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Correlation Structure of Time-Changed Generalized Mixed Fractional Brownian Motion

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Abstract: The generalized mixed fractional Brownian motion (gmfBm) is a Gaussian process with stationary increments that exhibits long-range dependence controlled by its Hurst indices. It is defined by taking linear combinations of a finite number of independent fractional Brownian motions with different Hurst indices. In this paper, we investigate the long-time behavior of gmfBm when it is time-changed by a tempered stable subordinator or a gamma process. As a main result, we show that the time-changed process exhibits a long-range dependence property under some conditions on the Hurst indices. The time-changed gmfBm can be used to model natural phenomena that exhibit long-range dependence, even when the underlying process is not itself long-range dependent.

Keywords: fractional Brownian motion; generalized mixed fractional Brownian motion; long-range dependence; tempered stable subordinator; gamma process

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1. Introduction

The fractional Brownian motion (fBm) $B^H = \{B_t^H, t \geq 0\}$ with parameter H, is a centered Gaussian process with the covariance function

$$Cov(B_t^H, B_s^H) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \quad s,t \geq 0, \tag{1}$$

where H is a real number in $(0,1)$, called the Hurst index. The case $H = \frac{1}{2}$ corresponds to the Brownian motion (Bm).

An extension of the fBm was introduced by Cheridito [1], called the mixed fractional Brownian motion (mfBm), which is a linear combination of a Bm and an independent fBm of Hurst index H, with stationary increments, that exhibit a long-range dependence for $H > \frac{1}{2}$. A mfBm of parameters a_1, a_2 and H is a process $M^H(a_1,a_2) = \{M_t^H(a_1,a_2),\ t \geq 0\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$M_t^H(a_1,a_2) = a_1 B_t + a_2 B_t^H, \quad t \geq 0,$$

where $B = \{B_t,\ t \geq 0\}$ is a Bm and $B^H = \{B_t^H,\ t \geq 0\}$ is an independent fBm of Hurst index $H \in (0,1)$. We refer also to [1–4] for further information on the mfBm process.

C. Elnouty [3] propose a generalization of the mfBm called fractional mixed fractional Brownian motion (fmfBm) of parameters a_1, a_2 and $H = (H_1, H_2)$. A fmfBm is a process $N^H(a_1,a_2) = \{N_t^H(a_1,a_2),\ t \geq 0\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$N_t^H(a_1,a_2) = a_1 B_t^{H_1} + a_2 B_t^{H_2}, \quad t \geq 0,$$

where $B^{H_i} = \{B_t^{H_i},\ t \geq 0\}$ are independent fBms of Hurst indices $H_i \in (0,1)$ for $i = 1,2$. In addition, the fmfBm was studied by Miao, Y et al. [5].

The fractional mixed fractional Brownian motion was further generalized by Thäle in 2009 [6] to the generalized mixed fractional Brownian motion. Let $a_1, \dots, a_n$, $n \in \mathbf{N}^*$ reel numbers and not all a_i equals zero. A generalized mixed fractional Brownian motion (gmfBm) of parameters $H = (H_1, H_2, \dots, H_n)$ and $a = (a_1, a_2, \dots, a_n)$; $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$ is a stochastic process $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$Z_t^{H,a} = \sum_{i=1}^{n} a_i B_t^{H_i},$$

where $\{B_t^{H_i},\ t \geq 0\}$ for $i = 1, 2, \dots, n$ are independent fBms of Hurst indices $H_i \in (0,1)$. The gmfBm is a generalization of both fractional Brownian motion and subfractional Brownian motion. The gmfBm is a centered Gaussian process with stationary increments that exhibits long-range dependence if and only if there exists some k with $H_k > \frac{1}{2}$, it can be used to model a wider range of natural phenomena than either fBm or sfBm. Internet traffic can be modeled using the gmfBm, as seen in [7]. The gmfBm market is a useful model for a variety of assets, including internet traffic. Internet traffic has been shown to exhibit long-range dependence, and the gmfBm model can be used to capture this dependence. The gmfBm market is a market where the underlying asset price satisfies the following stochastic differential equation:

$$dS_t = aS_t dt + bS_t dZ_t$$

where a and b stand for the standard deviation of the stock return and the volatility, see [8].

It should be noted that the gmfBm model is a generalization of all the fractional models studied in the literature. This generalized model degenerates into a single fBm model with $n = 1$, a Bm model with $n = 1$ and $H_1 = \frac{1}{2}$, an mfBm model with $n = 2$ and $H_1 = \frac{1}{2}$ and a fmfBm with $n = 2$. For a detailed survey on the properties of the gmfBm, we refer to [6,9,10].

The time-changed generalized mixed fractional Brownian motion is defined as

$$T_\beta^{H,a} = \{T_{\beta_t}^{H,a},\ t \geq 0\} = \{Z_{\beta_t}^{H,a},\ t \geq 0\},$$

where the parent process $T^{H,a}$ is a gmfBm with parameters $H = (H_1, H_2, \dots, H_n)$, $a = (a_1, a_2, \dots, a_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$ and the internal process is a subordinator $\beta = \{\beta_t, t \geq 0\}$ assumed to be independent of $B_t^{H_i}$, for $i = 1, 2, \dots, n$. If $H = (\frac{1}{2}, 0, \dots, 0)$ and $a = (1, 0, \dots, 0)$, the process $T_\beta^{H,a}$ is called subordinated Brownian motion. Also, the process $T_\beta^{H,\alpha}$, for $H = (H_1, 0, \dots, 0)$ and $a = (1, 0, \dots, 0)$ is called subordinated fractional Brownian motion it was considered in [11,12].

A time-changed process is a stochastic process that is constructed by taking the superposition of two independent stochastic systems. The first system is called the external process, and the second system is called the subordinator. The evolution of time in the external process is replaced by the subordinator, which is a non-decreasing stochastic process. The resulting time-changed process often retains important properties of the base process; however, certain characteristics may change. The idea of subordination was introduced by Bochner in 1949 [13], and it has been explored in many papers since then (e.g., [14–17]). Subordination is a versatile tool that can be used to construct a wide variety of stochastic processes. It is a powerful tool for modeling real-world phenomena, and it has been used in many different fields, including finance, insurance, and physics.

In the case that $H = (\frac{1}{2}, H_2, 0 \dots, 0)$ and $a = (a_1, a_2, 0, \dots, 0)$, the time-changed mixed fractional Brownian motion was discussed in [18] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets.

The time-changed processes have found many interesting applications, for example [18–26].

This study investigates the long-range dependency property of the time-changed gmfBm. We describe two processes that make up gmfBm's "operational time". In the first scenario, the internal process, which plays the role of time, is the tempered stable subordinator, whereas, in the second situation, it is the gamma process. As an application, we deduce the results concerned the long-range dependence property of the time-changed fBm by tempered stable subordinator and gamma process proved by Kumar et al. in [11,12], respectively.

2. Preliminaries

We define the tempered stable subordinator and gamma process in this section. Additionally, we quickly review the definitions of long-range dependence based on a process's correlation function.

A subordinator is a process with stationary and independent non-negative increments starting at zero. Subordinators are a special class of Lévy processes taking values in $[0, \infty)$ and their sample paths are non-decreasing; this is a type of stochastic process that is used to model random phenomena that have jumps (see [27,28] for more details). Let $\beta = \{\beta_t,\ t \geq 0\}$ be a subordinator. The infinite divisibility of the law of β implies that its Laplace transform can be expressed in the form

$$E(e^{-\lambda \beta_t}) = e^{-t\Phi(\lambda)}, \quad \lambda > 0,$$

where $\Phi : [0, \infty) \to [0, \infty)$, called the Laplace exponent, is a Bernstein function. Such that the Laplace exponent Φ can be expressed as

$$\Phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda\tau})\, \sigma(d\tau) < \infty,$$

which is known as the Lévy-Khintchine formula for the subordinator β. Where $a, b \geq 0$ and σ are a measure on the positive real half-line such that

$$\int_{(0,\infty)} (1 \wedge \tau)\, \sigma(d\tau) < \infty$$

2.1. Tempered Stable Subordinator

Tempered stable subordinator, where index $\alpha \in (0, 1)$ and tempering parameter $\lambda > 0$ (TSS) are the non-decreasing and non-negative Lévy process $S^{\lambda,\alpha} = \{S_t^{\lambda,\alpha},\ t \geq 0\}$ with density function:

$$f_{\lambda,\alpha}(x,t) = exp(-\lambda x + \lambda^\alpha t) f_\alpha(x,t), \;\; \lambda > 0, \;\; \alpha \in (0,1),$$

where

$$f_\alpha(x,t) = \frac{1}{\pi} \int_0^\infty e^{-xy} e^{-ty^\alpha \cos \alpha\pi} \sin(ty^\alpha \sin \alpha\pi) dy.$$

More details about TSS can be found in [11].

Lemma 1. *(see [11] for the proof)*
For $q > 0$, the asymptotic behavior of q-th order moments of $S_t^{\lambda,\alpha}$ satisfies

$$E(S_t^{\lambda,\alpha})^q \sim (\alpha\lambda^{\alpha-1} t)^q, \quad as\ t \to \infty.$$

2.2. Gamma Subordinator

Gamma subordinator $\Gamma = \{\Gamma_t, t \geq\}$ is a stationary and independent increments process with gamma distribution. More precisely, the increment $\Gamma_{t+s} - \Gamma_s$ has the density function

$$f(x,t) = \frac{1}{\Gamma(t/\nu)} x^{(t/\nu)-1} e^{-x}, \quad x > 0, \quad \nu > 0.$$

More details about gamma subordinator can be found in [12].

Lemma 2. *(see [12] for the proof)*
For $q > 0$, the asymptotic behavior of q-th order moments of Γ_t satisfies

$$E(\Gamma_t)^q \sim \left(\frac{t}{\nu}\right)^q, \quad \text{as } t \to \infty.$$

Lemma 3. *(see [12] for the proof)*
The covariance of Γ_t is

$$Cov(\Gamma_s, \Gamma_t) = \frac{t}{\nu} + \frac{t^2 - s^2}{\nu^2}, \quad \text{where} \quad s < t.$$

Then for fixed s and $t \to \infty$, it follows that

$$Cov(\Gamma_s, \Gamma_t) \sim \frac{t^2}{\nu^2}.$$

2.3. Long-Range Dependence

Notation 1. *Let X and Y be two random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. We denote the correlation coefficient $Corr(X,Y)$ by*

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}. \tag{2}$$

Definition 1. *Please note that a finite variance stationary process $\{X_t,\ t \geq 0\}$ is said to have a long-range dependence property (Cont and Tankov [29]), if $\sum_{k=0}^{\infty} \gamma_k = \infty$, where*

$$\gamma_k = Cov(X_t, X_{t+k}).$$

In the following definition, we give the equivalent definition for a non-stationary process $\{X_t,\ t \geq 0\}$.

Definition 2. *Let $s > 0$ be fixed and $t > s$. The process $\{X_t,\ t \geq 0\}$ is said to have a long-range dependence property if*

$$Corr(X_t, X_s) \sim c(s) t^{-d}, \ \text{ as } \ t \to \infty,$$

where $c(s)$ is a constant depending on s and $d \in (0,1)$. An equivalent definition given in [30].

Let $0 < s < t$ and s be fixed. Assume a stochastic process $\{X_t,\ t \geq 0\}$ has the correlation function $Corr(X(s), X(t))$ that satisfies $c_1(s)t^{-d} \leq Corr(X_s, X_t) \leq c_2(s)t^{-d}$ for large $t, d > 0$, $c_1(s) > 0$ and $c_2(s) > 0$., i.e.,

$$lim_{t \to \infty} \frac{Corr(X_s, X_t)}{t^{-d}} = c(s)$$

for some $c(s) > 0$ and $d > 0$. We say $\{X_t,\ t \geq 0\}$ has the long-range dependence property if $d \in (0,1)$ and has the short-range dependence property if $d \in (1,2)$.

Proposition 1. *The TSS with index $\alpha \in (0,1)$ and tempering parameter $\lambda > 0$ has LRD property.*

Proof. First, we compute the covariance function using the subordinator's independent increment characteristic. For $0 < s < t$, we have

$$\begin{aligned} Cov(S_s^{\lambda,\alpha}, S_t^{\lambda,\alpha}) &= Cov(S_s^{\lambda,\alpha}, (S_t^{\lambda,\alpha} - S_s^{\lambda,\alpha}) + S_s^{\lambda,\alpha}) \\ &= Cov(S_s^{\lambda,\alpha}, (S_t^{\lambda,\alpha} - S_s^{\lambda,\alpha})) + Cov(S_s^{\lambda,\alpha}, S_s^{\lambda,\alpha}) \\ &= Var(S_s^{\lambda,\alpha}). \end{aligned}$$

Thus, the correlation function is given by

$$\begin{aligned} Corr(S_s^{\lambda,\alpha}, S_t^{\lambda,\alpha}) &= \frac{Cov(S_s^{\lambda,\alpha}, S_t^{\lambda,\alpha})}{Var(S_s^{\lambda,\alpha})^{\frac{1}{2}} Var(S_t^{\lambda,\alpha})^{\frac{1}{2}}} \\ &= \frac{Var(S_s^{\lambda,\alpha})^{\frac{1}{2}}}{Var(S_t^{\lambda,\alpha})^{\frac{1}{2}}} \\ &= s^{\frac{1}{2}} t^{-\frac{1}{2}} \end{aligned}$$

Hence,

$$lim_{t \to \infty} \frac{Corr(S_s^{\lambda,\alpha}, S_t^{\lambda,\alpha})}{t^{-\frac{1}{2}}} = s^{\frac{1}{2}}$$

Therefore, the TSS $\{S_t^{\lambda,\alpha},\ t \geq 0\}$ has an LRD property. □

Similar to the proof of Proposition 1, we obtain

Proposition 2. *The gamma process has a long-range dependence property.*

Definition 3. *Let $a_1, \ldots, a_n$, reel numbers and not all a_i equals zero. A generalized mixed fractional Brownian motion (gmfBm) of parameters $H = (H_1, H_2, \ldots, H_n)$ and $a = (a_1, a_2, \ldots, a_n)$; $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$ is a stochastic process $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ by*

$$Z_t^{H,a} = \sum_{i=1}^{n} a_i B_t^{H_i},$$

where $\{B_t^{H_i},\ t \geq 0\}$ for $i = 1, 2, \ldots, n$ are independent fractional Brownian motions of Hurst indices $H_i \in (0,1)$.

Lemma 4. *(see [6] for the proof) The gmfBm has stationary increments and exhibits a long-range dependence property if, and only if , there exist some $k \in \{1, \ldots, n\}$ with $H_k > \frac{1}{2}$.*

3. gmfBm Time-Changed by Tempered Stable Subordinator

In this section, we will investigate the gmfBm time-changed by tempered stable subordinator.

Definition 4. *Let $n \in \mathbf{N}^*$. Let $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $a = (a_1, a_2, \ldots, a_n)$ and $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$. Let $S^{\lambda,\alpha} = \{S_t^{\lambda,\alpha},\ t \geq 0\}$ be a TSS with index $\alpha \in (0,1)$ and tempering parameter $\lambda > 0$. The time-changed process of $T^{H,a}$ by means of $S^{\lambda,\alpha}$ is the process $T_{S^{\lambda,\alpha}}^{H,a} = \{T_{S_t^{\lambda,\alpha}}^{H,a},\ t \geq 0\}$ defined by:*

$$T_{S_t^{\lambda,\alpha}}^{H,a} = \sum_{i=1}^{n} a_i B_{S_t^{\lambda,\alpha}}^{H_i}, \tag{3}$$

where the subordinator $S_t^{\lambda,\alpha}$ is assumed to be independent of all B^{H_i} for $i = 1, 2, \ldots, n$.

Proposition 3. *Let $T^{H,a} = \{T_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $a = (a_1, \ldots, a_n)$, and $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$. Let $T^{H,a}_{S^{\lambda,\alpha}}$ be the gmfBm time-changed by $S^{\lambda,\alpha}$. Then by Taylor's expansion we obtain, for fixed s and large t,*

$$Cov(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) \quad \sim \quad \sum_{i=1}^{n} {a_i}^2 H_i s(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i-1}, \ \ as \ \ t \to \infty. \tag{4}$$

Proof. Let $s > 0$ be fixed and let $s \leq t$. The covariance function of $T^{H,a}_{S_t^{\lambda,\alpha}}$ and $T^{H,a}_{S_s^{\lambda,\alpha}}$ is defined by

$$Cov(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) \quad = \quad E(T^{H,a}_{S_t^{\lambda,\alpha}} T^{H,a}_{S_s^{\lambda,\alpha}}) - E(T^{H,a}_{S_t^{\lambda,\alpha}})E(T^{H,a}_{S_s^{\lambda,\alpha}})$$

by observing that $E[T^{H,a}_{S_t^{\lambda,\alpha}}] = 0$, $t \geq 0$ and using ([11], p. 195), the process $T^{H,a}_{S_t^{\lambda,\alpha}}$ follows

$$\begin{aligned}
Cov(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) &= E[T^{H,a}_{S_t^{\lambda,\alpha}} T^{H,a}_{S_s^{\lambda,\alpha}}] \\
&= \frac{1}{2} E[(T^{H,a}_{S_t^{\lambda,\alpha}})^2 + (T^{H,a}_{S_s^{\lambda,\alpha}})^2 - (T^{H,a}_{S_t^{\lambda,\alpha}} - T^{H,a}_{S_s^{\lambda,\alpha}})^2] \\
&= \frac{1}{2} E[(Z^{H,a}_{S_t^{\lambda,\alpha}})^2 + (Z^{H,a}_{S_s^{\lambda,\alpha}})^2 - (Z^{H,a}_{S_t^{\lambda,\alpha}} - Z^{H,a}_{S_s^{\lambda,\alpha}})^2] \\
&= \frac{1}{2} E[(\sum_{i=1}^{n} a_i B^{H_i}_{S_t^{\lambda,\alpha}})^2 + (\sum_{i=1}^{n} a_i B^{H_i}_{S_s^{\lambda,\alpha}})^2] \\
&\quad -\frac{1}{2} E[(\sum_{i=1}^{n} a_i (B^{H_i}_{S_t^{\lambda,\alpha}} - B^{H_i}_{S_s^{\lambda,\alpha}}))^2].
\end{aligned}$$

Since the fractional Brownian motion has stationary increments, then

$$\begin{aligned}
Cov(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) &= \frac{1}{2} E[(\sum_{i=1}^{n} a_i B^{H_i}_{S_t^{\lambda,\alpha}})^2 + (\sum_{i=1}^{n} a_i B^{H_i}_{S_s^{\lambda,\alpha}})^2 - (\sum_{i=1}^{n} a_i B^{H_i}_{S_{t-s}^{\lambda,\alpha}})^2] \\
&= \frac{1}{2} E[\sum_{i=1}^{n} (a_i B^{H_1}_{S_t^{\lambda,\alpha}})^2 + 2\sum_{i \neq j} a_i a_j B^{H_i}_{S_t^{\lambda,\alpha}} B^{H_j}_{S_t^{\lambda,\alpha}}] \\
&\quad + \frac{1}{2} E[\sum_{i=1}^{n} (a_i B^{H_i}_{S_s^{\lambda,\alpha}})^2 + 2\sum_{i \neq j}^{n} a_i b_j B^{H_i}_{S_s^{\lambda,\alpha}} B^{H_j}_{S_s^{\lambda,\alpha}}] \\
&\quad - \frac{1}{2} E[\sum_{i=1}^{n} (a_i B^{H_i}_{S_{t-s}^{\lambda,\alpha}})^2 + 2\sum_{i \neq j} a_i b_j B^{H_i}_{S_{t-s}^{\lambda,\alpha}} B^{H_j}_{S_{t-s}^{\lambda,\alpha}}].
\end{aligned} \tag{5}$$

By the independence of the fBms' $B_t^{H_i}$ for $i = 1, \ldots, n$ and their independence of the $S^{\lambda,\alpha}$, we have

$$\begin{aligned}
E[B^{H_k}_{S_t^{\lambda,\alpha}} B^{H_l}_{S_t^{\lambda,\alpha}}] &= E[E(B_r^{H_k} B_r^{H_l} | S_t^{\lambda,\alpha})] \\
&= \int E[B_r^{H_k} B_r^{H_l}] f_{S_t^{\lambda,\alpha}}(dr) \\
&= 0
\end{aligned}$$

where $f_{S_t^{\lambda,\alpha}}(.)$ is the distribution function of $S_t^{\lambda,\alpha}$.
Thus, we obtain

$$\begin{aligned} Cov(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) &= \sum_{i=1}^{n} \frac{a_i^2}{2}[E(B^{H_i}_{S_t^{\lambda,\alpha}})^2 + E(B^{H_i}_{S_s^{\lambda,\alpha}})^2 - E(B^{H_i}_{S_{t-s}^{\lambda,\alpha}})^2] \\ &= \sum_{i=1}^{n} \frac{a_i^2}{2} E(B^{H_i}(1))^2[E(S_t^{\lambda,\alpha})^{2H_i} + E(S_s^{\lambda,\alpha})^{2H_i} - E(S_{t-s}^{\lambda,\alpha})^{2H_i}]. \end{aligned}$$

Hence for large t and using Lemma 1, we have

$$\begin{aligned} Cov(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) &\sim \sum_{i=1}^{n} \frac{a_i^2}{2}[(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i} + E(S_s^{\lambda,\alpha})^{2H_i} - (\alpha\lambda^{\alpha-1})^{2H_i}(t-s)^{2H_i}] \\ &= \sum_{i=1}^{n} \frac{a_i^2}{2}(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i}(2H_i\frac{s}{t} + E(S_s^{\lambda,\alpha})^{2H_i} t^{-2H_i} + O(t^{-2})) \\ &\sim \sum_{i=1}^{n} a_i^2 H_i s(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i-1}. \end{aligned}$$

□

Proposition 4. *Let $T^{H,a} = \{T_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $a = (a_1, \ldots, a_n)$, and $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$. Let $S^{\lambda,\alpha} = \{S_t^{\lambda,\alpha},\ t \geq 0\}$ be the TSS with index $\alpha \in (0,1)$ and tempering parameter $\lambda > 0$ and let $T^{H,a}_{S^{\lambda,\alpha}}$ be the gmfBm time-changed process by means of $S^{\lambda,\alpha}$. Then for fixed $s > 0$ and $t \to \infty$, we obtain*

$$\begin{aligned} E[(T^{H,a}_{S_t^{\lambda,\alpha}} - T^{H,a}_{S_s^{\lambda,\alpha}})^2] \sim &\sum_{i=1}^{n} a_i^2 H_i(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i} - \sum_{i=1}^{n} 2a_i^2 H_i s(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i-1} \\ &+ \sum_{i=1}^{n} a^2 H_i(\alpha\lambda^{\alpha-1})^{2H_i} s^{2H_i}. \end{aligned}$$

Proof. Let $s > 0$ be fixed and $t \to \infty$. Then, using Equation (4), we have

$$\begin{aligned} E[(T^{H,a}_{S_t^{\lambda,\alpha}} - T^{H,a}_{S_s^{\lambda,\alpha}})^2] &= E[(T^{H,a}_{S_t^{\lambda,\alpha}} - T^{H,a}_{S_s^{\lambda,\alpha}})(T^{H,a}_{S_t^{\lambda,\alpha}} - T^{H,a}_{S_s^{\lambda,\alpha}})] \\ &= E[(T^{H,a}_{S_t^{\lambda,\alpha}})^2 - T^{H,a}_{S_t^{\lambda,\alpha}} T^{H,a}_{S_s^{\lambda,\alpha}} - T^{H,a}_{S_t^{\lambda,\alpha}} T^{H,a}_{S_s^{\lambda,\alpha}} + (T^{H,a}_{S_s^{\lambda,\alpha}})^2] \\ &= E[(T^{H}_{S_t^{\lambda,\alpha}})^2 - 2T^{H,a}_{S_t^{\lambda,\alpha}} T^{H,a}_{S_s^{\lambda,\alpha}} + (T^{H,a}_{S_s^{\lambda,\alpha}})^2] \\ &\sim \sum_{i=1}^{n} \frac{a_i^2}{2}[(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i} + E(S_s^{\lambda,\alpha})^{2H_i} - (\alpha\lambda^{\alpha-1})^{2H_i}(t-s)^{2H_i}] \\ &= \sum_{i=1}^{n} \frac{a_i^2}{2}(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i}(2H_i\frac{s}{t} + E(S_s^{\lambda,\alpha})^{2H_i} t^{-2H_i} + O(t^{-2})) \\ &\sim \sum_{i=1}^{n} a_i^2 H_i s(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i-1}. \end{aligned}$$

□

Now we discuss the long-range dependence behavior of $T^{H,a}_{S^{\lambda,\alpha}}$.

Theorem 1. *Let $T^{H,a} = \{T_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $a = (a_1, \ldots, a_n)$ and $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$ with $H_i < H_n$ for $i = 1, 2, \ldots, n-1$. Let $S^{\lambda,\alpha} = \{S_t^{\lambda,\alpha},\ t \geq 0\}$ be the TSS with index $\alpha \in (0,1)$ and tempering parameter $\lambda > 0$. Then* *the time-changed gmfBm by means of $S^{\lambda,\alpha}$ exhibits a long-range dependence property for all Hurst indices satisfying $0 < 2H_i - H_n < 1$.*

Proof. Let $n \in \mathbf{N}^*$. Let $T^{H,a} = \{T_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $a = (a_1, \ldots, a_n)$ and $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$ with $H_i < H_n$ for $i = 1, 2, \ldots, n-1$. Let $S^{\lambda,\alpha} = \{S_t^{\lambda,\alpha},\ t \geq 0\}$ be the TSS with index $\alpha \in (0,1)$ and tempering parameter $\lambda > 0$. The process $T^{H,a}_{S^{\lambda,\alpha}}$ is not stationary, hence the Definition 2 will be used to establish the long-range dependence property.

Using Equations (2), (4) and by Taylor's expansion we obtain, as $t \to \infty$

$$
\begin{aligned}
Corr(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) &\sim \frac{\sum_{i=1}^{n} a_i^2 H_i s(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i-1}}{\sqrt{\left[\sum_{i=1}^{n} a_i^2 H_i (\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i}\right]}\sqrt{E(T^{H,a}_{S_s^{\lambda,\alpha}})^2}} \\
&= \frac{\sum_{i=1}^{n} a_i^2 H_i s(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i-1}}{\sqrt{a_n^2 H_n (\alpha\lambda^{\alpha-1})^{2H_n} \sigma^2 t^{2H_n}\left[\sum_{i=1}^{n-1}\left(\frac{H_i a_i^2}{a_n^2 H_n}(\alpha\lambda^{\alpha-1})^{2H_i-2H_n} t^{2H_i-2H_n}\right)+1\right]}} \\
&\sim \frac{\sum_{i=1}^{n} a_i^2 H_i s(\alpha\lambda^{\alpha-1})^{2H_i} t^{2H_i-1}}{\sqrt{a_n^2 H_n \sigma^2 (\alpha\lambda^{\alpha-1})^{2H_n} t^{2H_n}}}, \quad \textit{since } H_i < H_n, \\
&= \frac{\sum_{i=1}^{n-1} a_i^2 H_i H_n^{-\frac{1}{2}} s(\alpha\lambda^{\alpha-1})^{H_n-2H_i} t^{2H_i-H_n-1}}{|a_n|\sigma} + \frac{|a_n| H_n^{\frac{1}{2}} s(\alpha\lambda^{\alpha-1})^{H_n} t^{H_n-1}}{\sigma}.
\end{aligned}
$$

where $\sigma^2 = E(T^{H,a}_{S_s^{\lambda,\alpha}})^2$. Then, for fixed $s > 0$ and $t \to \infty$. For $H_i < H_n < 1$ for $i = 1, \ldots, n-1$, the correlation function is given by

$$
Corr(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) \sim c_1 t^{2H_i - H_n - 1} + c_2 t^{H_n - 1}.
$$

Therefore, for $0 < 2H_i - H_n < 1$ the correlation function of $T^{H,a}_{\Gamma_t}$ decays like a $t^{-(1-H_n)}$ for all $0 < H_n < 1$. Then, in the sense of Definition 2, the time-changed process $T^{H,a}_{S^{\lambda,\alpha}}$ exhibits a long-range dependence property for all $0 < 2H_i - H_n < 1$, $i = 1, 2, \ldots, n-1$. □

Remark 1. *When $n = 1$ in Equation (4) and using Equation (2) , we obtain*

$$
\begin{aligned}
Cov(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) &\sim H_1 s(\alpha\lambda^{\alpha-1})^{2H_1} t^{2H_1-1}, \quad \textit{as } t \to \infty, \\
Corr(T^{H,a}_{S_t^{\lambda,\alpha}}, T^{H,a}_{S_s^{\lambda,\alpha}}) &\sim H_1 s^{1-H_1} t^{H_1-1}, \quad \textit{as } t \to \infty.
\end{aligned}
$$

Hence we obtain the following results, proven in [11].

Corollary 1. *The fractional Brownian motion time-changed by TSS has long-range dependence for every $H \in (0,1)$.*

4. gmfBm Time-Changed by the Gamma Subordinator

This section looks into generalized mixed fractional Brownian motion time-changed by the gamma process.

Definition 5. *Let $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $a = (a_1, \ldots, a_n)$ and $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$. Let $\Gamma = \{\Gamma_t,\ t \geq 0\}$ be a gamma process. The time-changed process of $Z^{H,a}$ by means of Γ is the process $T_\Gamma^{H,a} = \{T_{\Gamma_t}^{H,a},\ t \geq 0\}$ defined by:*

$$
T_{\Gamma_t}^{H,a} = \sum_{i=1}^{n} a_i B_{\Gamma_t}^{H_i}, \tag{6}
$$

where the process Γ_t *is assumed to be independent of* $B_t^{H_i}$ *for* $i = 1, \ldots, n$.

Proposition 5. *Let* $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ *be a gmfBm of parameters* $a = (a_1, \ldots, a_n)$ *and* $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$. *Let* $T_\Gamma^{H,a}$ *be the gmfBm time-changed by* Γ. *Then we have*

1. *For* $t > s$, *the covariance function for the process* $T_{\Gamma_t}^{H,a}$ *follows*

$$Cov(T_{\Gamma_t}^{H,a}, T_{\Gamma_s}^{H,a}) = \sum_{i=1}^{n} \frac{a_i^2}{2} \left[\frac{\Gamma(2H_i + t/\nu)}{\Gamma(t/\nu)} + \frac{\Gamma(2H_i + s/\nu)}{\Gamma(s/\nu)} - \frac{\Gamma(2H_i + (t-s)/\nu)}{\Gamma((t-s)/\nu)} \right]$$

2. *For fixed* s *and large* t, *the process* $Y_{\Gamma_t}^{H}$ *follows*

$$Cov(T_{\Gamma_t}^{H,a}, T_{\Gamma_s}^{H,a}) \sim \sum_{i=1}^{n} \frac{2a_i^2 H_i s}{\nu^{2H_i}} t^{2H_i - 1}. \tag{7}$$

Proof.

1. Let s fixed. Let $s < t$. We use a similar procedure as in the proof of Equation (5). By the independence of the fBms' $B_t^{H_i}$ for $i = 1, \ldots, n$ and their independence of the Γ_t, we obtain

$$\begin{aligned} Cov(T_{\Gamma_t}^{H,a}, T_{\Gamma_s}^{H,a}) &= E(T_{\Gamma_t}^{H,a} T_{\Gamma_s}^{H,a}) \\ &= \sum_{i=1}^{n} \frac{a_i^2}{2} \left[E(B_{\Gamma_t}^{H_i})^2 + E(B_{\Gamma_s}^{H_i})^2 - E(B_{\Gamma_{t-s}}^{H_i})^2 \right] \\ &= \sum_{i=1}^{n} \frac{a_i^2}{2} \left[\frac{\Gamma(2H_i + t/\nu)}{\Gamma(t/\nu)} + \frac{\Gamma(2H_i + s/\nu)}{\Gamma(s/\nu)} - \frac{\Gamma(2H_i + (t-s)/\nu)}{\Gamma((t-s)/\nu)} \right]. \end{aligned}$$

2. Let $g(x) = \Gamma(x + 2H_i)/\Gamma(x)$ and $f(x) = \Gamma(x+1)/\Gamma(x)$. By Taylor expansion and [12] we have

$$\frac{g(x+h)}{g(x)} = 1 + 2H_i(h/x) + H(2H_i - 1)(h/x)^2 + O(x^{-3}), \tag{8}$$

and

$$\frac{f(x+h)}{f(x)} = 1 + (h/x) + O(x^{-2}). \tag{9}$$

For fixed s and large t, using Equations (7)–(9), $T_{\Gamma_t}^{H}$ follows

$$\begin{aligned} Cov(T_{\Gamma_t}^{H,a}, T_{\Gamma_s}^{H,a}) &= \textstyle\sum_{i=1}^{n} \frac{a_i^2}{2} g(t/\nu) \left[1 + \frac{g(s/\nu)}{g(t/\nu)} - \frac{g((t-s)/\nu)}{g(t/\nu)} \right] \\ &= \textstyle\sum_{i=1}^{n} \frac{a_i^2}{2} (t/\nu)^{2H_i} [1 + \frac{g(s/\nu)}{g(t/\nu)} - (1 - 2H_i(\frac{s}{t}) \\ &\quad + H_i(2H_i - 1)\left(\frac{s^2}{t^2}\right) + O(t^{-3}))] \\ &\sim \textstyle\sum_{i=1}^{n} \frac{2a_i^2 H_i s}{\nu^{2H_i}} t^{2H_i - 1}. \end{aligned}$$

□

Theorem 2. *Let* $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ *be a gmfBm of parameters* $a = (a_1, \ldots, a_n)$ *and* $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$, $n \in \mathbf{N}^*$ *with* $H_i < H_n$ *for* $i = 1, 2, \ldots, n-1$. *Let* $\Gamma = \{\Gamma_t,\ t \geq 0\}$ *be a gamma process with parameter* $\nu > 0$. *Let* $T_\Gamma^{H,a}$ *be the gmfBm time-changed by* Γ. *Then, the time-changed gmfBm by means of* Γ *has la ong-range dependence property for all Hurst indices satisfying* $0 < 2H_i - H_n < 1$.

Proof. Let $n \in \mathbf{N}^*$. Let $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $a = (a_1, \ldots, a_n)$ and $H = (H_1, \ldots, H_n)$, $H_i \in (0,1)$, $a_i \in \mathbf{R}$ with $H_i < H_n$ for $i = 1, 2, \ldots, n-1$. Let $\Gamma = \{\Gamma_t,\ t \geq 0\}$ be a gamma process with parameter $\nu > 0$. The process $T_{\Gamma}^{H,a}$ is not stationary, hence the Definition 2 will be used to establish the long-range dependence property.

Using Equations (2), (7) and by Taylor's expansion we obtain, as $t \to \infty$

$$
\begin{aligned}
Corr(T_{\Gamma_t}^{H,a}, T_{\Gamma_s}^{H,a}) &\sim \frac{\sum_{i=1}^{n} \frac{2a_i{}^2 H_i s}{\nu^{2H_i}} t^{2H_i - 1}}{\sqrt{(\sum_{i=1}^{n} \frac{2a_i{}^2 H_i}{\nu^{2H_i}} t^{2H_i})} \sqrt{E(T_{\Gamma_s}^{H,a})^2}} \\
&= \frac{\sum_{i=1}^{n} \frac{2a_i{}^2 H_i s}{\nu^{2H_i}} t^{2H_i - 1}}{\sqrt{\frac{2a_n^2 H_n t^{2H_n}}{\nu^{2H_n}} [\sum_{i=1}^{n-1} \frac{2a_i{}^2 H_i}{\nu^{2H_i}} t^{2H_i - 2H_n} + 1]} \sqrt{E(T_{\Gamma_s}^{H,a})^2}} \\
&\sim \frac{\sum_{i=1}^{n} \frac{2a_i{}^2 H_i s}{\nu^{2H_i}} t^{2H_i - 1}}{\sqrt{\frac{2a_n{}^2 H_n}{\nu^{2H_n}} t^{2H_n}} \sqrt{E(T_{\Gamma_s}^{H,a})^2}}, \quad \textit{since } H_i < H_n \\
&= \frac{\sum_{i=1}^{n-1} \frac{2a_i{}^2 H_i s}{\nu^{2H_i}}}{\sqrt{\frac{2a_n{}^2 H_n}{\nu^{2H_n}} E(T_{\Gamma_s}^{H,a})^2}} t^{2H_i - H_n - 1} + \frac{|a_n|(2H_n)^{\frac{1}{2}} \nu^{H_n} s}{\sqrt{E(T_{\Gamma_s}^{H,a})^2}} t^{H_n - 1}.
\end{aligned}
$$

Then, for fixed $s > 0$ and $t \to \infty$. For $H_i < H_n < 1$ for $i = 1, \ldots, n-1$, the correlation function is given by

$$
Corr(T_{\Gamma_t}^{H,a}, T_{\Gamma_s}^{H,a}) \sim c_1 t^{2H_i - H_n - 1} + c_2 t^{H_n - 1}. \tag{10}
$$

Therefore, for $0 < 2H_i - H_n < 1$ the correlation function of $T_{\Gamma_t}^{H,a}$ decays like a $t^{-(1-H_n)}$ for all $0 < H_n < 1$. Then, in the sense of Definition 2 the time-changed process $T_{S^{\lambda,\alpha}}^{H,a}$ exhibits the long-range dependence property for all $0 < 2H_i - H_n < 1$ for $i = 1, 2, \ldots, n-1$. □

Hence we obtain the following results, proved in [12].

Corollary 2. *The fractional Brownian motion time-changed by the gamma process has long-range dependence for all* $H \in (0,1)$.

5. Conclusions

The time-changed gmfBm is a versatile and powerful tool for modeling natural phenomena that exhibit long-range dependence. The ability to control the long-range dependence property through the Hurst indices is a key feature of this model, and it allows us to tailor the model to the specific characteristics of the phenomenon we are trying to model. In this paper, it is shown that the time-changed gmfBm exhibits a long-range dependence property under some conditions on the Hurst indices when it is time-changed by a tempered stable subordinator or a gamma process. This is a significant result, as it shows that the time-changed gmfBm can be used to model a wide variety of natural phenomena that exhibit long-range dependence, even when the underlying process is not itself long-range dependent. We deduce that the fractional Brownian motion time-changed by tempered stable subordinator or gamma process has long-range dependence for all $H \in (0,1)$.

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