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On Some New Maclaurin's Type Inequalities for Convex Functions in q-Calculus

Thanin Sitthiwirattham [1,2], Muhammad Aamir Ali [3] and Hüseyin Budak [4,*]

1 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand; thanin_sit@dusit.ac.th
2 Research Group for Fractional Calculus Theory and Applications, Science and Technology Research Institute, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
3 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China; mahr.muhammad.aamir@gmail.com
4 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Türkiye
* Correspondence: hsyn.budak@gmail.com

Abstract: This work establishes some new inequalities to find error bounds for Maclaurin's formulas in the framework of q-calculus. For this, we first prove an integral identity involving q-integral and q-derivative. Then, we use this new identity to prove some q-integral inequalities for q-differentiable convex functions. The inequalities proved here are very important in the literature because, with their help, we can find error bounds for Maclaurin's formula in both q and classical calculus.

Keywords: Maclaurin's inequalities; Hermite–Hadamard inequalities; convex functions; q-calculus

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1. Introduction

From the following Simpson's rules, many inequalities of Simpson's type were established:

(i.) Simpson's 1/3 rule:

$$\int_{\triangle_1}^{\triangle_2} \Upsilon(\tau)d\tau \approx \frac{\triangle_2-\triangle_1}{6}\left[\Upsilon(\triangle_1)+4\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)+\Upsilon(\triangle_2)\right].$$

(ii.) Simpson's 3/8 rule (cf. [1])):

$$\int_{\triangle_1}^{\triangle_2} \Upsilon(\tau)d\tau \approx \frac{\triangle_2-\triangle_1}{8}\left[\Upsilon(\triangle_1)+3\Upsilon\left(\frac{2\triangle_1+\triangle_2}{3}\right)+3\Upsilon\left(\frac{\triangle_1+2\triangle_2}{3}\right)+\Upsilon(\triangle_2)\right].$$

(iii.) Dual Simpson's 3/8 rule (cf. [1]):

$$\int_{\triangle_1}^{\triangle_2} \Upsilon(\tau)d\tau \approx \frac{\triangle_2-\triangle_1}{8}\left[3\Upsilon\left(\frac{5\triangle_1+\triangle_2}{6}\right)+2\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)+3\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right)\right].$$

Now, we mention the inequalities linked to the above formulas to recall the literature. The Simpson's inequality linked to the Simpson's 1/3 formula is given as:

Theorem 1 ([1]). *For a four times differentiable and continuous function* $\Upsilon : [\triangle_1, \triangle_2] \to \mathbb{R}$. *The following inequality holds*

$$\left|\frac{1}{6}\left[\Upsilon(\triangle_1)+4\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)+\Upsilon(\triangle_2)\right]-\frac{1}{\triangle_2-\triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)d\tau\right|$$
$$\leq \frac{1}{2880}\sup_{\tau\in(\triangle_1,\triangle_2)}\left|\Upsilon^{(4)}(\tau)\right|(\triangle_2-\triangle_1)^4,$$

$$\text{where} \sup_{\tau \in (\triangle_1, \triangle_2)} \left| \mathrm{Y}^{(4)}(\tau) \right| < \infty.$$

The Simpson's inequality linked to the Simpson's 3/8 formula is given as:

Theorem 2 ([1]). *For a four times differentiable and continuous function* $\mathrm{Y} : [\triangle_1, \triangle_2] \to \mathbb{R}$. *The following inequality holds*

$$\begin{aligned} &\left| \frac{1}{8} \left[\mathrm{Y}(\triangle_1) + 3\mathrm{Y}\left(\frac{2\triangle_1 + \triangle_2}{3} \right) + 3\mathrm{Y}\left(\frac{\triangle_1 + 2\triangle_2}{3} \right) + \mathrm{Y}(\triangle_2) \right] - \frac{1}{\triangle_2 - \triangle_1} \int_{\triangle_1}^{\triangle_2} \mathrm{Y}(\tau) d\tau \right| \\ \leq\ & \frac{1}{6480} \left\| \mathrm{Y}^{(4)} \right\|_\infty (\triangle_2 - \triangle_1)^4, \end{aligned}$$

$$\text{where} \sup_{\tau \in (\triangle_1, \triangle_2)} \left| \mathrm{Y}^{(4)}(\tau) \right| < \infty.$$

The Simpson's inequality linked to the dual Simpson's 3/8 formula is given as:

Theorem 3 ([1]). *For a four times differentiable and continuous function* $\mathrm{Y} : [\triangle_1, \triangle_2] \to \mathbb{R}$. *The following inequality holds*

$$\begin{aligned} &\left| \frac{1}{8} \left[3\mathrm{Y}\left(\frac{5\triangle_1 + \triangle_2}{6} \right) + 2\mathrm{Y}\left(\frac{\triangle_1 + \triangle_2}{2} \right) + 3\mathrm{Y}\left(\frac{\triangle_1 + 5\triangle_2}{6} \right) \right] - \frac{1}{\triangle_2 - \triangle_1} \int_{\triangle_1}^{\triangle_2} \mathrm{Y}(\tau) d\tau \right| \\ \leq\ & \frac{7}{51840} \left\| \mathrm{Y}^{(4)} \right\|_\infty (\triangle_2 - \triangle_1)^4, \end{aligned}$$

$$\text{where} \sup_{\tau \in (\triangle_1, \triangle_2)} \left| \mathrm{Y}^{(4)}(\tau) \right| < \infty.$$

Many authors have concentrated on obtaining new bounds for these quadrature formulas in recent years using a variety of justifications, including fractional integrals and convexities. For some of them, please refer to [2–8]. Several studies have been conducted on the subject of q-integral inequalities for various convexities. As an illustration, new inequalities of the Hermite–Hadamard, midpoint and trapezoidal type for q-integrals and q-differentiable convex functions were established in [9–13]. In order to prove Simpson's type inequality for q-differentiable convex and generic convex functions, the authors of [14–16] used q-integral. One can refer to [17–22] for more recent q-calculus inequalities.

The aim of this paper is to establish new inequalities that can be utilized to determine error bounds for Maclaurin's formulas within the framework of q-calculus. To achieve this objective, we begin by demonstrating an integral identity that incorporates both q-integral and q-derivative. Subsequently, we employ this novel identity to establish a series of q-integral inequalities specifically designed for q-differentiable convex functions. These inequalities hold significant importance in the existing literature since they enable us to determine error bounds for Maclaurin's formula in both q-calculus and classical calculus. By developing these new inequalities, this work contributes to the advancement and understanding of error estimation techniques in mathematical analysis.

2. q-Calculus

To the better understanding about q-calculus, we gave some concepts of q-calculus here (see, [23]) and q is a real number in $(0, 1)$:

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1}.$$

Definition 1 ([22]). *The left quantum or $q_{\triangle_1}$-derivative of $\Upsilon : [\triangle_1, \triangle_2] \to \mathbb{R}$ at $\tau \in [\triangle_1, \triangle_2]$ is expressed as:*

$$ {}_{\triangle_1}D_q\Upsilon(\tau) = \frac{\Upsilon(\tau) - \Upsilon(q\tau + (1-q)\triangle_1)}{(1-q)(\tau - \triangle_1)}, \ \tau \neq \triangle_1 . \tag{1} $$

Definition 2 ([10]). *The right quantum or $q^{\triangle_2}$-derivative of $\Upsilon : [\triangle_1, \triangle_2] \to \mathbb{R}$ at $\tau \in [\triangle_1, \triangle_2]$ is expressed as:*

$$ {}^{\triangle_2}D_q\Upsilon(\tau) = \frac{\Upsilon(q\tau + (1-q)\triangle_2) - \Upsilon(\tau)}{(1-q)(\triangle_2 - \tau)}, \ \tau \neq \triangle_2 . $$

Definition 3 ([22]). *The left quantum or $q_{\triangle_1}$-integral of $\Upsilon : [\triangle_1, \triangle_2] \to \mathbb{R}$ at $\tau \in [\triangle_1, \triangle_2]$ is defined as:*

$$ \int_{\triangle_1}^{\tau} \Upsilon(\varkappa) \ {}_{\triangle_1}d_q\varkappa = (1-q)(\tau - \triangle_1) \sum_{n=0}^{\infty} q^n \Upsilon(q^n\tau + (1-q^n)\triangle_1). $$

Definition 4 ([10]). *The right quantum or $q^{\triangle_2}$-integral of $\Upsilon : [\triangle_1, \triangle_2] \to \mathbb{R}$ at $\tau \in [\triangle_1, \triangle_2]$ is defined as:*

$$ \int_{\tau}^{\triangle_2} \Upsilon(\varkappa) \ {}^{\triangle_2}d_q\varkappa = (1-q)(\triangle_2 - \tau) \sum_{n=0}^{\infty} q^n \Upsilon(q^n\tau + (1-q^n)\triangle_2). $$

The following lemma will be used in our main results:

Lemma 1 ([16]). *For continuous functions $\Upsilon, g : [\triangle_1, \triangle_2] \to \mathbb{R}$, the following equality is true:*

$$ \begin{aligned} & \int_0^c g(\varkappa) \ {}_{\triangle_1}D_q\Upsilon(\varkappa\triangle_2 + (1-\varkappa)\triangle_1) d_q\varkappa \\ = \ & \frac{g(\varkappa)\Upsilon(\varkappa\triangle_2 + (1-\varkappa)\triangle_1)}{\triangle_2 - \triangle_1}\bigg|_0^c - \frac{1}{\triangle_2 - \triangle_1}\int_0^c D_q g(\varkappa)\Upsilon(q\varkappa\triangle_2 + (1-q\varkappa)\triangle_1) d_q\varkappa. \end{aligned} $$

3. Main Results

In this section, we first obtain an identity for q-differentiable functions. Then, by using this equality, we establish some new Maclaurin's type inequalities in q-calculus.

Lemma 2. *Let $\Upsilon : [\triangle_1, \triangle_2] \to \mathbb{R}$ be a q-differentiable function. If ${}_{\triangle_1}D_q\Upsilon$ is q-integrable function, then we have the following equality:*

$$ \begin{aligned} & \frac{1}{\triangle_2 - \triangle_1}\int_{\triangle_1}^{\triangle_2} \Upsilon(\tau) \ {}_{\triangle_1}d_q\tau - \frac{1}{8}\left[3\Upsilon\left(\frac{5\triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\triangle_2}{6}\right)\right] \\ = \ & \frac{(\triangle_2 - \triangle_1)}{36}\sum_{j=1}^{4} I_j, \end{aligned} \tag{2} $$

where

$$ \begin{aligned} I_1 &= \int_0^1 (1-q\varkappa) \ {}_{\triangle_1}D_q\Upsilon\left(\frac{1-\varkappa}{6}\triangle_1 + \frac{5+\varkappa}{6}\triangle_2\right) d_q\varkappa, \\ I_2 &= \int_0^1 \left(\frac{3}{2} - 4q\varkappa\right) \ {}_{\triangle_1}D_q\Upsilon\left(\frac{3-2\varkappa}{6}\triangle_1 + \frac{3+2\varkappa}{6}\triangle_2\right) d_q\varkappa, \\ I_3 &= \int_0^1 \left(\frac{5}{2} - 4q\varkappa\right) \ {}_{\triangle_1}D_q\Upsilon\left(\frac{5-2\varkappa}{6}\triangle_1 + \frac{1+2\varkappa}{6}\triangle_2\right) d_q\varkappa, \\ I_4 &= \int_0^1 (-q\varkappa) \ {}_{\triangle_1}D_q\Upsilon\left(\frac{6-\varkappa}{6}\triangle_1 + \frac{\varkappa}{6}\triangle_2\right) d_q\varkappa. \end{aligned} $$

Proof. By applying Lemma 1, we have:

$$\begin{aligned}
I_1 &= \int_0^1 (1-q\varkappa)\ {}_{\triangle_1}D_q \Upsilon\left(\frac{5+\varkappa}{6}\triangle_2 + \frac{1-\varkappa}{6}\triangle_1\right) d_q\varkappa \\
&= \int_0^1 (1-q\varkappa)\ {}_{\triangle_1}D_q \Upsilon\left(\varkappa\triangle_2 + (1-\varkappa)\frac{\triangle_1+5\triangle_2}{6}\right) d_q\varkappa \\
&= \frac{6}{\triangle_2-\triangle_1}(1-q)\Upsilon(\triangle_2) - \frac{6}{\triangle_2-\triangle_1}\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right) \\
&\quad + \frac{6q}{\triangle_2-\triangle_1}\int_0^1 \Upsilon\left(q\varkappa\triangle_2 + (1-q\varkappa)\frac{\triangle_1+5\triangle_2}{6}\right) d_q\varkappa \\
&= \frac{6}{\triangle_2-\triangle_1}(1-q)\Upsilon(\triangle_2) - \frac{6}{\triangle_2-\triangle_1}\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right) \\
&\quad + \frac{6q}{\triangle_2-\triangle_1}\left[(1-q)\sum_{n=0}^{\infty} q^n \Upsilon\left(q^{n+1}\triangle_2 + \left(1-q^{n+1}\right)\frac{\triangle_1+5\triangle_2}{6}\right)\right] \\
&= \frac{6}{\triangle_2-\triangle_1}(1-q)\Upsilon(\triangle_2) - \frac{6}{\triangle_2-\triangle_1}\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right) \\
&\quad + \frac{6}{\triangle_2-\triangle_1}\left[(1-q)\sum_{n=1}^{\infty} q^n \Upsilon\left(q^n\triangle_2 + (1-q^n)\frac{\triangle_1+5\triangle_2}{6}\right)\right] \\
&= \frac{6}{\triangle_2-\triangle_1}(1-q)\Upsilon(\triangle_1) - \frac{6}{\triangle_2-\triangle_1}\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right) \\
&\quad + \frac{6}{\triangle_2-\triangle_1}\left[(1-q)\sum_{n=0}^{\infty} q^n \Upsilon\left(q^n\triangle_2 + (1-q^n)\frac{\triangle_1+5\triangle_2}{6}\right) - (1-q)\Upsilon(\triangle_2)\right] \\
&= \frac{6(1-q)}{\triangle_2-\triangle_1}\sum_{n=0}^{\infty} q^n \Upsilon\left(q^n\triangle_2 + (1-q^n)\frac{\triangle_1+5\triangle_2}{6}\right) - \frac{6}{\triangle_2-\triangle_1}\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right).
\end{aligned}$$

By Definition 3, we can write:

$$I_1 = \frac{36}{(\triangle_2-\triangle_1)^2}\int_{\frac{\triangle_1+5\triangle_2}{6}}^{\triangle_2} \Upsilon(\tau)\ {}_{\frac{\triangle_1+5\triangle_2}{6}}d_q\tau - \frac{6}{\triangle_2-\triangle_1}\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right). \tag{3}$$

Similarly, by Lemma 1 and Definition 3, one can obtain

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{3}{2}-4q\varkappa\right)\ {}_{\triangle_1}D_q \Upsilon\left(\frac{3-2\varkappa}{6}\triangle_1 + \frac{3+2\varkappa}{6}\triangle_2\right) d_q\varkappa \\
&= \frac{36}{(\triangle_2-\triangle_1)^2}\int_{\frac{\triangle_1+\triangle_2}{2}}^{\frac{\triangle_1+5\triangle_2}{6}} \Upsilon(\tau)\ {}_{\frac{\triangle_1+\triangle_2}{2}}d_q\tau - \frac{15}{2}\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right) \\
&\quad - \frac{9}{2(\triangle_2-\triangle_1)}\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right),
\end{aligned} \tag{4}$$

$$\begin{aligned}
I_3 &= \int_0^1 \left(\frac{5}{2}-4q\varkappa\right)\ {}_{\triangle_1}D_q \Upsilon\left(\frac{5-2\varkappa}{6}\triangle_1 + \frac{1+2\varkappa}{6}\triangle_2\right) d_q\varkappa \\
&= \frac{36}{(\triangle_2-\triangle_1)^2}\int_{\frac{5\triangle_1+\triangle_2}{6}}^{\frac{\triangle_1+\triangle_2}{2}} \Upsilon(\tau)\ {}_{\frac{5\triangle_1+\triangle_2}{6}}d_q\tau - \frac{9}{2(\triangle_2-\triangle_1)}\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right) \\
&\quad - \frac{15}{2(\triangle_2-\triangle_1)}\Upsilon\left(\frac{5\triangle_1+\triangle_2}{6}\right)
\end{aligned} \tag{5}$$

and

$$I_4 = \int_0^1 (-q\varkappa)\ {}_{\triangle_1}D_q \Upsilon\left(\frac{6-\varkappa}{6}\triangle_1 + \frac{\varkappa}{6}\triangle_2\right) d_q\varkappa \tag{6}$$

$$= \frac{36}{(\triangle_2 - \triangle_1)^2} \int_{\triangle_1}^{\frac{5\triangle_1+\triangle_2}{6}} \Upsilon(\tau)\ {}_{\triangle_1}d_q\tau - \frac{6}{\triangle_2 - \triangle_1}\Upsilon\left(\frac{5\ \triangle_1 + \triangle_2}{6}\right).$$

Thus, we achieve the required equality by the adding equalities (3)–(6). □

Theorem 4. *Let all the conditions of Lemma 2 be hold. If $\left|{}_{\triangle_1}D_q\Upsilon\right|$ is a convex functions, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{1}{\triangle_2 - \triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)\ {}_{\triangle_1}d_q\tau - \frac{1}{8}\left[3\Upsilon\left(\frac{5\ \triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right]\right| \\
\leq\ & \frac{\triangle_2 - \triangle_1}{36}\left[\left(\frac{1}{[2]_q} + \frac{1}{6}(5A_1(q) + 3A_2(q) + 3A_3(q) + A_4(q))\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|\right. \\
&+\left(\frac{q}{[2]_q} + \frac{1}{6}(A_1(q) + 3A_2(q) + 3A_3(q) + 5A_4(q))\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|,
\end{aligned}$$

where

$$\begin{aligned}
A_1(q) &= \int_0^1\left|\frac{3}{2} - 4q\varkappa\right|\varkappa d_q\varkappa = \begin{cases} \frac{3-5q-5q^2}{2[2]_q[3]_q}, & 0 < q < \frac{3}{8} \\ \frac{160q^2+160q-69}{64[2]_q[3]_q}, & \frac{3}{8} < q < 1, \end{cases} \\
A_2(q) &= \int_0^1\left|\frac{3}{2} - 4q\varkappa\right|(1-\varkappa)d_q\varkappa = \begin{cases} \frac{3q+3q^2-5q^3}{2[2]_q[3]_q}, & 0 < q < \frac{3}{8} \\ \frac{160q^3-24q^2-24q+45}{64[2]_q[3]_q}, & \frac{3}{8} < q < 1 \end{cases} \\
A_3(q) &= \int_0^1\left|\frac{5}{2} - 4q\varkappa\right|\varkappa d_q\varkappa = \begin{cases} \frac{5-3q^2-3q}{2[2]_q[3]_q}, & 0 < q < \frac{5}{8} \\ \frac{96q^2+96q-35}{64[2]_q[3]_q}, & \frac{5}{8} < q < 1 \end{cases} \\
A_4(q) &= \int_0^1\left|\frac{5}{2} - 4q\varkappa\right|(1-\varkappa)d_q\varkappa = \begin{cases} \frac{5q+5q^2-3q^2}{2[2]_q[3]_q}, & 0 < q < \frac{5}{8} \\ \frac{96q^3+40q^2+40q+75}{64[2]_q[3]_q}, & \frac{5}{8} < q < 1. \end{cases}
\end{aligned}$$

Proof. By taking modulus in Lemma 2, we have:

$$\begin{aligned}
&\left|\frac{1}{\triangle_2 - \triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)\ {}_{\triangle_1}d_q\tau\right. \\
&\left.-\frac{1}{8}\left[3\Upsilon\left(\frac{5\ \triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right]\right| \\
\leq\ & \frac{\triangle_2 - \triangle_1}{36}[|I_1| + |I_2| + |I_3| + |I_4|].
\end{aligned} \tag{7}$$

Since $\left|{}_{\triangle_1}D_q\Upsilon\right|$ is convex, we obtain:

$$\begin{aligned}
|I_1| &\leq \int_0^1(1-q\varkappa)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5+\varkappa}{6}\ \triangle_2 + \frac{1-\varkappa}{6}\ \triangle_1\right)\right|d_q\varkappa \\
&= \int_0^1(1-q\varkappa)\left[\left|{}_{\triangle_1}D_q\Upsilon\left(\varkappa\ \triangle_2 + (1-\varkappa)\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right|\right]d_q\varkappa \\
&\leq \int_0^1(1-q\varkappa)\left[\varkappa\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right| + (1-\varkappa)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right|\right]d_q\varkappa \\
&= \left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|\left(\int_0^1\varkappa(1-q\varkappa)\ d_q\varkappa\right) \\
&\quad+\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right|\left(\int_0^1(1-\varkappa)(1-q\varkappa)d_q\varkappa\right)
\end{aligned} \tag{8}$$

$$
\begin{aligned}
&= \frac{\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|}{[2]_q[3]_q} + \frac{q\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right)\right|}{[3]_q} \\
&\leq \frac{\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|}{[2]_q[3]_q} + \frac{q\left(\frac{\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|+5\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|}{6}\right)}{[3]_q} \\
&= \frac{\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|}{[2]_q[3]_q} + \frac{q\left(\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + 5\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|\right)}{6[3]_q} \\
&= \left(\frac{5q^2+5q+6}{6[2]_q[3]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right| + \frac{q}{6[3]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|.
\end{aligned}
$$

Similarly, one can establish

$$
\begin{aligned}
|I_2| &\leq \int_0^1 \left|\frac{3}{2} - 4q\varkappa\right| \left|{}^{\triangle_2}D_q\Upsilon\left(\frac{3+2\varkappa}{6}\triangle_2 + \frac{3-2\varkappa}{6}\triangle_1\right)\right| d_q\varkappa \\
&\leq \int_0^1 \left|\frac{3}{2} - 4q\varkappa\right| \left[\varkappa\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right)\right| + (1-\varkappa)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)\right|\right] d_q\varkappa \\
&= A_1(q)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+5\triangle_2}{6}\right)\right| + A_2(q)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)\right| \\
&\leq A_1(q)\left[\frac{\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + 5\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|}{6}\right] + A_2(q)\left[\frac{\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + \left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|}{2}\right] \\
&= \left[\frac{5A_1(q)+3A_2(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right| + \left[\frac{A_1(q)+3A_2(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
|I_3| &\leq \int_0^1 \left|\frac{5}{2} - 4q\varkappa\right| \left|{}^{\triangle_2}D_q\Upsilon\left(\frac{1+2\varkappa}{6}\triangle_2 + \frac{5-2\varkappa}{6}\triangle_1\right)\right| d_q\varkappa \\
&\leq \int_0^1 \left|\frac{5}{2} - 4q\varkappa\right| \left[\varkappa\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)\right| + (1-\varkappa)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5\triangle_1+\triangle_2}{6}\right)\right|\right] d\varkappa \\
&= A_3(q)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)\right| + A_4(q)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5\triangle_1+\triangle_2}{6}\right)\right| \\
&\leq A_3(q)\left[\frac{\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + \left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|}{2}\right] + A_4(q)\left[\frac{5\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + \left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|}{6}\right] \\
&= \left[\frac{3A_3(q)+A_4(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right| + \left[\frac{3A_3(q)+5A_4(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
|I_4| &\leq \int_0^1 q\varkappa\left|{}^{\triangle_2}D_q\Upsilon\left(\frac{\varkappa}{6}\triangle_2 + \frac{6-\varkappa}{6}\triangle_1\right)\right| d_q\varkappa \\
&\leq \int_0^1 q\varkappa\left[(1-\varkappa)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + \varkappa\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5\triangle_1+\triangle_2}{6}\right)\right|\right] \\
&= \frac{q^3\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|}{[2]_q[3]_q} + \frac{q\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5\triangle_1+\triangle_2}{6}\right)\right|}{[3]_q} \\
&\leq \frac{q^3\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|}{[2]_q[3]_q} + \left[\frac{q\left(5\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + \left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|\right)}{6[3]_q}\right] \\
&= \left[\frac{6q^3+5q^2+5q}{6[2]_q[3]_q}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| + \frac{q}{6[3]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|.
\end{aligned} \tag{11}
$$

Thus, by using (8)–(11) in (7), we obtain the resultant inequality. □

Example 1. *Let consider the function* $\Upsilon : [1,2] \to \mathbb{R}$, $\Upsilon(\varkappa) = \varkappa^3$ *with* $q = \frac{3}{4}$. *Since*

$$_{\triangle_1}D_q\Upsilon(\tau) = {}_1D_{\frac{3}{4}}\tau^3 = \frac{\tau^3 - \left(\frac{4}{3}\tau + \frac{1}{4}\right)}{\frac{1}{4}(\tau - 1)} = \frac{37}{16}\tau^2 + \frac{5}{8}\tau + \frac{1}{16},$$

then $\left|{}_{\triangle_1}D_q\Upsilon\right|$ *is convex on* $[1,2]$. *Therefore, Theorem 4 can be applied for the function* $\Upsilon(\varkappa) = \varkappa^3$. *By Definition 3,*

$$\begin{aligned}
\int_{\triangle_1}^{\triangle_2} \Upsilon(\varkappa)\ {}_{\triangle_1}d_q\varkappa &= \int_1^2 \varkappa^3\ {}_1d_{\frac{3}{4}}\varkappa = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(\left(\frac{3}{4}\right)^n + 1\right)^3 \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(\left(\frac{3}{4}\right)^{3n} + 3\left(\frac{3}{4}\right)^{2n} + 3\left(\frac{3}{4}\right)^n + 1\right) \\
&= \frac{1}{4}\left[\frac{1}{1-\frac{81}{256}} + \frac{3}{1-\frac{27}{64}} + \frac{1}{1-\frac{9}{16}} + \frac{1}{1-\frac{3}{4}}\right] \\
&= \frac{20943}{6475}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{8}\left[3\Upsilon\left(\frac{5\,\triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\,\triangle_2}{6}\right)\right] \\
&= \frac{1}{8}\left[3\left(\frac{7}{6}\right)^3 + 2\left(\frac{1}{2}\right)^3 + \left(\frac{11}{6}\right)^3\right] \\
&= \frac{1207}{864}.
\end{aligned}$$

Thus, the left hand side of Theorem 4 reduces

$$\begin{aligned}
&\left|\frac{1}{\triangle_2 - \triangle_1}\int_{\triangle_1}^{\triangle_2} \Upsilon(\tau)\ {}_{\triangle_1}d_q\tau - \frac{1}{8}\left[3\Upsilon\left(\frac{5\,\triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\,\triangle_2}{6}\right)\right]\right| \\
&= \left|\frac{20943}{6475} - \frac{1207}{864}\right| \\
&\cong 1.8375.
\end{aligned}$$

On the other hand, for $q = \frac{3}{4}$, *we have*

$$\begin{aligned}
A_1\left(\frac{3}{4}\right) &= \frac{160\left(\frac{3}{4}\right)^2 + 160\frac{3}{4} - 69}{64\left(1+\frac{3}{4}\right)\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2\right)} = \frac{141}{259} \\
A_2\left(\frac{3}{4}\right) &= \frac{160\left(\frac{3}{4}\right)^3 - 24\left(\frac{3}{4}\right)^2 - 24\frac{3}{4} + 45}{64\left(1+\frac{3}{4}\right)\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2\right)} = \frac{81}{259} \\
A_3\left(\frac{3}{4}\right) &= \frac{96\left(\frac{3}{4}\right)^2 + 96\frac{3}{4} - 35}{64\left(1+\frac{3}{4}\right)\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2\right)} = \frac{13}{37} \\
A_4\left(\frac{3}{4}\right) &= \frac{96\left(\frac{3}{4}\right)^3 + 40\left(\frac{3}{4}\right)^2 + 40\frac{3}{4} + 75}{64\left(1+\frac{3}{4}\right)\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2\right)} = \frac{24}{37}.
\end{aligned}$$

Moreover, since

$$ {}_{\triangle_1}D_q\Upsilon(\tau) = \frac{37}{16}\tau^2 + \frac{5}{8}\tau + \frac{1}{16}, $$

we obtain

$$ \begin{aligned} \left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right| &= \left|{}_1D_{\frac{3}{4}}\Upsilon(2)\right| = \frac{633}{16} \\ \left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right| &= \left|{}_1D_{\frac{3}{4}}\Upsilon(1)\right| = 3. \end{aligned} $$

Consequently, we can calculate the right hand side of Theorem 4 as

$$ \begin{aligned} & \frac{\triangle_2 - \triangle_1}{36}\left[\left(\frac{1}{[2]_q} + \frac{1}{6}(5A_1(q) + 3A_2(q) + 3A_3(q) + A_4(q))\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right| \right. \\ & \left. + \left(\frac{q}{[2]_q} + \frac{1}{6}(A_1(q) + 3A_2(q) + 3A_3(q) + 5A_4(q))\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|\right] \\ = & \frac{1}{36}\left[3\left(\frac{4}{7} + \frac{1}{6}\left(5\frac{141}{259} + 3\frac{81}{259} + 3\frac{13}{37} + \frac{24}{37}\right)\right)\right. \\ & \left. + \frac{633}{16}\left(\frac{3}{7} + \frac{1}{6}\left(\frac{141}{259} + 3\frac{81}{259} + 3\frac{13}{37} + 5\frac{24}{37}\right)\right)\right] \\ = & \frac{1}{36}\left(\frac{57}{7} + \frac{5697}{56}\right) \\ = & \frac{293}{96} \\ \cong & 3.052. \end{aligned} $$

Since $1.8375 \le 3.052$, *it is clear that Theorem 4 is valid for the function* $\Upsilon(\varkappa) = \varkappa^3$.

Remark 1. *If we set limit as* $q \to 1^-$ *in Theorem 4, then we obtain the following inequality:*

$$ \begin{aligned} & \left|\frac{1}{8}\left[3\Upsilon\left(\frac{5\triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\triangle_2}{6}\right)\right] - \frac{1}{\triangle_2 - \triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)d\tau\right| \\ \le & \frac{25(\triangle_2 - \triangle_1)}{576}\left[\left|\Upsilon'(\triangle_1)\right| + \left|\Upsilon'(\triangle_2)\right|\right]. \end{aligned} $$

This inequality can be found as a special case of [7].

Theorem 5. *Let all the conditions of Lemma 2 be hold. If* $\left|{}_{\triangle_1}D_q\Upsilon\right|^r$, $r \ge 1$ *is a convex function, then the following inequality holds*

$$ \begin{aligned} & \left|\frac{1}{\triangle_2 - \triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)\,{}_{\triangle_1}d_q\tau - \frac{1}{8}\left[3\Upsilon\left(\frac{5\triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\triangle_2}{6}\right)\right]\right| \\ \le & \frac{\triangle_2 - \triangle_1}{36}\left[\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left(\left(\frac{5q^2 + 5q + 6}{6[2]_q[3]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r + \frac{q}{6[3]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}}\right. \\ & + (A_5(q))^{1-\frac{1}{r}}\left(\left[\frac{5A_1(q) + 3A_2(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r + \left[\frac{A_1(q) + 3A_2(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}} \\ & + (A_6(q))^{1-\frac{1}{r}}\left(\left[\frac{3A_3(q) + A_4(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r + \left[\frac{3A_3(q) + 5A_4(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}} \\ & \left. + \left(\frac{q}{[2]_q}\right)^{1-\frac{1}{r}}\left(\left[\frac{6q^3 + 5q^2 + 5q}{6[2]_q[3]_q}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r + \frac{q}{6[3]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r\right)^{\frac{1}{r}}\right], \end{aligned} $$

where $A_1(q) - A_4(q)$ are same as defined in the Theorem 4 and

$$A_5(q) = \begin{cases} \frac{3-5q}{2[2]_q}, & 0 < q < \frac{3}{8} \\ \frac{20q-3}{8[2]_q}, & \frac{3}{8} < q < 1, \end{cases}$$

$$A_6(q) = \begin{cases} \frac{5-3q}{2[2]_q}, & 0 < q < \frac{5}{8} \\ \frac{37-20q}{8[2]_q}, & \frac{5}{8} < q < 1. \end{cases}$$

Proof. By using power mean inequality in (7), we obtain:

$$\begin{aligned} &\left| \frac{1}{\triangle_2 - \triangle_1} \int_{\triangle_1}^{\triangle_2} \Upsilon(\tau)\ {}_{\triangle_1}d_q\tau - \frac{1}{8}\left[3\Upsilon\left(\frac{5\ \triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right]\right| \\ \leq\ & \frac{\triangle_2 - \triangle_1}{36}[|I_1| + |I_2| + |I_3| + |I_4|] \\ \leq\ & \frac{\triangle_2 - \triangle_1}{36}\left[\left(\int_0^1 (1-q\varkappa)d_q\varkappa\right)^{1-\frac{1}{r}} \left(\int_0^1 (1-q\varkappa)\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5+\varkappa}{6}\ \triangle_2 + \frac{1-\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}}\right. \\ &+ \left(\int_0^1 \left|\frac{3}{2} - 4q\varkappa\right| d_q\varkappa\right)^{1-\frac{1}{r}} \left(\int_0^1 \left|\frac{3}{2} - 4q\varkappa\right| \left|{}_{\triangle_1}D_q\Upsilon\left(\frac{3+2\varkappa}{6}\ \triangle_2 + \frac{3-2\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}} \\ &+ \left(\int_0^1 \left|\frac{5}{2} - 4q\varkappa\right| d_q\varkappa\right)^{1-\frac{1}{r}} \left(\int_0^1 \left|\frac{5}{2} - 4q\varkappa\right| \left|{}_{\triangle_1}D_q\Upsilon\left(\frac{1+2\varkappa}{6}\ \triangle_2 + \frac{5-2\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}} \\ &\left.+ \left(\int_0^1 q\varkappa\, d_q\varkappa\right)^{1-\frac{1}{r}} \left(\int_0^1 q\varkappa \left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\varkappa}{6}\ \triangle_2 + \frac{6-\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}}\right]. \end{aligned}$$

By using the convexity of $\left|{}_{\triangle_1}D_q\Upsilon\right|^r$, we have:

$$\begin{aligned} &\left| \frac{1}{\triangle_2 - \triangle_1} \int_{\triangle_1}^{\triangle_2} \Upsilon(\tau)\ {}_{\triangle_1}d_q\tau - \frac{1}{8}\left[3\Upsilon\left(\frac{5\ \triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right]\right| \\ \leq\ & \frac{\triangle_2 - \triangle_1}{36}\left[\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}} \left(\left(\frac{5q^2+5q+6}{6[2]_q[3]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r + \frac{q}{6[3]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}}\right. \\ &+ (A_5(q))^{1-\frac{1}{r}} \left(\left[\frac{5A_1(q) + 3A_2(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r + \left[\frac{A_1(q) + 3A_2(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}} \\ &+ (A_6(q))^{1-\frac{1}{r}} \left(\left[\frac{3A_3(q) + A_4(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r + \left[\frac{3A_3(q) + 5A_4(q)}{6}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}} \\ &\left.+ \left(\frac{q}{[2]_q}\right)^{1-\frac{1}{r}} \left(\left[\frac{6q^3+5q^2+5q}{6[2]_q[3]_q}\right]\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r + \frac{q}{6[3]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r\right)^{\frac{1}{r}}\right]. \end{aligned}$$

Thus, the proof is completed. □

Remark 2. *If we set the limit as $q \to 1^-$ in Theorem 5, then we have the following inequality:*

$$\begin{aligned} &\left|\frac{1}{8}\left[3\Upsilon\left(\frac{5\ \triangle_1 + \triangle_2}{6}\right) + 2\Upsilon\left(\frac{\triangle_1 + \triangle_2}{2}\right) + 3\Upsilon\left(\frac{\triangle_1 + 5\ \triangle_2}{6}\right)\right] - \frac{1}{\triangle_2 - \triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)d\tau\right| \\ \leq\ & \frac{\triangle_2 - \triangle_1}{36}\left[\frac{1}{2}\left\{\left(\frac{8|\Upsilon'(\triangle_1)|^r + |\Upsilon'(\triangle_2)|^r}{9}\right)^{\frac{1}{r}} + \left(\frac{|\Upsilon'(\triangle_1)|^r + 8|\Upsilon'(\triangle_2)|^r}{9}\right)^{\frac{1}{r}}\right\}\right. \end{aligned}$$

$$+\frac{17}{16}\left\{\left(\frac{1726|\Upsilon'(\triangle_1)|^r+722|\Upsilon'(\triangle_2)|^r}{9}\right)^{\frac{1}{r}}+\left(\frac{722|\Upsilon'(\triangle_1)|^r+1726|\Upsilon'(\triangle_2)|^r}{9}\right)^{\frac{1}{r}}\right\}.$$

This inequality can be found as a special case of [7].

Theorem 6. *Let all the conditions of Lemma 2 be hold. If $\left|{}_{\triangle_1}D_q\Upsilon\right|^r$, $r>1$ is a convex function, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{1}{\triangle_2-\triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)\ {}_{\triangle_1}d_q\tau-\frac{1}{8}\left[3\Upsilon\left(\frac{5\ \triangle_1+\triangle_2}{6}\right)+2\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)+3\Upsilon\left(\frac{\triangle_1+5\ \triangle_2}{6}\right)\right]\right|\\
\leq\ &\frac{\triangle_2-\triangle_1}{36}\left[\left(\frac{1-(1-q)^{s+1}}{q(s+1)}\right)^{\frac{1}{s}}\left(\left(\frac{6+5q}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r+\frac{q}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}}\right.\\
&+(A_7(q,s))^{\frac{1}{s}}\left(\left(\frac{5+3q}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r+\frac{1+3q}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}}\\
&+(A_8(q,s))^{\frac{1}{s}}\left(\left(\frac{3+q}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r+\frac{3+5q}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}}\\
&\left.+\left(\frac{q^s}{(s+1)}\right)^{\frac{1}{s}}\left(\left(\frac{6+5q}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r+\frac{q}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r\right)^{\frac{1}{r}}\right],
\end{aligned}$$

where $r^{-1}+s^{-1}=rs$ and

$$A_7(q,s)=\begin{cases}\frac{3^{s+1}-(3-8q)^{s+1}}{2^{s+3}q(s+1)}, & 0<q<\frac{3}{8}\\ \frac{3^{s+1}-(8q-3)^{s+1}}{2^{s+3}q(s+1)}, & \frac{3}{8}<q<1,\end{cases}$$

$$A_8(q,s)=\begin{cases}\frac{5^{s+1}-(5-8q)^{s+1}}{2^{s+3}q(s+1)}, & 0<q<\frac{5}{8}\\ \frac{5^{s+1}-(8q-5)^{s+1}}{2^{s+3}q(s+1)}, & \frac{5}{8}<q<1.\end{cases}$$

Proof. Taking modulus in (2) and using Hölder's inequality, we have:

$$\begin{aligned}
&\left|\frac{1}{\triangle_2-\triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)\ {}_{\triangle_1}d_q\tau-\frac{1}{8}\left[3\Upsilon\left(\frac{5\ \triangle_1+\triangle_2}{6}\right)+2\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)+3\Upsilon\left(\frac{\triangle_1+5\ \triangle_2}{6}\right)\right]\right|\\
\leq\ &\frac{\triangle_2-\triangle_1}{36}[|I_1|+|I_2|+|I_3|+|I_4|]\\
\leq\ &\frac{\triangle_2-\triangle_1}{36}\left[\left(\int_0^1(1-q\varkappa)^s d_q\varkappa\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5+\varkappa}{6}\ \triangle_2+\frac{1-\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}}\right.\\
&+\left(\int_0^1\left|\frac{3}{2}-4q\varkappa\right|^s d_q\varkappa\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{3+2\varkappa}{6}\ \triangle_2+\frac{3-2\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|\frac{5}{2}-4q\varkappa\right|^s d_q\varkappa\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{1+2\varkappa}{6}\ \triangle_2+\frac{5-2\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}}\\
&\left.+\left(\int_0^1(q\varkappa)^s d_q\varkappa\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\varkappa}{6}\ \triangle_2+\frac{6-\varkappa}{6}\ \triangle_1\right)\right|^r d_q\varkappa\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Now from convexity, we have

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3\Upsilon\left(\frac{5\,\triangle_1+\triangle_2}{6}\right)+2\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)+3\Upsilon\left(\frac{\triangle_1+5\,\triangle_2}{6}\right)\right]-\frac{1}{\triangle_2-\triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)\ {}^{\triangle_2}d_q\tau\right| \\
\leq\ &\frac{\triangle_2-\triangle_1}{36}\left[\left(\frac{1-(1-q)^{s+1}}{q(s+1)}\right)^{\frac{1}{s}}\left(\frac{1}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r+\frac{q}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+5\,\triangle_2}{6}\right)\right|^r\right)^{\frac{1}{r}}\right. \\
&+(A_7(q,s))^{\frac{1}{s}}\left(\frac{1}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+5\,\triangle_2}{6}\right)\right|^r+\frac{q}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)\right|^r\right)^{\frac{1}{r}} \\
&+(A_8(q,s))^{\frac{1}{s}}\left(\frac{1}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)\right|^r+\frac{q}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5\,\triangle_1+\triangle_2}{6}\right)\right|^r\right)^{\frac{1}{r}} \\
&\left.+\left(\frac{q^s}{s+1}\right)^{\frac{1}{s}}\left(\frac{q}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r+\frac{1}{[2]_q}\left|{}_{\triangle_1}D_q\Upsilon\left(\frac{5\,\triangle_1+\triangle_2}{6}\right)\right|^r\right)^{\frac{1}{r}}\right] \\
\leq\ &\frac{\triangle_2-\triangle_1}{36}\left[\left(\frac{1-(1-q)^{s+1}}{q(s+1)}\right)^{\frac{1}{s}}\left(\left(\frac{6+5q}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r+\frac{q}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}}\right. \\
&+(A_7(q,s))^{\frac{1}{s}}\left(\left(\frac{5+3q}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r+\frac{1+3q}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}} \\
&+(A_8(q,s))^{\frac{1}{s}}\left(\left(\frac{3+q}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r+\frac{3+5q}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r\right)^{\frac{1}{r}} \\
&\left.+\left(\frac{q^s}{s+1}\right)^{\frac{1}{s}}\left(\left(\frac{6q+5}{6[2]_q}\right)\left|{}_{\triangle_1}D_q\Upsilon(\triangle_1)\right|^r+\frac{1}{6[2]_q}\left|{}_{\triangle_1}D_q\Upsilon(\triangle_2)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Remark 3. *If we set the limit as $q \to 1^-$ in Theorem 6, then we have the following inequality:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3\Upsilon\left(\frac{5\,\triangle_1+\triangle_2}{6}\right)+2\Upsilon\left(\frac{\triangle_1+\triangle_2}{2}\right)+3\Upsilon\left(\frac{\triangle_1+5\,\triangle_2}{6}\right)\right]-\frac{1}{\triangle_2-\triangle_1}\int_{\triangle_1}^{\triangle_2}\Upsilon(\tau)d\tau\right| \\
\leq\ &\frac{\triangle_2-\triangle_1}{36}\left[\left(\frac{1}{s+1}\right)^{\frac{1}{s}}\left\{\left(\frac{11|\Upsilon'(\triangle_1)|^r+|\Upsilon'(\triangle_2)|^r}{12}\right)^{\frac{1}{r}}+\left(\frac{|\Upsilon'(\triangle_1)|^r+11|\Upsilon'(\triangle_2)|^r}{12}\right)^{\frac{1}{r}}\right\}\right. \\
&\left.+\frac{1}{2}\left(\frac{3^{s+1}+5^{s+1}}{8}\right)^{\frac{1}{s}}\left\{\left(\frac{2|\Upsilon'(\triangle_1)|^r+|\Upsilon'(\triangle_2)|^r}{3}\right)^{\frac{1}{r}}+\left(\frac{|\Upsilon'(\triangle_1)|^r+2|\Upsilon'(\triangle_2)|^r}{3}\right)^{\frac{1}{r}}\right\}\right].
\end{aligned}
$$

This inequality can be found as a special case of [7].

4. Conclusions

In this paper, we used the left q-derivative and integral to prove some new Maclaurin's formula type inequalities for q-differentiable convex functions. It is also shown that the newly established inequalities are extensions of some existing inequalities in the literature. The inequalities presented in this work are very important because, with the help of these inequalities, we can find error bounds for Maclaurin's formula in classical and q-calculus. It is a very interesting and novel problem, and it is hoped that future researchers will be able to establish similar inequalities for co-ordinated convex functions.

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References

1. Davis, P.J.; Rabinowitz, P. *Methods of Numerical Integration*; Academic Press: New York, NY, USA; San Francisco, CA, USA; London, UK, 1975.
2. Dragomir, S.S. On Simpson's quadrature formula for mappings of bounded variation and applications. *Tamkang J. Math.* **1999**, *30*, 53–58. [CrossRef]
3. Erden, S.; Iftikhar, S.; Kumam, P.; Awan, M.U. Some Newton's like inequalities with applications. *Rev. R. Acad. Cienc. Exactas F ´ıs. Nat.* **2022**, *114*, 195. [CrossRef]
4. Gao, S.; Shi, W. On new inequalities of Newton's type for functions whose second derivatives absolute values are convex. *Int. J. Pure Appl. Math.* **2012**, *74*, 33–41.
5. Noor, M.A.; Noor, K.I.; Iftikhar, S. Some Newton's type inequalities for harmonic convex functions. *J. Adv. Math. Stud.* **2016**, *9*, 7–16.
6. Sitthiwirattham, T.; Nonlaopon, K.; Ali, M.A.; Budak, H. Riemann–Liouville fractional Newton's type inequalities for differentiable convex functions. *Fractal Fract.* **2022**, *6*, 175. [CrossRef]
7. Mefthah, B.; Allel, N. Maclaurin's inequalities for functions whose first derivatives are preinvex. *J. Math. Anal Model.* **2022**, *3*, 52–64.
8. Hezenci, F. Fractional inequalities of Corrected Euler–Maclaurin-type for twice-differentiable functions. *Comput. Appl. Math.* **2023**, *42*, 2. [CrossRef]
9. Alp, N.; Sarikaya, M.Z.; Kunt, M.; İşcan, İ. q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ. Sci.* **2018**, *30*, 193–203. [CrossRef]
10. Bermudo, S.; Kórus, P.; Valdés, J.N. On q-Hermite–Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [CrossRef]
11. Ali, M.A.; Budak, H.; Fečkon, M.; Khan, S. A new version of q-Hermite-Hadamard's midpoint and trapezoid type inequalities for convex functions. *Math. Slovaca* **2023**, *73*, 369–386. [CrossRef]
12. Sitthiwirattham, T.; Ali, M.A.; Ali, A.; Budak, H. A new q-Hermite-Hadamard's inequality and estimates for midpoint type inequalities for convex functions. *Miskolc. Math. Notes* **2022**, *in press*.
13. Budak, H. Some trapezoid and midpoint type inequalities for newly defined quantum integrals. *Proyecciones* **2021**, *40*, 199–215. [CrossRef]
14. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Meth. Appl. Sci.* **2020**, *44*, 378–390. [CrossRef]
15. Sial, I.B.; Mei, S.; Ali, M.A.; Nonlaopon, K. On Some Generalized Simpson's and Newton's Inequalities for (α, m)-Convex Functions in Quantum Calculus. *Mathematics* **2021**, *9*, 3266. [CrossRef]
16. Soontharanon, J.; Ali, M.A.; Budak, H.; Nonlaopon, K.; Abdullah, Z. Simpson's and Newton's Type Inequalities for (α, m)-convex functions via Quantum Calculus. *Symmetry* **2022**, *14*, 736. [CrossRef]
17. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [CrossRef]
18. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.* **2015**, *269*, 242–251. [CrossRef]
19. Sudsutad, W.; Ntouyas, S.K.; Tariboon, J. Quantum integral inequalities for convex functions. *J. Math. Inequal.* **2015**, *9*, 781–793. [CrossRef]
20. Zhuang, H.; Liu, W.; Park, J. Some quantum estimates of Hermite-Hadmard inequalities for quasi-convex functions. *Mathematics* **2019**, *7*, 152. [CrossRef]
21. Gauchman, H. Integral inequalities in q-calculus. *Comput. Math. Appl.* **2004**, *47*, 281–300. [CrossRef]

22. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *282*, 1–19. [CrossRef]
23. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Berlin/Heidelberg, Germany, 2001.