fractal and fractional

Article

Application of Analytical Techniques for Solving Fractional Physical Models Arising in Applied Sciences

Mashael M. AlBaidani [1], Abdul Hamid Ganie [2], Fahad Aljuaydi [1] and Adnan Khan [3,*]

[1] Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Al Kharj 11942, Saudi Arabia; m.albaidani@psau.edu.sa (M.M.A.); f.aljuaydi@psau.edu.sa (F.A.)
[2] Basic Science Department, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia; a.ganie@seu.edu.sa
[3] Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan
* Correspondence: adnanmummand@gmail.com

Abstract: In this paper, we examined the approximations to the time-fractional Kawahara equation and modified Kawahara equation, which model the creation of nonlinear water waves in the long wavelength area and the transmission of signals. We implemented two novel techniques, namely the homotopy perturbation transform method and the Elzaki transform decomposition method. The derivative having fractional-order is taken in Caputo sense. The Adomian and He's polynomials make it simple to handle the nonlinear terms. To illustrate the adaptability and effectiveness of derivatives with fractional order to represent the water waves in long wavelength regions, numerical data have been given graphically. A key component of the Kawahara equation is the symmetry pattern, and the symmetrical nature of the solution may be observed in the graphs. The importance of our suggested methods is illustrated by the convergence of analytical solutions to the precise solutions. The techniques currently in use are straightforward and effective for solving fractional-order issues. The offered methods reduced computational time is their main advantage. It will be possible to solve fractional partial differential equations using the study's findings as a tool.

Keywords: Elzaki transform; Kawahara and modified Kawahara equations; homotopy perturbation method; Adomian decomposition method; Caputo operator

check for updates

Citation: AlBaidani, M.M.; Ganie, A.H.; Aljuaydi, F.; Khan, A. Application of Analytical Techniques for Solving Fractional Physical Models Arising in Applied Sciences. *Fractal Fract.* **2023**, *7*, 584. https://doi.org/10.3390/fractalfract7080584

Academic Editors: Yusuf Gürefe, Nguyen Huy Tuan and Haci Mehmet Baskonus

Received: 14 June 2023
Revised: 23 July 2023
Accepted: 25 July 2023
Published: 28 July 2023

1. Introduction

A 300-year-old mathematical field known as fractional calculus (FC) was later advanced by Liouville, Euler, and Abel in 1823 and was first defined by Rieman and Liouville in the 19th century as "the generalization of the integer derivative to fractional order". Researcher interest in FC has increased significantly over the past few decades. It was discovered that fractional derivatives can be used to perfectly model a variety of applications, particularly interdisciplinary events, for example, signal processing, control theory, viscoelasticity, nonlinear earthquake oscillation, robotics, and so on [1–4]. We recommend the reader to read [5–9] for additional information and applications of FC. Fractional differential equations (FDEs) have garnered considerable prominence recently, and they are employed a lot in economics, astronomy, physics, ecology, and a plethora of other domains [10,11]. As a result, scholars have given FDEs a lot of attention for their ability to interpret real-world phenomena realistically, and the field of mathematics as a whole has grown in popularity in areas such as heat conduction [12], probability and statistics [13], circuit systems [14], disclaimer fluid flow [15], optics and signal processing [16], inviscid fluid [17], and so on. In the previous works, renowned scholars have presented and developed a number of definitions for fractional derivatives, including Riemann–Liouville (RL), Atangana–Baleanu, Weyl, Abel, Caputo, Caputo–Fabrizio, and Riesz. The most well-known, Caputo and RL fractional derivatives, offer a greater degree of freedom in the explanation and modeling of physical processes than regular derivatives. To learn about these fractional derivatives, see [18–21].

Fractional differential equations have been used by many scientists and engineers to study diverse biological and physical systems. Scientists from a variety of fields have shown that solving these equations is an interesting and beneficial research topic. In the recent areas of applied mathematics and engineering, numerous effective methods for handling such models have been developed. The iterative Laplace transform method [22], finite difference method [23], fractional sub-equation method [24], monotone iterative technique [25], homotopy analysis method [26], wavelets optimization method [27], fractional Newton method [28], reproducing kernel Hilbert space method [29], (G′/G)-expansion method [30], variational iteration method [31], modified Adams–Bashforth method [32], and several others [33–35].

The investigation of traveling wave solutions for nonlinear equations has greatly advanced modern studies on nonlinear physical phenomena. Nonlinear wave processes are studied in a wide range of scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid-state physics, and geology. The Kawahara equation is one of the important equations in physics and ocean engineering. The Kawahara equation, which plays a significant role in explaining the creation of nonlinear water waves in the long wavelength region, is the most exciting equation since it has weaker dispersion but high nonlinearity. Physicists and mathematicians both use dispersive wave equations extensively. The Kawahara equation (KE) and modified Kawahara equation (MKE) have attracted a lot of attention and have been the subject of active research in recent years [36–38]. In 1972, Kawahara [39] suggested the KE for describing solitary-wave propagation in medium. Kawahara analyzed this type of equation numerically and observed that it has solitary wave solutions that are both monotone and oscillatory. Two more crucial aspects of the Kawahara equation are the symmetry pattern and collection of conservation laws. A generalization of the Kawahara equation's symmetry and conservation principles were looked at in [40]. Both shallow water waves with surface tension and plasma magneto–acoustic wave theory prove the issue. Additionally, the MKE has a wide range of uses in disciplines such as plasma waves, capillary-gravity water waves, and other ones [37,41–43]. This study aims to examine the analytical framework and effectiveness of applying the homotopy perturbation transform method (HPTM) and the Elzaki transform decomposition method (ETDM) to find the approximate solutions of the time-fractional Kawahara equation (TFKE) and modified Kawahara equation (TFME) as

$$D_{\tau}^{\varpi}\mathbb{U}(\mu,\tau)+\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau)+\mathbb{U}_{\mu\mu\mu}(\mu,\tau)-\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)=0, \quad 0<\varpi\leq 1, \tag{1}$$

with

$$\mathbb{U}(\mu,0)=f(\mu),$$

$$D_{\tau}^{\varpi}\mathbb{U}(\mu,\tau)+\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau)+\alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau)+\beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)=0, \quad 0<\varpi\leq 1, \tag{2}$$

with

$$\mathbb{U}(\mu,0)=g(\mu),$$

where $\alpha>0,\beta<0$ are constants that are not zero. $\mathbb{U}(\mu,\tau)$ is a function of space and time that vanishes at values of $\mu<0$ and $\tau<0$. The functions $f(\mu)$ and $g(\mu)$ are typically defined on the interval $-\infty<\mu<\infty$ as hyperbolic functions. Dispersive wave equations are essential in practical physics and math.

Recently, a number of scholars have looked at the TFKE and TFMKE utilizing a variety of approaches and methods, including the new iterative method [44], the homotopy analysis method [45], the residual power series method [46], and the Laplace iterative transform method [47]. To the best of our knowledge, this is the first time to apply the HPTM and ETDM to examine the Kawahara equations within the Caputo operator.

This work aims to use HPTM and ETDM to solve the TFKE and TFMKE. The Elzaki transform (ET), homotopy perturbation method, and Adomian decomposition method are three powerful methods used in the development of the HPTM and ETDM. It is easier to calculate the series terms utilizing the suggested methods than the traditional Adomian and perturbation techniques since they do not involve computing the derivative or integrals of fractional-order in the recursive formula. Therefore, it is considered that these techniques can be used to quickly and simply solve particular classes of nonlinear partial differential equations (PDEs). These techniques provide a solution that can be precise or approximative using a quick convergence series. As a result, a variety of linear and nonlinear PDEs are increasingly being solved using the HPTM and ETDM. Numerous physical problems, including fractional-order problems have been studied using HPTM and ETDM, such as the fractional partial differential equations [48], approximate analytical view of physical and biological models in the setting of Caputo operator [49], fractional-order multi-dimensional telegraph equation [50], family of Fisher's reaction-diffusion equation [51], and nonlinear fractional heat-like equations [52].

The following describes the paper's layout. Section 2 presents the Elzaki transform (ET) of essential definitions as well as some other results helpful in the research. The fundamental concepts of HPTM and ETDM are obtained using fractional Caputo derivative in Sections 3 and 4. Section 5 contains solutions for TFKE and TFMKE using HPTM and ETDM. In Section 6, we discuss the numerical simulations for present methods. Finally, we describe our findings in Section 7.

2. Preliminaries

The fractional derivative definition of Caputo and some properties of ET are illustrated below.

Definition 1. *The arbitrary order RL operator is stated as [53–55]*

$$D^{\varpi}\mathbb{U}(\mu) = \begin{cases} \frac{d^{\varsigma}}{d\mu^{\varsigma}}\mathbb{U}(\mu), & \varpi = \varsigma, \\ \frac{1}{\Gamma(\varsigma-\varpi)}\frac{d}{d\mu^{\varsigma}}\int_0^{\mu}\mathbb{U}(\mu)(\mu-\xi)^{\varsigma-\varpi-1}d\xi, & \varsigma-1<\varpi<\varsigma, \end{cases}$$

where $\varsigma \in Z^{+}$, $\varpi \in R^{+}$ *and*

$$D^{-\varpi}\mathbb{U}(\mu) = \frac{1}{\Gamma(\varpi)}\int_0^{\mu}(\mu-\xi)^{\varpi-1}\mathbb{U}(\xi)d\xi, \ \ 0<\varpi\leq 1.$$

Definition 2. *The arbitrary order RL integral operator is stated as [53–55]*

$$J^{\varpi}\mathbb{U}(\mu) = \frac{1}{\Gamma(\varpi)}\int_0^{\mu}(\mu-1)^{\varpi-1}\mathbb{U}(\tau)d\tau, \ \ \tau>0, \ \ \mu>0,$$

having the below properties:

$$J^{\varpi}\mu^{\varsigma} = \frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma+\varpi+1)}\mu^{\varsigma+\varpi},$$
$$D^{\varpi}\mu^{\varsigma} = \frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma-\varpi+1)}\mu^{\varsigma-\varpi}.$$

Definition 3. *The arbitrary order Caputo derivative is stated as [53–55]*

$$D^{\varpi}\mathbb{U}(\mu) = \begin{cases} \frac{1}{\Gamma(\varsigma-\varpi)}\int_0^{\mu}\mathbb{U}^{\varsigma}(\xi)(\mu-\xi)^{\varsigma-\varpi-1}d\xi, & \varsigma-1<\varpi<\varsigma, \\ \frac{d^{\varsigma}}{d\mu^{\varsigma}}\mathbb{U}(\mu), & \varsigma=\varpi. \end{cases} \tag{3}$$

having the below properties:

$$\begin{aligned}&J_\mu^\varpi D_\mu^\varpi \mathbb{U}(\mu) = g(\mu) - \sum_{k=0}^{\varsigma-1} g^k(0^+)\frac{\mu^k}{k!},\ for\ \ \mu > 0,\ \ and\ \ \varsigma - 1 < \varpi \le \varsigma,\ \ \varsigma \in \mathbf{N}. \\ &D_\mu^\varpi J_\mu^\varpi \mathbb{U}(\mu) = g(\mu).\end{aligned} \tag{4}$$

Definition 4. *The ET of the given function $\mathbb{U}(\tau)$ is stated as [56]*

$$\mathbb{E}[\mathbb{U}(\tau)] = M(u) = u\int_u^\infty e^{\frac{-\tau}{u}}\mathbb{U}(\tau)d\tau,\ \ \tau > 0. \tag{5}$$

Here, u is the transform variable.

Definition 5. *The inverse ET is defined by*

$$\mathbb{E}^{-1}[\mathbb{E}(u)] = \mathbb{U}(\tau) = \frac{1}{2\pi\iota}\int_{\varpi-\iota\infty}^{\varpi+\iota\infty}\mathcal{T}\left(\frac{1}{u}\right)e^{u\tau}udu = \Sigma\ \ residues\ \ of\ \ \mathbb{U}\left(\frac{1}{u}\right)e^{u\tau}u. \tag{6}$$

Definition 6. *The ET of Caputo arbitrary order derivative is stated as [57]*

$$\mathbb{E}[D_\mu^\varpi \mathbb{U}(\mu)] = u^{-\varpi}\mathbb{E}[\mathbb{U}(\mu)] - \sum_{i=0}^{\varsigma-1} u^{2-\varpi+i}\mathbb{U}^{(i)}(0),\ \ where\ \ \varsigma - 1 < \varpi < \varsigma. \tag{7}$$

Integration by parts can be applied in order to find ET of partial derivatives.

1. $\mathbb{E}[\tau^n] = n!u^{n+2}$.
2. $\mathbb{E}[\mathbb{U}'] = \dfrac{M(u)}{u} - u\mathbb{U}(0)$.
3. $\mathbb{E}[\mathbb{U}''] = \dfrac{M(u)}{u^2} - \mathbb{U}(0) - u\mathbb{U}'(0)$.
4. $\mathbb{E}[\mathbb{U}^n] = \dfrac{M(u)}{u^2} - \sum_{i=0}^{n-1} u^{2-n+i}\mathbb{U}^{(i)}(0)$.

3. Methodology of the Homotopy Perturbation Transform Method

To illustrate the core notion of HPTM, we solve the following general nonlinear problem of the form:

$$D_\tau^\varpi \mathbb{U}(\mu,\tau) = [\mathcal{M} + \mathcal{N}]\mathbb{U}(\mu,\tau),\ \ \tau > 0,\ \ 0 < \varpi \le 1, \tag{8}$$

having initial guess

$$\mathbb{U}(\mu,0) = \vartheta(\mu).$$

Here, $D_\tau^\varpi = \frac{\partial^\varpi}{\partial\zeta^\varpi}$ symbolizes the Caputo noninteger derivative, and $\mathcal{M}, \mathcal{N}$ are linear and nonlinear differential operators.

By means of Definition 6 at $\varsigma = 1$, we obtain

$$\frac{1}{u^\varpi}\{M(u) - u^2\mathbb{U}(\mu,0)\} = \mathbb{E}[[\mathcal{M} + \mathcal{N}]\mathbb{U}(\mu,\tau)], \tag{9}$$

and after, we have

$$M(u) = u^2\mathbb{U}(\mu,0) + u^\varpi\mathbb{E}[[\mathcal{M} + \mathcal{N}]\mathbb{U}(\mu,\tau)], \tag{10}$$

with M(u)=$\mathbb{E}[\mathbb{U}(\mu,\tau)]$.

In terms of inverse ET, we obtain

$$\mathbb{U}(\mu,\tau) = \mathbb{U}(\mu,0) + \mathbb{E}^{-1}[u^\varpi\mathbb{E}[[\mathcal{M} + \mathcal{N}]\mathbb{U}(\mu,\tau)]]. \tag{11}$$

By applying homotopy perturbation method (HPM) to (11), we have

$$\mathbb{U}(\mu,\tau) = \mathbb{U}(\mu,0) + \epsilon(\mathbb{E}^{-1}[u^{\varpi}\mathbb{E}[[\mathcal{M}+\mathcal{N}]\mathbb{U}(\mu,\tau)]]) \tag{12}$$

with $\epsilon \in [0,1]$ is the homotopy parameter

The solution is expanded as a result of the homotopy parameter ϵ as

$$\mathbb{U}(\mu,\tau) = \sum_{m=0}^{\infty} \epsilon^m \mathbb{U}_m(\mu,\tau). \tag{13}$$

The nonlinear term is determined as

$$\mathcal{N}[\mathbb{U}(\mu,\tau)] = \sum_{n=0}^{\infty} \epsilon^n H_n(\mathbb{U}), \tag{14}$$

by means of homotopy polynomial and is illustrated as

$$H_n(\mathbb{U}_0,\mathbb{U}_1,\cdots,\mathbb{U}_n) = \frac{1}{n!}D_{\epsilon}^n\left[\mathcal{N}\left(\sum_{i=0}^{\infty}\epsilon^i\mathbb{U}_i\right)\right], n = 0,1,2,\cdots \tag{15}$$

with $D_{\epsilon}^n = \frac{\partial^n}{\partial\epsilon^n}$.

By utilizing (13) and (14) in (12), we obtain

$$\sum_{n=0}^{\infty}\epsilon^n\mathbb{U}_n(\mu,\tau) = \mathbb{U}(\mu,0) + \epsilon \times \left[\mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left\{\mathcal{M}\left(\sum_{n=0}^{\infty}\epsilon^n\mathbb{U}_n(\mu,\tau)\right) + \sum_{n=0}^{\infty}\epsilon^n H_n(\mu,\tau)\right\}\right]\right]. \tag{16}$$

On equating both sides' ϵ coefficient, we have

$$\begin{aligned}
&\epsilon^0 : \mathbb{U}_0(\mu,\tau) = \mathbb{U}(\mu,0),\\
&\epsilon^1 : \mathbb{U}_1(\mu,\tau) = \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}(\mathcal{M}(\mathbb{U}_0(\mu,\tau)) + H_0(\mu,\tau))\right],\\
&\epsilon^2 : \mathbb{U}_2(\mu,\tau) = \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}(\mathcal{M}(\mathbb{U}_1(\mu,\tau)) + H_1(\mu,\tau))\right],\\
&.\\
&.\\
&.\\
&\epsilon^n : \mathbb{U}_n(\mu,\tau) = \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}(\mathcal{M}(\mathbb{U}_{n-1}(\mu,\tau)) + H_{n-1}(\mu,\tau))\right],\ \ n > 0, n \in \mathbf{N}.
\end{aligned} \tag{17}$$

At the end, our approximate solution in terms of series is

$$\mathbb{U}(\mu,\tau) = \mathbb{U}_0(\mu,\tau) + \mathbb{U}_1(\mu,\tau) + \mathbb{U}_2(\mu,\tau) + \cdots \tag{18}$$

4. Methodology of the Elzaki Transform Decomposition Method

To illustrate the core notion of ETDM, we solve the following general nonlinear problem of the form:

$$D_{\tau}^{\varpi}\mathbb{U}(\mu,\tau) = [\mathcal{M}+\mathcal{N}]\mathbb{U}(\mu,\tau),\ \ \tau > 0,\ \ 0 < \varpi \leq 1, \tag{19}$$

having initial guess

$$\mathbb{U}(\mu,0) = \vartheta(\mu).$$

Here, $D_{\tau}^{\varpi} = \frac{\partial^{\varpi}}{\partial\zeta^{\varpi}}$ symbolizes the Caputo noninteger derivative, and $\mathcal{M}, \mathcal{N}$ are linear and nonlinear differential operators.

By means of Definition 6 at $\varsigma = 1$, we obtain

$$\frac{1}{u^{\varpi}}\{M(u) - u^2\mathbb{U}(\mu,0)\} = \mathbb{E}[[\mathcal{M}+\mathcal{N}]\mathbb{U}(\mu,\tau)], \tag{20}$$

and after, we have

$$M(u) = u\mathbb{U}(\mu, 0) + u^{\varpi}\mathbb{E}[[\mathcal{M} + \mathcal{N}]\mathbb{U}(\mu, \tau)], \tag{21}$$

with M(u) = $\mathbb{E}[\mathbb{U}(\mu, \tau)]$.

In terms of inverse ET, we obtain

$$\mathbb{U}(\mu, \tau) = \mathbb{U}(\mu, 0) + \mathbb{E}^{-1}[u^{\varpi}\mathbb{E}[[\mathcal{M} + \mathcal{N}]\mathbb{U}(\mu, \tau)]. \tag{22}$$

In addition, the solution in series form is stated as

$$\mathbb{U}(\mu, \tau) = \sum_{n=0}^{\infty} \mathbb{U}_n(\mu, \tau). \tag{23}$$

The nonlinear term is determined as

$$\mathcal{N}[\mathbb{U}(\mu, \tau)] = \sum_{n=0}^{\infty} \mathcal{A}_n. \tag{24}$$

with

$$\mathcal{A}_n = \frac{1}{n!}\left[\frac{\partial^n}{\partial \ell^n}\left\{\mathcal{N}\left(\sum_{i=0}^{\infty} \ell^i \mathbb{U}_i\right)\right\}\right]_{\ell=0}, \quad n = 0, 1, 2, \cdots \tag{25}$$

By utilizing (23) and (24) in (22), we obtain

$$\sum_{n=0}^{\infty} \mathbb{U}_n(\mu, \tau) = \mathbb{U}(\mu, 0) + \mathbb{E}^{-1}u^{\varpi}\left[\mathbb{E}\left\{\mathcal{M}\left(\sum_{n=0}^{\infty} \mathbb{U}_n(\mu, \tau)\right) + \sum_{n=0}^{\infty} \mathcal{A}_n\right\}\right]. \tag{26}$$

On comparison of both sides,

$$\mathbb{U}_0(\mu, \tau) = \mathbb{U}(\mu, 0), \tag{27}$$

$$\mathbb{U}_1(\mu, \tau) = \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\{\mathcal{M}(\mathbb{U}_0(\mu, \tau)) + \mathcal{A}_0\}\right].$$

At the end, we can write the solution in general for $m \geq 1$ as

$$\mathbb{U}_{m+1}(\mu, \tau) = \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\{\mathcal{M}(\mathbb{U}_m(\mu, \tau)) + \mathcal{A}_m\}\right].$$

5. Applications

Problem 1. *Assume TFKE as given below:*

$$D_{\tau}^{\varpi}\mathbb{U}(\mu, \tau) + \mathbb{U}(\mu, \tau)\mathbb{U}_{\mu}(\mu, \tau) + \mathbb{U}_{\mu\mu\mu}(\mu, \tau) - \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu, \tau) = 0, \quad 0 < \varpi \leq 1, \tag{28}$$

having initial guess

$$\mathbb{U}(\mu, 0) = \frac{105}{169}\,\text{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right).$$

When $\varpi = 1$, we obtain the exact solution as

$$\mathbb{U}(\mu, \tau) = \frac{105}{169}\,\text{sech}^4\left(\frac{1}{2\sqrt{13}}\left(\mu - \frac{36\tau}{169}\right)\right). \tag{29}$$

By means of Definition 6 at $\varsigma = 1$, we obtain

$$\mathbb{E}\left(\frac{\partial^{\varpi}\mathbb{U}}{\partial \tau^{\varpi}}\right) = \mathbb{E}\left[-\mathbb{U}(\mu, \tau)\mathbb{U}_{\mu}(\mu, \tau) - \mathbb{U}_{\mu\mu\mu}(\mu, \tau) + \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu, \tau)\right], \tag{30}$$

and after, we have

$$\frac{1}{u^{\varpi}}\{M(u) - u^2\mathbb{U}(\mu,0)\} = \mathbb{E}\big[-\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\big], \tag{31}$$

$$M(u) = u\mathbb{U}(\mu,0) + u^{\varpi}\mathbb{E}\big[-\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\big]. \tag{32}$$

In terms of inverse ET, we obtain

$$\begin{aligned}
\mathbb{U}(\mu,\tau) &= \mathbb{U}(\mu,0) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right],\\
\mathbb{U}(\mu,\tau) &= \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right].
\end{aligned} \tag{33}$$

By utilizing HPM, we have

$$\begin{aligned}
\sum_{m=0}^{\infty}\epsilon^m\mathbb{U}_m(\mu,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right) - \epsilon\left[\mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\left(\sum_{m=0}^{\infty}\epsilon^m H_m(\mathbb{U})\right) + \left(\sum_{m=0}^{\infty}\epsilon^m\mathbb{U}_m(\mu,\tau)\right)_{\mu\mu\mu} -\right.\right.\right.\\
\left.\left.\left.\left(\sum_{m=0}^{\infty}\epsilon^m\mathbb{U}_m(\mu,\tau)\right)_{\mu\mu\mu\mu\mu}\right]\right]\right].
\end{aligned} \tag{34}$$

In addition, the nonlinear terms by means of He's polynomial $H_m(\mathbb{U})$ is presented as below. Some nonlinear terms are calculated as follows:

$$\begin{aligned}
&H_0(\mathbb{U}) = \mathbb{U}_0(\mathbb{U}_0)_{\mu},\\
&H_1(\mathbb{U}) = \mathbb{U}_1(\mathbb{U}_0)_{\mu} + \mathbb{U}_0(\mathbb{U}_1)_{\mu},\\
&\vdots
\end{aligned}$$

On equating both sides ϵ coefficient, we have

$$\begin{aligned}
&\epsilon^0 : \mathbb{U}_0(\mu,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right),\\
&\epsilon^1 : \mathbb{U}_1(\mu,\tau) = \frac{7560\sqrt{13}\sinh\left(\frac{\sqrt{13}\mu}{26}\right)}{371,293\cosh^5\left(\frac{\sqrt{13}\mu}{26}\right)}\frac{\tau^{\varpi}}{\Gamma(\varpi+1)},\\
&\epsilon^2 : \mathbb{U}_2(\mu,\tau) = \frac{136,080\left(2\sinh\left(\frac{\sqrt{13}\mu}{26}-1\right)2\sinh\left(\frac{\sqrt{13}\mu}{26}+1\right)\right)}{62,748,517\left(\sinh^2\left(\frac{\sqrt{13}\mu}{26}+1\right)\right)^3}\frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)},\\
&\vdots
\end{aligned}$$

At the end, our approximate solution in terms of series is as

$$\mathbb{U}(\mu,\tau) = \mathbb{U}_0(\mu,\tau) + \mathbb{U}_1(\mu,\tau) + \mathbb{U}_2(\mu,\tau) + \cdots$$

$$\mathbb{U}(\mu,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right) + \frac{7560\sqrt{13}\sinh\left(\frac{\sqrt{13}\mu}{26}\right)}{371,293\cosh^5\left(\frac{\sqrt{13}\mu}{26}\right)}\frac{\tau^{\varpi}}{\Gamma(\varpi+1)} +$$

$$\frac{136,080\left(2\sinh\left(\frac{\sqrt{13}\mu}{26}-1\right)2\sinh\left(\frac{\sqrt{13}\mu}{26}+1\right)\right)}{62,748,517\left(\sinh^2\left(\frac{\sqrt{13}\mu}{26}+1\right)\right)^3}\frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)} + \cdots$$

Solution in terms of ETDM

By means of Definition 6 at $\varsigma = 1$, we obtain

$$\mathbb{E}\left\{\frac{\partial^{\varpi}\mathbb{U}}{\partial\tau^{\varpi}}\right\} = \mathbb{E}\left[-\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]. \tag{35}$$

After, we have

$$\frac{1}{u^{\varpi}}\{M(u) - u^2\mathbb{U}(\mu,0)\} = \mathbb{E}\left[-\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right], \tag{36}$$

$$M(u) = u^2\mathbb{U}(\mu,0) + u^{\varpi}\mathbb{E}\left[-\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]. \tag{37}$$

In terms of inverse ET, we obtain

$$\begin{aligned}\mathbb{U}(\mu,\tau) &= \mathbb{U}(\mu,0) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right],\\ \mathbb{U}(\mu,\tau) &= \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) + \mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right].\end{aligned} \tag{38}$$

Also, the series form solution is stated as

$$\mathbb{U}(\mu,\tau) = \sum_{m=0}^{\infty}\mathbb{U}_m(\mu,\tau). \tag{39}$$

By putting Equation (39) and the Adomian polynomials for nonlinear term as define in Equation (24), thus Equation (38) turns into

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{U}_m(\tau) &= \mathbb{U}(\mu,0) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\sum_{m=0}^{\infty}\mathcal{A}_m + \mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right],\\ \sum_{m=0}^{\infty}\mathbb{U}_m(\tau) &= \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\sum_{m=0}^{\infty}\mathcal{A}_m + +\mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right].\end{aligned} \tag{40}$$

Some nonlinear terms are presented as follows:

$$\begin{aligned}
\mathcal{A}_0 &= \mathbb{U}_0(\mathbb{U}_0)_\mu, \\
\mathcal{A}_1 &= \mathbb{U}_1(\mathbb{U}_0)_\mu + \mathbb{U}_0(\mathbb{U}_1)_\mu, \\
&\vdots
\end{aligned}$$

On comparison of both sides

$$\mathbb{U}_0(\mu,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right).$$

On $m = 0$,

$$\mathbb{U}_1(\mu,\tau) = \frac{7560\sqrt{13}\sinh\left(\frac{\sqrt{13}\mu}{26}\right)}{371,293\cosh^5\left(\frac{\sqrt{13}\mu}{26}\right)}\frac{\tau^\varpi}{\Gamma(\varpi+1)}.$$

On $m = 1$,

$$\mathbb{U}_2(\mu,\tau) = \frac{136,080\left(2\sinh\left(\frac{\sqrt{13}\mu}{26} - 1\right)2\sinh\left(\frac{\sqrt{13}\mu}{26} + 1\right)\right)}{62,748,517\left(\sinh^2\left(\frac{\sqrt{13}\mu}{26} + 1\right)\right)^3}\frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)}.$$

At the end, our approximate solution in terms of series is

$$\mathbb{U}(\mu,\tau) = \sum_{m=0}^{\infty}\mathbb{U}_m(\mu,\tau) = \mathbb{U}_0(\mu,\tau) + \mathbb{U}_1(\mu,\tau) + \mathbb{U}_2(\mu,\tau) + \cdots$$

$$\begin{aligned}
\mathbb{U}(\mu,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\mu}{2\sqrt{13}}\right) + \frac{7560\sqrt{13}\sinh\left(\frac{\sqrt{13}\mu}{26}\right)}{371,293\cosh^5\left(\frac{\sqrt{13}\mu}{26}\right)}\frac{\tau^\varpi}{\Gamma(\varpi+1)} + \\
\frac{136,080\left(2\sinh\left(\frac{\sqrt{13}\mu}{26} - 1\right)2\sinh\left(\frac{\sqrt{13}\mu}{26} + 1\right)\right)}{62,748,517\left(\sinh^2\left(\frac{\sqrt{13}\mu}{26} + 1\right)\right)^3}\frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)} + \cdots
\end{aligned}$$

Problem 2. *Assume TFMKE as given below:*

$$D_\tau^\varpi\mathbb{U}(\mu,\tau) + \mathbb{U}^2(\mu,\tau)\mathbb{U}_\mu(\mu,\tau) + \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau) = 0, \quad 0 < \varpi \leq 1, \tag{41}$$

having initial guess

$$\mathbb{U}(\mu,0) = \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right).$$

When $\varpi = 1$, we obtain the exact solution as

$$\mathbb{U}(\mu,\tau) = \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\left(\mu - \frac{25\beta - 4\alpha^2}{25\beta}\tau\right)\right). \tag{42}$$

By means of Definition 6 at $\varsigma = 1$, we obtain

$$\mathbb{E}\left(\frac{\partial^{\varpi}\mathbb{U}}{\partial\tau^{\varpi}}\right) = \mathbb{E}\Big[-\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\Big], \tag{43}$$

after, we have

$$\frac{1}{u^{\varpi}}\{M(u) - u^2\mathbb{U}(\mu,0)\} = \mathbb{E}\Big[-\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\Big], \tag{44}$$

$$M(u) = u\mathbb{U}(\mu,0) + u^{\varpi}\mathbb{E}\Big[-\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) - \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) - \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\Big]. \tag{45}$$

In terms of inverse ET, we obtain

$$\begin{aligned}
\mathbb{U}(\mu,\tau) &= \mathbb{U}(\mu,0) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) + \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right],\\
\mathbb{U}(\mu,\tau) &= \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau) + \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right].
\end{aligned} \tag{46}$$

By utilizing HPM, we have

$$\begin{aligned}
\sum_{m=0}^{\infty}\epsilon^m\mathbb{U}_m(\mu,\tau) &= \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right) - \epsilon\left[\mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\left(\sum_{m=0}^{\infty}\epsilon^m H_m(\mathbb{U})\right) + \alpha\left(\sum_{m=0}^{\infty}\epsilon^m\mathbb{U}_m(\mu,\tau)\right)_{\mu\mu\mu} +\right.\right.\right.\\
&\left.\left.\left.\beta\left(\sum_{m=0}^{\infty}\epsilon^m\mathbb{U}_m(\mu,\tau)\right)_{\mu\mu\mu\mu\mu}\right]\right]\right].
\end{aligned} \tag{47}$$

In addition, the nonlinear terms by means of He's polynomial $H_m(\mathbb{U})$ is presented as below. Some nonlinear terms are calculated as follows:

$$\begin{aligned}
H_0(\mathbb{U}) &= \mathbb{U}_0^2(\mathbb{U}_0)_{\mu},\\
H_1(\mathbb{U}) &= \mathbb{U}_0^2(\mathbb{U}_1)_{\mu} + 2\mathbb{U}_0\mathbb{U}_1(\mathbb{U}_0)_{\mu},\\
&\vdots
\end{aligned}$$

On equating both sides ϵ coefficient, we have

$$\begin{aligned}
\epsilon^0 : \mathbb{U}_0(\mu,\tau) &= \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right),\\
\epsilon^1 : \mathbb{U}_1(\mu,\tau) &= \frac{6\sqrt{2}\alpha^3\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}}{125(-\beta)^{\frac{3}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha}\mu}{10\sqrt{\beta}}\right)^3}\frac{\tau^{\varpi}}{\Gamma(\varpi+1)},\\
\epsilon^2 : \mathbb{U}_2(\mu,\tau) &= \frac{6\sqrt{10}\alpha^{\frac{11}{2}}}{15625(-\beta)^{\frac{7}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha}\mu}{10\sqrt{\beta}}\right)^8}\left(45\sqrt{\alpha}\sinh^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right) - 2\sqrt{\alpha}\right.\\
&+ 51\sqrt{\alpha}\sin^2\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) - 51\sqrt{\alpha}\sin^4\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) + 2\sqrt{\alpha}\sin^6\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) + 6\\
&\sqrt{\beta}\sin\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}} - 57\sqrt{\beta}\sin^3\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\times\\
&\left.\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}} + 6\sqrt{\beta}\sin^5\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}\right)\\
&\frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)},\\
&\vdots
\end{aligned}$$

At the end, our approximate solution in terms of series is

$$\mathbb{U}(\mu,\tau)=\mathbb{U}_0(\mu,\tau)+\mathbb{U}_1(\mu,\tau)+\mathbb{U}_2(\mu,\tau)+\cdots$$

$$\mathbb{U}(\mu,\tau)=\frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right)+\frac{6\sqrt{2}\alpha^3\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}}{125(-\beta)^{\frac{3}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha}\mu}{10\sqrt{\beta}}\right)^3}$$
$$\frac{\tau^{\varpi}}{\Gamma(\varpi+1)}+\frac{6\sqrt{10}\alpha^{\frac{11}{2}}}{15625(-\beta)^{\frac{7}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha}\mu}{10\sqrt{\beta}}\right)^8}\left(45\sqrt{\alpha}\sinh^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right)-2\sqrt{\alpha}\right.$$
$$+51\sqrt{\alpha}\sin^2\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)-51\sqrt{\alpha}\sin^4\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)+2\sqrt{\alpha}\sin^6\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)+6$$
$$\sqrt{\beta}\sin\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}-57\sqrt{\beta}\sin^3\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\times$$
$$\left.\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}+6\sqrt{\beta}\sin^5\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}\right)$$
$$\frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)}+\cdots$$

Solution in terms of ETDM

By means of Definition 6 at $\varsigma=1$, we obtain

$$\mathbb{E}\left\{\frac{\partial^{\varpi}\mathbb{U}}{\partial\tau^{\varpi}}\right\}=\mathbb{E}\Big[-\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau)-\alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau)-\beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\Big]. \tag{48}$$

After, we have

$$\frac{1}{u^{\varpi}}\{M(u)-u^2\mathbb{U}(\mu,0)\}=\mathbb{E}\Big[-\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau)-\alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau)-\beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\Big], \tag{49}$$

$$M(u)=u^2\mathbb{U}(\mu,0)+u^{\varpi}\mathbb{E}\Big[-\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau)-\alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau)-\beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\Big]. \tag{50}$$

In terms of inverse ET, we obtain

$$\begin{aligned}\mathbb{U}(\mu,\tau)&=\mathbb{U}(\mu,0)-\mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau)+\alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau)+\beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right],\\ \mathbb{U}(\mu,\tau)&=\frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right)-\mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\mathbb{U}^2(\mu,\tau)\mathbb{U}_{\mu}(\mu,\tau)+\alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau)+\beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right].\end{aligned} \tag{51}$$

Also, the series form solution is stated as

$$\mathbb{U}(\mu,\tau)=\sum_{m=0}^{\infty}\mathbb{U}_m(\mu,\tau). \tag{52}$$

By putting Equation (52) and the Adomian polynomials for nonlinear terms as defined in Equation (24), thus Equation (51) turns into

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{U}_m(\tau) &= \mathbb{U}(\mu,0) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\sum_{m=0}^{\infty}\mathcal{A}_m + \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right], \\ \sum_{m=0}^{\infty}\mathbb{U}_m(\tau) &= \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right) - \mathbb{E}^{-1}\left[u^{\varpi}\mathbb{E}\left[\sum_{m=0}^{\infty}\mathcal{A}_m + \alpha\mathbb{U}_{\mu\mu\mu}(\mu,\tau) + \beta\mathbb{U}_{\mu\mu\mu\mu\mu}(\mu,\tau)\right]\right].\end{aligned} \tag{53}$$

Some nonlinear terms are presented as follows:

$$\begin{aligned}\mathcal{A}_0 &= \mathbb{U}_0^2(\mathbb{U}_0)_{\mu}, \\ \mathcal{A}_1 &= \mathbb{U}_0^2(\mathbb{U}_1)_{\mu} + 2\mathbb{U}_0\mathbb{U}_1(\mathbb{U}_0)_{\mu}, \\ &\vdots\end{aligned}$$

On comparison of both sides

$$\mathbb{U}_0(\mu,\tau) = \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right).$$

On $m = 0$,

$$\mathbb{U}_1(\mu,\tau) = \frac{6\sqrt{2}\alpha^3\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}}{125(-\beta)^{\frac{3}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha\mu}}{10\sqrt{\beta}}\right)^3}\frac{\tau^{\varpi}}{\Gamma(\varpi+1)}.$$

On $m = 1$,

$$\begin{aligned}\mathbb{U}_2(\mu,\tau) = {}& \frac{6\sqrt{10}\alpha^{\frac{11}{2}}}{15625(-\beta)^{\frac{7}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha\mu}}{10\sqrt{\beta}}\right)^8}\left(45\sqrt{\alpha}\sinh^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right) - 2\sqrt{\alpha}\right. \\ & + 51\sqrt{\alpha}\sin^2\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) - 51\sqrt{\alpha}\sin^4\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) + 2\sqrt{\alpha}\sin^6\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) + 6 \\ & \sqrt{\beta}\sin\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}} - 57\sqrt{\beta}\sin^3\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\times \\ & \left.\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}} + 6\sqrt{\beta}\sin^5\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}\right) \\ & \frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)},\end{aligned}$$

At the end, our approximate solution in terms of series is

$$\mathbb{U}(\mu,\tau) = \sum_{m=0}^{\infty}\mathbb{U}_m(\mu,\tau) = \mathbb{U}_0(\mu,\tau) + \mathbb{U}_1(\mu,\tau) + \mathbb{U}_2(\mu,\tau) + \cdots$$

$$\mathbb{U}(\mu,\tau) = \frac{3\alpha}{\sqrt{-10\beta}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right) + \frac{6\sqrt{2}\alpha^3 \sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}}{125(-\beta)^{\frac{3}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha}\mu}{10\sqrt{\beta}}\right)^3}$$
$$\frac{\tau^{\varpi}}{\Gamma(\varpi+1)} + \frac{6\sqrt{10}\alpha^{\frac{11}{2}}}{15625(-\beta)^{\frac{7}{2}}\cos\left(\frac{\sqrt{5}\sqrt{-\alpha}\mu}{10\sqrt{\beta}}\right)^8}\left(45\sqrt{\alpha}\sinh^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{5\beta}}\mu\right) - 2\sqrt{\alpha}\right.$$
$$+ 51\sqrt{\alpha}\sin^2\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) - 51\sqrt{\alpha}\sin^4\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) + 2\sqrt{\alpha}\sin^6\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right) + 6$$
$$\sqrt{\beta}\sin\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}} - 57\sqrt{\beta}\sin^3\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\times$$
$$\left.\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}} + 6\sqrt{\beta}\sin^5\left(\frac{\sqrt{5}\sqrt{\alpha}\mu}{10\sqrt{\beta}}\right)\sinh\left(\frac{\sqrt{5}\mu\sqrt{-\frac{\alpha}{\beta}}}{10}\right)\sqrt{-\frac{\alpha}{\beta}}\right)$$
$$\frac{\tau^{2\varpi}}{\Gamma(2\varpi+1)} + \cdots$$

6. Numerical Simulation Studies

This part of the work deals with the numerical behavior of nonlinear FDEs using the HPTM and the ETDM. By utilizing Maple, the aforementioned issues can be analyzed through tables and graphs. The graphs in Figure 1a illustrate the nature of the precise solution, and Figure 1b illustrates the nature of the proposed approaches solution at $\varpi = 1$. Figure 2a,b show the outcomes of suggested techniques at different orders of $\varpi = 0.80, 0.60$. The nature in terms of absolute error for the derived equation obtained by both procedures at $0 < \tau \leq 0.01$ is shown in Figure 3. The absolute error is calculated by the difference of exact and our method's solution. Table 1 displays the estimated and accurate values of the equation $\mathbb{U}(\mu,\tau)$ at various values of ϖ of problem 1. The graphs in Figure 4a,b illustrate the nature of the precise and proposed approaches solution at $\alpha = 0.001$, $\beta = -1$ and $\varpi = 1$. Figure 5a,b show the outcomes of suggested techniques at different orders of $\varpi = 0.80, 0.60$. The nature in terms of absolute error for the derived equation obtained by both procedures at $0 < \tau \leq 0.01$ is shown in Figure 6. The absolute error is calculated by the difference of exact and our method's solution. Table 2 displays the estimated and accurate values of the equation $\mathbb{U}(\mu,\tau)$ at various values of ϖ of problem 2. Since the HPTM and ETDM gave the same solution, we draw a single solution graph for both methods. It must be renowned that we utilized third-order approximation during the computations and that we generated a virtuous estimate with the precise solution of the given problems. The graphical nature also illustrates that the precise solution and the proposed approaches have a strong rate of agreement. It has been proven that the suggested approaches are the most effective means of resolving FPDEs.

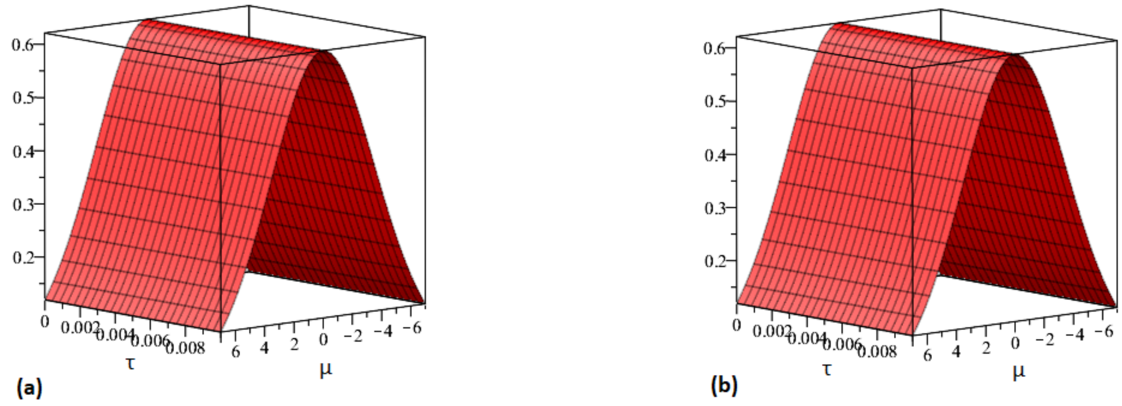

Figure 1. Plot (**a**) illustrating the exact solution, (**b**) illustrating the HPTM and ETDM solution for Problem 1.

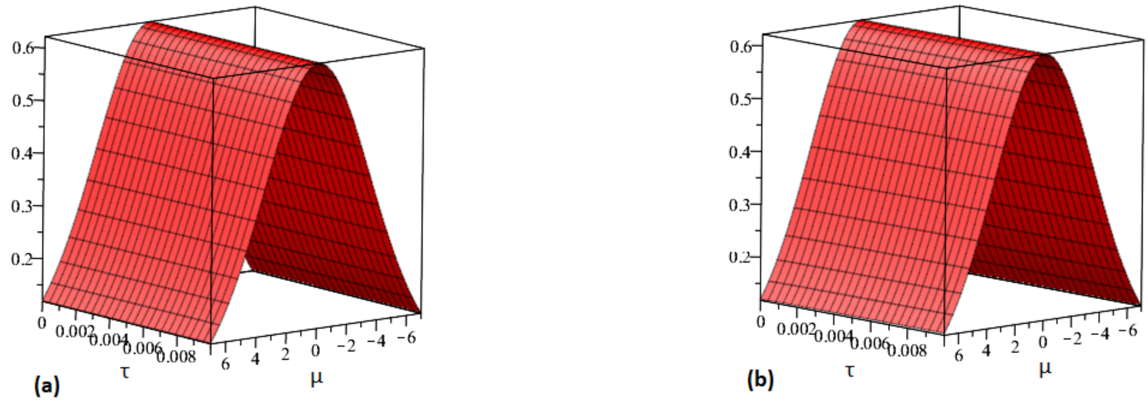

Figure 2. Plot (**a**) illustrating the HPTM and ETDM solution at $\varpi = 0.80$, (**b**) illustrating the HPTM and ETDM solution at $\varpi = 0.60$ for Problem 1.

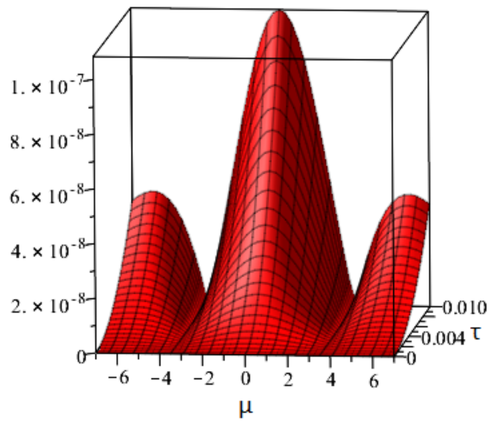

Figure 3. Plot illustrating the HPTM and ETDM solution in terms of absolute error for Problem 1.

Table 1. Approximate solution of our techniques at different values of ϖ with respect to the accurate solution for Problem 1.

τ	$\varpi = \mathbf{0.85}$	$\varpi = \mathbf{0.90}$	$\varpi = \mathbf{0.95}$	$\varpi = \mathbf{1}\ (appro)$	$\varpi = \mathbf{1}\ (exact)$
00.0	00.62130177	00.62130177	00.62130177	00.62130177	00.62130166
00.1	00.62108433	00.62107963	00.62107594	00.62107304	00.62107293
00.2	00.62038966	00.62038027	00.62037289	00.62036710	00.62036700
00.3	00.61921963	00.61920557	00.61919453	00.61918586	00.61918576
00.4	00.61757737	00.61755869	00.61754401	00.61753249	00.61753239
00.5	00.61546729	00.61544403	00.61542575	00.61541141	00.61541131
00.6	00.61289501	00.61286723	00.61284541	00.61282829	00.61282819
00.7	00.60986737	00.60983516	00.60980984	00.60978999	00.60978988
00.8	00.60639239	00.60635581	00.60632707	00.60630454	00.60630444
00.9	00.60247919	00.60243835	00.60240627	00.60238110	00.60238101
01.0	00.59813800	00.59809301	00.59805766	00.59802994	00.59802984

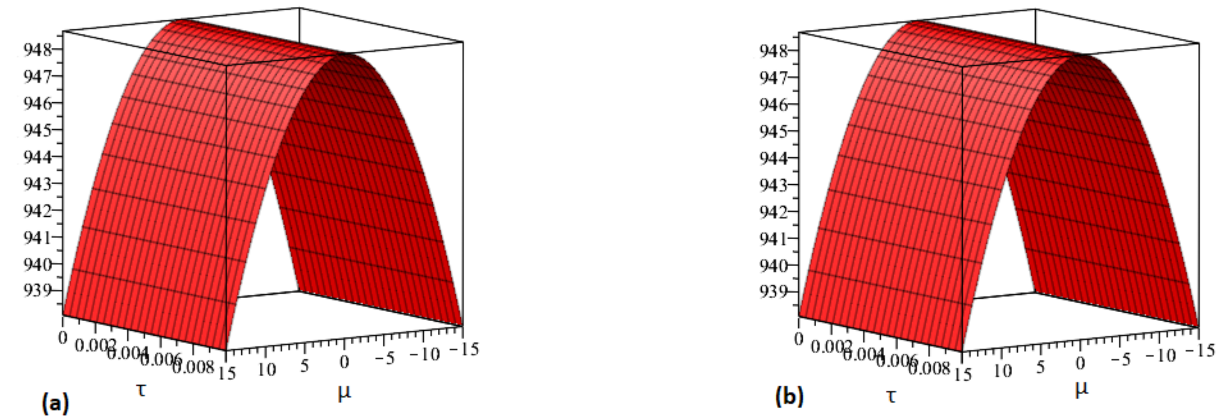

Figure 4. Plot (**a**) illustrating the exact solution, (**b**) illustrating the HPTM and ETDM solution for Problem 2.

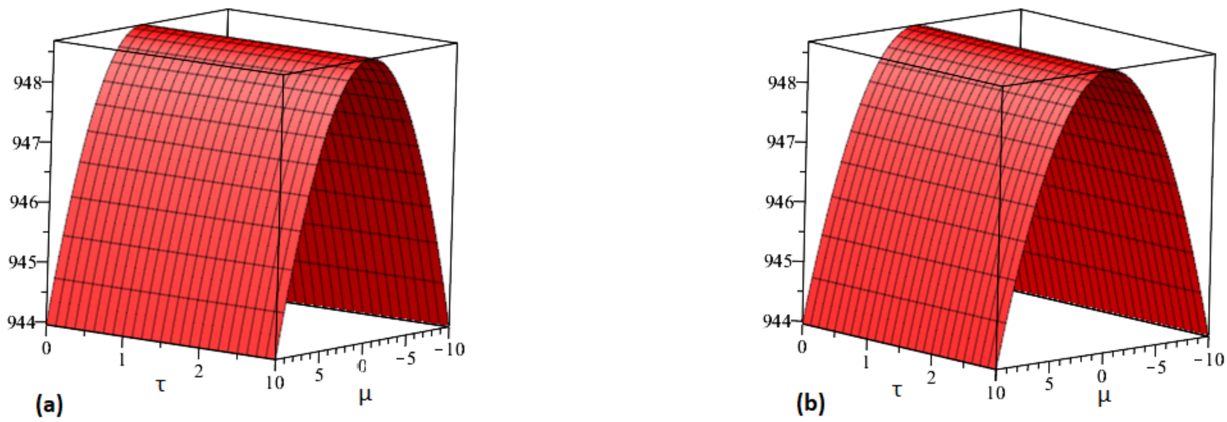

Figure 5. Plot (**a**) illustrating the HPTM and ETDM solution at $\varpi = 0.80$, (**b**) illustrating the HPTM and ETDM solution at $\varpi = 0.60$ for Problem 2.

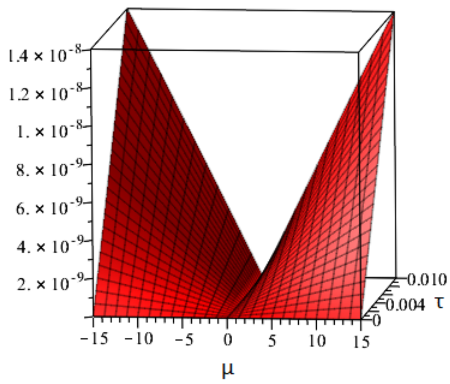

Figure 6. Plot illustrating the HPTM and ETDM solution in terms of absolute error for Problem 2.

Table 2. Approximate solution of our techniques at different values of ϖ with respect to the accurate solution for Problem 2.

τ	$\varpi = 0.25$	$\varpi = 0.50$	$\varpi = 0.75$	$\varpi = 1\ (appro)$	$\varpi = 1\ (exact)$
00.0	00.009486832980	00.009486832980	00.009486832980	00.009486832980	00.009486785543
00.1	00.009486785555	00.009486785551	00.009486785549	00.009486785548	00.009486832980
00.2	00.009486643266	00.009486643258	00.009486643253	00.009486643250	00.009486785549
00.3	00.009486406113	00.009486406101	00.009486406094	00.009486406090	00.009486643250
00.4	00.009486074112	00.009486074096	00.009486074086	00.009486074080	00.009486406091
00.5	00.009485647274	00.009485647254	00.009485647242	00.009485647235	00.009486074081
00.6	00.009485125608	00.009485125584	00.009485125570	00.009485125561	00.009485647233
00.7	00.009484509151	00.009484509123	00.009484509106	00.009484509096	00.009485125565
00.8	00.009483797917	00.009483797885	00.009483797865	00.009483797854	00.009484509098
00.9	00.009482991935	00.009482991899	00.009482991876	00.009482991864	00.009483797854
01.0	00.009482091239	00.009482091199	00.009482091174	00.009482091160	00.009482991863

7. Conclusions

In this study, we used HPTM and ETDM to analyze the approximations of TFKE and TFMKE solutions based on the Caputo fractional derivative operator. The suggested approaches overcome the majority of the restrictions by combining three effective techniques. The numerical simulation is displayed to show that the fractional order goes to classical order and to confirm the accuracy of the results. The fact that the derived solutions converge at the actual solutions quite quickly shows how close approximations are to exact results. For different fractional orders, the characteristics of the generated series solution have been depicted in the form of 3D representations. These figures reveal that the proposed methods solutions are highly effective and conforming, utilizing simulation studies, as seen in the tables. We furthermore assure you that the generated implementations will converge to the precise result as the order of the result rises. The numerical outcomes imply that the current methods are simple, efficient, and precise. The effect of all relevant variables was addressed and shown using graphs and tables. The modified Korteweg–De Vries equation, as well as fuzzy partial differential equations, can all be approximated using these reliable, accessible, and efficient approaches. These fractional physical models are common in engineering and science.

Author Contributions: Conceptualization, M.M.A., A.H.G., F.A. and A.K.; methodology, M.M.A., A.H.G., F.A. and A.K.; software, A.K.; validation, M.M.A., A.H.G., F.A. and A.K.; formal analysis, A.K.; investigation, A.K.; resources, M.M.A., A.H.G., F.A. and A.K.; data curation, A.H.G. and A.K.; writing—original draft preparation, A.K.; writing—review and editing, A.K.; visualization, A.H.G. and A.K.; supervision, A.H.G. and M.M.A.; project administration, M.M.A.; funding acquisition, M.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The numerical data used to support the findings of this study are included within the article.

Acknowledgments: We are thankful to the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University for funding this work with award number (2022/01/22758).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; p. 204.
2. Inc, M.; Yusuf, A.; Aliyu, A.I.; Baleanu, D. Soliton solutions and stability analysis for some conformable nonlinear partial differential equations in mathematical physics. *Opt. Quantum Electron.* **2018**, *50*, 190. [CrossRef]
3. Akdemir, A.O.; Dutta, H.; Atangana, A. *Fractional Order Analysis: Theory, Methods and Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2020.
4. Bagley R.L.; Torvik, P.J. Fractional calculus in the transient analysis of viscoelastically damped structures. *AIAA J.* **1985**, *23*, 918–925. [CrossRef]
5. Al-Smadi, M.; Djeddi, N.; Momani, S.; Al-Omari, S.; Araci, S. An attractive numerical algorithm for solving nonlinear Caputo-Fabrizio fractional Abel differential equation in a Hilbert space. *Adv. Differ. Equations* **2021**, *2021*, 271. [CrossRef]
6. Guo, B.; Pu, X.; Huang, F. *Fractional Partial Differential Equations and Their Numerical Solutions*; World Scientific: Singapore, 2015.
7. Nonlaopon, K.; Alsharif, A.M.; Zidan, A.M.; Khan, A.; Hamed, Y.S.; Shah, R. Numerical investigation of fractional-order Swift-Hohenberg equations via a Novel transform. *Symmetry* **2021**, *13*, 1263. [CrossRef]
8. Yang, Q.; Liu, F.; Turner, I. Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. *Appl. Math. Model.* **2010**, *34*, 200–218. [CrossRef]
9. Fathima, D.; Alahmadi, R.A.; Khan, A.; Akhter, A.; Ganie, A.H. An Efficient Analytical Approach to Investigate Fractional Caudrey-Dodd-Gibbon Equations with Non-Singular Kernel Derivatives. *Symmetry* **2023**, *15*, 850. [CrossRef]
10. Eskandari, Z.; Avazzadeh, Z.; Khoshsiar Ghaziani, R.; Li, B. Dynamics and bifurcations of a discrete-time Lotka-Volterra model using nonstandard finite difference discretization method. *Math. Methods Appl. Sci.* **2022**. [CrossRef]
11. Xu, C.; Liao, M.; Li, P.; Guo, Y.; Liu, Z. Bifurcation properties for fractional order delayed BAM neural networks. *Cogn. Comput.* **2021**, *13*, 322–356. [CrossRef]
12. Povstenko, Y.Z. Thermoelasticity that uses fractional heat conduction equation. *J. Math. Sci.* **2009**, *162*, 296–305. [CrossRef]
13. Bapna, I.B.; Mathur, N. Application of fractional calculus in statistics. *Int. J. Contemp. Math. Sci.* **2012**, *7*, 849–856.
14. Hartley, T.T.; Lorenzo, C.F.; Qammer, H.K. Chaos in a fractional order Chua's system. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **1995**, *42*, 485–490. [CrossRef]
15. Kulish, V.V.; Lage, J.L. Application of fractional calculus to fluid mechanics. *J. Fluids Eng.* **2002**, *124*, 803–806. [CrossRef]
16. Baskin, E.; Iomin, A. Electro-chemical manifestation of nanoplasmonics in fractal media. *Cent. Eur. J. Phys.* **2013**, *11*, 676–684. [CrossRef]
17. Goncalvès, E.; Zeidan, D. Numerical simulation of unsteady cavitation in liquid hydrogen flows. *Int. J. Eng. Syst. Model. Simul.* **2017**, *9*, 41–52. [CrossRef]
18. Sunthrayuth, P.; Alyousef, H.A.; El-Tantawy, S.A.; Khan, A.; Wyal, N. Solving fractional-order diffusion equations in a plasma and fluids via a novel transform. *J. Funct. Spaces* **2022**, *2022*, 1899130. [CrossRef]
19. Mishra, N.K.; AlBaidani, M.M.; Khan, A.; Ganie, A.H. Numerical Investigation of Time-Fractional Phi-Four Equation via Novel Transform. *Symmetry* **2023**, *15*, 687. [CrossRef]
20. Kolokoltsov, V.N. The probabilistic point of view on the generalized fractional partial differential equations. *Fract. Calc. Appl. Anal.* **2019**, *22*, 543–600. [CrossRef]
21. Zidan, A.M.; Khan, A.; Shah, R.; Alaoui, M.K.; Weera, W. Evaluation of time-fractional Fisher's equations with the help of analytical methods. *AIMS Math.* **2022**, *7*, 18746–18766. [CrossRef]
22. Jafari, H.; Nazari, M.; Baleanu, D.; Khalique, C.M. A new approach for solving a system of fractional partial differential equations. *Comput. Math. Appl.* **2013**, *66*, 838–843. [CrossRef]
23. Zhang, Y. A finite difference method for fractional partial differential equation. *Appl. Math. Comput.* **2009**, *215*, 524–529. [CrossRef]
24. Zheng, B.; Wen, C. Exact solutions for fractional partial differential equations by a new fractional sub-equation method. *Adv. Differ. Equations* **2013**, *2013*, 199. [CrossRef]

25. Dhaigude, D.B.; Kiwne, S.B.; Dhaigude, R.M. Monotone iterative scheme for weakly coupled system of finite difference reaction-diffusion equations. *Commun. Appl. Anal.* **2008**, *12*, 161.
26. Abbasbandy, S. The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Phys. Lett. A* **2006**, *360*, 109–113. [CrossRef]
27. Ara, A.; Khan, N.A.; Razzaq, O.A.; Hameed, T.; Raja, M.A.Z. Wavelets optimization method for evaluation of fractional partial differential equations: an application to financial modelling. *Adv. Differ. Equations* **2018**, *2018*, 8. [CrossRef]
28. Akgül, A.; Cordero, A.; Torregrosa, J.R. A fractional Newton method with 2αth-order of convergence and its stability. *Appl. Math. Lett.* **2019**, *98*, 344–351. [CrossRef]
29. Boutarfa, B.; Akgül, A.; Inc, M. New approach for the Fornberg-Whitham type equations. *J. Comput. Appl. Math.* **2017**, *312*, 13–26. [CrossRef]
30. Zheng, B. (G′/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics. *Commun. Theor. Phys.* **2012**, *58*, 623. [CrossRef]
31. Sontakke, B.R.; Shelke, A.S.; Shaikh, A.S. Solution of non-linear fractional differential equations by variational iteration method and applications. *Far East J. Math. Sci.* **2019**, *110*, 113–129. [CrossRef]
32. ur Rahman, M.; Arfan, M.; Shah, Z.; Alzahrani, E. Evolution of fractional mathematical model for drinking under Atangana-Baleanu Caputo derivatives. *Phys. Scr.* **2021**, *96*, 115203. [CrossRef]
33. Ganie, A.H.; Mofarreh, F.; Khan, A. A Fractional Analysis of Zakharov-Kuznetsov Equations with the Liouville-Caputo Operator. *Axioms* **2023**, *12*, 609. [CrossRef]
34. Botmart, T.; Agarwal, R.P.; Naeem, M.; Khan, A.; Shah, R. On the solution of fractional modified Boussinesq and approximate long wave equations with non-singular kernel operators. *AIMS Math.* **2022**, *7*, 12483–12513. [CrossRef]
35. Mishra, N.K.; AlBaidani, M.M.; Khan, A.; Ganie, A.H. Two Novel Computational Techniques for Solving Nonlinear Time-Fractional Lax's Korteweg-de Vries Equation. *Axioms* **2023**, *12*, 400. [CrossRef]
36. Kaya, D.; Al-Khaled, K. A numerical comparison of a Kawahara equation. *Phys. Lett. A* **2007**, *363*, 433–439. [CrossRef]
37. Jin, L. Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation. *Math. Comput. Model.* **2009**, *49*, 573–578. [CrossRef]
38. Kudryashov, N.A. A note on new exact solutions for the Kawahara equation using Exp-function method. *J. Comput. Appl. Math.* **2010**, *234*, 3511–3512. [CrossRef]
39. Kawahara, T. Oscillatory solitary waves in dispersive media. *J. Phys. Soc. Jpn.* **1972**, *33*, 260–264. [CrossRef]
40. Vašíček, J. Symmetries and conservation laws for a generalization of Kawahara equation. *J. Geom. Phys.* **2020**, *150*, 103579. [CrossRef]
41. Jabbari, A.; Kheiri, H. New exact traveling wave solutions for the Kawahara and modified Kawahara equations by using modified tanh-coth method. *Acta Univ. Apulensis* **2010**, *23*, 21–38.
42. Wazwaz, A.M. New solitary wave solutions to the modified Kawahara equation. *Phys. Lett. A* **2007**, *360*, 588–592. [CrossRef]
43. Kurulay, M. Approximate analytic solutions of the modified Kawahara equation with homotopy analysis method. *Adv. Differ. Equations* **2012**, *2012*, 178. [CrossRef]
44. Sontakke, B.R.; Shaikh, A. Approximate solutions of time fractional Kawahara and modified Kawahara equations by fractional complex transform. *Commun. Numer. Anal.* **2012**, *2*, 218–229. [CrossRef]
45. Zafar, H.; Ali, A.; Khan, K.; Sadiq, M.N. Analytical Solution of Time Fractional Kawahara and Modified Kawahara Equations by Homotopy Analysis Method. *Int. J. Appl. Comput. Math.* **2022**, *8*, 94. [CrossRef]
46. Mahmood, B.A.; Yousif, M.A. A novel analytical solution for the modified Kawahara equation using the residual power series method. *Nonlinear Dyn.* **2017**, *89*, 1233–1238. [CrossRef]
47. Dhaigude, D.B.; Bhadgaonkar, V.N. A novel approach for fractional Kawahara and modified Kawahara equations using Atangana-Baleanu derivative operator. *Math. Comput. Sci.* **2021**, *11*, 2792–2813.
48. Areshi, M.; Khan, A.; Shah, R.; Nonlaopon, K. Analytical investigation of fractional-order Newell-Whitehead-Segel equations via a novel transform. *AIMS Math.* **2022**, *7*, 6936–6958. [CrossRef]
49. Rashid, S.; Kubra, K.T.; Sultana, S.; Agarwal, P.; Osman, M.S. An approximate analytical view of physical and biological models in the setting of Caputo operator via Elzaki transform decomposition method. *J. Comput. Appl. Math.* **2022**, *413*, 114378. [CrossRef]
50. Shah, N.A.; Dassios, I.; Chung, J.D. A decomposition method for a fractional-order multi-dimensional telegraph equation via the Elzaki transform. *Symmetry* **2020**, *13*, 8. [CrossRef]
51. Loyinmi, A.C.; Akinfe, T.K. Exact solutions to the family of Fisher's reaction-diffusion equation using Elzaki homotopy transformation perturbation method. *Eng. Rep.* **2020**, *2*, e12084. [CrossRef]
52. Sedeeg, A.K.H. A coupling Elzaki transform and homotopy perturbation method for solving nonlinear fractional heat-like equations. *Am. Math. Comput. Model* **2016**, *1*, 15–20.
53. Elzaki, T.M. The new integral transform Elzaki transform. *Glob. J. Pure Appl. Math.* **2011**, *7*, 57–64.
54. Alshikh, A.A.; Mahgob, M.M.A. A comparative study between Laplace transform and two new integrals "Elzaki" transform and "Aboodh" transform. *Pure Appl. Math. J.* **2016**, *5*, 145–150. [CrossRef]
55. Elzaki, T.M.; Alkhateeb, S.A. Modification of Sumudu transform "Elzaki transform" and Adomian decomposition method. *Appl. Math. Sci.* **2015**, *9*, 603–611. [CrossRef]

56. Elzaki, T.M. On the connections between Laplace and Elzaki transforms. *Adv. Theor. Appl. Math.* **2011**, *6*, 1–11.
57. Ganie, A.H.; AlBaidani, M.M.; Khan, A. A Comparative Study of the Fractional Partial Differential Equations via Novel Transform. *Symmetry* **2023**, *15*, 1101. [CrossRef]