



Article

Local Stability, Global Stability, and Simulations in a Fractional Discrete Glycolysis Reaction–Diffusion Model

Tareq Hamadneh ¹, Amel Hioual ^{2,*} , Omar Alsayyed ³, Yazan Alaya AL-Khassawneh ⁴ , Abdallah Al-Husban ⁵ and Adel Ouannas ⁶

- ¹ Department of Mathematics, Faculty of Science, Al Zaytoonah University of Jordan, Amman 11733, Jordan; t.hamadneh@zuj.edu.jo
- ² Laboratory of Dynamical Systems and Control, University of Oum EL-Bouaghi, Oum El Bouaghi 04000, Algeria
- ³ Department of Mathematics, Faculty of Science, The Hashemite University, Zarqa 13133, Jordan; Omarm_re@hu.edu.jo
- ⁴ Data Science and Artificial Intelligence Department, Zarqa University, Zarqa 13110, Jordan; ykhassawneh@zu.edu.jo
- ⁵ Department of Mathematics, Faculty of Science and Technology, Irbid National University, Irbid 21110, Jordan; dralhosban@inu.edu.jo
- ⁶ Department of Mathematics and Computer Science, University of Oum EL-Bouaghi, Oum El Bouaghi 04000, Algeria; ouannas.adel@univ-oeb.dz
- * Correspondence: amel.hioual@univ-oeb.dz

Abstract: In the last few years, reaction–diffusion models associated with discrete fractional calculus have risen in prominence in scientific fields, not just due to the requirement for numerical simulation but also due to the described biological phenomena. This work investigates a discrete equivalent of the fractional reaction–diffusion glycolysis model. The discrete fractional calculus tool is introduced to the discrete modeling of diffusion problems in the Caputo-like delta sense, and a fractional discretization diffusion model is described. The local stability of the equilibrium points in the proposed discrete system is examined. We additionally investigate the global stability of the equilibrium point by developing a Lyapunov function. Furthermore, this study indicates that the L1 finite difference scheme and the second-order central difference scheme can successfully preserve the characteristics of the associated continuous system. Finally, an equivalent summation representing the model’s numerical formula is shown. The diffusion concentration is further investigated for different fractional orders, and examples with simulations are presented to corroborate the theoretical findings.

Keywords: fractional discrete-time reaction–diffusion systems; L1 finite difference scheme; local stability; Lyapunov function; global stability

MSC: 39A12; 39A30; 39A60; 39B82



Citation: Hamadneh, T.; Hioual, A.; Alsayyed, O.; AL-Khassawneh, Y.A.; Al-Husban, A.; Ouannas, A. Local Stability, Global Stability, and Simulations in Fractional Discrete Glycolysis Reaction–Diffusion Model. *Fractal Fract.* **2023**, *7*, 587. <https://doi.org/10.3390/fractalfract7080587>

Academic Editors: Jiafa Xu, Yaohong Li and Akbar Zada

Received: 29 May 2023

Revised: 2 July 2023

Accepted: 24 July 2023

Published: 29 July 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Reaction–diffusion equations have recently gained popularity due to their applicability to a wide range of biological and chemical processes as well as their unique characteristics in terms of the propagation speed and the new oscillations not seen in standard two-species reaction–diffusion systems. The biochemical process that converts glucose to pyruvate is known as glycolysis, which is present in the vast majority of living creatures today [1]. Throughout this process, two molecules of pyruvate are generally created for every molecule of glucose. The prevalence of glycolysis in a wide range of animals on Earth suggests that it is a primitive metabolic route [2]. Glycolysis is a series of reactions that divide pyruvates (three-carbon molecules) into two components in order to obtain energy from glucose. The same reaction strategy underpins all glycolysis models. The model differs due to differences in the mechanism for the main enzyme reaction. Because such systems have spatial variables,

they are widely utilized to analyze certain unusual behaviors, which can include self-replicating spikes, self-excitation, and spatiotemporal chaos. The need to comprehend dynamics is emphasized for this model in order to actualize a broader range of real-world scenarios. Numerous significant attempts have been made to explore the behavior of the reaction–diffusion glycolysis model. The application of a linear control rule to synchronize the reaction–diffusion glycolysis model is discussed in [3]. In [4], the numerical solution of a general glycolysis reaction–diffusion system was studied. In addition, the stability of the Selkov–Schnakenberg reaction–diffusion system was addressed in [5].

In modern-world experiences, fractional-order nonlinear equations are often employed to simulate a variety of physical phenomena [6–9]. Due to its numerous applications in the fields of chemistry, biology, physics, technology, thermal technology, electricity, mechanics, data processing, electrohydraulics, population modeling, machine learning, and control, the concept of fractional calculus continues to attract the attention of scientists. [10–13]. Due to its use in and relevance to many fields of research and engineering, fractional reaction–diffusion models have gained attention in recent years. In this field, for example, in [14], a pair fractional reaction–diffusion model was explored. In fractional reaction–diffusion systems, they examined nonlinear oscillations and the stability domain. In [15], a solution to a specific fractional reaction–diffusion problem was proposed. In addition, several breakthroughs have been made in the research on the dynamics of the fractional reaction diffusion glycolysis model, for example, see [16–18]. In the works described earlier, the fractional component is dedicated to the time partial derivative; nevertheless, in some circumstances, the standard Laplacian cannot imitate the diffusion, and the mean square movement of the diffusing components does not progress linearly over time. Following numerous findings in biological systems, emphasis has been placed on fractional subdiffusion, which is defined as the mean square movement of a population propagating as a sublinear power law in time. It is now widely known that if particles are confined for arbitrarily extended periods of time, subdiffusion occurs (see [19–21]).

Discrete reaction–diffusion systems have received less attention than continuous reaction–diffusion systems. Discrete reaction–diffusion systems are formed of cells or lattices that contain chemical processes and are linked together via mass transfer across their membranes. Discrete reaction diffusion systems are more tightly linked to biological systems than continuous systems, and in reaction to a small disturbance, they may exhibit extremely intriguing and diverse patterns in a two-dimensional space [22–24]. However, fractional discrete reaction diffusion equations have received little attention [25]. In [26], a fractional discrete diffusion equation was proposed. The chaotic behavior of a variable-order fractional diffusion equation on discontinuous time scales was examined in [27]. In [28], the global dynamics for a class of discrete fractional epidemic models with reaction–diffusion were also investigated. Apparently, we have a gap in our understanding of the dynamics of such systems.

In recent decades, both integer and noninteger fractional orders of the well-known glycolysis reaction diffusion model have received extensive study. The majority of this work has focused on the dynamics and behavior of solutions ([29–32]). However, to the authors' knowledge, this is the first investigation of the dynamics and stability of the discrete fractional reaction–diffusion glycolysis model. This prompted us to study the stability of the equilibrium state in a discrete fractional-order reaction–diffusion glycolysis model. A discrete fractional glycolysis reaction diffusion model in the Caputo-like delta sense is described in this study, employing a second-order central difference scheme and an L1 finite difference scheme. Both the local and global stability were explored, and simulations were conducted to test the model's applicability.

The following is a summary of the paper's content. Section 2 provides an introduction to fractional discrete calculus, as well as the fundamental concepts associated with the delta Caputo h-discrete difference operator. The mathematical model under examination is described in Section 3. The analysis of the local stability of the equilibrium state for the presence and absence of diffusion is highlighted in Section 4. The global stability of

the equilibrium state, which relies on the fractional orders of the investigated model, is discussed in Section 5. Section 6 includes numerical simulations that back up the findings. Section 7 outlines a number of results.

2. Preliminaries

This section provides the necessary terminology and stability theory for the fractional discrete calculus.

Definition 1 ([33]). Given that $z : \mathbb{N} \rightarrow \mathbb{R}$, the forward difference operator Δ is stated by

$$\Delta z(\tau) = z(\tau + 1) - z(\tau); \quad \tau \in \mathbb{N}. \quad (1)$$

Also, the recursive identification of the operator Δ^n , $n = 1, 2, 3, \dots$ is $\Delta^n z(\tau) = \Delta(\Delta^{n-1} z(\tau))$, $\tau \in \mathbb{N}$. In particular, the second-order difference operator of the function $z(t)$ is provided by

$$\Delta^2 z(\tau) = z(\tau + 2) - 2z(\tau + 1) + z(\tau). \quad (2)$$

Theorem 1 ([33]). We have the summation by parts formulae for the following two functions: $z; \eta : \mathbb{R} \rightarrow \mathbb{R}$ and $a; b \in \mathbb{N}; \quad a < b$;

$$\sum_{j=a}^{b-1} z(j) \Delta \eta(j) = z(j) \eta(j) \Big|_a^b - \sum_{j=a}^{b-1} \eta(j+1) \Delta z(j), \quad (3)$$

$$\sum_{j=a}^{b-1} z(j+1) \Delta \eta(j) = z(j) \eta(j) \Big|_a^b - \sum_{j=a}^{b-1} \eta(j) \Delta z(j). \quad (4)$$

Definition 2 ([34,35]). Let $z \in (\hbar\mathbb{N})_a \rightarrow \mathbb{R}$. For a given $\vartheta > 0$, the ϑ -th order \hbar -sum is given by

$${}_{\hbar} \Delta_a^{-\vartheta} z(t) = \frac{\hbar}{\Gamma(\vartheta)} \sum_{s=\frac{t}{\hbar}}^{\frac{a}{\hbar}} (t - \sigma(s\hbar))^{\vartheta-1} z(s\hbar), \quad \sigma(s\hbar) = (s+1)\hbar, \quad t \in (\hbar\mathbb{N})_{a+\vartheta\hbar},$$

the initial value is $a \in \mathbb{R}$, and the \hbar -falling factorial function is specified by

$$t_{\hbar}^{(\vartheta)} = \hbar^{\vartheta} \frac{\Gamma(\frac{t}{\hbar} + 1)}{\Gamma(\frac{t}{\hbar} + 1 - \vartheta)},$$

while

$$(\hbar\mathbb{N})_{a+\vartheta\hbar} = \{a + (1 - \vartheta)\hbar, a + (2 - \vartheta)\hbar, \dots\}.$$

Definition 3 ([36,37]). Considering a function $z(t)$ given on $(\hbar\mathbb{N})_a$ and for a particular $\vartheta > 0$, the Caputo \hbar -difference operator is represented by

$${}_{\hbar}^C \Delta_a^{\vartheta} z(t) = {}_{\hbar} \Delta_a^{-(n-\vartheta)} \Delta_{\hbar}^n z(t), \quad (5)$$

where $\Delta_{\hbar}^n z(t) = \frac{z(t + \hbar) - z(t)}{\hbar}$.

Lemma 1 ([34]). The following inequality holds

$${}_{\hbar}^C \Delta_a^{\vartheta} z^2(t) \leq 2z(t + \vartheta\hbar) {}_{\hbar}^C \Delta_a^{\vartheta} z(t), \quad t \in (\hbar\mathbb{N})_{a+\vartheta\hbar}, \quad (6)$$

where $0 < \vartheta \leq 1$.

Let us consider the nonlinear fractional-order difference system:

$${}_{\hbar}^C \Delta_a^{\vartheta} z(t) = \psi(t + \hbar\vartheta, z(t + \hbar\vartheta)), \quad t \in (\hbar\mathbb{N})_{a+\vartheta\hbar}. \quad (7)$$

Theorem 2 ([38]). Assume that the equilibrium point of (21) is z^* . z^* is asymptotically stable if all of the eigenvalues of $\psi'(z^*)$ are found in S_h^θ , where

$$S_h^\theta = \left\{ w \in \mathbb{C} : |\text{Arg}(w)| > \frac{\theta\pi}{2} \quad \text{or} \quad |w| > \frac{2^\theta}{h^\theta} \cos^\theta \left(\frac{\text{Arg}(w)}{\theta} \right) \right\}. \tag{8}$$

Theorem 3 ([34]). Assume that the equilibrium point of the system (21) is $z = 0$. If there is a positively definite and decreasing scalar function, the equilibrium point is asymptotically stable, such that ${}^C_h \Delta_a^\theta V(t, z(t)) \leq 0$.

3. The Fractional Discrete Glycolysis Reaction–Diffusion System

The model under consideration is approximated employing two commonly used approaches in this section. This is, to our knowledge, the first discrete model in the literature. The glycolysis reaction–diffusion system, as previously stated, was provided as a model. Refs. [4,5] cite it as follows:

$$\begin{cases} \frac{\partial z}{\partial t} = d_1 \Delta z + bw - z + z^2 w, & z \in \Omega, t > 0, \\ \frac{\partial w}{\partial t} = d_2 \Delta w + a - bw - z^2 w, & z \in \Omega, t > 0, \\ \partial_z = \partial_w = 0, & z \in \partial\Omega, t > 0, \\ u(z, 0) = z_0(z) > 0, \quad w(z, 0) = w_0(z) > 0, & z \in \Omega. \end{cases} \tag{9}$$

In this case, the variables $z(z, t)$ and $w(z, t)$ denote the chemical concentrations, d_1 and d_2 are the diffusion coefficients, a is the dimensionless input flow, and b is the dimensionless constant rate for the low activity state. Ω is a bounded domain in \mathbb{R}^n with sufficiently smooth boundaries. For a two-cell system, this model has some stability findings (see [39,40]). The Sel’klov model, which has attracted a lot of interest recently ([41–43]), is known to exist for $b = 0$.

Researchers have investigated time-fractional systems extensively; hence, the following fractional-time Glycolysis reaction-diffusion system was presented:

$$\begin{cases} {}^C_0 D_t^\delta z - d_1 \Delta z = bw - z + z^2 w, & z \in \Omega, t > 0 \\ {}^C_0 D_t^\delta w - d_2 \Delta w = a - bw - z^2 w, & z \in \Omega, t > 0, \end{cases} \tag{10}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$, $0 < \delta \leq 1$ is the fractional order, and ${}^C_0 D_t^\delta$ describes the Caputo fractional derivative. d_1, d_2 , and σ are strictly positive constants with the same initial conditions and Neumann boundary conditions.

Relying on the model (10) and the method of discretization employed in [26], provided that $z \in [0, L]$, we obtain $z_{i+1} = z_i + k$, $i = 0, \dots, m$, and by applying (1), $\frac{\partial^2 z(z,t)}{\partial z^2}$ and $\frac{\partial^2 w(z,t)}{\partial z^2}$ may be approximated as

$$\begin{cases} \frac{\partial^2 z(z, t)}{\partial z^2} \approx \frac{z_{i+1}(t) - 2z_i(t) + z_{i-1}(t)}{k^2}, \\ \frac{\partial^2 w(z, t)}{\partial z^2} \approx \frac{w_{i+1}(t) - 2w_i(t) + w_{i-1}(t)}{k^2}. \end{cases}$$

By defining the second-order difference operator of z_i and w_i , we obtain

$$\begin{cases} \frac{\partial^2 z(z, t)}{\partial z^2} \approx \frac{\Delta^2 z_{i-1}(t)}{k^2}, \\ \frac{\partial^2 w(z, t)}{\partial z^2} \approx \frac{\Delta^2 w_{i-1}(t)}{k^2}. \end{cases}$$

Thus, we introduce the following discrete-time reaction–diffusion fractional glycolysis system:

$$\begin{cases} {}^C_{\hbar} \Delta_{t_0}^{\vartheta} z_i(t) = \frac{d_1}{k^2} \Delta^2 z_{i-1}(t + \hbar\vartheta) + b w_i(t + \hbar\vartheta) - z_i(t + \hbar\vartheta) + z_i^2(t + \hbar\vartheta) w_i(t + \hbar\vartheta), \\ {}^C_{\hbar} \Delta_{t_0}^{\vartheta} w_i(t) = \frac{d_2}{k^2} \Delta^2 w_{i-1}(t + \hbar\vartheta) + a - b w_i(t + \hbar\vartheta) - z_i^2(t + \hbar\vartheta) w_i(t + \hbar\vartheta), \end{cases} \quad (11)$$

with the periodic boundary conditions

$$\begin{cases} z_0(t) = z_m(t), & z_1(t) = z_{m+1}(t), \\ w_0(t) = w_m(t), & w_1(t) = w_{m+1}(t), \end{cases} \quad (12)$$

and the initial condition

$$z_i(t_0) = \phi_1(z_i) \geq 0, \quad w_i(t_0) = \phi_2(z_i) \geq 0.$$

4. Local Stability

To analyze the asymptotic stability of the previously stated discrete fractional glycolysis model, we consider the single equilibrium produced by this system:

$$\begin{cases} \frac{d_1}{k^2} \Delta^2 z^* + b w^* - z^* + z^{2*} w^* = 0, \\ \frac{d_2}{k^2} \Delta^2 w^* + a - b w^* - z^{2*} w^* = 0. \end{cases} \quad (13)$$

Considering Theorem 1, the unique equilibrium point of system (11) is provided by

$$(z^*, w^*) = \left(a, \frac{a}{a^2 + b} \right). \quad (14)$$

4.1. Local Stability of the Free Diffusions System

In this section, we provide sufficient conditions for the system's local asymptotic stability.

$$\begin{cases} {}^C_{\hbar} \Delta_{t_0}^{\vartheta} z(t) = b w(t + \hbar\vartheta) - w(t + \hbar\vartheta) + u^2(t + \hbar\vartheta) w(t + \hbar\vartheta), \\ {}^C_{\hbar} \Delta_{t_0}^{\vartheta} w(t) = a - b z(t + \hbar\vartheta) - z^2(t + \hbar\vartheta) w(t + \hbar\vartheta). \end{cases} \quad (15)$$

Theorem 4. *The conditions that follow must be satisfied for system (15) to be locally asymptotically stable at the positive steady state (z^*, w^*) :*

- If $\left(\frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2) \right)^2 \geq 4(a^2 + b^2)$ and $\frac{a^2 - b^2}{a^2 + b^2} < a^2 + b^2$.
- If $\left(\frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2) \right)^2 < 4(a^2 + b^2)$ and $\frac{a^2 - b^2}{a^2 + b^2} \leq a^2 + b^2$.

Proof. The characteristic equation for the eigenvalues is discovered around this stable state using linear stability analysis:

$$J(z^*, w^*) = \begin{pmatrix} \frac{\partial \psi}{\partial z} & \frac{\partial \psi}{\partial w} \\ \frac{\partial \Psi}{\partial z} & \frac{\partial \Psi}{\partial w} \end{pmatrix} = \begin{pmatrix} \frac{2a^2}{a^2 + b^2} - 1 & a^2 + b^2 \\ -\frac{2a^2}{a^2 + b^2} & -(a^2 + b^2) \end{pmatrix}, \quad (16)$$

with

$$\psi(z, w) = b w(t + \hbar\vartheta) - z(t + \hbar\vartheta) + z^2(t + \hbar\vartheta) w(t + \hbar\vartheta), \quad (17)$$

and

$$\Psi(z, w) = a - b w(t + \hbar\vartheta) - z^2(t + \hbar\vartheta) w(t + \hbar\vartheta). \quad (18)$$

From the Jacobian matrix, we can see that

$$\text{tr}(J) = \frac{2a^2}{a^2 + b^2} - (a^2 + b^2) - 1, \quad \det(J) = a^2 + b^2, \tag{19}$$

and $J(\mathfrak{z}^*, \mathfrak{w}^*)$ has the characteristic equation given by

$$\Lambda^2 - \text{tr}(J)\Lambda + \det(J) = 0, \tag{20}$$

or its discriminant is

$$\mathfrak{N}_\Lambda = \text{tr}^2(J) - 4\det(J) = \left(\frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2) \right)^2 - 4(a^2 + b^2). \tag{21}$$

As a consequence, we may conclude the following according to the sign of (21).

- If $\mathfrak{N}_\Lambda > 0$, it is evident that $\det(J) > 0$. Thus, the sign of $\text{tr}(J)$ determines the eigenvalues' negativity, and the eigenvalues Λ_1 and Λ_2 are real and may be expressed as

$$\Lambda_1 = \frac{\text{tr}(J) - \sqrt{\mathfrak{N}_\Lambda}}{2}, \quad \Lambda_2 = \frac{\text{tr}(J) + \sqrt{\mathfrak{N}_\Lambda}}{2}. \tag{22}$$

- If $\text{tr}(J) < 0$, we have

$$\Lambda_1 = \frac{\text{tr}(J) - \sqrt{\mathfrak{N}_\Lambda}}{2} < 0. \tag{23}$$

Hence, $\text{Arg}(\Lambda_1) = \pi$. The two eigenvalues are real; therefore, $\text{Arg}(\Lambda_1) = \text{Arg}(\Lambda_2) = \pi$. As a result, $(\mathfrak{z}^*, \mathfrak{w}^*)$ is asymptotically stable according to Theorem 2.

- If $\text{tr}(J) > 0$, then we have

$$\Lambda_2 = \frac{\text{tr}(J) + \sqrt{\mathfrak{N}_\Lambda}}{2} > 0. \tag{24}$$

Therefore, $\text{Arg}(\Lambda_2) = 0$, and based on Theorem 2, system (15) is unstable.

- If $\mathfrak{N}_\Lambda < 0$, then

$$\Lambda_1 = \frac{\text{tr}(J) - i\sqrt{-\mathfrak{N}_\Lambda}}{2}, \quad \Lambda_2 = \frac{\text{tr}(J) + i\sqrt{-\mathfrak{N}_\Lambda}}{2}. \tag{25}$$

We may examine the solutions relying on the sign of $\text{tr}(J)$.

- If $\text{tr}(J) < 0$ or $\text{tr}(J) > 0$, consequently, system (15) is asymptotically stable following the same case that was previously analyzed.
- If $\text{tr}(J) = 0$, then

$$\text{Arg}\left(\frac{-i\sqrt{-\mathfrak{N}_\Lambda}}{2}\right) = \text{Arg}\left(\frac{i\sqrt{-\mathfrak{N}_\Lambda}}{2}\right) = \frac{\pi}{2},$$

and system (15) is asymptotically stable.

- If $\mathfrak{N}_\Lambda = 0$, it is impossible for $\text{tr}(J)$ to equal zero, since $\det(J) > 0$. The sign of the eigenvalues and that of $\text{tr}(J)$ are the same. As a consequence, $(\mathfrak{z}^*, \mathfrak{w}^*)$ is unstable if $\text{tr}(J) > 0$, and it is asymptotically stable for all $\vartheta \in (0, 1]$, if $\text{tr}(J) < 0$.

The proof is complete. \square

4.2. Local Stability of the Diffusion System

The steady state $(\mathfrak{z}^*, \mathfrak{w}^*)$ may be stable in the presence of diffusion under specific parameter conditions. We follow the same procedure as in [44], beginning by examining the eigenvalues of this equation:

$$\Delta^2 \mathfrak{z}_{i-1}(t + \hbar\vartheta) + \Lambda_i \mathfrak{z}_i(t + \hbar\vartheta) = 0, \tag{26}$$

with the periodic boundary conditions:

$$z_0(t) = z_m(t), \quad z_1(t) = z_{m+1}(t). \tag{27}$$

We obtain

$$\begin{cases} {}^C \Delta_{t_0}^\vartheta z_i(t) = \frac{d_1}{k^2} \Lambda_i z_i(t + \hbar\vartheta) + b w_i(t + \hbar\vartheta) - z_i(t + \hbar\vartheta) + z_i^2(t + \hbar\vartheta) w_i(t + \hbar\vartheta), \\ {}^C \Delta_{t_0}^\vartheta w_i(t) = \frac{d_2}{k^2} \Lambda_i w_i(t + \hbar\vartheta) + a - b w_i(t + \hbar\vartheta) - z_i^2(t + \hbar\vartheta) w_i(t + \hbar\vartheta). \end{cases} \tag{28}$$

When linearizing system (28) about the steady state (z^*, w^*) , we obtain the following.

$$J_i = \begin{pmatrix} -\frac{d_1}{k^2} \Lambda_i + \frac{2a^2}{a^2 + b^2} - 1 & a^2 + b^2 \\ -\frac{2a^2}{a^2 + b^2} & -\frac{d_2}{k^2} \Lambda_i - (a^2 + b^2) \end{pmatrix}. \tag{29}$$

Then, the following result is obtained.

Theorem 5. We suppose that

$$\left(\frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2) \right)^2 > 4(a^2 + b^2) \quad \text{and} \quad \frac{a^2 - b^2}{a^2 + b^2} < a^2 + b^2;$$

system (11) is asymptotically stable at the steady state (z^*, w^*) , if the following hold:

- If $d_1 < d_2$ and $\frac{d_1}{k^2} \Lambda_i \geq \frac{2a^2}{a^2 + b^2} - 1$.
- If $d_1 > d_2$ and $\frac{d_1}{k^2} \Lambda_i \geq \frac{2a^2}{a^2 + b^2} - 1$,
and in addition, the eigenvalues

$$\mu_j(\Lambda_i) = \frac{\text{tr}(J_i) \pm \sqrt{\text{tr}(J_i)^2 - 4\text{det}(J_i)}}{2}, \quad j = 1, 2,$$

satisfy $\text{Arg}(\mu_j(\Lambda_i)) > \frac{\vartheta\pi}{2}$.

Proof. We linearize the system to investigate its local asymptotic stability. By applying basic linear operator theory and taking the system’s fractional nature into account, we may argue that (u^*, v^*) is asymptotically stable if the eigenvalues of the linearized system satisfy the conditions of Theorem 2. We obtain

$$J_i = \begin{pmatrix} 6\frac{d_1}{k^2} \Lambda_i + \frac{2a^2}{a^2 + b^2} - 1 & a^2 + b^2 \\ -\frac{2a^2}{a^2 + b^2} & 6\frac{d_2}{k^2} \Lambda_i - (a^2 + b^2) \end{pmatrix} = J - \mu(\Lambda_i)I, \tag{30}$$

which has the eigenvalue equation

$$\mu^2(\Lambda_i) - \text{tr}(J_i)\mu(\Lambda_i) + \text{det}(J_i) = 0, \tag{31}$$

where

$$\text{tr}(J_i) = -\left(\frac{d_1}{k^2} + \frac{d_2}{k^2}\right) \Lambda_i + \frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2), \tag{32}$$

and

$$\text{det}(J_i) = \frac{d_1}{k^2} \frac{d_2}{k^2} \Lambda_i^2 + \left(\frac{d_1}{k^2}(a^2 + b^2) - \frac{d_2}{k^2} \left(\frac{a^2 - b^2}{a^2 + b^2}\right)\right) \Lambda_i + a^2 + b^2, \tag{33}$$

and its discriminant is

$$\begin{aligned}
\mathfrak{N}_i &= \text{tr}^2(J_i) - 4\det(J_i) \\
&= \left(-\left(\frac{d_1}{k^2} + \frac{d_2}{k^2}\right)\Lambda_i + \frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2) \right)^2 \\
&\quad - 4\left(\frac{d_1 d_2}{k^2 k^2}\Lambda_i^2 + \left(\frac{d_1}{k^2}(a^2 + b^2) - \frac{d_2}{k^2}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)\right)\Lambda_i + a^2 + b^2\right), \\
&= \left(\frac{d_1}{k^2} - \frac{d_2}{k^2}\right)^2 \Lambda_i^2 - 2\left(\frac{d_1}{k^2} - \frac{d_2}{k^2}\right)\left(\frac{a^2 - b^2}{a^2 + b^2} + (a^2 + b^2)\right)\Lambda_i + \Delta_\Lambda.
\end{aligned}$$

The stability of $(\mathfrak{z}^*, \mathfrak{w}^*)$ depends on the sign of $\Delta 1_i$. The discriminant of $\Delta 1_i$ in relation to Λ_i is

$$\begin{aligned}
\mathfrak{N}_{\Lambda_i} &= 4\left(\frac{d_1}{k^2} - \frac{d_2}{k^2}\right)^2 \left(\frac{a^2 - b^2}{a^2 + b^2} + (a^2 + b^2)\right)^2 \\
&\quad - 4\left(\frac{d_1}{k^2} - \frac{d_2}{k^2}\right)^2 \left(\left(\frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2)\right)^2 - 4(a^2 + b^2)\right) \\
&= 32\left(a\left(\frac{d_1}{k^2} - \frac{d_2}{k^2}\right)\right)^2.
\end{aligned}$$

Clearly, $\mathfrak{N}_{\Lambda_i} > 0$, and because $d_1 \neq d_2$, we distinguish two cases,

- If $d_1 < d_2$, then $\left(\frac{a^2 - b^2}{a^2 + b^2} - (a^2 + b^2)\right)^2 > 4(a^2 + b^2)$, and the two solutions of the equation $\mathfrak{N}_{\Lambda_i} = 0$ are both negative. Thus, $\mathfrak{N}_{\Lambda_i} > 0$, and the roots of (31) are

$$\begin{cases} \mu_1(\Lambda_i) = \frac{\text{tr}(J_i) + \sqrt{\text{tr}(J_i)^2 - 4\det(J_i)}}{2}, \\ \mu_2(\Lambda_i) = \frac{\text{tr}(J_i) - \sqrt{\text{tr}(J_i)^2 - 4\det(J_i)}}{2}. \end{cases} \quad (34)$$

It should be noted that the solutions are real, and $\mu(\Lambda_i)_1 < 0$. In addition, if $\frac{d_1}{k^2}\Lambda_1 \geq \frac{2a^2}{a^2 + b^2} - 1$, then $\mu(\Lambda_i)_2 < 0$. This leads to

$$|\text{Arg}(\mu_1(\Lambda_i))| = |\text{Arg}(\mu_2(\Lambda_i)_2)| = \pi, \quad (35)$$

which ensures $(\mathfrak{z}^*, \mathfrak{w}^*)$ is asymptotic stability.

- If $d_1 > d_2$, we have $\frac{d_1}{k^2}\Lambda_1 \geq \frac{2a^2}{a^2 + b^2} - 1$. This returns us to the previous scenario. Again, for $\frac{d_1}{k^2}\Lambda_1 - \frac{3\left(\frac{a}{b}\right)^2 - 5}{1 + \left(\frac{a}{b}\right)^2}$, $\det(J_i) > 0$; hence, λ_1 and λ_2 are negative and must meet the conditions of Theorem 2.

□

5. Global Stability

In this section, we show how the constant steady-state solution is globally asymptotically stable.

Theorem 6. *If*

$$-b\mathfrak{w}^* + 2\mathfrak{w}^*\mathfrak{z}^{*2} \leq 0, \quad (36)$$

then system (11) is globally asymptotically stable.

Proof. To demonstrate this result, we employ the identical Lyapunov function as in [28]. We consider the following function for this purpose:

$$l(z) = z - 1 - \ln(z),$$

where this function has a strict global minimum, i.e., $l(1) = 0$. Let the following be a nonnegative function:

$$\mathfrak{L}(t) = \mathfrak{L}_1(t) + \mathfrak{L}_2(t),$$

where

$$\mathfrak{L}_1(t) = \sum_{i=1}^m \mathfrak{z}^* l\left(\frac{\mathfrak{z}_i}{\mathfrak{z}^*}\right), \quad \mathfrak{L}_2(t) = \sum_{i=1}^m \mathfrak{w}^* l\left(\frac{\mathfrak{w}_i}{\mathfrak{w}^*}\right).$$

First, we calculate $L_1(t)$

$$\begin{aligned} {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}_1(t) &= \sum_{i=1}^m {}^C_{\hbar} \Delta_{t_0}^{\gamma} \mathfrak{z}^* l\left(\frac{\mathfrak{z}_i(t)}{\mathfrak{z}^*}\right), \\ &\leq \sum_{i=1}^m \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) {}^C_{\hbar} \Delta_{t_0}^{\gamma} \mathfrak{z}_i(t), \\ &\leq \sum_{i=1}^m \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) \left(\frac{d_1}{k^2} \Delta^2 \mathfrak{z}_{i-1}(t + \hbar\vartheta) + b\mathfrak{w}_i(t + \hbar\vartheta) - \mathfrak{z}_i(t + \hbar\vartheta)\right) \\ &\quad + \mathfrak{z}_i^2(t + \hbar\vartheta)\mathfrak{w}_i(t + \hbar\vartheta), \\ &\leq \sum_{i=1}^m \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) (b\mathfrak{w}_i(t + \hbar\vartheta) - b\mathfrak{w}^* - \mathfrak{z}_i(t + \hbar\vartheta) + \mathfrak{z}^* \\ &\quad + \mathfrak{z}_i^2(t + \hbar\vartheta)\mathfrak{w}_i(t + \hbar\vartheta) - \mathfrak{z}^{2*}\mathfrak{w}^*) \\ &\quad + \frac{d_1}{k^2} \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) (\mathfrak{z}_{i+1}(t + \hbar\vartheta) - 2\mathfrak{z}_i(t + \hbar\vartheta) + \mathfrak{z}_{i-1}(t + \hbar\vartheta)). \end{aligned}$$

Using (6), we obtain

$$\begin{aligned} {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}_1(t) &\leq \sum_{i=1}^m b\mathfrak{w}^* \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) \left(1 - \frac{\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{w}^*}\right) \\ &\quad - \mathfrak{z}^{2*}\mathfrak{w}^* \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) \left(1 - \frac{\mathfrak{z}_i^2(t + \hbar\vartheta)\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{z}^{2*}\mathfrak{w}^*}\right) \\ &\quad + \mathfrak{z}^* \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) \left(1 - \frac{\mathfrak{z}_i(t + \hbar\vartheta)}{\mathfrak{z}^*}\right) + \sum_{i=1}^m \frac{d_1}{k^2} (\mathfrak{z}_{i+1}(t + \hbar\vartheta) \\ &\quad - 2\mathfrak{z}_i(t + \hbar\vartheta) + \mathfrak{z}_{i-1}(t + \hbar\vartheta)) - \frac{d_1}{k^2} \mathfrak{z}^* \left(\frac{\mathfrak{z}_{i+1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)} + \frac{\mathfrak{z}_{i-1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)} - 2\right), \\ &\leq \sum_{i=1}^m -b\mathfrak{w}^* \left(l\left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) + l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{w}^*}\right) - l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}^*}{\mathfrak{w}^*\mathfrak{z}_i(t + \hbar\vartheta)}\right)\right) \\ &\quad + \mathfrak{z}^{2*}\mathfrak{w}^* \left(l\left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) - l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}_i(t + \hbar\vartheta)}{\mathfrak{z}^*\mathfrak{w}^*}\right) + l\left(\frac{\mathfrak{z}^2(t + \hbar\vartheta)\mathfrak{w}(t + \hbar\vartheta)}{\mathfrak{z}^{2*}\mathfrak{w}^*}\right)\right) \\ &\quad - \mathfrak{z}^* \left(l\left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) + l\left(\frac{\mathfrak{z}_i(t + \hbar\vartheta)}{\mathfrak{z}^*}\right)\right) \\ &\quad + \frac{d_1}{k^2} (\mathfrak{z}_{m+1}(t + \hbar\vartheta) - 2\mathfrak{z}_m(t + \hbar\vartheta) + \mathfrak{z}_0(t + h\gamma) - \mathfrak{z}_1(t + \hbar\vartheta)) \\ &\quad - \mathfrak{z}^* \frac{d_1}{k^2} \left(\frac{\mathfrak{z}_{i+1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{z}_{i-1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)}\right), \\ &\leq \sum_{i=1}^m (-b\mathfrak{w}^* + \mathfrak{z}^{2*}\mathfrak{w}^* - \mathfrak{z}^*) l\left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) - b\mathfrak{w}^* l\left(\frac{\mathfrak{z}_i(t + \hbar\vartheta)}{\mathfrak{z}^*}\right) - \mathfrak{z}^* l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{w}^*}\right) \\ &\quad + \mathfrak{z}^{2*}\mathfrak{w}^* l\left(\frac{\mathfrak{z}^2(t + \hbar\vartheta)\mathfrak{w}(t + \hbar\vartheta)}{\mathfrak{z}^{2*}\mathfrak{w}^*}\right) - \mathfrak{z}^{2*}\mathfrak{w}^* l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}_i(t + \hbar\vartheta)}{\mathfrak{z}^*\mathfrak{w}^*}\right) + \mathfrak{z}^* l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}^*}{\mathfrak{w}^*\mathfrak{z}_i(t + \hbar\vartheta)}\right) \\ &\quad - \mathfrak{z}^* \frac{d_1}{k^2} \left(\frac{\mathfrak{z}_{i+1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{z}_{i-1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)}\right). \end{aligned}$$

Now, we have

$$\begin{aligned}
 {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}_2(t) &= \sum_{i=1}^m {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{w}^* l\left(\frac{\mathfrak{w}_i(t)}{\mathfrak{w}^*}\right), \\
 &\leq \sum_{i=1}^m \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{w}_i(t), \\
 &\leq \sum_{i=1}^m \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) \left(\frac{d_2}{k^2} \Delta^2 \mathfrak{w}_{i-1}(t + \hbar\vartheta) + a - b\mathfrak{w}_i(t + \hbar\vartheta) - \mathfrak{z}_i^2(t + \hbar\vartheta)\mathfrak{w}_i(t + \hbar\vartheta)\right), \\
 &\leq \sum_{i=1}^m \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) \left(b\mathfrak{w}^* + \mathfrak{z}^{2*}\mathfrak{w}^* - b\mathfrak{w}_i(t + \hbar\vartheta) - \mathfrak{z}_i^2(t + \hbar\vartheta)\mathfrak{w}_i(t + \hbar\vartheta)\right), \\
 &+ \frac{d_2}{k^2} \sum_{i=1}^m \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) (\mathfrak{z}_{i+1}(t) - 2\mathfrak{z}_i(t) + \mathfrak{z}_{i-1}(t)) + o(1), \\
 &\leq \sum_{i=1}^m b\mathfrak{w}^* \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) \left(1 - \frac{\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{w}^*}\right) \\
 &+ \mathfrak{w}^* \mathfrak{z}^{2*} \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + h\gamma)}\right) \left(1 - \frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}_i^2(t + \hbar\vartheta)}{\mathfrak{w}^* u^{2*}}\right) \\
 &+ \sum_{i=1}^m \frac{d_2}{k^2} (\mathfrak{w}_{i+1}(t + \hbar\vartheta) - 2\mathfrak{w}_i(t + \hbar\vartheta) + \mathfrak{w}_{i-1}(t + \hbar\vartheta)) \\
 &- \frac{d_2}{k^2} \mathfrak{w}^* \left(\frac{\mathfrak{w}_{i+1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)} + \frac{\mathfrak{w}_{i-1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)} - 2\right), \\
 &\leq \sum_{i=1}^m -b\mathfrak{w}^* \left(l\left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) + l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{w}^*}\right)\right) \\
 &+ \mathfrak{w}^* \mathfrak{z}^{2*} \left(-l\left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) + l\left(\frac{\mathfrak{z}^{2*}}{\mathfrak{z}_i^2(t + \hbar\vartheta)}\right) - l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}_i^2(t + \hbar\vartheta)}{\mathfrak{w}^* \mathfrak{z}^{2*}}\right)\right) \\
 &+ \frac{d_2}{k^2} (\mathfrak{w}_{m+1}(t + \hbar\vartheta) - 2\mathfrak{w}_m(t + \hbar\vartheta) + \mathfrak{w}_0(t + \hbar\vartheta) - \mathfrak{w}_1(t + \hbar\vartheta)) \\
 &- \mathfrak{w}^* \frac{d_2}{k^2} \left(\frac{\mathfrak{w}_{i+1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{w}_{i-1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)}\right), \\
 &\leq \sum_{i=1}^m (-b\mathfrak{w}^* - \mathfrak{w}^* \mathfrak{z}^{2*}) l\left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) - b\mathfrak{w}^* l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{w}^*}\right) + \mathfrak{w}^* \mathfrak{z}^{2*} l\left(\frac{\mathfrak{z}_i^2(t + \hbar\vartheta)}{\mathfrak{z}^{2*}}\right) \\
 &- \mathfrak{w}^* \mathfrak{z}^{2*} l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}_i^2(t + \hbar\vartheta)}{\mathfrak{w}^* \mathfrak{z}^{2*}}\right) - \mathfrak{w}^* \frac{d_2}{k^2} \left(\frac{\mathfrak{w}_{i+1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{w}_{i-1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)}\right).
 \end{aligned}$$

According to [28], we have

$$\begin{aligned}
 \frac{\mathfrak{z}_{i+1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{z}_{i-1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)} &\geq 0, \\
 \frac{\mathfrak{w}_{i+1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{w}_{i-1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)} &\geq 0.
 \end{aligned}$$

We conclude that

$$\begin{aligned}
 {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}(t) &= {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}_1(t) + {}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}_2(t), \\
 &\leq \sum_{i=1}^m (-b\mathfrak{w}^* + \mathfrak{z}^{2*}\mathfrak{w}^* - \mathfrak{z}^*) l\left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) + (-b\mathfrak{w}^* + 2\mathfrak{w}^*\mathfrak{z}^{*2}) l\left(\frac{\mathfrak{z}_i(t + \hbar\vartheta)}{\mathfrak{z}^*}\right) \\
 &\quad - (b\mathfrak{w}^* + \mathfrak{z}^*) l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)}{\mathfrak{w}^*}\right) - (b\mathfrak{w}^* + \mathfrak{w}^*\mathfrak{z}^{*2}) l\left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \hbar\vartheta)}\right) \\
 &\quad - \mathfrak{z}^{2*}\mathfrak{w}^* l\left(\frac{\mathfrak{w}_i(t + \hbar\vartheta)\mathfrak{z}_i(t + \hbar\vartheta)}{\mathfrak{z}^*\mathfrak{w}^*}\right) - \mathfrak{z}^* \frac{d_1}{k^2} \left(\frac{\mathfrak{z}_{i+1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{z}_{i-1}(t + \hbar\vartheta)}{\mathfrak{z}_i(t + \hbar\vartheta)}\right) \\
 &\quad - \mathfrak{w}^* \frac{d_2}{k^2} \left(\frac{\mathfrak{w}_{i+1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)} - 2 + \frac{\mathfrak{w}_{i-1}(t + \hbar\vartheta)}{\mathfrak{w}_i(t + \hbar\vartheta)}\right).
 \end{aligned}$$

Hence, since $-b\mathfrak{w}^* + 2\mathfrak{w}^*\mathfrak{z}^{*2} \leq 0$, then,

$${}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}(t) \leq 0$$

and ${}^C_{\hbar} \Delta_{t_0}^{\vartheta} \mathfrak{L}(t) = 0$ if and only if $(\mathfrak{z}_i, \mathfrak{w}_i) = (0, 0)$. $(\mathfrak{z}^*, \mathfrak{w}^*)$ is therefore globally asymptotically stable under condition (36) according to Theorem 3. \square

6. Simulations

Here, we demonstrate several simulations that illustrate certain theoretical aspects of the stability of the discrete fractional glycolysis reaction–diffusion system. Matlab was used to carry out all of the simulations in this section.

By changing the characteristics and order of the system, we are able to observe how it behaves. When we use the numerical solution shown below, system (11) looks like:

$$\left\{ \begin{aligned}
 \mathfrak{z}_i(n\hbar) &= \phi_1(\mathfrak{z}_i) + \frac{\hbar^\vartheta}{\Gamma(\vartheta)} \sum_{\tau=1}^n \frac{\Gamma(n - \tau + \vartheta)}{\Gamma(n - \tau + 1)} [d_1 \frac{\mathfrak{z}_{i+1}((\tau - 1)\hbar) - 2\mathfrak{z}_i((\tau - 1)\hbar) + \mathfrak{z}_{i-1}((\tau - 1)\hbar)}{k^2} \\
 &\quad + b\mathfrak{z}_i((\tau - 1)\hbar) + \mathfrak{z}_i^2((\tau - 1)\hbar)\mathfrak{w}_i((\tau - 1)\hbar)], \\
 \mathfrak{w}_i(n\hbar) &= \phi_2(\mathfrak{z}_i) + \frac{\hbar^\vartheta}{\Gamma(\vartheta)} \sum_{\tau=1}^n \frac{\Gamma(n - \tau + \vartheta)}{\Gamma(n - \tau + 1)} [d_2 \frac{\mathfrak{w}_{i+1}((\tau - 1)\hbar) - 2\mathfrak{w}_i((\tau - 1)\hbar) + \mathfrak{w}_{i-1}((\tau - 1)\hbar)}{k^2} \\
 &\quad + a - b\mathfrak{z}_i((\tau - 1)\hbar) - \mathfrak{w}_i((\tau - 1)\hbar) - \mathfrak{z}_i^2((\tau - 1)\hbar)\mathfrak{w}_i((\tau - 1)\hbar)], \\
 1 &\leq i \leq m, \\
 n &> 0.
 \end{aligned} \right. \tag{37}$$

Example 1. To demonstrate our point, we consider the following parameter values: $(d_1, d_2, a, b) = (1, 2, 0.1, 0.01)$, $N = 100$, $\hbar = 0.1$, $t \in [0, 10]$, and $\mathfrak{z} \in [0, 20]$ and the boundary conditions $(\mathfrak{z}_0(t), \mathfrak{w}_0(t)) = (3, 1)$, $(\mathfrak{z}_1(t), \mathfrak{w}_1(t)) = (3, 1)$. Also, we provide the following appropriate initial conditions:

$$\begin{cases} \phi_1(\mathfrak{z}_i) = 3.5 + \frac{\cos(\pi\mathfrak{z}_i)}{5}, \\ \phi_2(\mathfrak{z}_i) = 2 + \frac{\cos(\pi\mathfrak{z}_i)}{5}. \end{cases}$$

We can see that all of the solutions to our model ultimately reach the same unique positive equilibrium $(\mathfrak{z}^*, \mathfrak{w}^*) = (0.1, 4)$. The unique equilibrium is hence asymptotically stable. For a , which denotes the dimensionless input flow, and b , which denotes the dimensionless constant rate for the low activity state, the conditions in Theorem 6 are clearly demonstrated, and this numerical finding is consistent with our earlier theoretical conclusions.

The results stated previously for various fractional orders are supported by Figures 1–3, which describe the dynamic behavior of the system (11) and show that as the fractional order ϑ approaches 1, the model’s solution becomes more stable as it rapidly converges to the unique equilibrium.

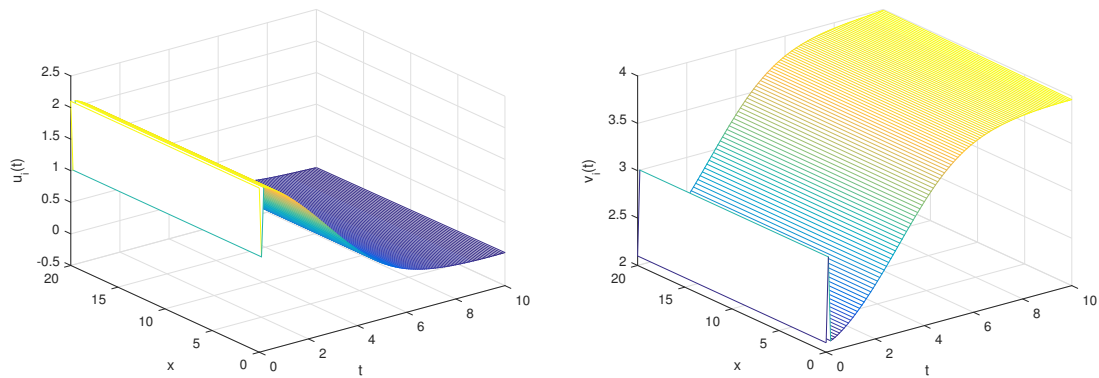


Figure 1. The state trajectories of $r \delta_i(t)$ and $w_i(t)$ for $\vartheta = 0.1$.

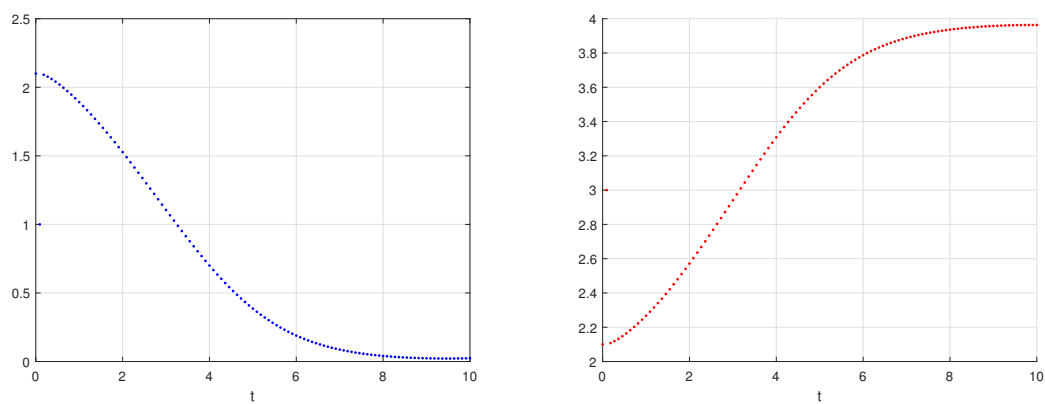


Figure 2. The dynamic behaviors of $\delta_i(t)$ and $w_i(t)$ for $N = 10$, $(a, b, d_1, d_2) = (1, 2, 0.1, 0.01)$ and $\vartheta = 0.1$.

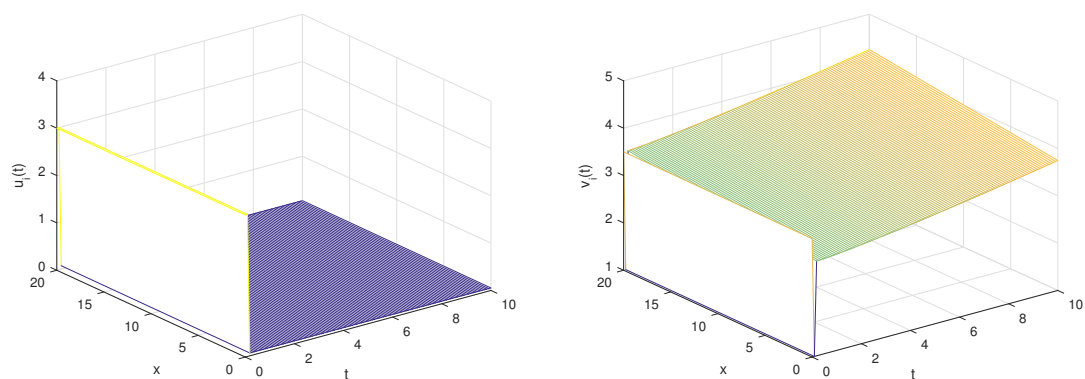


Figure 3. The dynamic behaviors of $\delta_i(t)$ and $w_i(t)$ for $N = 10$, $(a, b, d_1, d_2) = (1, 2, 0.1, 0.01)$ and $\vartheta = 0.9$.

7. Conclusions

In this study, a discrete equivalent of the fractional reaction–diffusion glycolysis model was investigated using the second-order central difference and L1 finite difference schemes. This unique approach’s capacity to keep the system confined and positive should be emphasized. Moreover, the local stability of the discrete system was explored. The results were consistent with those of the continuous system that serves as a continuous analog. This suggests that the discrete system generated by the considered schemes can preserve as continuous system characteristics like the positivity, boundedness, and global stability of the equilibrium point, which are first evaluated as a way to assess the stability of

discrete fractional reaction diffusion systems using Lyapunov functions. Additionally, computations and simulations show the effectiveness and viability of the suggested system. In addition, the Lyapunov functional and the linearization approach may be utilized to solve the stability issue in discrete fractional reaction–diffusion models. More importantly, the results of this study might be easily extended to a variety of reaction–diffusion discrete fractional spatiotemporal systems, as well as other dynamical problems including chaos and synchronization control.

Author Contributions: Formal analysis, T.H. and O.A.; Investigation, A.H.; Software, A.H.; Supervision, A.O.; Validation, Y.A.A.-K.; Visualization, A.A.-H.; Writing—original draft, A.H.; Writing—review and editing, A.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Mojikon, F.D.; Kasimin, M.E.; Molujin, A.M.; Gansau, J.A.; Jawan, R. Probiotication of Nutritious Fruit and Vegetable Juices: An Alternative to Dairy-Based Probiotic Functional Products. *Nutrients* **2022**, *14*, 3457. [[CrossRef](#)] [[PubMed](#)]
- Romano, A.H.; Conway, T. Evolution of carbohydrate metabolic pathways. *Res. Microbiol.* **1996**, *147*, 448. [[CrossRef](#)] [[PubMed](#)]
- Ouannas, A.; Batiha, I.M.; Bekiros, S.; Liu, J.; Jahanshahi, H.; Aly, A.A.; Alghtani, A.H. Synchronization of the glycolysis reaction-diffusion model via linear control law. *Entropy* **2021**, *23*, 1516. [[CrossRef](#)] [[PubMed](#)]
- Ahmed, N.; Ss, T.; Imran, M.; Rafiq, M.; Rehman, M.A.; Younis, M. Numerical analysis of auto-catalytic glycolysis model. *AIP Adv.* **2019**, *9*, 085213. [[CrossRef](#)]
- Al Noufaey, K.S. Stability analysis for Selkov-Schnakenberg reaction-diffusion system. *Open Math.* **2021**, *19*, 46–62. [[CrossRef](#)]
- Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: Hoboken, NJ, USA, 1993.
- Oldham, K.; Spanier, J. *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*; Elsevier: Amsterdam, The Netherlands, 1974.
- Dababneh, A.; Djenina, N.; Ouannas, A.; Grassi, G.; Batiha, I. M.; Jebri, I. H. A new incommensurate fractional-order discrete COVID-19 model with vaccinated individuals compartment. *Fractal Fract.*, **2022**, *6*, 456. [[CrossRef](#)]
- Saadeh, R.; Abbes, A.; Al-Husban, A.; Ouannas, A.; Grassi, G. The Fractional Discrete Predator–Prey Model: Chaos, Control and Synchronization. *Fractal Fract.* **2023**, *7*, 120. [[CrossRef](#)]
- Hilfer, R. (Ed.) *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
- Silva, M.F.; Machado, J.T. Fractional order PD δ joint control of legged robots. *J. Vib. Control* **2006**, *12*, 1483–1501. [[CrossRef](#)]
- Duarte, F.B.; Machado, J.T. Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators. *Nonlinear Dyn.* **2002**, *29*, 315–342. [[CrossRef](#)]
- Tenreiro Machado, J.A.; Silva, M.F.; Barbosa, R.S.; Jesus, I.S.; Reis, C.M.; Marcos, M.G.; Galhano, A.F. Some applications of fractional calculus in engineering. *Math. Probl. Eng.* **2010**, *2010*, 639801. [[CrossRef](#)]
- Gafiychuk, V.; Datsko, B.; Meleshko, V. Mathematical modeling of time fractional reaction–diffusion systems. *J. Comput. Appl. Math.* **2008**, *220*, 215–225. [[CrossRef](#)]
- Haubold, H.J.; Mathai, A.M.; Saxena, R.K. Further solutions of fractional reaction–diffusion equations in terms of the H-function. *J. Comput. Appl. Math.* **2011**, *235*, 1311–1316. [[CrossRef](#)]
- Che, H.; Wang, Y.L.; Li, Z.Y. Novel patterns in a class of fractional reaction–diffusion models with the Riesz fractional derivative. *Math. Comput. Simul.* **2022**, *202*, 149–163. [[CrossRef](#)]
- Belmahi, N.; Shawagfeh, N. A new mathematical model for the glycolysis phenomenon involving Caputo fractional derivative: Well posedness, stability and bifurcation. *Solitons Fractals* **2021**, *142*, 110520. [[CrossRef](#)]
- Abdel-Aty, A.H.; Khater, M.M.; Baleanu, D.; Abo-Dahab, S.M.; Bouslimi, J.; Omri, M. Oblique explicit wave solutions of the fractional biological population (BP) and equal width (EW) models. *Adv. Differ. Equ.* **2020**, *2020*, 552. [[CrossRef](#)]
- Angstmann, C.N.; Henry, B.I. Time Fractional Fisher–KPP and Fitzhugh–Nagumo Equations. *Entropy* **2020**, *22*, 1035. [[CrossRef](#)]
- Abad, E.; Yuste, S.B.; Lindenberg, K. Reaction-subdiffusion and reaction-superdiffusion equations for evanescent particles performing continuous-time random walks. *Phys. Rev. E* **2010**, *81*, 031115. [[CrossRef](#)]
- Angstmann, C.N.; Donnelly, I.C.; Henry, B.I. Continuous time random walks with reactions forcing and trapping. *Math. Model. Nat. Phenomena* **2013**, *8*, 17–27. [[CrossRef](#)]
- Klein, C.T.; Mayer, B. A model for pattern formation in gap-junction coupled cells. *J. Theor. Biol.* **1997**, *186*, 107–115. [[CrossRef](#)]
- Maselko, J. Mosaic Pattern Formations in Multicellular Chemical Systems. *J. Phys. Chem.* **1995**, *99*, 2949–2952. [[CrossRef](#)]
- Anakira, N.; Hioual, A.; Ouannas, A.; Oussaeif, T.E.; Batiha, I.M. Global Asymptotic Stability for Discrete-Time SEI Reaction-Diffusion Model. In Proceedings of the International Conference on Mathematics and Computations, Zarqa, Jordan, 11–13 May 2022; Springer Nature: Singapore, 2022; pp. 345–357.

25. Almatroud, O.A.; Hioual, A.; Ouannas, A.; Grassi, G. On Fractional-Order Discrete-Time Reaction Diffusion Systems. *Mathematics* **2023**, *11*, 2447. [[CrossRef](#)]
26. Wu, G.C.; Baleanu, D.; Zeng, S.D.; Deng, Z.G. Discrete fractional diffusion equation. *Nonlinear Dyn.* **2015**, *80*, 281. [[CrossRef](#)]
27. Mesdoui, F.; Ouannas, A.; Shawagfeh, N.; Grassi, G.; Pham, V.T. Synchronization methods for the Degrn-Harrison reaction-diffusion systems. *IEEE Access* **2020**, *8*, 91829–91836. [[CrossRef](#)]
28. Zhenzhen, L.; Yongguang, Y.; Weiyi, X.; Guojian, R.; Xiangyun, M. Global dynamics for a class of discrete fractional epidemic model with reaction-diffusion. *arXiv* **2022**, arXiv:2208.06548.
29. Bhargava, S.C. On the higgins model of glycolysis. *Bull. Math. Biol.* **1980**, *42*, 829–836. [[CrossRef](#)] [[PubMed](#)]
30. Peng, R. Qualitative analysis of steady states to the Sel'kov model. *J. Differ. Equ.* **2007**, *241*, 386–398. [[CrossRef](#)]
31. Lee, K.J.; Swinney, H.L. Replicating spots in reaction-diffusion systems. *Int. J. Bifurc. Chaos* **1997**, *7*, 1149–1158. [[CrossRef](#)]
32. You, Y. Asymptotical dynamics of Selkov equations. *Discret. Contin. Dyn. Syst.-S* **2008**, *2*, 193–219. [[CrossRef](#)]
33. Kelley, W.G.; Peterson, A.C. *Difference Equations: An Introduction with Applications*; Academic Press: Cambridge, MA, USA, 2001.
34. Baleanu, D.; Wu, G.C.; Bai, Y.R.; Chen, F.L. Stability analysis of Caputo-like discrete fractional systems. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *48*, 520–530. [[CrossRef](#)]
35. Hioual, A.; Ouannas, A.; Oussaeif, T.E.; Grassi, G.; Batiha, I.M.; Momani, S. On Variable-Order Fractional Discrete Neural Networks: Solvability and Stability. *Fractal Fract.* **2022**, *6*, 119. [[CrossRef](#)]
36. Abdeljawad, T. On Riemann and Caputo fractional differences. *Comput. Math. Appl.* **2011**, *62*, 1602–1611. [[CrossRef](#)]
37. Abdeljawad, T. Different type kernel h—Fractional differences and their fractional h—Sums. *Solitons Fractals* **2018**, *116*, 146–156. [[CrossRef](#)]
38. Cermák, J.; Nechvátal, L. On a problem of linearized stability for fractional difference equations. *Nonlinear Dyn.* **2021**, *104*, 1253–1267. [[CrossRef](#)]
39. Ashkenazi, M.; Othmer, H.G. Spatial patterns in coupled biochemical oscillators. *J. Math. Biol.* **1977**, *5*, 305–350. [[CrossRef](#)]
40. Tyson, J.; Kauffman, S. Control of mitosis by a continuous biochemical oscillation: Synchronization; spatially inhomogeneous oscillations. *J. Math. Biol.* **1975**, *1*, 289–310. [[CrossRef](#)]
41. Davidson, F.A.; Rynne, B.P. A priori bounds and global existence of solutions of the steady-state Sel'kov model. *Proc. R. Soc. Edinb. Sect. A Math.* **2000**, *130*, 507–516. [[CrossRef](#)]
42. Furter, J.E.; Eilbeck, J.C. Analysis of bifurcations in reaction–diffusion systems with no-flux boundary conditions: The Sel'kov model. *Proc. R. Soc. Edinb. Sect. A Math.* **1995**, *125*, 413–438. [[CrossRef](#)]
43. Peng, R.; Wang, M.; Yang, M. Positive steady-state solutions of the Sel'kov model. *Math. Comput. Model.* **2006**, *44*, 945–951. [[CrossRef](#)]
44. Xu, L.; Zhao, L.J.; Chang, Z.X.; Feng, J.T.; Zhang, G. Turing instability and pattern formation in a semi-discrete Brusselator model. *Mod. Phys. Lett. B* **2013**, *27*, 1350006. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.