$$+E_{\rho,1}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}+t^\rho E^2_{\rho,1+\rho}(-\alpha t^\rho)\varphi_{0k}v_{k_0}+\int_0^t(t-\tau)^{2\rho-1}E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho)f_{k_0}(\tau)v_{k_0}d\tau$$

$$+\frac{1}{2}\int_0^t(t-\tau)^{\rho-1}\Big[R^{-1}\tilde{E}_{\rho,\rho}(-S^-(t-\tau)^\rho)-R^{-1}\tilde{E}_{\rho,\rho}(-S^+(t-\tau)^\rho)\Big]f(\tau)d\tau,$$

which we can denote by using the following equation:

$$\tilde{E}_{\rho,\mu}(-St^\rho)g=\sum_{k\neq k_0}E_{\rho,\mu}\left(-\left(\alpha\pm\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)g_k v_k.$$

In the case when there are several indices, $k \in \mathbb{N}$, such that $\alpha^2 = \lambda_k$, we can repeat the same argument with a slight modification in the finite number of terms (as noted above, the multiplicity of each eigenvalue, λ_k, is finite).

We show that $u(t)$ is a solution to problem (2), in the above cases, according to Definition 1.

Since both cases are studied in a precisely similar way, we will consider only Case II. According to (6), we write (32) as follows:

$$u(t) = \frac{1}{2}\Big[\tilde{E}_{\rho,1}(-S^-t^\rho)+\tilde{E}_{\rho,1}(-S^+t^\rho)\Big]\varphi_1 + \frac{\alpha}{2}R^{-1}\tilde{E}_{\rho,1}(-S^-t^\rho)\varphi_1 \tag{33}$$

$$-\frac{\alpha}{2}R^{-1}\tilde{E}_{\rho,1}(-S^+t^\rho)\varphi_1+\frac{1}{2}\Big[R^{-1}\tilde{E}_{\rho,1}(-S^-t^\rho)-R^{-1}\tilde{E}_{\rho,1}(-S^+t^\rho)\Big]\varphi_0$$

$$+E_{\rho,1}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}+\frac{\alpha t^\rho}{\rho}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}+\frac{2\alpha t^\rho}{\rho}E_{\rho,1+\rho}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}$$

$$+\frac{2t^\rho}{\rho}E_{\rho,1+\rho}(-\alpha t^\rho)\varphi_{0k_0}v_{k_0}+\frac{1}{\rho}\int_0^t(t-\tau)^{2\rho-1}E_{\rho,2\rho-1}(-\alpha(t-\tau)^\rho)f_{k_0}(\tau)v_{k_0}d\tau$$

$$+\frac{t^\rho}{\rho}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{0k_0}v_{k_0}+\frac{1+\rho}{\rho}\int_0^t(t-\tau)^{2\rho-1}E_{\rho,2\rho}(-\alpha(t-\tau)^\rho)f_{k_0}(\tau)v_{k_0}d\tau$$

$$+\frac{1}{2}\int_0^t(t-\tau)^{\rho-1}\Big[R^{-1}\tilde{E}_{\rho,\rho}(-S^-(t-\tau)^\rho)-R^{-1}\tilde{E}_{\rho,\rho}(-S^+(t-\tau)^\rho)\Big]f(\tau)d\tau.$$

Estimate $\|u(t)\|_{C(H)}$ using (16) and (20), Corollary 1, (25):

$$||u(t)||_{C(H)}\leq(M+\alpha C_4)||\varphi_1||+C_4||\varphi_0||+\left(M+\frac{3\alpha MT^\rho}{\rho}\right)|\varphi_{1k_0}|+\frac{3MT^\rho}{\rho}|\varphi_{0k_0}|$$

$$+\frac{MT^{2\rho}}{\rho^2}(2+\rho)\max_{0\leq t\leq T}|f_{k_0}(t)|+C\max_{0\leq t\leq T}||f(t)||_\epsilon.$$

Next, we prove that this series converges after applying the operator, A, and the derivatives, $(D^\rho)^2_t$ and D^ρ_t.

Let us estimate $Au(t)$. If $S_j(t)$ is a partial sum of (33), then

$$AS_j(t) = \frac{1}{2}\sum_{\substack{k=1\\k\neq k_0}}^{j}\Bigg[E_{\rho,1}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k} + E_{\rho,1}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k}$$

$$+\frac{\alpha}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k} - \frac{\alpha}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k}$$

$$+\frac{1}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{0k} - \frac{1}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{0k}$$

$$+\frac{1}{\sqrt{\alpha^2-\lambda_k}}\int_0^t (t-\tau)^{\rho-1}E_{\rho,\rho}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)(t-\tau)^\rho\right)f_k(\tau)d\tau$$

$$-\frac{1}{\sqrt{\alpha^2-\lambda_k}}\int_0^t (t-\tau)^{\rho-1}E_{\rho,\rho}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)(t-\tau)^\rho\right)f_k(\tau)d\tau\Bigg]\lambda_k v_k$$

$$+\frac{2\alpha t^\rho}{\rho}E_{\rho,1+\rho}(-\alpha t^\rho)\varphi_{1k_0}\lambda_{k_0}v_{k_0} + \frac{t^\rho}{\rho}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{0k_0}\lambda_{k_0}v_{k_0} + \frac{2t^\rho}{\rho}E_{\rho,1+\rho}(-\alpha t^\rho)\varphi_{0k_0}\lambda_{k_0}v_{k_0}$$

$$+E_{\rho,1}(-\alpha t^\rho)\varphi_{1k_0}\lambda_{k_0}v_{k_0} + \frac{1}{\rho}\int_0^t (t-\tau)^{2\rho-1}E_{\rho,2\rho-1}(-\alpha(t-\tau)^\rho)f_{k_0}(\tau)\lambda_{k_0}v_{k_0}d\tau$$

$$+\frac{\alpha t^\rho}{\rho}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{1k_0}\lambda_{k_0}v_{k_0} + \frac{1+\rho}{\rho}\int_0^t (t-\tau)^{2\rho-1}E_{\rho,2\rho}(-\alpha(t-\tau)^\rho)f_{k_0}(\tau)\lambda_{k_0}v_{k_0}d\tau$$

Using Equations (19) and (22) and Corollary 1 and (23) consequently for the above-given expression, we obtain:

$$||AS_j(t)|| \leq C_3 t^{-\rho}||\varphi_1||_{\frac{1}{2}} + \alpha C_6 t^{-\rho}||\varphi_1|| + C_6 t^{-\rho}||\varphi_0|| + \alpha^2\Big(M + \frac{3\alpha M T^\rho}{\rho}\Big)|\varphi_{1k_0}|$$
$$+\frac{3\alpha^2 M T^\rho}{\rho}|\varphi_{0k_0}| + \frac{\alpha^2 M T^{2\rho}}{\rho^2}(2+\rho)\max_{0\leq t\leq T}|f_{k_0}(t)| + C\max_{0\leq t\leq T}||f(t)||_\epsilon, \quad t>0.$$

Hence, it is sufficient to have $\varphi_0 \in H$, $\varphi_1 \in D(A^{\frac{1}{2}})$ and $f(t) \in C([0,T];D(A^\epsilon))$ for having $Au(t) \in C((0,T];H)$.

Let us now estimate $D_t^\rho u(t)$. If $S_j(t)$ is a partial sum of (33), then, by (8), (14), and (15), we see that

$$D_t^\rho S_j(t) = \frac{1}{2}\sum_{\substack{k=1\\k\neq k_0}}^{j}\Bigg[-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)E_{\rho,1}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k}$$

$$-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)E_{\rho,1}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k}$$

$$-\frac{\alpha\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k}$$

$$+\frac{\alpha\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{1k}$$

$$-\frac{\alpha-\sqrt{\alpha^2-\lambda_k}}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{0k}$$

$$+\frac{\alpha+\sqrt{\alpha^2-\lambda_k}}{\sqrt{\alpha^2-\lambda_k}}E_{\rho,1}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)t^\rho\right)\varphi_{0k}$$

$$-\frac{\alpha-\sqrt{\alpha^2-\lambda_k}}{\sqrt{\alpha^2-\lambda_k}}\int_0^t (t-\tau)^{\rho-1}E_{\rho,\rho}\left(-\left(\alpha-\sqrt{\alpha^2-\lambda_k}\right)(t-\tau)^\rho\right)f_k(\tau)d\tau$$

$$+\frac{\alpha+\sqrt{\alpha^2-\lambda_k}}{\sqrt{\alpha^2-\lambda_k}}\int_0^t (t-\tau)^{\rho-1}E_{\rho,\rho}\left(-\left(\alpha+\sqrt{\alpha^2-\lambda_k}\right)(t-\tau)^\rho\right)f_k(\tau)d\tau\Bigg]v_k$$

$$-\frac{t^\rho\alpha}{\rho}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{0k_0}v_{k_0}-\frac{\alpha^2t^\rho}{\rho}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}-\frac{2\alpha^2t^\rho}{\rho}E_{\rho,\rho+1}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}$$

$$-\alpha E_{\rho,1}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}-\frac{2\alpha t^\rho}{\rho}E_{\rho,\rho+1}(-\alpha t^\rho)\varphi_{0k_0}v_{k_0}$$

$$-2\alpha\int_0^t (t-\tau)^{2\rho-1}E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho)f_{k_0}(\tau)v_{k_0}d\tau$$

$$-\alpha^2 J_t^{-\rho}\left(\int_0^t (t-\tau)^{2\rho-1}E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho)f_{k_0}(\tau)v_{k_0}d\tau\right)+J_t^{-\rho}f_{k_0}(t)v_{k_0}$$

Applying Equations (17) and (21), Corollary 1 and (24), and (26), for the corresponding terms of the above expression, we have:

$$||D_t^\rho S_j(t)||\le C_5||\varphi_0||+(C_1t^{-\rho}+\alpha C_5)||\varphi_1||+\frac{3M\alpha^2T^\rho}{\rho}|\varphi_{1k_0}|+\frac{3M\alpha T^\rho}{\rho}|\varphi_{0k_0}|+\frac{TM^\rho}{\rho}\max_{0\le t\le T}|f_{k_0}|$$

$$+\frac{2M\alpha^2T^{2\rho}}{2\rho^2}(2+\rho)\max_{0\le t\le T}|f_{k_0}|+\frac{MT^{3\rho}(2+\rho)}{\Gamma(\rho)2\rho^3}\max_{0\le t\le T}|f_{k_0}|+C\max_{0\le t\le T}||f(t)||_\epsilon,\quad t>0.$$

If $\varphi_1,\varphi_0\in H$ and $f(t)\in C([0,T];D(A^\epsilon))$, then we have $D_t^\rho u(t)\in C((0,T];H))$. Further, Equation (2) implies $(D_t^\rho)^2u(t)=-2\alpha D_t^\rho u(t)-Au(t)+f(t)$. Therefore, arguing as above, we find that $(D_t^\rho)^2u(t)\in C((0,T];H)$.

Using Equation (18) and similar ideas as in the proof of the above estimate, we have

$$||D_t^\rho S_j(t)||\le C_5||\varphi_0||+C_2||\varphi_1||_{\frac{1}{2}}+\alpha C_5||\varphi_1||+\frac{3M\alpha^2T^\rho}{\rho}|\varphi_{1k_0}|+\frac{3M\alpha T^\rho}{\rho}|\varphi_{0k_0}|$$

$$+\frac{2M\alpha^2T^{2\rho}}{2\rho^2}(2+\rho)\max_{0\le t\le T}|f_{k_0}|+\frac{MT^{3\rho}(2+\rho)}{\Gamma(\rho)2\rho^3}\max_{0\le t\le T}|f_{k_0}|+\frac{T^\rho}{\rho}\max_{0\le t\le T}|f_{k_0}|+C\max_{0\le t\le T}||f(t)||_\epsilon.$$

If $\varphi_1\in D(A^{\frac{1}{2}})$,$\varphi_0\in H$ and $f(t)\in C([0,T];D(A^\epsilon))$, then we have $D_t^\rho u(t)\in C(H)$.

Let us prove the uniqueness of the solution. We use a standard technique based on the completeness of the set of eigenfunctions, $\{v_k\}$, in H.

Let $u(t)$ be a solution to the problem

$$\begin{cases}(D_t^\rho)^2u(t)+2\alpha D_t^\rho u(t)+Au(t)=0,\quad 0<t\le T;\\ \lim\limits_{t\to 0}D_t^\rho u(t)=0,\\ u(0)=0.\end{cases}\tag{34}$$

Set $u_k(t)=(u(t),v_k)$. Then, by virtue of Equation (34) and the selfadjointness of operator A,

$$(D_t^\rho)^2u_k(t)=((D_t^\rho)^2u(t),v_k)=(-2\alpha D_t^\rho u(t)-Au(t),v_k)=$$

$$=(-2\alpha D_t^\rho u(t),v_k)-(Au(t),v_k)=-2\alpha(D_t^\rho u(t),v_k)-(u(t),Av_k)=$$

$$=-2\alpha D_t^\rho u_k(t)-\lambda_k u_k(t).$$

Hence, we have the following problem for $u_k(t)$:

$$\begin{cases} (D_t^\rho)^2 u_k(t) + 2\alpha D_t^\rho u_k(t) + \lambda_k u(t) = 0, \quad 0 < t \leq T; \\ \lim_{t\to 0} D_t^\rho u_k(t) = 0, \\ u_k(0) = 0. \end{cases}$$

Lemma 3 implies that $u_k(t) \equiv 0$ for all k. Consequently, due to the completeness of the system of eigenfunctions $\{v_k\}$, we have $u(t) \equiv 0$, as required. □

4. Conclusions

As noted above, fractional telegraph equations model various physical and biological processes. Therefore, the numerical solution of Cauchy problems for such equations is important. For this, in turn, it is necessary to make sure that the solution depends continuously on the initial data and the right side of the equation. Moreover, one should have a representation for the solution, for example, in the form of a Fourier series in terms of eigenfunctions of the elliptic part of the equation. This is exactly what is conducted in Theorem 1.

In addition, in Theorem 1, conditions are found on the initial functions and the right side of the equation that guarantee both the existence and the uniqueness of the solution of the Cauchy problem. It should be emphasized that these conditions turned out to be less stringent than expected in a well-known article by R. Cascaval et al. [3], where a similar problem was considered for a homogeneous equation and with some restriction on the spectrum of the operator, A.

It should be noted that despite the importance for applications, inverse problems for fractional telegraph equations have not yet been studied. In our next works, we will study inverse problems of finding the right side of the equation, i.e., source functions.

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