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Article

An Accurate Approach to Simulate the Fractional Delay Differential Equations

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Abstract: The fractional Legendre polynomials (FLPs) that we present as an effective method for solving fractional delay differential equations (FDDEs) are used in this work. The Liouville–Caputo sense is used to characterize fractional derivatives. This method uses the spectral collocation technique based on FLPs. The proposed method converts FDDEs into a set of algebraic equations. We lay out a study of the convergence analysis and figure out the upper bound on error for the approximate solution. Examples are provided to demonstrate the precision of the suggested approach.

Keywords: FDDEs; orthogonal system; spectral collocation technique; convergence analysis; fractional-order Legendre polynomials

MSC: 41A30; 65N20

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1. Introduction

In the hope of simulating a variety of physical events, it is crucial to use differential equations based on fractional-order operators [1,2]. Some applications of these equations include earthquake simulation, damping principles, fluid dynamics, biology, physics, and engineering [3–7]. Numerous techniques have been employed to locate appropriate solutions to these equations depending on their physical structure. The majority of the current systems are flawed and unworkable considering the actual physical existence of these problems. The numerical findings show the full reliability of the proposed algorithms [8–10]. Since explicit exact solutions for FDEs are still lacking, approximation and numerical techniques such as the spectral collocation method (SCM), FEM, FVM, and many others have been used to solve many fractional models [11–15]. The paper's primary goal is to use the mentioned algorithm ([16,17]) to obtain the numerical solution of the following FDDE using SCM depending on the fractional Legendre polynomials [18]:

$$D^{\alpha}u(x) = f(x, u(x), u(g(x))), \qquad 0 < x < 1, \qquad 0 < \alpha \le 2, \tag{1}$$

where

$$u(0) = \lambda_0, \qquad u(1) = \lambda_1, \tag{2}$$

where the order of the Liouville–Caputo fractional is α. The delay function is a function g that satisfies condition $a \le g(x) \le x$, $x \in [a,b]$ and $g(x) \in C[a,b]$. Also, we shall calculate an upper bound for the estimated solution's resulting error and the convergence will be studied. In ([19,20]), the authors used spline functions to study both the error and stability analysis for first-order delay differential equations. We plan to discretize Equation (1) using the Legendre collocation method to produce an algebraic system of equations (linear/nonlinear) and solve it.

The manuscript is structured as follows: Section 2 presents the definitions and approximate formula for the fractional derivative. Section 3 provides some thoughts on the recently revealed fractional-order Legendre polynomials. The polynomials approximation and error estimation are discussed in Section 4. The implementation of the Legendre spectral method for solving FDDE is introduced in Section 5. Section 6 also discusses numerical simulation. We provide the conclusions and some observations in Section 7.

2. Preliminaries and Notations

Definition 1. *The following formula defines the fractional derivative of order α of Liouville–Caputo ([15,21]):*

$$D^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\xi)}{(t-\xi)^{\alpha-m+1}} d\xi, & m-1 < \alpha < m, \\ f^{(m)}(t), & \alpha = m \in \mathbb{N}, \end{cases}$$

where

$$D^{\alpha} t^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \alpha \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, t^{n-\alpha}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geq \lceil \alpha \rceil. \end{cases} \tag{3}$$

Additionally, a generalization of Taylor's formula is introduced based on the fractional derivative of Liouville–Caputo [22]. Assume that

$$D^{k\alpha} f(t) \in C(0,a], \quad \text{for} \quad k = 0,1,\dots,n+1, \ \text{where} \quad 0 < \alpha \leq 1,$$

then,

$$f(t) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{(D^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)}\, t^{(n+1)\alpha}, \quad 0 \leq \xi \leq t, \quad \forall\, t \in (0,a]. \tag{4}$$

3. Legendre Polynomials of Fractional Order

Since the well-known Legendre polynomials $L_k(z)$ are defined on $[-1,1]$ ([23,24]), we introduce the shifted Legendre polynomials $L_k^*(t)$ by inserting $z = 2t-1$ to use these polynomials on $[0,1]$.

We shall introduce the fractional-order Legendre polynomials in this section. By changing the variables $t = x^{\nu}$ and $\nu > 0$ on $L_k^*(t)$, these polynomials will be obtained. We represent FLPs, $L_k^*(x^{\nu})$ by $P_k^{\nu}(x)$. We note that the recurrence formula for $P_k^{\nu}(x)$ may be deduced from the recurrence relation of $L_k^*(t)$ [18]:

$$P_{k+1}^{\nu}(x) = \frac{(2k+1)(2x^{\nu}-1)}{(k+1)}\, P_k^{\nu}(x) - \frac{k}{k+1}\, P_{k-1}^{\nu}(x), \quad P_0^{\nu}(x) = 1, \quad P_1^{\nu}(x) = 2x^{\nu}-1, \quad k = 1,2,\dots,.$$

The analytic form of the FLPs, $P_k^{\nu}(x)$ of degree νk is given by:

$$P_k^{\nu}(x) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2}\, x^{i\,\nu}. \tag{5}$$

Note that $P_k^{\nu}(0) = (-1)^k$ and $P_k^{\nu}(1) = 1$.

Lemma 1 ([25]). *According to the weight function $w^{\nu}(x) = x^{\nu-1}$, the fractional-order polynomials $P_i^{\nu}(x)$ are orthogonal over $[0,q]$. Therefore, we obtain*

$$\int_0^1 P_i^{\nu}(x) P_j^{\nu}(x) w^{\nu}(x) dx = \frac{\delta_{ij}}{\nu(2i+1)}.$$

Proof. The formula can be obtained directly by taking $t = x^{\nu}$, in the orthogonality condition $\int_0^1 L_i^*(t) L_j^*(t) dt = \frac{\delta_{ij}}{2i+1}$. □

4. Approximating Polynomials and Estimating Errors

Let a non-negative, integrable, real-valued function spanning the range $(0,1)$ be denoted by $w^{\nu}(x) = x^{\nu-1}$, $\nu > 0$. In the first $(m+1)$-terms of FLPs, the function $u(x) \in L^2_{w^{\nu}}$, which is a square-integrable function in $[0,1]$, can be represented and approximated as follows:

$$u_m^{\nu}(x) = \sum_{i=0}^{m} c_i P_i^{\nu}(x). \tag{6}$$

The error estimate for this approximation in w^{ν}-norm is provided by the following theorem.

Theorem 1. *The error bound can be calculated using the following formula if u_m^{ν} is the best estimation to $u(x)$ out of $V_m^{\nu} := span\{P_0^{\nu}(x), P_1^{\nu}(x), \ldots, P_m^{\nu}(x)\}$, and $D^{i\,\nu}u(x) \in C(0,1]$ for $i = 0,1,\ldots,m+1$:*

$$\|u - u_m^{\nu}\|_{w^{\nu}} \leq \frac{e^{(m+1)\nu - \frac{\theta}{12(m+1)\nu}}}{((m+1)\nu)^{(m+1)\nu+\frac{1}{2}}} \frac{\gamma_{\nu}}{\sqrt{2\pi(2m+3)\nu}},$$

where $0 < \theta < 1$, $\gamma_{\nu} = \sup_{x\in(0,1]} |D^{(m+1)\,\nu}u(x)|$.

Proof. As an approximate representation of $u(x)$ and denoted by $y(x)$, consider the generalized Taylor's polynomial as follows:

$$y(x) = \sum_{i=0}^{m} \frac{x^{i\,\nu}}{\Gamma(i\nu+1)} D^{i\,\nu}u(0^+), \qquad x \in (0,1],$$

where it is known that the following error bound exists and:

$$|u(x) - y(x)| = \left| \frac{x^{(m+1)\nu}}{\Gamma((m+1)\nu+1)} D^{(m+1)\nu}u(\xi_x) \right| \leq \gamma_{\nu} \frac{x^{(m+1)\nu}}{\Gamma((m+1)\nu+1)} = \frac{\gamma_{\nu}\, x^{(m+1)\nu}}{((m+1)\nu)!}.$$

Since u_m^{ν}, $y(x) \in V_m^{\nu}$ and u_m^{ν} is the best approximation to $u(x)$ out of V_m^{ν}, then

$$\begin{aligned}\|u - u_m^{\nu}\|^2_{w^{\nu}} \leq \|u - y\|^2_{w^{\nu}} &\leq \left(\frac{\gamma_{\nu}}{((m+1)\nu)!}\right)^2 \int_0^1 x^{\nu-1} x^{2(m+1)\nu} dx \\ &= \left(\frac{\gamma_{\nu}}{((m+1)\nu)!}\right)^2 \frac{1}{(2m+3)\nu},\end{aligned}$$

this together with the well-known Stirling formula ($n! = \sqrt{2\pi}\, n^{n+1/2}\, e^{-n+\theta/12n}$, for some $(0 < \theta < 1)$ lead to the desired result. □

The primary formula for calculating an approximation of the Liouville–Caputo fractional derivative is given in the following theorem.

Theorem 2. *Let $u(x)$ be as described in (6) and $\alpha > 0$, then:*

$$D^{\alpha}(u_m^{\nu}(x)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\nu,\alpha)} x^{\nu k-\alpha}, \qquad w_{i,k}^{(\nu,\alpha)} = \frac{(-1)^{(i+k)}(i+k)!\Gamma(\nu k+1)}{(i-k)!((k)!)^2\Gamma(\nu k-\alpha+1)}. \tag{7}$$

Proof. We can obtain this formula directly by using the linearity properties of the Liouville–Caputo operation and then employing Equation (3) in Formula (5) with some simplification. □

Theorem 3. *The error* $|E_T(m)| = |D^\alpha u(x) - D^\alpha u_m^\nu(x)|$ *in approximating* $D^\alpha u(x)$ *by* $D^\alpha u_m^\nu(x)$ *is bounded by:*

$$|E_T(m)| \leq \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} \Big) \Big|, \tag{8}$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i+k}(i+k)!(2j+1)}{(i-k)!((k)!)^2\Gamma(\nu k-\alpha+1)} \times \sum_{r=0}^{j} \frac{(-1)^{j+r}(j+r)!}{(j-r)!\,(r!)^2(\nu k-\alpha+r+1)}, \qquad j=0,1,\ldots.$$

Proof. The proof can be found in [24]. □

5. Implementation of Legendre-SCM for Solving FDDE

Consider the FDDE (1), we first write $u(x)$ as:

$$u_m^\nu(x) = \sum_{i=0}^{m} c_i P_i^\nu(x). \tag{9}$$

From Equations (1) and (9) and Theorem 2, we have:

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\nu,\alpha)} x^{\nu k-\alpha} = f\left(x, \sum_{i=0}^{m} c_i P_i^\nu(x), \sum_{i=0}^{m} c_i P_i^\nu(g(x)) \right). \tag{10}$$

We now collocate Equation (10) at $(m+1-\lceil\alpha\rceil)$ points $x_p,\ p=0,1,\ldots,m-\lceil\alpha\rceil$ as:

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\nu,\alpha)} x_p^{\nu k-\alpha} = f\left(x_p, \sum_{i=0}^{m} c_i P_i^\nu(x_p), \sum_{i=0}^{m} c_i P_i^\nu(g(x_p)) \right). \tag{11}$$

We utilize the roots of the fractional Legendre polynomial $P_{m+1-\lceil\alpha\rceil}^\nu(x)$ for suitable collocation nodes.

From (9) in (2), and use the properties of the Formula (5), we can obtain the following equations:

$$\sum_{i=0}^{m}(-1)^i c_i = \lambda_0, \qquad \sum_{i=0}^{m} c_i = \lambda_1. \tag{12}$$

Using Equation (11) and $\lceil\alpha\rceil$ equations of B.Cs gives $(m+1)$ algebraic equations in the unknown $c_i,\ i=0,1,\ldots,m$, which can be solved. Consequently, $u(x)$ for (1) can be found.

6. Numerical Results

Example 1. *Consider the following differential equation for a linear fractional delay:*

$$D^{0.5}u(x) + u(x/4) + 2\sqrt{x}u(x) = 2x + 0.5(\sqrt{x} + \sqrt{\pi}), \tag{13}$$

with the initial condition:

$$u(0) = 0. \tag{14}$$

Using the provided method and $m=2$, *we arrive at the following approximation of the solution:*

$$u_2^\nu(x) = \sum_{n=0}^{2} c_n P_n^\nu(x). \tag{15}$$

Using Equation (11) with $\alpha = 0.5$, *we have:*

$$\sum_{n=\lceil\alpha\rceil}^{2}\sum_{k=\lceil\alpha\rceil}^{n} c_n\, w_{n,k}^{(\nu,\alpha)} x_p^{\nu\, k-\alpha} + \sum_{n=0}^{2} c_n P_n^{\nu}(x_p/4) + 2\sqrt{x_p}\sum_{n=0}^{2} c_n P_n^{\nu}(x_p) = 2x_p + 0.5(\sqrt{x_p}+\sqrt{\pi}), \tag{16}$$

with $p = 0, 1$ where x_p are roots of $P_2^{\nu}(x)$ and $x_0 = 0.0446582$, $x_1 = 0.622008$, at $\nu = 0.5$. Using Equations (14) and (15) we get:

$$c_0 - c_1 + c_2 = 0. \tag{17}$$

Now, solving Equations (16) and (17), we get:

$$c_0 = 1/2, \qquad c_1 = 1/2, \qquad c_2 = 0.0.$$

Thus, using Equation (15) we can get:

$$u(x) \simeq \tfrac{1}{2}P_0^{\nu}(x) + \tfrac{1}{2}P_1^{\nu}(x) + 0.0P_2^{\nu}(x) = \sqrt{x} = \text{The exact solution}.$$

Example 2. *Consider the following differential equation for a linear fractional delay:*

$$D^{\alpha}u(x) + e^{-\sqrt{x}/4}\, u(x/16) - \frac{1}{2\sqrt{x}}u(x) = 1, \qquad 0 < \alpha \le 1, \tag{18}$$

with

$$u(0) = 1. \tag{19}$$

The exact solution of Equation (18) at $\alpha = 1$ is $u(x) = e^{\sqrt{x}}$. Using the provided method and $m = 4$, we arrive at the following approximation of the solution:

$$u_4^{\nu}(x) = \sum_{n=0}^{4} c_n P_n^{\nu}(x). \tag{20}$$

Using Equation (11), we have:

$$\sum_{n=\lceil\alpha\rceil}^{4}\sum_{k=\lceil\alpha\rceil}^{n} c_n\, w_{n,k}^{(\nu,\alpha)} x_p^{\nu\, k-\alpha} + e^{-\sqrt{x_p}/4}\sum_{n=0}^{4} c_n P_n^{\nu}(x_p/16) - \frac{1}{2\sqrt{x_p}}\sum_{n=0}^{4} c_n P_n^{\nu}(x_p) = 1, \tag{21}$$

with $p = 0, 1, 2, 3$ where x_p are roots of $P_4^{\nu}(x)$ and $x_0 = 0.004821$, $x_1 = 0.108961$, $x_2 = 0.448887$, $x_3 = 0.865957$, at $\nu = 0.5$.

Using Equations (19) and (20), we obtain:

$$c_0 - c_1 + c_2 - c_3 + c_4 = 1. \tag{22}$$

Now, solving Equations (21) and (22), we can obtain:

$$c_0 = 1.718288, \quad c_1 = 0.845162, \quad c_2 = 0.139858, \quad c_3 = 0.013982, \quad c_4 = 0.000998.$$

Thus, using Equation (20) we can obtain the approximate solution $u(x)$ of Example 2. The approximate solution and the exact one are given in Figure 1 at $\alpha = 1$ and $\nu = 0.5$. But, the approximate solution with $\alpha(= 0.75, 0.85)$ is given in Figure 2. These graphs demonstrate that the exact solution and numerical solution are in very good agreement, and the approximate solution exhibits the same behavior.

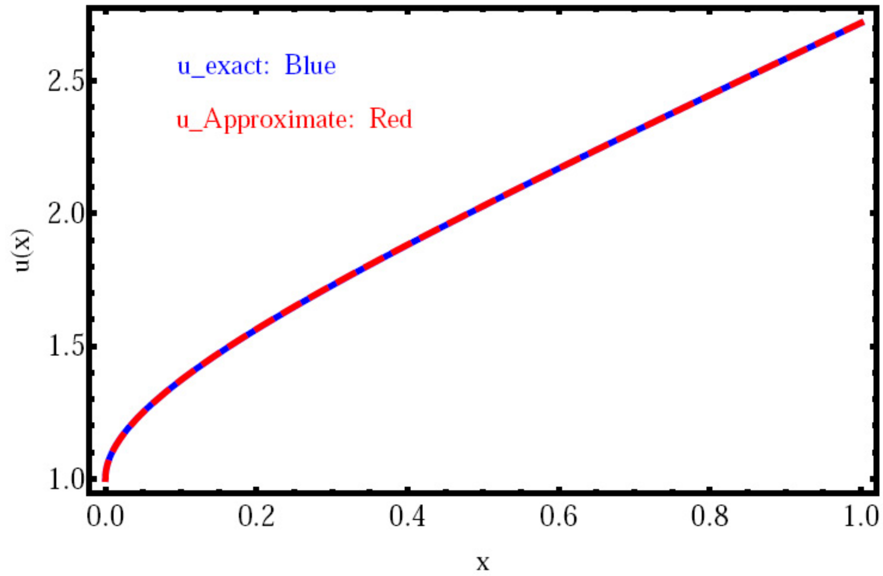

Figure 1. The numerical solution and the exact solution of Example 2.

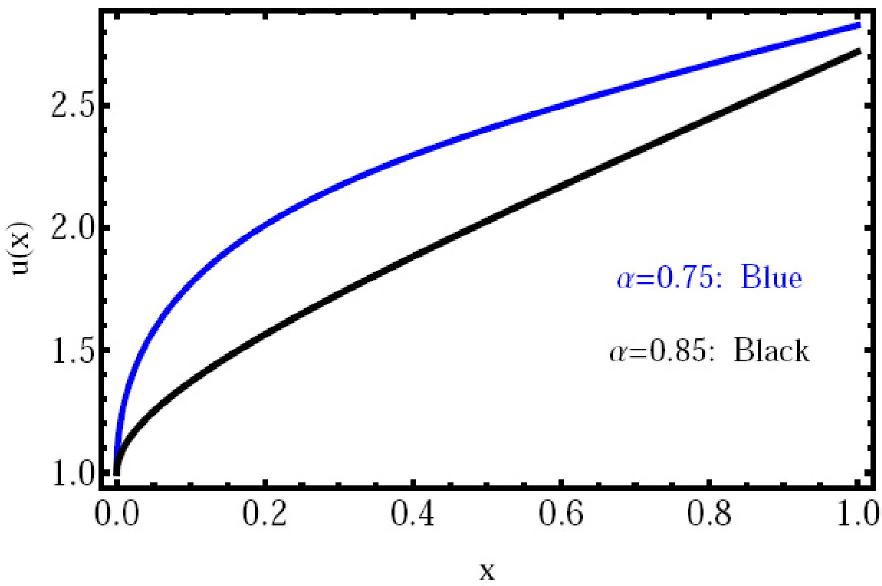

Figure 2. The approximate solution with different values of α of Example 2.

Example 3. *Consider the following non-linear fractional delay differential equation:*

$$D^{1.5}u(x) = u(x-0.5) + u^3(x) + h(x), \tag{23}$$

where $h(x) = \frac{\Gamma(2\nu+1)}{\Gamma(2\nu-0.5)} x^{2\nu-1.5} - (x-0.5)^{2\nu} - x^{6\nu}$, *with*

$$u(0) = 0, \qquad u(1) = 1. \tag{24}$$

Using the provided method and $m = 3$, *we arrive at the following approximation of the solution:*

$$u_m^\nu(x) = \sum_{n=0}^{m} c_n P_n^\nu(x). \tag{25}$$

Using Equation (11), we have:

$$\sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\nu,\alpha)} x_p^{\nu k-\alpha} = \sum_{n=0}^{3} c_n P_n^{\nu}(x_p - 0.5) + \left(\sum_{n=0}^{3} c_n P_n^{\nu}(x_p)\right)^3 + h(x_p), \tag{26}$$

with $p = 0,1,2$ *where* x_p *are roots of* $P_3^{\nu}(x)$. *Using Equations (24) and (25), we obtain:*

$$c_0 - c_1 + c_2 - c_3 = 0, \tag{27}$$

$$c_0 + c_1 + c_2 + c_3 = 1. \tag{28}$$

Solving (26)–(28), we obtain:

$$c_0 = 1/3, \qquad c_1 = 1/2, \qquad c_2 = 1/6, \qquad c_3 = 0.0.$$

Thus, using Equation (25) at $\nu = 1.25$, *we can obtain:*

$$u(x) = \tfrac{1}{3}P_0^{\nu}(x) + \tfrac{1}{2}P_1^{\nu}(x) + \tfrac{1}{6}P_2^{\nu}(x) + 0.0P_3^{\nu}(x) = x^{2.5} = x^{2\nu} = \textit{The exact solution}.$$

Example 4 ((Ikeda system-One delay)). *Consider the Ikeda delay system with one delay, which is described in ([26,27]):*

$$D^{\alpha}u(t) = -\kappa\, u(t) + \lambda\, \sin(\mu\, u(t-\tau)), \qquad 1 < \alpha \le 2, \tag{29}$$

where κ, λ, μ, *and* τ *are constants; with*

$$u(0) = u_0, \qquad u(1) = u_1. \tag{30}$$

Using the provided method and $m = 10$, *we arrive at the approximation of the solution as in Equation (25). Using Equation (11), we have*

$$\sum_{n=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{n} c_n\, w_{n,k}^{(\nu,\alpha)} t_p^{\nu k-\alpha} = -\kappa \sum_{n=0}^{m} c_n P_n^{\nu}(t_p) + \lambda\, \sin\left(\mu \sum_{n=0}^{m} c_n P_n^{\nu}(t_p - \tau)\right), \tag{31}$$

with $p = 0(1)8$ *where* t_p *are roots of* $P_9^{\nu}(t)$. *Using Equations (25) and (30), we can obtain:*

$$\sum_{n=0}^{m} (-1)^n\, c_n = u_0, \qquad \sum_{n=0}^{m} c_n = u_1. \tag{32}$$

Now, solving the system of Equations (31)–(32), we obtain the approximate solution of this model (29).

To conclude this numerical analysis with the simulation; we define the residual error function (REF) as:

$$REF(t,m) = \sum_{n=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{n} c_n\, w_{n,k}^{(\nu,\alpha)}\, t^{\nu k-\alpha} + \kappa \sum_{n=0}^{m} c_n P_n^{\nu}(t) - \lambda\sin\left(\mu \sum_{n=0}^{m} c_n P_n^{\nu}(t-\tau)\right). \tag{33}$$

The absolute relative error drops to zero in all situations when the residual is minimal ($REF(t,m) \to 0$)*, indicating that the solution tends to the exact one. Since the precise answer is unknown in the situation of* α *in fractional order, we shall employ this form of error. Finally, REF includes other forms; for further information, see [28].*

Using the provided method, we present a numerical simulation of this model (29) in Figures 3–5 with $\kappa = 3$, $\lambda = 24$, $\mu = 1$ and initial values $u_0 = u_1 = 0.1$. In Figure 3, the REF at $\tau = 0.1$, $\alpha = 1.98$ with $m = 10$ (a) and $m = 15$ (b) is given. In Figure 4, the REF at $\tau = 0.01$, $m = 12$ with $\alpha = 1.95$ (a) and $\alpha = 1.99$ (b) is given. Finally, in Figure 5, we provide a numerical simulation of this model using $m = 15$; with $\tau = 0.3, 0.6, 0.9, 1.2$ (a) and $\alpha = 1.2, 1.4, 1.6, 1.8, 2.0$ (b). These Figures demonstrate that the solution depends on α and τ, demonstrating the effectiveness of the numerical technique for solving the posed issue in fractional derivatives.

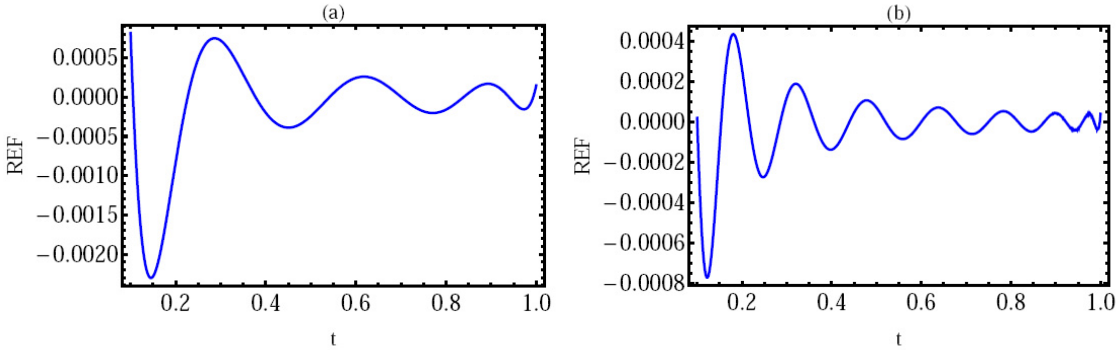

Figure 3. The REF of Example 4 with $\tau = 0.1$, $\alpha = 1.98$; at $m = 10$ (**a**) and $m = 15$ (**b**).

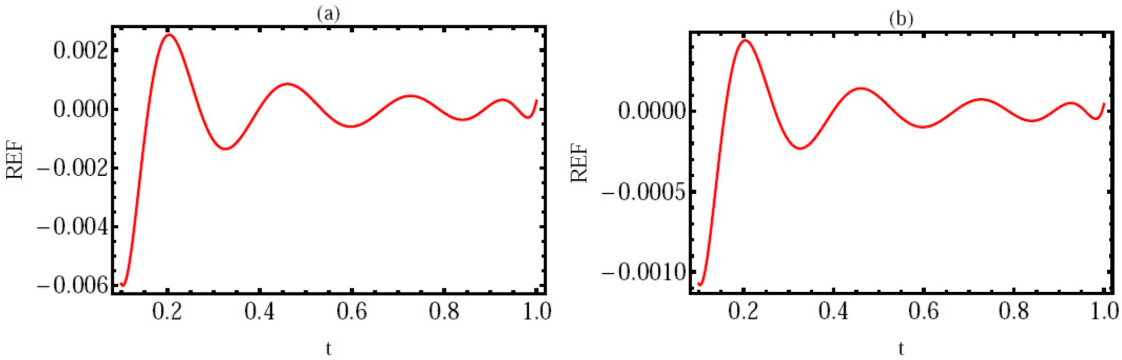

Figure 4. The REF of Example 4 with $\tau = 0.01$, $m = 12$; at $\alpha = 1.95$ (**a**) and $\alpha = 1.99$ (**b**).

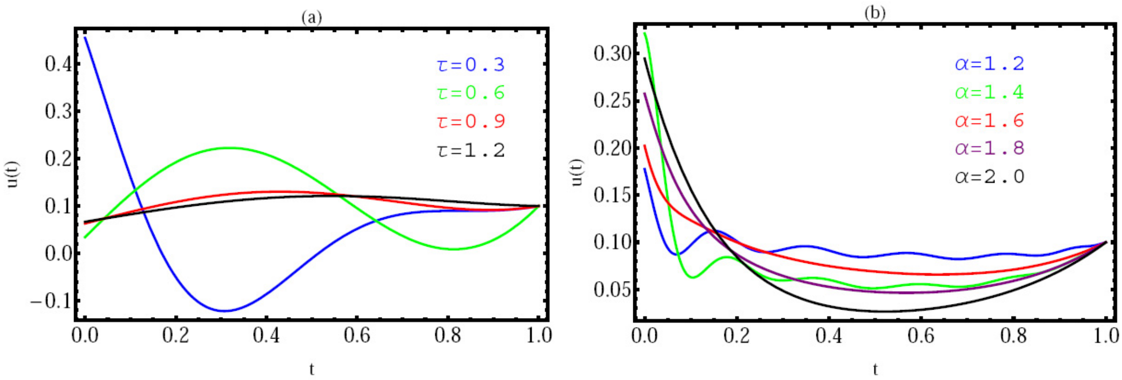

Figure 5. The numerical solution of Example 4, with $m = 15$; with different values of τ (**a**) and α (**b**).

Example 5 ((Ikeda system-two delays)). *Consider the Ikeda delay system with two delays which is described in [29]:*

$$D^{\alpha}u(t) = -\kappa\, u(t - \tau_1) + \lambda\, \sin(\mu\, u(t - \tau_2)), \qquad 1 < \alpha \leq 2, \tag{34}$$

where κ, λ, μ, τ_1, and τ_2 are constants and with:

$$u(0) = u_0, \qquad u(1) = u_1. \tag{35}$$

Using the provided method and $m = 10$*, we arrive at the approximation of the solution as in (25). Using Equation (11), we have*

$$\sum_{n=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{n} c_n\, w_{n,k}^{(\nu,\alpha)}\, t_p^{\nu k-\alpha} = -\kappa \sum_{n=0}^{m} c_n P_n^{\nu}(t_p-\tau_1) + \lambda\, \sin\left(\mu \sum_{n=0}^{m} c_n P_n^{\nu}(t_p-\tau_2)\right), \quad (36)$$

with $p = 0(1)8$ *where* t_p *are roots of* $P_9^{\nu}(t)$*. Using (25) and (35), we get:*

$$\sum_{n=0}^{m} (-1)^n\, c_n = u_0, \qquad \sum_{n=0}^{m} c_n = u_1. \quad (37)$$

Now, solving the system of Equations (36)–(37), we obtain the approximate solution to this model (34).

Here, we define the REF to conduct a full numerical study with the simulation:

$$REF(t,m) = \sum_{n=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{n} c_n\, w_{n,k}^{(\nu,\alpha)}\, x^{\nu k-\alpha} + \kappa \sum_{n=0}^{m} c_n P_n^{\nu}(t-\tau_1) - \lambda \sin\left(\mu \sum_{n=0}^{m} c_n P_n^{\nu}(t-\tau_2)\right). \quad (38)$$

Using the provided method, we present a numerical simulation of this model (34) using the suggested approach in Figures 6–8 with $\kappa = 3$, $\lambda = 24$, $\mu = 1$ and initial values $u_0 = u_1 = 0.1$. In Figure 6, the REF at $\tau_1 = \tau_2 = 0.1$, $\alpha = 1.98$ with $m = 10$ (a) and $m = 15$ (b) is given. In Figure 7, the REF at $\tau_1 = \tau_2 = 0.01$, $m = 12$ with $\alpha = 1.95$ (a) and $\alpha = 1.99$ (b) is given. Finally, in Figure 8, the solution at $m = 10$, $\alpha = 1.98$; with $\tau_1 = 0.3, 0.6, 0.9, 1.2$ (a) and $\tau_2 = 0.3, 0.6, 0.9, 1.2$ (b) is given. These Figures demonstrate that the solution depends on α, τ_1 and τ_2, demonstrating the effectiveness of the numerical technique for solving this issue in fractional derivatives.

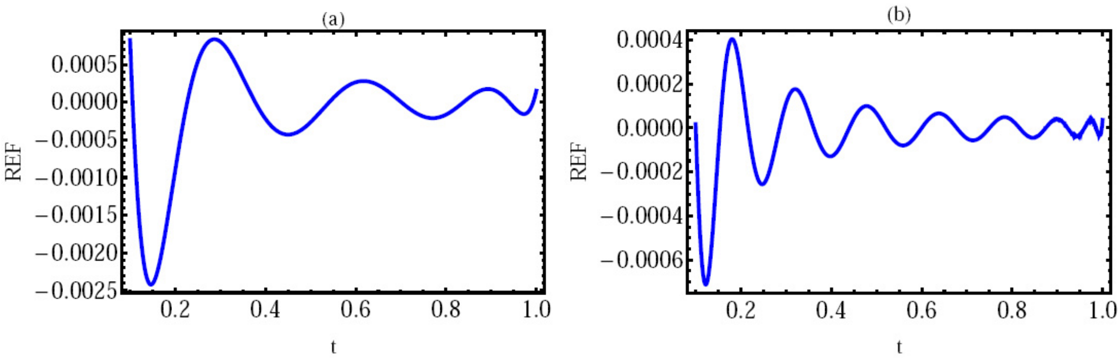

Figure 6. The REF of Example 5 with $\tau_1 = \tau_2 = 0.1$, $\alpha = 1.98$; at $m = 10$ (**a**) and $m = 15$ (**b**).

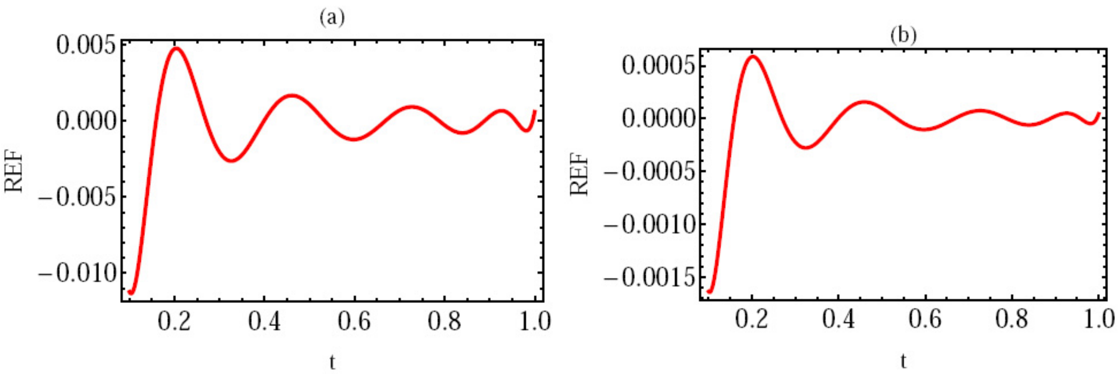

Figure 7. The REF of Example 5 with $\tau_1 = \tau_2 = 0.01$, $m = 12$; at $\alpha = 1.95$ (**a**) and $\alpha = 1.99$ (**b**).

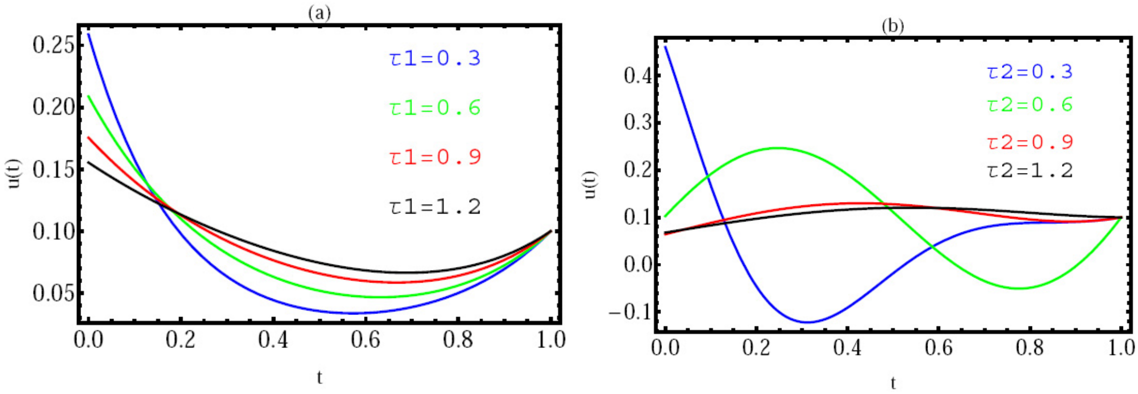

Figure 8. The numerical solution of Example 5, with $m = 10$, $\alpha = 1.98$; with different values of τ_1 (**a**) and τ_2 (**b**).

7. Conclusions and Remarks

This work's primary focus is on presenting fractional Legendre polynomials with states and demonstrating some of their characteristics. In addition, the Legendre SCM is used in this article to solve fractional delay differential equations. The suggested problem is reduced to a system of algebraic equations, which can be solved using an appropriate numerical method, leveraging the properties of the new fractional Legendre polynomials. We examine the convergence analysis, the generated formula's accuracy, and the approximation of the solution. The numerical simulations show that this approach to applying the Liouville–Caputo derivative is an effective way to solve FDDE. In addition, Only a few shifted fractional Legendre polynomials are required to produce a good outcome. By including new terms in the series, overall error rates can be reduced.

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