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Computational Study for Fiber Bragg Gratings with Dispersive Reflectivity Using Fractional Derivative

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Abstract: In this paper, the new representations of optical wave solutions to fiber Bragg gratings with cubic–quartic dispersive reflectivity having the Kerr law of nonlinear refractive index structure are retrieved with high accuracy. The residual power series technique is used to derive power series solutions to this model. The fractional derivative is taken in Caputo's sense. The residual power series technique (RPST) provides the approximate solutions in truncated series form for specified initial conditions. By using three test applications, the efficiency and validity of the employed technique are demonstrated. By considering the suitable values of parameters, the power series solutions are illustrated by sketching 2D, 3D, and contour profiles. The analysis of the obtained results reveals that the RPST is a significant addition to exploring the dynamics of sustainable and smooth optical wave propagation across long distances through optical fibers.

Keywords: Caputo's fractional derivative; fiber Bragg gratings; residual power series technique; Kerr law

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1. Introduction

Over the last few decades, the investigation of optical wave propagation has caught the attention of different scientists in applied mathematics. Many significant developments have been made in the field of nonlinear optics [1–5]. In the realm of nonlinear optics, it is widely known that the propagation of an optical pulse in nonlinear media, including Kerr law and non-Kerr law, may be represented by the nonlinear Schrödinger equation [6]. The Kerr law of nonlinearity arises when a light wave in an optical fiber experiences nonlinear responses due to nonharmonic motion of electrons caused by an external electric field [7–9]. Low chromatic dispersion is one of the fundamental issues with soliton transmission over intercontinental distances. To manage and lessen this effect, a number of strategies and approaches have been applied. The project was then successfully completed by adding gratings to optical fibers [10–12].

One of the most inventive technologies applied to nonlinear optics is fiber Bragg grating (FBG) [13]. FBG technology has been embraced by the field of fiber optics since its discovery. Presently, most fiber optic sensor systems use FBG technology. Due to the inherent benefits of FBGs, including their compact size, quick reaction, dispersed sensing, and immunity to the electromagnetic field, FBG-based sensing has attracted a lot of research interest [14]. Measurements of numerous physical parameters, including temperature, pressure, and strain are frequently performed using FBG technology [15]. For a wide range of applications, FBGs are also useful in signal shaping as well as filtering components [16].

Occasionally, another difficulty comes when the chromatic dispersion is gradually depleted, and hence, there is an unstable balance between the chromatic dispersion and the nonlinearity, being followed by a possible pulse collapse. To overcome this difficulty, the chromatic dispersion is replaced by a combination of third-order and fourth-order dispersions.

Optical waves in FBGs with different nonlinear refractive indices, where the chromatic dispersion is replaced by the third- and fourth-order dispersions together with the dispersive reflectivity have been retrieved analytically in the earlier studies. For example, Wang et al. [17] investigated this model by implementing the trial equation approach to retrieve cubic–quartic optical solitons in FBGs. Yildirim et al. [18] retrieved the soliton solutions of this model with the sine-Gordon equation method. Malik et al. [19] investigated the nonlinear Schrödinger equation, which considered the cubic–quartic form of nonlinear refractive index with parabolic law of nonlinearity by Lie symmetry. Arnous et al. [20] derived the optical solitons in FBGs with cubic–quartic dispersive reflectivity with the aid of enhanced Kudryashov's technique. Zayed et al. [21] used the auxiliary equation approach to explore solitons in FBGs with cubic–quartic dispersive reflectivity having the Kerr law of nonlinear refractive index.

In this paper, we consider a model in FBGs with cubic–quartic dispersive reflectivity having a Kerr nonlinear refractive index and Caputo's time-fractional derivative of order ϱ $(0 < \varrho \leq 1)$. The mathematical form of the considered model is expressed as

$$\begin{aligned} \imath D_t^{\varrho} n + \imath a_1 o_{\varpi\varpi\varpi} + b_1 o_{\varpi\varpi\varpi\varpi} + (c_1|n|^2 + d_1|o|^2)n + \imath\alpha_1 n_{\varpi} + \beta_1 o = 0, \\ \imath D_t^{\varrho} o + \imath a_2 n_{\varpi\varpi\varpi} + b_2 n_{\varpi\varpi\varpi\varpi} + (c_2|n|^2 + d_2|n|^2)o + \imath\alpha_2 o_{\varpi} + \beta_2 n = 0, \end{aligned} \tag{1}$$

where $n(\varpi, t)$ and $o(\varpi, t)$ denote wave profiles, and t and ϖ denote the time in dimensionless form and the non-dimensional distance, respectively. The coefficients $a_j, b_j, \alpha_j, c_j, d_j$ and β_j correspond to third-order dispersion, fourth-order dispersion, intermodal dispersion, self-phase modulation, cross-phase modulation, and detuning parameters, respectively. The exact solution of this model at $\varrho = 1$ is reported by Ming-Yue-Wang et al. [17].

The primary goal of this work is to present the optical wave solutions and their graphical observations for the considered model in FBGs with Caputo's time-fractional derivative using the residual power series technique (RPST). The obtained results are novel, and the proposed technique is utilized for the first time to study the considered fractional model in this work.

The discipline of fractional calculus has seen tremendous advancements in study at the intersection of probability, chaos, differential equations, and mathematical physics [22–26]. The nonlocal property of fractional differential equations (FDEs) is a significant advantage of employing them in various mathematical modeling. Retrieving the analytical solutions for FDEs is sometimes difficult to achieve due to the computational complexities of fractional operators. In this respect, numerous methods have been constructed and motivated to investigate the approximate solution for FDEs, among which are the Sinc-collocation method [27], the predictor–corrector compact difference scheme [28], the variational iteration method [29], the homotopy analysis technique [30], the homotopy asymptotic method [31], the homotopy perturbation scheme [32], the differential transform method [33], the linearly compact scheme [34], the Adomian decomposition method [35] etc.

One of the most efficient techniques for exploring approximate analytic solutions for linear and nonlinear FDEs is the residual power series technique (RPST) [36–40]. The RPST was introduced by the Jordan mathematician Abu Arqub [41]. The RPST has been successfully used to construct the approximate analytic solutions of FDEs without implementing linearization, perturbation, or discretization techniques, showing the reliability and simplicity of this technique. This proposed technique has been successfully applied to investigate the solutions of time-fractional Whitham–Broer–Kaup equations [42], Black–Scholes European option pricing equations [43], the KdV equation [44], the nonlinear Schrödinger equation [45], the Biswas–Milovic equation [46], the Caudrey–Dodd–Gibbon–

Sawada–Kotera equation [47], etc. The suggested technique is a reliable, practical, and astonishingly effective tool for examining the approximate solutions of many types of real-life nonlinear models.

The RPST has several advantages. It is well-known for its accuracy and simplicity to obtain the desired results. It does not require variable discretization, and there is no requirement for huge computer memory or time to reach the solutions. Its nature is global regarding the approximate analytical solutions, which makes it useful to investigate various mathematical, engineering, and physical problems. Being analytic expressions, the results obtained through the RPST can be further explored with derivative calculations.

The arrangement of this paper is given in the following manner. Some preliminaries of fractional calculus and the RPST are presented in Section 2. A description of the fundamental steps of the proposed method and the construction of a solution to the suggested problem is illustrated in Section 3. Graphical findings for different applications are exhibited in Section 4. The conclusion is presented in the last section.

2. Preliminaries

This section introduces the fundamental properties of fractional calculus theory, allowing us to follow the solutions of a coupled nonlinear Schrödinger equation for the Kerr law of nonlinear refractive index in FBGs.

The Caputo technique is chosen in this study because it is appropriate for real-world physical problems, and it specifies integer-order initial conditions for FDEs.

Definition 1 ([22])**.** *Caputo's time-fractional derivative of order ϱ of $\delta(\varpi, t)$ is defined as*

$$D_t^{\varrho}\delta(\varpi,t) = \begin{cases} \frac{1}{\Gamma(\zeta-\varrho)}\int_0^t (t-\psi)^{\zeta-\varrho-1}\frac{\partial^{\zeta}\delta(\varpi,\psi)}{\partial\psi^{\zeta}}d\psi, & 0 \le \zeta-1 < \varrho < \zeta,\ t > \psi > s \ge 0, \varpi \in I, \\ \frac{\partial^{\zeta}\delta(\varpi,t)}{\partial t^{\zeta}}, & \varrho = \zeta \in N. \end{cases}$$

Theorem 1 ([22])**.** *If $\zeta - 1 < \varrho \le \zeta, \zeta \in N$, then*

(a) $D_t^{\varrho} I_t^{\varrho}\delta(\varpi,t) = \delta(\varpi,t)$,

(b) $I_t^{\varrho} D_t^{\varrho}\delta(\varpi,t) = \delta(\varpi,t) - \sum_{i=0}^{n-1}\frac{\partial^i\delta(\varpi,s^+)}{\partial t^i}\frac{t^i}{i!}$.

Lemma 1 ([22])**.** *If $f(\varpi)$ is a continuous function and $\varrho, \vartheta > 0$, then the following result holds:*

$$I_a^{\varrho} I_a^{\vartheta} f(\varpi) = I_a^{\varrho} I_a^{\vartheta} f(\varpi) = I_a^{\varrho+\vartheta} f(\varpi). \quad (2)$$

Other basic properties of fractional-order derivatives are shown in [22–24]. Some results from [48] that are important for the RPST are as follows:

Definition 2. *A power series of the following form*

$$\sum_{k=0}^{\infty} e_k (t-t_0)^{k\varrho} = e_0 + e_1(t-t_0)^{\varrho} + e_2(t-t_0)^{2\varrho} + \ldots, \quad 0 \le \zeta - 1 < \varrho \le \zeta,\ t \ge t_0, \quad (3)$$

is a fractional power series about $t = t_0$.

Definition 3. *A multiple fractional power series about $t = t_0$, for $0 \le \zeta - 1 < \varrho \le \zeta$ is defined as follows:*

$$\sum_{k=0}^{\infty} \widetilde{\delta}_k(\varpi)(t-t_0)^{k\varrho} = \widetilde{\delta}_0(\varpi) + \widetilde{\delta}_1(\varpi)(t-t_0)^{\varrho} + \widetilde{\delta}_2(\varpi)(t-t_0)^{2\varrho} + \ldots, \quad t \ge t_0, \quad (4)$$

where the coefficients of the series are $\widetilde{\delta}_k$ and are functions of ϖ.

Theorem 2. *Assume that δ has a fractional power series representation of the form at $t = t_0$,*

$$\delta(t) = \sum_{k=0}^{\infty} e_k (t - t_0)^{k\varrho}, \quad 0 \le \zeta - 1 < \varrho \le \zeta,\ t_0 \le t < t_0 + R. \tag{5}$$

If $D^{k\varrho}\delta(t)$ are continuous on $(t_0, t_0 + R)$, $k = 0, 1, 2, \ldots$, then the coefficients "e_k" appearing in Equation (5) can be determined as

$$e_k = \frac{D^{k\varrho}\delta(t_0)}{\Gamma(k\varrho + 1)},$$

where R is considered as the series's radius of convergence.

Other results related to the RPST can be found in [37,48,49].

3. RPST to FBGs for Cubic-Quartic Dispersive Reflectivity with Time-Fractional Derivative

Consider a coupled nonlinear Schrödinger equation with the cubic–quartic dispersive reflectivity having the Kerr law of nonlinear refractive index as

$$\begin{aligned} \iota D_t^{\varrho} n + \iota a_1 o_{\varpi\varpi\varpi} + b_1 o_{\varpi\varpi\varpi\varpi} + (c_1|n|^2 + d_1|o|^2)n + \iota\alpha_1 n_{\varpi} + \beta_1 o = 0, \\ \iota D_t^{\varrho} o + \iota a_2 n_{\varpi\varpi\varpi} + b_2 n_{\varpi\varpi\varpi\varpi} + (c_2|n|^2 + d_2|n|^2)o + \iota\alpha_2 o_{\varpi} + \beta_2 n = 0, \end{aligned} \tag{6}$$

subject to the initial conditions

$$\begin{aligned} n(\varpi, 0) = \tilde{q}(\varpi), \\ o(\varpi, 0) = \tilde{r}(\varpi), \end{aligned} \tag{7}$$

where D_t^{ϱ} represents Caputo's time-fractional derivative of order ϱ. The fundamental goal of this work is to develop the solution for Equations (6) and (7) by its power series expansion among its truncated residual function.

For the construction of power series solutions of Equations (6) and (7), let

$$\begin{aligned} n(\varpi, t) = p(\varpi, t) + \iota s(\varpi, t), \\ o(\varpi, t) = u(\varpi, t) + \iota w(\varpi, t), \end{aligned} \tag{8}$$

where $n(\varpi, 0) = p(\varpi, 0) + \iota s(\varpi, 0)$, and $o(\varpi, 0) = u(\varpi, 0) + \iota w(\varpi, 0)$. The following system can be obtained using the above equations in (6) and (7) as

$$\begin{aligned} D_t^{\varrho} s + a_1 w_{\varpi\varpi\varpi} - b_1 u_{\varpi\varpi\varpi\varpi} - c_1 p(p^2 + s^2) - d_1 p(u^2 + w^2) + \alpha_1 s_{\varpi} - \beta_1 u = 0, \\ D_t^{\varrho} p + a_1 u_{\varpi\varpi\varpi} + b_1 w_{\varpi\varpi\varpi\varpi} + c_1 s(p^2 + s^2) + d_1 s(u^2 + w^2) + \alpha_1 p_{\varpi} + \beta_1 w = 0, \\ D_t^{\varrho} w + a_2 s_{\varpi\varpi\varpi} - b_2 p_{\varpi\varpi\varpi\varpi} - c_2 u(u^2 + w^2) - d_2 u(p^2 + s^2) + \alpha_2 w_{\varpi} - \beta_2 p = 0, \\ D_t^{\varrho} u + a_2 p_{\varpi\varpi\varpi} + b_2 s_{\varpi\varpi\varpi\varpi} + c_2 w(u^2 + w^2) + d_2 w(p^2 + s^2) + \alpha_2 u_{\varpi} + \beta_2 s = 0, \end{aligned} \tag{9}$$

with initial conditions

$$\begin{aligned} p(\varpi, 0) = \sigma_1(\varpi), \\ s(\varpi, 0) = \sigma_2(\varpi), \\ u(\varpi, 0) = \xi_1(\varpi), \\ w(\varpi, 0) = \xi_2(\varpi), \end{aligned} \tag{10}$$

where $\tilde{q}(\varpi) = \sigma_1(\varpi) + \iota\sigma_2(\varpi)$, and $\tilde{r}(\varpi) = \xi_1(\varpi) + \iota\xi_2(\varpi)$.

3.1. General Procedure of the RPST

The following are the primary steps for the recommended process.
Step A. Assume the fractional power series solutions of above system regarding the initial point $(t = 0)$ as

$$\begin{aligned} p(\varpi,t) &= \sum_{k=0}^{\infty} g_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ s(\varpi,t) &= \sum_{k=0}^{\infty} f_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ u(\varpi,t) &= \sum_{k=0}^{\infty} l_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ w(\varpi,t) &= \sum_{k=0}^{\infty} m_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}. \end{aligned} \tag{11}$$

For the numerical values of fractional power series solutions for the system of equations with initial conditions, take the $z-th$ truncated series of $p(\varpi,t), s(\varpi,t), u(\varpi,t)$, and $w(\varpi,t)$ as $p_z(\varpi,t), s_z(\varpi,t), u_z(\varpi,t)$, and $w_z(\varpi,t)$, respectively, i.e.,

$$\begin{aligned} p_z(\varpi,t) &= \sum_{k=0}^{z} g_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ s_z(\varpi,t) &= \sum_{k=0}^{z} f_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ u_z(\varpi,t) &= \sum_{k=0}^{z} l_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ w_z(\varpi,t) &= \sum_{k=0}^{z} m_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}. \end{aligned} \tag{12}$$

Using the initial conditions, the $0-th$ fractional power series solutions can be obtained as

$$\begin{aligned} p_0(\varpi,t) &= g_0(\varpi) = \sigma_1(\varpi), \\ s_0(\varpi,t) &= f_0(\varpi) = \sigma_2(\varpi), \\ u_0(\varpi,t) &= l_0(\varpi) = \xi_1(\varpi), \\ w_0(\varpi,t) &= m_0(\varpi) = \xi_2(\varpi). \end{aligned} \tag{13}$$

The system of Equations (12) can be rewritten as

$$\begin{aligned} p_z(\varpi,t) &= \sigma_1(\varpi) + \sum_{k=1}^{z} g_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ s_z(\varpi,t) &= \sigma_2(\varpi) + \sum_{k=1}^{z} f_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ u_z(\varpi,t) &= \xi_1(\varpi) + \sum_{k=1}^{z} l_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}, \\ w_z(\varpi,t) &= \xi_2(\varpi) + \sum_{k=1}^{z} m_k(\varpi)\frac{t^{k\varrho}}{\Gamma(k\varrho+1)}. \end{aligned} \tag{14}$$

The required expressions for $z-th$ fractional power series solutions will be obtained by varying $k = 1,2,3,\ldots,z$ in $p(\varpi,t), s(\varpi,t), u(\varpi,t)$, and $w(\varpi,t)$.

Step B. For Equations (9) and (10), the residual functions will be constructed. This construction will be made in the following manner as

$$
\begin{aligned}
Res_p(\varpi,t) &= \frac{\partial^\varrho s}{\partial t^\varrho} + a_1\frac{\partial^3 w}{\partial \varpi^3} - b_1\frac{\partial^4 u}{\partial \varpi^4} - c_1 p(p^2+s^2) - d_1 p(u^2+w^2) + \alpha_1\frac{\partial s}{\partial \varpi} - \beta_1 u,\\
Res_s(\varpi,t) &= \frac{\partial^\varrho p}{\partial t^\varrho} + a_1\frac{\partial^3 u}{\partial \varpi^3} + b_1\frac{\partial^4 w}{\partial \varpi^4} + c_1 s(p^2+s^2) + d_1 s(u^2+w^2) + \alpha_1\frac{\partial p}{\partial \varpi} + \beta_1 w,\\
Res_u(\varpi,t) &= \frac{\partial^\varrho w}{\partial t^\varrho} + a_2\frac{\partial^3 s}{\partial \varpi^3} - b_2\frac{\partial^4 p}{\partial \varpi^4} - c_2 u(u^2+w^2) - d_2 p(p^2+s^2) + \alpha_2\frac{\partial w}{\partial \varpi} - \beta_2 p,\\
Res_w(\varpi,t) &= \frac{\partial^\varrho u}{\partial t^\varrho} + a_2\frac{\partial^3 p}{\partial \varpi^3} + b_2\frac{\partial^4 s}{\partial \varpi^4} + c_2 w(u^2+w^2) + d_2 w(p^2+s^2) + \alpha_2\frac{\partial u}{\partial \varpi} + \beta_2 s.
\end{aligned}
\tag{15}
$$

Taking $p_z = p_z(\varpi,t), s_z = s_z(\varpi,t), u_z = u_z(\varpi,t)$, and $w_z = w_z(\varpi,t)$, then the $z-th$ residual functions $z = 1,2,3,\dots$ can be defined as

$$
Res_{p,z}(\varpi,t) = \frac{\partial_z^\varrho s}{\partial t^\varrho} + a_1\frac{\partial^3 w_z}{\partial \varpi^3} - b_1\frac{\partial^4 u_z}{\partial \varpi^4} - c_1 p_z(p_z^2+s_z^2) - d_1 p_z(u_z^2+w_z^2) + \alpha_1\frac{\partial s_z}{\partial \varpi} - \beta_1 u_z,
$$

$$
Res_{s,z}(\varpi,t) = \frac{\partial^\varrho p_z}{\partial t^\varrho} + a_1\frac{\partial^3 u_z}{\partial \varpi^3} + b_1\frac{\partial^4 w_z}{\partial \varpi^4} + c_1 s_z(p_z^2+s_z^2) + d_1 s_z(u_z^2+w_z^2) + \alpha_1\frac{\partial p_z}{\partial \varpi} + \beta_1 w_z,
$$

$$
Res_{u,z}(\varpi,t) = \frac{\partial^\varrho w_z}{\partial t^\varrho} + a_2\frac{\partial^3 s_z}{\partial \varpi^3} - b_2\frac{\partial^4 p_z}{\partial \varpi^4} - c_2 u_z(u_z^2+w_z^2) - d_2 p_z(p_z^2+s_z^2) + \alpha_2\frac{\partial w_z}{\partial \varpi} - \beta_2 p_z,
$$

$$
Res_{w,z}(\varpi,t) = \frac{\partial^\varrho u_z}{\partial t^\varrho} + a_2\frac{\partial^3 p_z}{\partial \varpi^3} + b_2\frac{\partial^4 s_z}{\partial \varpi^4} + c_2 w_z(u_z^2+w_z^2) + d_2 w_z(p_z^2+s_z^2) + \alpha_2\frac{\partial u_z}{\partial \varpi} + \beta_2 s_z. \tag{16}
$$

Some important results of $Res_p(\varpi,t), Res_s(\varpi,t), Res_u(\varpi,t)$, and $Res_w(\varpi,t)$ that are useful for the residual power series solutions for $i = 1,2,\dots,z$ are stated below:

1. $Res_p(\varpi,t) = 0, Res_s(\varpi,t) = 0, Res_u(\varpi,t) = 0, Res_w(\varpi,t) = 0$
2. $\lim\limits_{z\to\infty} Res_{p,z}(\varpi,t) = Res_p(\varpi,t), \lim\limits_{z\to\infty} Res_{s,z}(\varpi,t) = Res_s(\varpi,t),$
   $\lim\limits_{z\to\infty} Res_{u,z}(\varpi,t) = Res_u(\varpi,t), \lim\limits_{z\to\infty} Res_{w,z}(\varpi,t) = Res_w(\varpi,t)$
   for each $\varpi \in I$ and $t = 0$,
3. $D^{i\varrho} Res_p(\varpi,0) = D^{i\varrho} Res_{p,z}(\varpi,0) = 0,$
   $D^{i\varrho} Res_s(\varpi,0) = D^{i\varrho} Res_{s,z}(\varpi,0) = 0,$
   $D^{i\varrho} Res_u(\varpi,0) = D^{i\varrho} Res_{u,z}(\varpi,0) = 0,$
   $D^{i\varrho} Res_w(\varpi,0) = D^{i\varrho} Res_{w,z}(\varpi,0) = 0.$

Step C. Substituting $p_z(\varpi,t), s_z(\varpi,t), u_z(\varpi,t)$, and $w_z(\varpi,t)$ into (16) and calculating the fractional derivative $D_t^{(z-1)\varrho}$ of $Res_{p,z}(\varpi,t), Res_{s,z}(\varpi,t), Res_{u,z}(\varpi,t)$, and $Res_{w,z}(\varpi,t)$, $z = 1,2,3,\dots$ at the initial point $(t = 0)$, together with results mentioned in Step B, the resulting algebraic systems are as follows:

$$
\begin{aligned}
D_t^{(z-1)\varrho} Res_{p,z}(\varpi,0) &= 0,\\
D_t^{(z-1)\varrho} Res_{s,z}(\varpi,0) &= 0,\\
D_t^{(z-1)\varrho} Res_{u,z}(\varpi,0) &= 0,\\
D_t^{(z-1)\varrho} Res_{w,z}(\varpi,0) &= 0.
\end{aligned}
\tag{17}
$$

Step D. The required values of $g_k(\varpi), f_k(\varpi), l_k(\varpi)$, and $m_k(\varpi), k = 1,2,\dots,z$ can be derived by solving Systems (17). Finally, the $z-th$ residual power series solutions can be obtained.

In the next discussion, the first, second, and third residual power series solutions of the suggested problem are developed in detail by following the above steps.

3.2. Residual Power Series Solutions of Proposed Model

For $z = 1$, the first approximated residual power series solutions can be written as

$$\begin{aligned}
p_1(\varpi,t) &= \sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(\varrho+1)},\\
s_1(\varpi,t) &= \sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(\varrho+1)},\\
u_1(\varpi,t) &= \xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(\varrho+1)},\\
w_1(\varpi,t) &= \xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(\varrho+1)}.
\end{aligned}$$

Following are the first residual functions:

$$\begin{aligned}
Res_{p,1}(\varpi,t) &= \frac{\partial^{\varrho} s_1}{\partial t^{\varrho}} + a_1\frac{\partial^3 w_1}{\partial\varpi^3} - b_1\frac{\partial^4 u_1}{\partial\varpi^4} - c_1 p_1(p_1^2+s_1^2) - d_1 p_1(u_1^2+w_1^2) + \alpha_1\frac{\partial s_1}{\partial\varpi} - \beta_1 u_1,\\
Res_{s,1}(\varpi,t) &= \frac{\partial^{\varrho} p_1}{\partial t^{\varrho}} + a_1\frac{\partial^3 u_1}{\partial\varpi^3} + b_1\frac{\partial^4 w_1}{\partial\varpi^4} + c_1 s_1(p_1^2+s_1^2) + d_1 s_1(u_1^2+w_1^2) + \alpha_1\frac{\partial p_1}{\partial\varpi} + \beta_1 w_1,\\
Res_{u,1}(\varpi,t) &= \frac{\partial^{\varrho} w_1}{\partial t^{\varrho}} + a_2\frac{\partial^3 s_1}{\partial\varpi^3} - b_2\frac{\partial^4 p_1}{\partial\varpi^4} - c_2 u_1(u_1^2+w_1^2) - d_2 u_1(p_1^2+s_1^2) + \alpha_2\frac{\partial w_1}{\partial\varpi} - \beta_2 p_1,\\
Res_{w,1}(\varpi,t) &= \frac{\partial^{\varrho} u_1}{\partial t^{\varrho}} + a_2\frac{\partial^3 p_1}{\partial\varpi^3} + b_2\frac{\partial^4 s_1}{\partial\varpi^4} + c_2 w_1(u_1^2+w_1^2) + d_2 w_1(p_1^2+s_1^2) + \alpha_2\frac{\partial u_1}{\partial\varpi} + \beta_2 s_1,
\end{aligned} \tag{18}$$

where $p_1 = p_1(\varpi,t), s_1 = s_1(\varpi,t), u_1 = u_1(\varpi,t)$, and $w_1 = w_1(\varpi,t)$. Substitute the $1st$ truncated series $p_1(\varpi,t), s_1(\varpi,t), u_1(\varpi,t)$, and $w_1(\varpi,t)$ into the first residual functions, $Res_{p,1}(\varpi,t)$, $Res_{s,1}(\varpi,t)$, $Res_{u,1}(\varpi,t)$, and $Res_{w,1}(\varpi,t)$, respectively, as

$$\begin{aligned}
Res_{p,1}(\varpi,t) =& f_1(\varpi) + a_1\frac{\partial^3}{\partial\varpi^3}\left(\xi_1(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(\varrho+1)}\right) - b_1\frac{\partial^4}{\partial\varpi^4}\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(\varrho+1)}\right)\\
&- c_1\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)\left[\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2 + \left(\sigma_2(\varpi) + f_1(\varpi)\right.\right.\\
&\left.\left.\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2\right] - d_1\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)\left[\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2\right.\\
&\left.+\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2\right] + \alpha_1\frac{\partial}{\partial\varpi}\left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)\\
&- \beta_1\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right),
\end{aligned}$$

$$\begin{aligned}
Res_{s,1}(\varpi,t) =& g_1(\varpi) + a_1\frac{\partial^3}{\partial\varpi^3}\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right) + b_1\frac{\partial^4}{\partial\varpi^4}\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)\\
&+ c_1\left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)\left[\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2 + \left(\sigma_2(\varpi) + f_1(\varpi)\right.\right.\\
&\left.\left.\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2\right] + d_1\left(\sigma_2(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)\left[\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2 +\right.\\
&\left.\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)^2\right] + \alpha_1\frac{\partial}{\partial\varpi}\left(\sigma_2(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right)\\
&+ \beta_1\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right),
\end{aligned}$$

$$
\begin{aligned}
Res_{u,1}(\varpi,t) =& m_1(\varpi) + a_2\frac{\partial^3}{\partial\varpi^3}\left(\sigma_2(\varpi)+f_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right) - b_2\frac{\partial^4}{\partial\varpi^4}\left(\sigma_1(\varpi)+g_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\\
&- c_2\left(\xi_1(\varpi)+l_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\left[\left(\xi_1(\varpi)+l_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2+\left(\xi_2(\varpi)+m_1(\varpi)\right.\right.\\
&\left.\left.\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2\right] - d_2\left(\sigma_1(\varpi)+g_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\left[\left(\sigma_1(\varpi)+g_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2+\right.\\
&\left.\left(\sigma_2(\varpi)+f_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2\right] + \alpha_2\frac{\partial}{\partial\varpi}\left(\xi_2(\varpi)+m_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\\
&- \beta_2\left(\sigma_1(\varpi)+g_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right),
\end{aligned}
$$

$$
\begin{aligned}
Res_{w,1}(\varpi,t) =& l_1(\varpi) + a_2\frac{\partial^3}{\partial\varpi^3}\left(\sigma_1(\varpi)+g_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right) + b_2\frac{\partial^4}{\partial\varpi^4}\left(\sigma_2(\varpi)+f_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\\
&+ c_2\left(\xi_2(\varpi)+m_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\left[\left(\xi_1(\varpi)+l_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2+\left(\xi_2(\varpi)+m_1(\varpi)\right.\right.\\
&\left.\left.\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2\right] + d_2\left(\xi_2(\varpi)+m_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\left[\left(\sigma_1(\varpi)+g_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2+\right.\\
&\left.\left(\sigma_2(\varpi)+f_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)^2\right] + \alpha_2\frac{\partial}{\partial\varpi}\left(\xi_1(\varpi)+l_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right)\\
&+ \beta_2\left(\sigma_2(\varpi)+f_1(\varpi)\frac{t^\varrho}{\Gamma(1+\varrho)}\right).
\end{aligned} \tag{19}
$$

By using Equations (17) and (19), the required values of $f_1(\varpi), g_1(\varpi), l_1(\varpi)$, and $m_1(\varpi)$ are given as

$$
\begin{aligned}
f_1(\varpi) &= \xi_1(\varpi)+\xi_1(\varpi)^2\sigma_1(\varpi)+\xi_2(\varpi)^2\sigma_1(\varpi)+\sigma_1(\varpi)^3+\sigma_1(\varpi)\sigma_2(\varpi)^2-\sigma_2'(\varpi)-\xi_2^{(3)}(\varpi)+\xi_1^{(4)}(\varpi),\\
g_1(\varpi) &= -\xi_2(\varpi)-\xi_1(\varpi)^2\sigma_2(\varpi)-\xi_2(\varpi)^2\sigma_2(\varpi)-\sigma_2(\varpi)^3-\sigma_1(\varpi)^2\sigma_2(\varpi)-\sigma_1'(\varpi)-\xi_1^{(3)}(\varpi)-\xi_2^{(4)}(\varpi),\\
m_1(\varpi) &= \sigma_1(\varpi)+\xi_1(\varpi)\xi_2(\varpi)^2+\xi_1(\varpi)\sigma_1(\varpi)^2+\xi_1(\varpi)^3+\xi_1(\varpi)\sigma_2(\varpi)^2-\xi_2'(\varpi)-\sigma_2^{(3)}(\varpi)+\sigma_1^{(4)}(\varpi),\\
l_1(\varpi) &= -\sigma_2(\varpi)-\xi_1(\varpi)^2\xi_2(\varpi)-\xi_2(\varpi)\sigma_1(\varpi)^2-\xi_2(\varpi)^3+\xi_2(\varpi)\sigma_2(\varpi)^2-\xi_1'(\varpi)-\sigma_1^{(3)}(\varpi)-\sigma_2^{(4)}(\varpi).
\end{aligned}
$$

The approximate solutions with $z=1$ can be written as

$$
\begin{aligned}
p_1(\varpi,t) = \sigma_1(\varpi) + \frac{t^\varrho}{\Gamma(1+\varrho)}\Big(&-\xi_2(\varpi)-\xi_1(\varpi)^2\sigma_2(\varpi)-\xi_2(\varpi)^2\sigma_2(\varpi)-\sigma_2(\varpi)^3-\sigma_1(\varpi)^2\sigma_2(\varpi)\\
&-\sigma_1'(\varpi)-\xi_1^{(3)}(\varpi)-\xi_2^{(4)}(\varpi)\Big),
\end{aligned}
$$

$$
\begin{aligned}
s_1(\varpi,t) = \sigma_2(\varpi) + \frac{t^\varrho}{\Gamma(1+\varrho)}\Big(&\xi_1(\varpi)+\xi_1(\varpi)^2\sigma_1(\varpi)+\xi_2(\varpi)^2\sigma_1(\varpi)+\sigma_1(\varpi)^3+\sigma_1(\varpi)\sigma_2(\varpi)^2\\
&-\sigma_2'(\varpi)-\xi_2^{(3)}(\varpi)+\xi_1^{(4)}(\varpi)\Big),
\end{aligned}
$$

$$
\begin{aligned}
u_1(\varpi,t) = \xi_1(\varpi) + \frac{t^\varrho}{\Gamma(1+\varrho)}\Big(&-\sigma_2(\varpi)-\xi_1(\varpi)^2\xi_2(\varpi)-\xi_2(\varpi)\sigma_1(\varpi)^2-\xi_2(\varpi)^3+\xi_2(\varpi)\sigma_2(\varpi)^2\\
&-\xi_1'(\varpi)-\sigma_1^{(3)}(\varpi)-\sigma_2^{(4)}(\varpi)\Big),
\end{aligned}
$$

$$
\begin{aligned}
w_1(\varpi,t) = \xi_2(\varpi) + \frac{t^\varrho}{\Gamma(1+\varrho)}\Big(&\sigma_1(\varpi)+\xi_1(\varpi)\xi_2(\varpi)^2+\xi_1(\varpi)\sigma_1(\varpi)^2+\xi_1(\varpi)^3+\xi_1(\varpi)\sigma_2(\varpi)^2\\
&-\xi_2'(\varpi)-\sigma_2^{(3)}(\varpi)+\sigma_1^{(4)}(\varpi)\Big).
\end{aligned}
$$

For $z = 2$, the second approximated residual power series solutions can be written as

$$\begin{aligned}
p_2(\varpi,t) &= \sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)},\\
s_2(\varpi,t) &= \sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)},\\
u_2(\varpi,t) &= \xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)},\\
w_2(\varpi,t) &= \xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}.
\end{aligned}$$

Following are the second residual functions:

$$\begin{aligned}
Res_{p,2}(\varpi,t) &= \frac{\partial^{\varrho} s_2}{\partial t^{\varrho}} + a_1\frac{\partial^3 w_2}{\partial\varpi^3} - b_1\frac{\partial^4 u_2}{\partial\varpi^4} - c_1 p_2(p_2^2+s_2^2) - d_1 p_2(u_2^2+w_2^2) + \alpha_1\frac{\partial s_2}{\partial\varpi} - \beta_1 u_2,\\
Res_{s,2}(\varpi,t) &= \frac{\partial^{\varrho} p_2}{\partial t^{\varrho}} + a_1\frac{\partial^3 u_2}{\partial\varpi^3} + b_1\frac{\partial^4 w_2}{\partial\varpi^4} + c_1 s_2(p_2^2+s_2^2) + d_1 s_2(u_2^2+w_2^2) + \alpha_1\frac{\partial p_2}{\partial\varpi} + \beta_1 w_2,\\
Res_{u,2}(\varpi,t) &= \frac{\partial^{\varrho} w_2}{\partial t^{\varrho}} + a_2\frac{\partial^3 s_2}{\partial\varpi^3} - b_2\frac{\partial^4 u_2}{\partial\varpi^4} - c_2 u_2(u_2^2+w_2^2) - d_2 u_2(p_2^2+s_2^2) + \alpha_2\frac{\partial w_2}{\partial\varpi} - \beta_2 p_2,\\
Res_{w,2}(\varpi,t) &= \frac{\partial^{\varrho} u_2}{\partial t^{\varrho}} + a_2\frac{\partial^3 p_2}{\partial\varpi^3} + b_2\frac{\partial^4 s_2}{\partial\varpi^4} + c_2 w_2(u_2^2+w_2^2) + d_2 w_2(p_2^2+s_2^2) + \alpha_2\frac{\partial u_2}{\partial\varpi} + \beta_2 s_2.
\end{aligned}$$

Substitute the second truncated series $p_2(\varpi,t), s_2(\varpi,t), u_2(\varpi,t)$, and $w_2(\varpi,t)$ into the second residual functions $Res_{p,2}(\varpi,t), Res_{s,2}(\varpi,t), Res_{u,2}(\varpi,t)$, and $Res_{w,2}(\varpi,t)$, respectively, as

$$\begin{aligned}
Res_{p,2}(\varpi,t) &= f_1(\varpi) + f_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + a_1\frac{\partial^3}{\partial\varpi^3}\left(\xi_1(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\\
&- b_1\frac{\partial^4}{\partial\varpi^4}\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right) - c_1\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right.\\
&\left.+ g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\left[\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2 + \left(\sigma_2(\varpi) + f_1(\varpi)\right.\right.\\
&\left.\left.\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] - d_1\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\\
&\left[\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2 + \left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\right.\right.\\
&\left.\left.\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] + \alpha_1\frac{\partial}{\partial\varpi}\left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\\
&- \beta_1\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right),
\end{aligned}$$

$$\begin{aligned}
Res_{s,2}(\varpi,t) =& g_1(\varpi) + g_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + a_1\frac{\partial^3}{\partial\varpi^3}\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\\
&+ b_1\frac{\partial^4}{\partial\varpi^4}\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right) + c_1\left(\sigma_2(\varpi) + f_1\right.\\
&\left.(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\left[\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right.\\
&\left.+ \left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] + d_1\left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\right.\\
&\left.\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\left[\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2 + \left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right.\right.\\
&\left.\left.+ m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] + \alpha_1\frac{\partial}{\partial\varpi}\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\\
&+ \beta_1\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right),
\end{aligned}$$

$$\begin{aligned}
Res_{u,2}(\varpi,t) =& m_1(\varpi) + m_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + a_2\frac{\partial^3}{\partial\varpi^3}\left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\\
&- b_2\frac{\partial^4}{\partial\varpi^4}\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right) - c_2\left(\xi_1(\varpi) + l_1(\varpi)\right.\\
&\left.\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\left[\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right.\\
&\left.+ \left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] - d_2\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right.\\
&\left.+ g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\left[\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2 + \left(\sigma_2(\varpi)\right.\right.\\
&\left.\left.+ f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] + \alpha_2\frac{\partial}{\partial\varpi}\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right.\\
&\left.+ m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right) - \beta_2\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right),
\end{aligned}$$

$$\begin{aligned}
Res_{w,2}(\varpi,t) =& l_1(\varpi) + l_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + a_2\frac{\partial^3}{\partial\varpi^3}\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\\
&+ b_2\frac{\partial^4}{\partial\varpi^4}\left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right) + c_2\left(\xi_2(\varpi) + m_1\right.\\
&\left.(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\left[\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right.\\
&\left.+ \left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] + d_2\left(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right.\\
&\left.+ m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)\left[\left(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2 + \left(\sigma_2(\varpi)\right.\right.\\
&\left.\left.+ f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right)^2\right] + \alpha_2\frac{\partial}{\partial\varpi}\left(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}\right.\\
&\left.+ l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right) + \beta_2\left(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\right).
\end{aligned} \tag{20}$$

By using Equations (17) and (20), the required values of $f_2(\varpi), g_2(\varpi), l_2(\varpi)$, and $m_2(\varpi)$ are given as

$$\begin{aligned}f_2(\varpi) =& l_1(\varpi) + g_1(\varpi)\xi_1^2(\varpi) + g_1(\varpi)\xi_2(\varpi)^2 + 2l_1(\varpi)\xi_1(\varpi)\sigma_1(\varpi) + 2m_1(\varpi)\xi_2(\varpi)\sigma_1(\varpi)\\ &+ 3g_1(\varpi)\sigma_1(\varpi)^2 + 2f_1(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) + g_1(\varpi)\sigma_2(\varpi)^2 - f_1'(\varpi) - m_1^{(3)}(\varpi) + l_1^{(4)}(\varpi),\end{aligned}$$

$$\begin{aligned}g_2(\varpi) =& - m_1(\varpi) - f_1(\varpi)\xi_1^2(\varpi) - f_1(\varpi)\xi_2(\varpi)^2 - 2l_1(\varpi)\xi_1(\varpi)\sigma_2(\varpi) - 2m_1(\varpi)\xi_2(\varpi)\sigma_2(\varpi)\\ &- 3f_1(\varpi)\sigma_2(\varpi)^2 - 2g_1(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) - g_1(\varpi)\sigma_2(\varpi)^2 - g_1'(\varpi) - l_1^{(3)}(\varpi) - m_1^{(4)}(\varpi),\end{aligned}$$

$$\begin{aligned}m_2(\varpi) =& g_1(\varpi) + l_1(\varpi)\xi_2^2(\varpi) + l_1(\varpi)\sigma_1(\varpi)^2 + 2g_1(\varpi)\xi_1(\varpi)\sigma_1(\varpi) + 2f_1(\varpi)\xi_1(\varpi)\sigma_2(\varpi)\\ &+ 3l_1(\varpi)\xi_1(\varpi)^2 + 2m_1(\varpi)\xi_1(\varpi)\xi_2(\varpi) + l_1(\varpi)\sigma_2(\varpi)^2 - m_1'(\varpi) - f_1^{(3)}(\varpi) + g_1^{(4)}(\varpi),\end{aligned}$$

$$\begin{aligned}l_2(\varpi) =& - f_1(\varpi) - m_1(\varpi)\sigma_1^2(\varpi) - m_1(\varpi)\xi_1(\varpi)^2 - 2g_1(\varpi)\xi_2(\varpi)\sigma_1(\varpi) - 2f_1(\varpi)\xi_2(\varpi)\sigma_2(\varpi)\\ &- 3m_1(\varpi)\xi_2(\varpi)^2 - 2l_1(\varpi)\xi_1(\varpi)\xi_2(\varpi) - m_1(\varpi)\sigma_2(\varpi)^2 - l_1'(\varpi) - g_1^{(3)}(\varpi) - f_1^{(4)}(\varpi).\end{aligned}$$

The following are approximate solutions with $z = 2$:

$$\begin{aligned}p_2(\varpi,t) =& \sigma_1(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\Big(- \xi_2(\varpi) - \xi_1(\varpi)^2\sigma_2(\varpi) - \xi_2(\varpi)^2\sigma_2(\varpi) - \sigma_2(\varpi)^3 - \sigma_1(\varpi)^2\sigma_2(\varpi) - \sigma_1'(\varpi)\\ &- \xi_1^{(3)}(\varpi) - \xi_2^{(4)}(\varpi)\Big) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\Big(- m_1(\varpi) - f_1(\varpi)\xi_1^2(\varpi) - f_1(\varpi)\xi_2(\varpi)^2 - 2l_1(\varpi)\xi_1(\varpi)\sigma_2(\varpi)\\ &- 2m_1(\varpi)\xi_2(\varpi)\sigma_2(\varpi) - 3f_1(\varpi)\sigma_2(\varpi)^2 - 2g_1(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) - g_1(\varpi)\sigma_2(\varpi)^2\\ &- g_1'(\varpi) - l_1^{(3)}(\varpi) - m_1^{(4)}(\varpi)\Big),\end{aligned}$$

$$\begin{aligned}s_2(\varpi,t) =& \sigma_2(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\Big(\xi_1(\varpi) + \xi_1(\varpi)^2\sigma_1(\varpi) + \xi_2(\varpi)^2\sigma_1(\varpi) + \sigma_1(\varpi)^3 + \sigma_1(\varpi)\sigma_2(\varpi)^2 - \sigma_2'(\varpi)\\ &- \xi_2^{(3)}(\varpi) + \xi_1^{(4)}(\varpi)\Big) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\Big(l_1(\varpi) + g_1(\varpi)\xi_1^2(\varpi) + g_1(\varpi)\xi_2(\varpi)^2 + 2l_1(\varpi)\xi_1(\varpi)\sigma_1(\varpi)\\ &+ 2m_1(\varpi)\xi_2(\varpi)\sigma_1(\varpi) + 3g_1(\varpi)\sigma_1(\varpi)^2 + 2f_1(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) + g_1(\varpi)\sigma_2(\varpi)^2\\ &- f_1'(\varpi) - m_1^{(3)}(\varpi) + l_1^{(4)}(\varpi)\Big),\end{aligned}$$

$$\begin{aligned}u_2(\varpi,t) =& \xi_1(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\Big(- \sigma_2(\varpi) - \xi_1(\varpi)^2\xi_2(\varpi) - \xi_2(\varpi)\sigma_1(\varpi)^2 - \xi_2(\varpi)^3 + \xi_2(\varpi)\sigma_2(\varpi)^2 - \xi_1'(\varpi)\\ &- \sigma_1^{(3)}(\varpi) - \sigma_2^{(4)}(\varpi)\Big) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\Big(- f_1(\varpi) - m_1(\varpi)\sigma_1^2(\varpi) - m_1(\varpi)\xi_1(\varpi)^2 - 2g_1(\varpi)\xi_2(\varpi)\sigma_1(\varpi)\\ &- 2f_1(\varpi)\xi_2(\varpi)\sigma_2(\varpi) - 3m_1(\varpi)\xi_2(\varpi)^2 - 2l_1(\varpi)\xi_1(\varpi)\xi_2(\varpi) - m_1(\varpi)\sigma_2(\varpi)^2\\ &- l_1'(\varpi) - g_1^{(3)}(\varpi) - f_1^{(4)}(\varpi)\Big),\end{aligned}$$

$$\begin{aligned}w_2(\varpi,t) =& \xi_2(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\Big(\sigma_1(\varpi) + \xi_1(\varpi)\xi_2(\varpi)^2 + \xi_1(\varpi)\sigma_1(\varpi)^2 + \xi_1(\varpi)^3 + \xi_1(\varpi)\sigma_2(\varpi)^2 - \xi_2'(\varpi)\\ &- \sigma_2^{(3)}(\varpi) + \sigma_1^{(4)}(\varpi)\Big) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\Big(g_1(\varpi) + l_1(\varpi)\xi_2^2(\varpi) + l_1(\varpi)\sigma_1(\varpi)^2 + 2g_1(\varpi)\xi_1(\varpi)\sigma_1(\varpi)\\ &+ 2f_1(\varpi)\xi_1(\varpi)\sigma_2(\varpi) + 3l_1(\varpi)\xi_1(\varpi)^2 + 2m_1(\varpi)\xi_1(\varpi)\xi_2(\varpi) + l_1(\varpi)\sigma_2(\varpi)^2\\ &- m_1'(\varpi) - f_1^{(3)}(\varpi) + g_1^{(4)}(\varpi)\Big).\end{aligned}$$

For $z = 3$, the third approximated residual power series solutions can be written as

$$\begin{aligned}
p_3(\varpi, t) &= \sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}, \\
s_3(\varpi, t) &= \sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}, \\
u_3(\varpi, t) &= \xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}, \\
w_3(\varpi, t) &= \xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}.
\end{aligned}$$

Following are the third residual functions:

$$\begin{aligned}
Res_{p,3}(\varpi, t) &= \frac{\partial^{\varrho} s_3}{\partial t^{\varrho}} + a_1\frac{\partial^3 w_3}{\partial \varpi^3} - b_1\frac{\partial^4 u_3}{\partial \varpi^4} - c_1 p_3(p_3^2 + s_3^2) - d_1 p_3(u_3^2 + w_3^2) + \alpha_1\frac{\partial s_3}{\partial \varpi} - \beta_1 u_3, \\
Res_{s,3}(\varpi, t) &= \frac{\partial^{\varrho} p_3}{\partial t^{\varrho}} + a_1\frac{\partial^3 u_3}{\partial \varpi^3} + b_1\frac{\partial^4 w_3}{\partial \varpi^4} + c_1 s_3(p_2^2 + s_2^2) + d_1 s_3(u_3^2 + w_3^2) + \alpha_1\frac{\partial p_3}{\partial \varpi} + \beta_1 w_3, \\
Res_{u,3}(\varpi, t) &= \frac{\partial^{\varrho} w_3}{\partial t^{\varrho}} + a_2\frac{\partial^3 s_3}{\partial \varpi^3} - b_2\frac{\partial^4 u_3}{\partial \varpi^4} - c_2 u_3(u_3^2 + w_3^2) - d_2 u_3(p_3^2 + s_3^2) + \alpha_2\frac{\partial w_3}{\partial \varpi} - \beta_2 p_3, \\
Res_{w,3}(\varpi, t) &= \frac{\partial^{\varrho} u_3}{\partial t^{\varrho}} + a_2\frac{\partial^3 p_3}{\partial \varpi^3} + b_2\frac{\partial^4 s_3}{\partial \varpi^4} + c_2 w_3(u_3^2 + w_3^2) + d_2 w_3(p_3^2 + s_3^2) + \alpha_2\frac{\partial u_3}{\partial \varpi} + \beta_2 s_3.
\end{aligned}$$

Substitute the third truncated series $p_3(\varpi, t), s_3(\varpi, t), u_3(\varpi, t)$, and $w_3(\varpi, t)$ into the third residual functions, $Res_{p,3}(\varpi, t), Res_{s,3}(\varpi, t), Res_{u,3}(\varpi, t)$, and $Res_{w,3}(\varpi, t)$, respectively, as

$$\begin{aligned}
Res_{p,3}(\varpi, t) = {} & f_1(\varpi) + f_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_3(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + a_1\frac{\partial^3}{\partial \varpi^3}\bigg(\xi_1(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} \\
& + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) - b_1\frac{\partial^4}{\partial \varpi^4}\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi) \\
& \frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) - c_1\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} \\
& + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)\bigg[\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi) \\
& \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2 + \bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi) \\
& \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] - d_1\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi) \\
& \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)\bigg[\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + l_3(\varpi) \\
& \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2 + \bigg(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi) \\
& \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] + \alpha_1\frac{\partial}{\partial \varpi}\bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi) \\
& \frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) - \beta_1\bigg(\xi_1(x) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi) \\
& \frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg),
\end{aligned}$$

$$Res_{s,3}(\varpi,t) = g_1(\varpi) + g_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_3(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + a_1\frac{\partial^3}{\partial\varpi^3}\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}$$
$$+ l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) + b_1\frac{\partial^4}{\partial\varpi^4}\bigg(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}$$
$$+ m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) + c_1\bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)$$
$$\bigg[\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2 + \bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}$$
$$+ f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] + d_1\bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)}$$
$$+ f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)\bigg[\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2 + \bigg(\xi_2(\varpi)$$
$$+ m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] + \alpha_1\frac{\partial}{\partial\varpi}\bigg(\sigma_1(\varpi) + g_1(\varpi)$$
$$\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) + \beta_1\bigg(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)$$
$$\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg),$$

$$Res_{u,3}(\varpi,t) = m_1(\varpi) + m_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_3(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + a_2\frac{\partial^3}{\partial\varpi^3}\bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}$$
$$+ f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) - b_2\frac{\partial^4}{\partial\varpi^4}\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)$$
$$\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) - c_2\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} +$$
$$l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)\bigg[\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2$$
$$+ \bigg(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] - d_2\bigg(\sigma_1(\varpi)$$
$$+ g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)\bigg[\bigg(\sigma_1(\varpi) + g_1(\varpi)$$
$$\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2 + \bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}$$
$$+ f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] + \alpha_2\frac{\partial}{\partial\varpi}\bigg(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)}$$
$$+ m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) - \beta_2\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + g_2(\varpi)$$
$$\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg),$$

$$\begin{aligned}
Res_{w,3}(\varpi,t) = {} & l_1(\varpi) + l_2(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_3(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + a_2\frac{\partial^3}{\partial\varpi^3}\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} \\
& + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) + b_2\frac{\partial^4}{\partial\varpi^4}\bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi) \\
& \frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) + c_2\bigg(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + \\
& m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)\bigg[\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2 \\
& + \bigg(\xi_2(\varpi) + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] + d_2\bigg(\xi_2(\varpi) \\
& + m_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + m_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + m_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)\bigg[\bigg(\sigma_1(\varpi) + g_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} \\
& + g_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + g_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2 + \bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} \\
& + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg)^2\bigg] + \alpha_2\frac{\partial}{\partial\varpi}\bigg(\xi_1(\varpi) + l_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + l_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} \\
& + l_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg) + \beta_2\bigg(\sigma_2(\varpi) + f_1(\varpi)\frac{t^{\varrho}}{\Gamma(1+\varrho)} + f_2(\varpi)\frac{t^{2\varrho}}{\Gamma(1+2\varrho)} + f_3(\varpi)\frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg).
\end{aligned} \tag{21}$$

By using Equations (17) and (21), we get the values of $f_3(\varpi), g_3(\varpi), l_3(\varpi)$, and $m_3(\varpi)$ as,

$$\begin{aligned}
f_3(\varpi) = {} & l_2(\varpi) + g_2(\varpi)\xi_1(\varpi)^2 + g_2(\varpi)\xi_2(\varpi)^2 + 2l_2(\varpi)\xi_1(\varpi)\sigma_1(\varpi) + 2m_2(\varpi)\xi_2(\varpi)\sigma_1(\varpi) \\
& + 3g_2(\varpi)\sigma_1(\varpi)^2 + 2f_2(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) + g_2(\varpi)\sigma_2(\varpi)^2 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\bigg(2g_1(\varpi) \\
& l_1(\varpi)\xi_1(\varpi) + 2g_1(\varpi)m_1(\varpi)\xi_2(\varpi) + f_1(\varpi)^2\sigma_1(\varpi) + 3g_1(\varpi)^2\sigma_1(\varpi) + l_1(\varpi)^2\sigma_1(\varpi) \\
& + m_1(\varpi)^2\sigma_1(\varpi) + 2f_1(\varpi)g_1(\varpi)\sigma_2(\varpi)\bigg) - f_2'(\varpi) - m_2^{(3)}(\varpi) + l_2^{(4)}(\varpi),
\end{aligned}$$

$$\begin{aligned}
g_3(\varpi) = {} & -m_2(\varpi) - f_2(\varpi)\xi_1(\varpi)^2 - f_2(\varpi)\xi_2(\varpi)^2 - 2l_2(\varpi)\xi_1(\varpi)\sigma_2(\varpi) - 2m_2(\varpi)\xi_2(\varpi)\sigma_2(\varpi) \\
& - 3f_2(\varpi)\sigma_2(\varpi)^2 - 2g_2(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) + f_2(\varpi)\sigma_1(\varpi)^2 - \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\bigg(2f_1(\varpi) \\
& l_1(\varpi)\xi_1(\varpi) + 2f_1(\varpi)m_1(\varpi)\xi_2(\varpi) + m_1(\varpi)^2\sigma_2(\varpi) + 3f_1(\varpi)^2\sigma_2(\varpi) + g_1(\varpi)^2\sigma_2(\varpi) \\
& + l_1(\varpi)^2\sigma_2(\varpi) + 2f_1(\varpi)g_1(\varpi)\sigma_1(\varpi)\bigg) - g_2'(\varpi) - l_2^{(3)}(\varpi) - m_2^{(4)}(\varpi),
\end{aligned}$$

$$\begin{aligned}
m_3(\varpi) = {} & g_2(\varpi) + l_2(\varpi)\sigma_1(\varpi)^2 + l_2(\varpi)\sigma_2(\varpi)^2 + 2f_2(\varpi)\xi_1(\varpi)\sigma_2(\varpi) + 2g_2(\varpi)\xi_1(\varpi)\sigma_1(\varpi) \\
& + 3l_2(\varpi)\xi_1(\varpi)^2 + 2m_2(\varpi)\xi_1(\varpi)\xi_2(\varpi) + l_2(\varpi)\xi_2(\varpi)^2 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\bigg(2l_1(\varpi) \\
& m_1(\varpi)\xi_2(\varpi) + 2f_1(\varpi)l_1(\varpi)\sigma_2(\varpi) + m_1(\varpi)^2\xi_1(\varpi) + 3l_1(\varpi)^2\xi_1(\varpi) + g_1(\varpi)^2\xi_1(\varpi) \\
& + f_1(\varpi)^2\xi_1(\varpi) + 2g_1(\varpi)l_1(\varpi)\sigma_1(\varpi)\bigg) - m_2'(\varpi) - f_2^{(3)}(\varpi) + g_2^{(4)}(\varpi),
\end{aligned}$$

$$\begin{aligned}
l_3(\varpi) = {} & -f_2(\varpi) - m_2(\varpi)\xi_1(\varpi)^2 - 2l_2\xi_1(\varpi)\xi_2(\varpi) - 3m_2(\varpi)\xi_2(\varpi)^2 - 2f_2(\varpi)\xi_2(\varpi)\sigma_2(\varpi) \\
& - 2g_2(\varpi)^2\xi_2(\varpi)\sigma_1(\varpi) - m_2(\varpi)\sigma_1(\varpi)^2 + m_2(\varpi)\sigma_2(\varpi)^2 - \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\bigg(2f_1(\varpi) \\
& m_1(\varpi)\xi_2(\varpi)2f_1(\varpi)m_1(\varpi)\sigma_2(\varpi) + l_1(\varpi)m_1(\varpi)\xi_1(\varpi) + f_1(\varpi)^2\xi_2(\varpi) + l_1(\varpi)^2\xi_2(\varpi) \\
& + g_1(\varpi)^2\xi_2(\varpi)m_1(\varpi)^2\xi_2(\varpi) + 3m_1(\varpi)^2\xi_1(\varpi) + 2g_1(\varpi)m_1(\varpi)\sigma_1(\varpi)\bigg) \\
& + l_2'(\varpi) + g_2^{(3)}(\varpi) + f_2^{(4)}(\varpi).
\end{aligned}$$

Thus, following are the approximate solutions with $z = 3$:

$$
\begin{aligned}
p_3(\varpi,t) =&\sigma_1(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\bigg(- \xi_2(\varpi) - \xi_1(\varpi)^2\sigma_2(\varpi) - \xi_2(\varpi)^2\sigma_2(\varpi) - \sigma_2(\varpi)^3 - \sigma_1(\varpi)^2\sigma_2(\varpi) \\
& - \sigma_1'(\varpi) - \xi_1^{(3)}(\varpi) - \xi_2^{(4)}(\varpi)\bigg) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\bigg(- m_1(\varpi) - f_1(\varpi)\xi_1^2(\varpi) - f_1(\varpi)\xi_2(\varpi)^2 \\
& - 2l_1(\varpi)\xi_1(\varpi)\sigma_2(\varpi) - 2m_1(\varpi)\xi_2(\varpi)\sigma_2(\varpi) - 3f_1(\varpi)\sigma_2(\varpi)^2 - 2g_1(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) \\
& - g_1(\varpi)\sigma_2(\varpi)^2 - g_1'(\varpi) - l_1^{(3)}(\varpi) - m_1^{(4)}(\varpi)\bigg) + \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg(- m_2(\varpi) \\
& - f_2(\varpi)\xi_1(\varpi)^2 - f_2(\varpi)\xi_2(\varpi)^2 - 2l_2(\varpi)\xi_1(\varpi)\sigma_2(\varpi) - 2m_2(\varpi)\xi_2(\varpi)\sigma_2(\varpi) \\
& - 3f_2(\varpi)\sigma_2(\varpi)^2 - 2g_2(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) + f_2(\varpi)\sigma_1(\varpi)^2 - \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\bigg[2f_1(\varpi) \\
& l_1(\varpi)\xi_1(\varpi) + 2f_1(\varpi)m_1(\varpi)\xi_2(\varpi) + m_1(\varpi)^2\sigma_2(\varpi) + 3f_1(\varpi)^2\sigma_2(\varpi) + g_1(\varpi)^2\sigma_2(\varpi) \\
& + l_1(\varpi)^2\sigma_2(\varpi) + 2f_1(\varpi)g_1(\varpi)\sigma_1(\varpi)\bigg] - g_2'(\varpi) - l_2^{(3)}(\varpi) - m_2^{(4)}(\varpi)\bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_3(\varpi,t) =&\sigma_2(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\bigg(\xi_1(\varpi) + \xi_1(\varpi)^2\sigma_1(\varpi) + \xi_2(\varpi)^2\sigma_1(\varpi) + \sigma_1(\varpi)^3 + \sigma_1(\varpi)\sigma_2(\varpi)^2 \\
& - \sigma_2'(\varpi) - \xi_2^{(3)}(\varpi) + \xi_1^{(4)}(\varpi)\bigg) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\bigg(l_1(\varpi) + g_1(\varpi)\xi_1^2(\varpi) + g_1(\varpi)\xi_2(\varpi)^2 \\
& + 2l_1(\varpi)\xi_1(\varpi)\sigma_1(\varpi) + 2m_1(\varpi)\xi_2(\varpi)\sigma_1(\varpi) + 3g_1(\varpi)\sigma_1(\varpi)^2 + 2f_1(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) \\
& + g_1(\varpi)\sigma_2(\varpi)^2 - f_1'(\varpi) - m_1^{(3)}(\varpi) + l_1^{(4)}(\varpi)\bigg) + \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg(l_2(\varpi) \\
& + g_2(\varpi)\xi_1(\varpi)^2 + g_2(\varpi)\xi_2(\varpi)^2 + 2l_2(\varpi)\xi_1(\varpi)\sigma_1(\varpi) + 2m_2(\varpi)\xi_2(\varpi)\sigma_1(\varpi) \\
& + 3g_2(\varpi)\sigma_1(\varpi)^2 + 2f_2(\varpi)\sigma_1(\varpi)\sigma_2(\varpi) + g_2(\varpi)\sigma_2(\varpi)^2 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\bigg[2g_1(\varpi) \\
& l_1(\varpi)\xi_1(\varpi) + 2g_1(\varpi)m_1(\varpi)\xi_2(\varpi) + f_1(\varpi)^2\sigma_1(\varpi) + 3g_1(\varpi)^2\sigma_1(\varpi) + l_1(\varpi)^2\sigma_1(\varpi) \\
& + m_1(\varpi)^2\sigma_1(\varpi) + 2f_1(\varpi)g_1(\varpi)\sigma_2(\varpi)\bigg] - f_2'(\varpi) - m_2^{(3)}(\varpi) + l_2^{(4)}(\varpi))\bigg),
\end{aligned}
$$

$$
\begin{aligned}
u_3(\varpi,t) =&\xi_1(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\bigg(- \sigma_2(\varpi) - \xi_1(\varpi)^2\xi_2(\varpi) - \xi_2(\varpi)\sigma_1(\varpi)^2 - \xi_2(\varpi)^3 + \xi_2(\varpi)\sigma_2(\varpi)^2 \\
& - \xi_1'(\varpi) - \sigma_1^{(3)}(\varpi) - \sigma_2^{(4)}(\varpi)\bigg) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\bigg(- f_1(\varpi) - m_1(\varpi)\sigma_1^2(\varpi) - m_1(\varpi)\xi_1(\varpi)^2 \\
& - 2g_1(\varpi)\xi_2(\varpi)\sigma_1(\varpi) - 2f_1(\varpi)\xi_2(\varpi)\sigma_2(\varpi) - 3m_1(\varpi)\xi_2(\varpi)^2 - 2l_1(\varpi)\xi_1(\varpi)\xi_2(\varpi) \\
& - m_1(\varpi)\sigma_2(\varpi)^2 - l_1'(\varpi) - g_1^{(3)}(\varpi) - f_1^{(4)}(\varpi)\bigg) + \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\bigg(- f_2(\varpi) \\
& - m_2(\varpi)\xi_1(\varpi)^2 - 2l_2\xi_1(\varpi)\xi_2(\varpi) - 3m_2(\varpi)\xi_2(\varpi)^2 - 2f_2(\varpi)\xi_2(\varpi)\sigma_2(\varpi) \\
& - 2g_2(\varpi)^2\xi_2(\varpi)\sigma_1(\varpi) - m_2(\varpi)\sigma_1(\varpi)^2 + m_2(\varpi)\sigma_2(\varpi)^2 - \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\bigg[2f_1(\varpi) \\
& m_1(\varpi)\xi_2(\varpi)2f_1(\varpi)m_1(\varpi)\sigma_2(\varpi) + l_1(\varpi)m_1(\varpi)\xi_1(\varpi) + f_1(\varpi)^2\xi_2(\varpi) + l_1(\varpi)^2\xi_2(\varpi) \\
& + g_1(\varpi)^2\xi_2(\varpi)m_1(\varpi)^2\xi_2(\varpi) + 3m_1(\varpi)^2\xi_1(\varpi) + 2g_1(\varpi)m_1(\varpi)\sigma_1(\varpi)\bigg] \\
& + l_2'(\varpi) + g_2^{(3)}(\varpi) + f_2^{(4)}(\varpi)\bigg),
\end{aligned}
$$

$$\begin{aligned} w_3(\varpi,t) =& \xi_2(\varpi) + \frac{t^{\varrho}}{\Gamma(1+\varrho)}\Big(\sigma_1(\varpi) + \xi_1(\varpi)\xi_2(\varpi)^2 + \xi_1(\varpi)\sigma_1(\varpi)^2 + \xi_1(\varpi)^3 + \xi_1(\varpi)\sigma_2(\varpi)^2 \\ & - \xi_2'(\varpi) - \sigma_2^{(3)}(\varpi) + \sigma_1^{(4)}(\varpi)\Big) + \frac{t^{2\varrho}}{\Gamma(1+2\varrho)}\Big(g_1(\varpi) + l_1(\varpi)\xi_2^2(\varpi) + l_1(\varpi)\sigma_1(\varpi)^2 \\ & + 2g_1(\varpi)\xi_1(\varpi)\sigma_1(\varpi) + 2f_1(\varpi)\xi_1(\varpi)\sigma_2(\varpi) + 3l_1(\varpi)\xi_1(\varpi)^2 + 2m_1(\varpi)\xi_1(\varpi)\xi_2(\varpi) \\ & + l_1(\varpi)\sigma_2(\varpi)^2 - m_1'(\varpi) - f_1^{(3)}(\varpi) + g_1^{(4)}(\varpi)\Big) + \frac{t^{3\varrho}}{\Gamma(1+3\varrho)}\Big(g_2(\varpi) \\ & + l_2(\varpi)\sigma_1(\varpi)^2 + l_2(\varpi)\sigma_2(\varpi)^2 + 2f_2(\varpi)\xi_1(\varpi)\sigma_2(\varpi) + 2g_2(\varpi)\xi_1(\varpi)\sigma_1(\varpi) \\ & + 3l_2(\varpi)\xi_1(\varpi)^2 + 2m_2(\varpi)\xi_1(\varpi)\xi_2(\varpi) + l_2(\varpi)\xi_2(\varpi)^2 + \frac{\Gamma(1+2\varrho)}{\Gamma(1+\varrho)^2}\Big[2l_1(\varpi) \\ & m_1(\varpi)\xi_2(\varpi) + 2f_1(\varpi)l_1(\varpi)\sigma_2(\varpi) + m_1(\varpi)^2\xi_1(\varpi) + 3l_1(\varpi)^2\xi_1(\varpi) + g_1(\varpi)^2\xi_1(\varpi) \\ & + f_1(\varpi)^2\xi_1(\varpi) + 2g_1(\varpi)l_1(\varpi)\sigma_1(\varpi)\Big] - m_2'(\varpi) - f_2^{(3)}(\varpi) + g_2^{(4)}(\varpi)\Big). \end{aligned}$$

4. Graphical Representation

To depict the reliability of the RPST to obtain the optical solutions to fiber Bragg gratings with cubic–quartic dispersive reflectivity having the Kerr law of nonlinear refractive index structures, three applications are examined. The fractional derivative is defined by Caputo's sense.

Example 1. *Examine the system,*

$$\begin{aligned} \iota D_t^{\varrho} n + \iota a_1 o_{\varpi\varpi\varpi} + b_1 o_{\varpi\varpi\varpi\varpi} + (c_1|n|^2 + d_1|o|^2)n + \iota\alpha_1 n_{\varpi} + \beta_1 o = 0, \\ \iota D_t^{\varrho} o + \iota a_2 n_{\varpi\varpi\varpi} + b_2 n_{\varpi\varpi\varpi\varpi} + (c_2|n|^2 + d_2|n|^2)o + \iota\alpha_2 o_{\varpi} + \beta_2 n = 0, \end{aligned} \tag{22}$$

where $0 < \varrho < 1$, *subject to the initial conditions,*

$$n(\varpi,0) = \left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\left[(\gamma_1-\gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\xi - \xi_0\right)\right) + \gamma_2\right]\right)\exp^{\iota(-\kappa\varpi+\theta)}, \tag{23}$$

$$o(\varpi,0) = 2\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\left[(\gamma_1-\gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\xi - \xi_0\right)\right) + \gamma_2\right]\right)\exp^{\iota(-\kappa\varpi+\theta)}. \tag{24}$$

This problem has an analytical solution for $\varrho = 1$ that is provided in [17] as

$$n(\varpi,t) = \left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\left[(\gamma_1-\gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\xi - \xi_0\right)\right) + \gamma_2\right]\right)\exp^{\iota(-\kappa\varpi+\omega t+\theta)}, \tag{25}$$

$$o(\varpi,t) = 2\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\left[(\gamma_1-\gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}}\xi - \xi_0\right)\right) + \gamma_2\right]\right)\exp^{\iota(-\kappa\varpi+\omega t+\theta)}. \tag{26}$$

For graphical observations, 2D plotting, contour plotting, and 3D plotting are constructed for Example 1 and shown in the Figures 1–6. It can be seen that the power series solutions very accurately agreed with the analytical solutions. Finally, by employing a large number of terms from the residual power series approximations, a lower error can be attained. The 2D graphs for Example 1 illustrate that the solution had nearly identical behavior for the standard case $\varrho = 1$ and various values of ϱ in terms of accuracy.

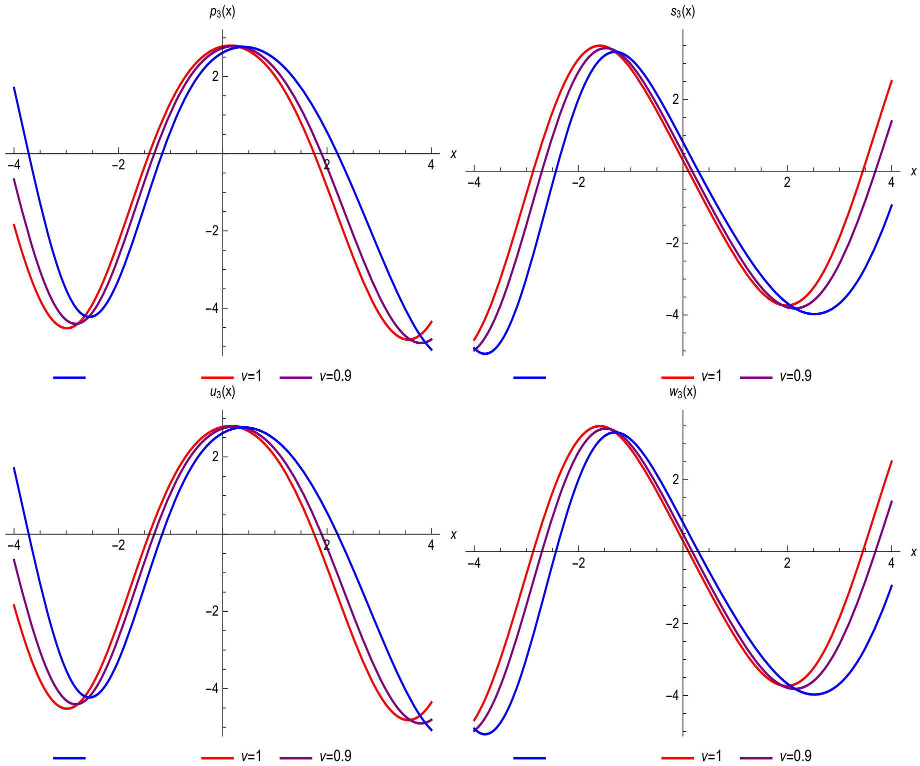

Figure 1. Two-dimensional (2D) graphs of the 3rd residual power series solutions with various values of ϱ for Example 1 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1, t = 0.01$, and $\theta = 0$.

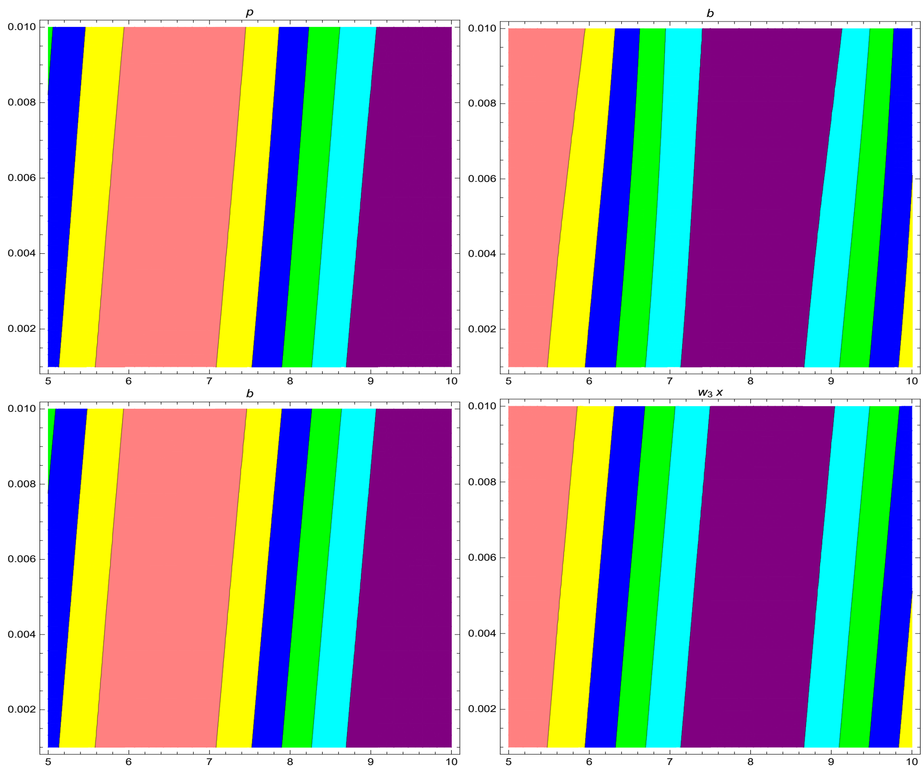

Figure 2. Contour plots for the 3rd residual power series solutions $p_3(\varpi, t), s_3(\varpi, t), u_3(\varpi, t)$, and $w_3(\varpi, t)$ with various values of ϱ for Example 1 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1, t = 0.01$, and $\theta = 0$.

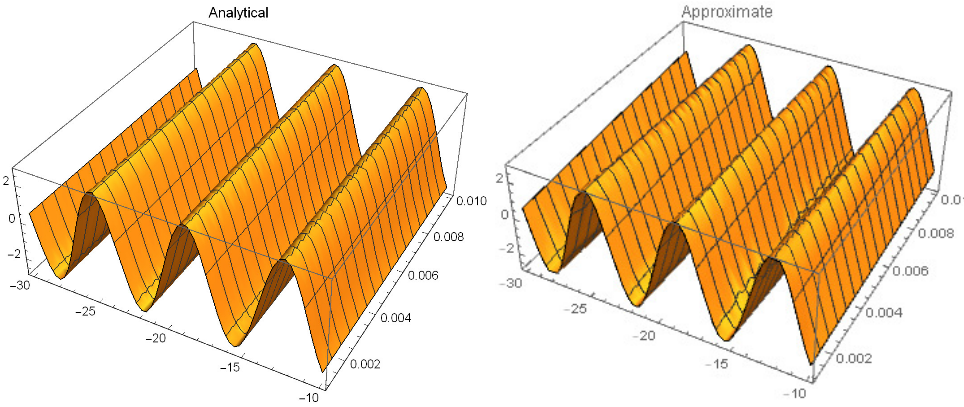

Figure 3. Three-dimensional (3D) surface plots of analytical solution and approximate solutions $p_3(\varpi, t)$ at $\varrho = 1$ of Example 1 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1$, and $\theta = 0$.

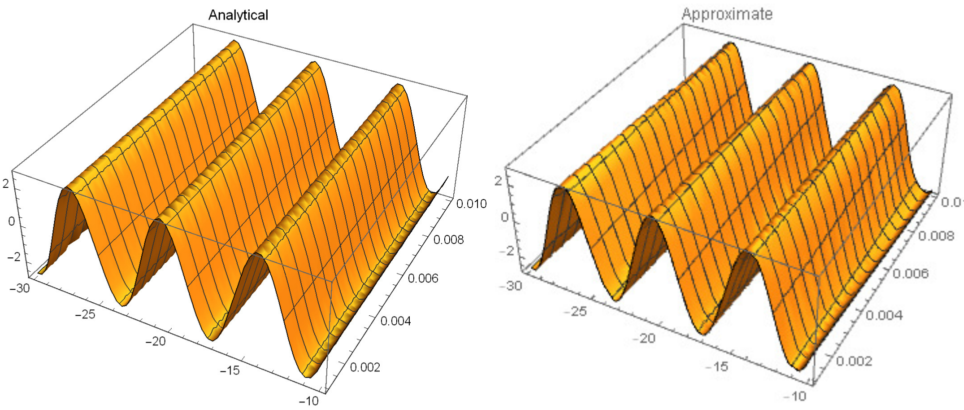

Figure 4. Three-dimensional (3D) surface plots of analytical solution and approximate solutions $s_3(\varpi, t)$ at $\varrho = 1$ of Example 1 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1$, and $\theta = 0$.

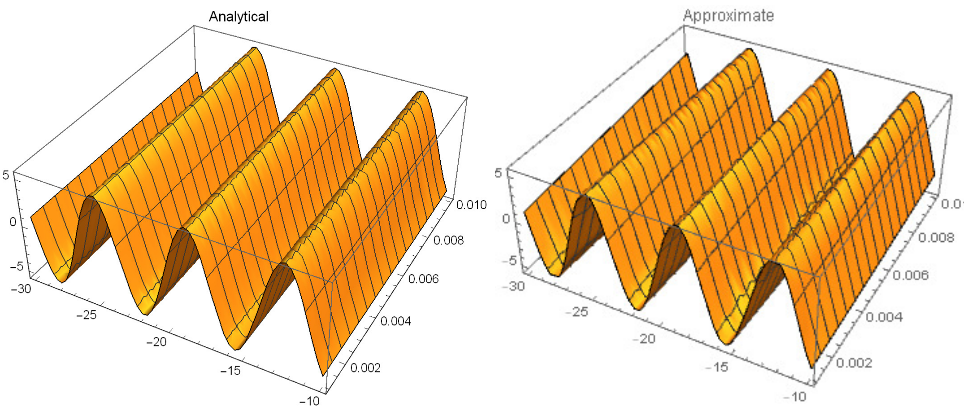

Figure 5. Three-dimensional (3D) surface plots of analytical solution and approximate solutions $u_3(\varpi, t)$ at $\varrho = 1$ of Example 1 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1$, and $\theta = 0$.

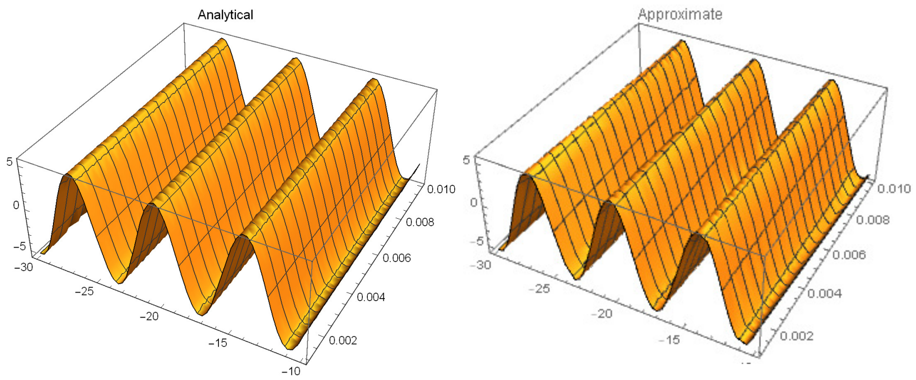

Figure 6. Three-dimensional (3D) surface plots of analytical solution and approximate solutions $w_3(\varpi, t)$ at $\varrho = 1$ of Example 1 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1$, and $\theta = 0$.

Example 2. *Examine the system*

$$\begin{aligned} &\iota D_t^{\varrho} n + \iota a_1 o_{\varpi\varpi\varpi} + b_1 o_{\varpi\varpi\varpi\varpi} + (c_1|n|^2 + d_1|o|^2)n + \iota\alpha_1 n_{\varpi} + \beta_1 o = 0, \\ &\iota D_t^{\varrho} o + \iota a_2 n_{\varpi\varpi\varpi} + b_2 n_{\varpi\varpi\varpi\varpi} + (c_2|n|^2 + d_2|n|^2)o + \iota\alpha_2 o_{\varpi} + \beta_2 n = 0, \end{aligned} \tag{27}$$

where $0 < \varrho < 1$*, subject to the initial conditions*

$$n(\varpi, 0) = \left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \left[(\gamma_1 - \gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \xi - \xi_0\right)\right) + \gamma_2\right]\right) tanh(-\kappa\varpi + \theta)\exp^{\iota(-\kappa\varpi+\theta)}, \tag{28}$$

$$o(\varpi, 0) = 2\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \left[(\gamma_1 - \gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \xi - \xi_0\right)\right) + \gamma_2\right]\right) tanh(-\kappa\varpi + \theta)\exp^{\iota(-\kappa\varpi+\theta)}. \tag{29}$$

The advantage of using the RPST is that it establishes the continuous approximated solutions. In the Figures 7–9, the power series generated solutions of (27) are depicted geometrically by 3D plotting, contour plotting and 2D plotting. Error can be reduced by taking many terms, and the efficiency of the method can be increased.

Example 3. *Examine the system,*

$$\begin{aligned} &\iota D_t^{\varrho} n + \iota a_1 o_{\varpi\varpi\varpi} + b_1 o_{\varpi\varpi\varpi\varpi} + (c_1|n|^2 + d_1|o|^2)n + \iota\alpha_1 n_{\varpi} + \beta_1 o = 0, \\ &\iota D_t^{\varrho} o + \iota a_2 n_{\varpi\varpi\varpi} + b_2 n_{\varpi\varpi\varpi\varpi} + (c_2|n|^2 + d_2|n|^2)o + \iota\alpha_2 o_{\varpi} + \beta_2 n = 0, \end{aligned} \tag{30}$$

where $0 < \varrho < 1$*, subject to the initial conditions,*

$$n(\varpi, 0) = \left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \left[(\gamma_1 - \gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \xi - \xi_0\right)\right) + \gamma_2\right]\right) sinh(-\kappa\varpi + \theta)\exp^{\iota(-\kappa\varpi+\theta)}, \tag{31}$$

$$o(\varpi, 0) = 2\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \left[(\gamma_1 - \gamma_2)\tanh^2\left(\frac{\sqrt{(\gamma_1-\gamma_2)}}{2}\left(\left(\frac{-2A_3}{15k^2}\right)^{\frac{-1}{6}} \xi - \xi_0\right)\right) + \gamma_2\right]\right) sinh(-\kappa\varpi + \theta)\exp^{\iota(-\kappa\varpi+\theta)}. \tag{32}$$

Figure 10 represents the mathematical behavior of Equation (30) subject to initial conditions Equations (31) and (32). It can be clearly observed that the obtained solution also depends on the initial conditions. By varying the initial conditions, the mathematical behavior of a problem also changes. Consequently, the proposed technique is an effective technique for the approximated solutions of different fractional-order models.

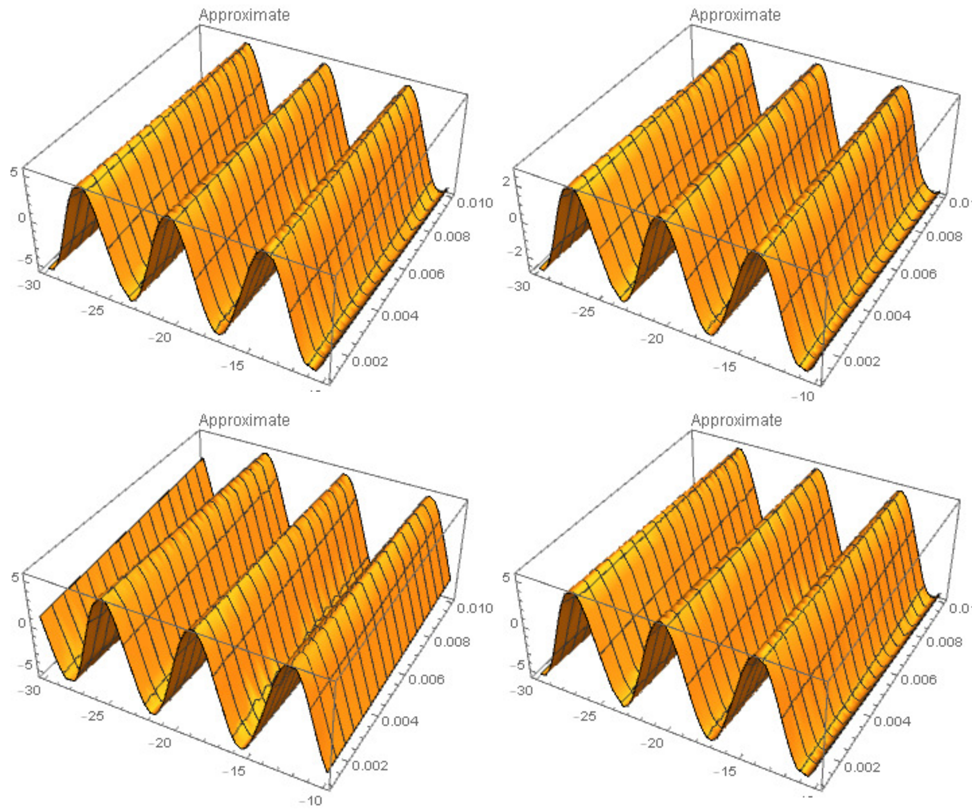

Figure 7. Three-dimensional (3D) surface plots approximate solutions $p_3(\varpi, t)$, $s_3(\varpi, t)$, $u_3(\varpi, t)$, and $w_3(\varpi, t)$ at $\varrho = 1$ of Example 2 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1$, and $\theta = 0$.

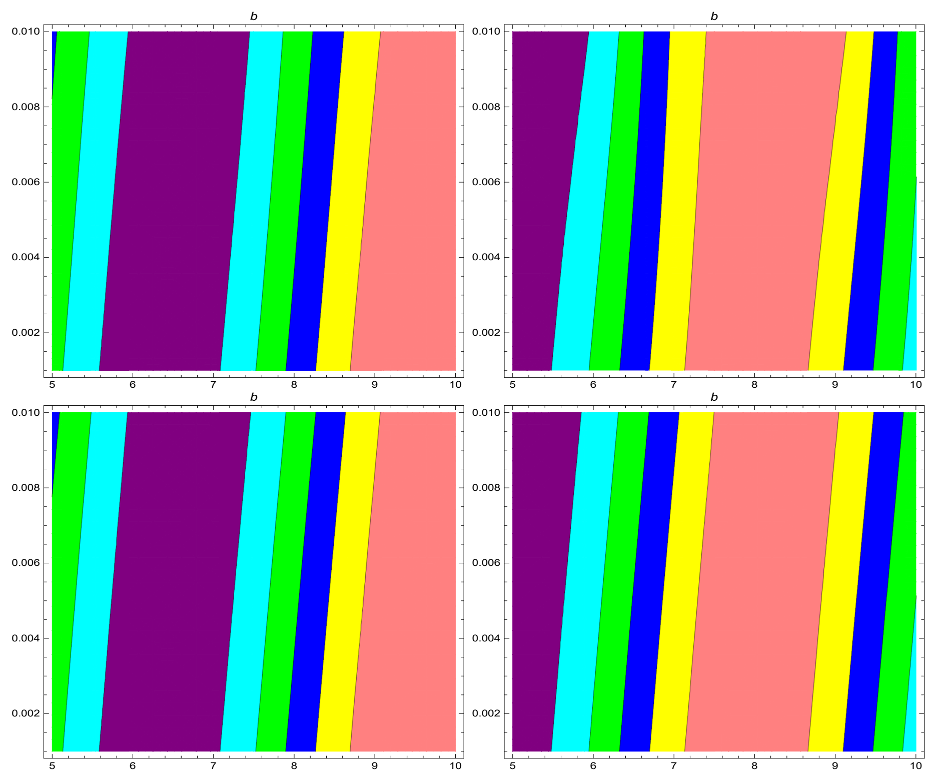

Figure 8. Contour plots for the 3rd residual power series solutions with various values of ϱ for Example 2 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1, t = 0.01$, and $\theta = 0$.

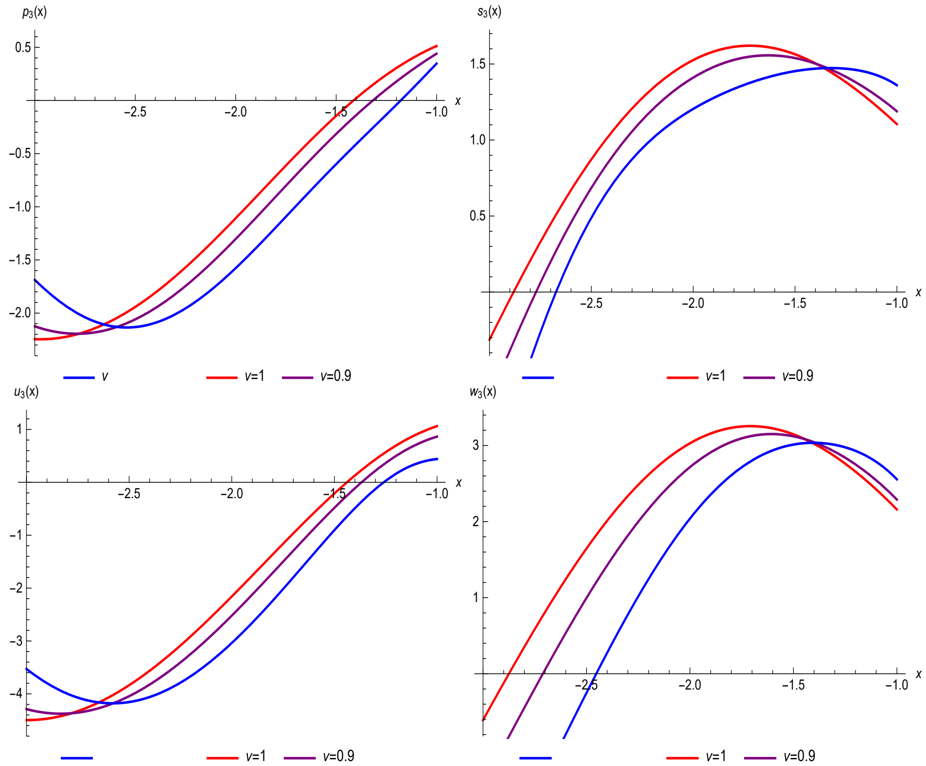

Figure 9. Two-dimensional (2D) graphs of the 3rd residual power series solutions $p_3(\varpi, t), s_3(\varpi, t), u_3(\varpi, t)$, and $w_3(\varpi, t)$ with various values of ϱ for Example 2 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1, t = 0.01$, and $\theta = 0$.

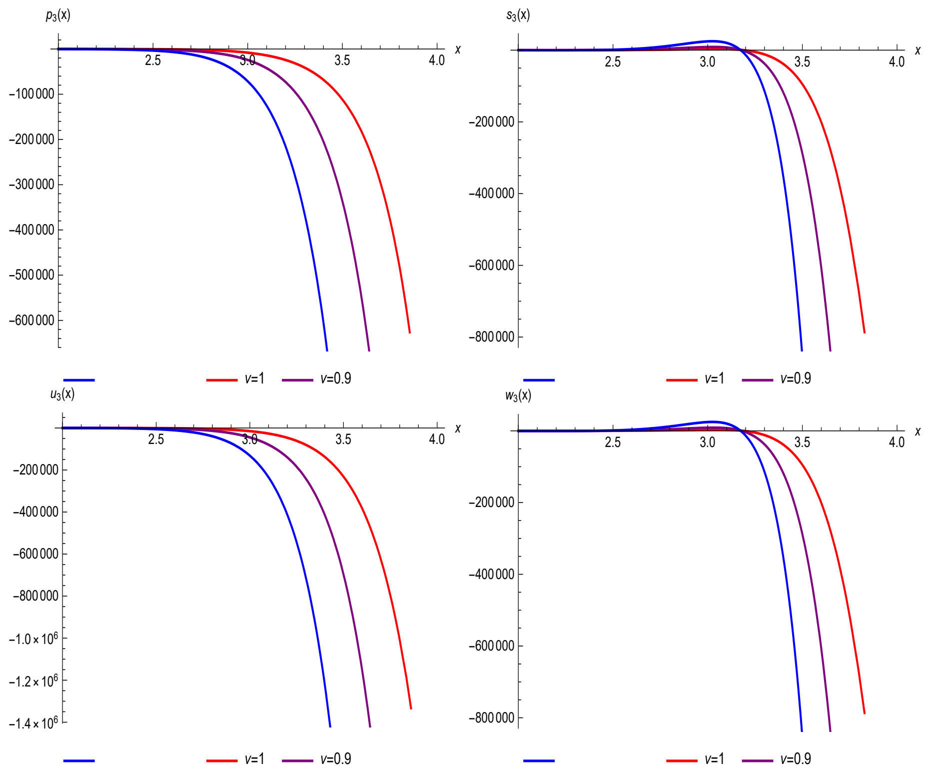

Figure 10. Two-dimensional (2D) graphs of the 3rd residual power series solutions with various values of ϱ for Example 3 with $A_3 = -1, k = 1, \gamma_1 = 2, \gamma_2 = 1, \xi_0 = 0, \kappa = 1, t = 0.01$, and $\theta = 0$.

5. Conclusions

In this paper, the RPST with Caputo's time-fractional derivative is applied to develop the approximate solutions to FBGs with cubic–quartic dispersive reflectivity having the Kerr law of nonlinear refractive index, with high accuracy for the first time. The efficiency and reliability of the RPST are established by three test applications with different initial conditions. It is worth mentioning that the RPST structure has a rapidly converging series with components that are easily calculated with symbolic calculation software. The obtained results are also demonstrated through 2D and 3D representations by taking different values of ϱ. Since the exact solution of the problem for $\varrho = 1$ is available in the literature, a graphical comparison is made to confirm that the RPS approximation solutions are in agreement with the exact solutions for $\varrho = 1$. The variation in the solutions for change in the value of ϱ is also observed. The consequence found by using the active RPST is that it can be successfully used to investigate the dynamics of nonlinear models in the field of optical fibers.

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