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Article

On the Generalization of Ostrowski-Type Integral Inequalities via Fractional Integral Operators with Application to Error Bounds

Gauhar Rahman [1], Miguel Vivas-Cortez [2,*], Çetin Yildiz [3,*], Muhammad Samraiz [4], Shahid Mubeen [4] and Mansour F. Yassen [5,6]

[1] Department of Mathematics and Statistics, Hazara University, Mansehra 21300, Pakistan; gauhar55uom@gmail.com or drgauhar.rahman@hu.edu.pk
[2] Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Av. 12 de Octubre 1076 y Roca, Quito 170143, Ecuador
[3] Department of Mathematics, K.K. Education Faculty, Atatürk University, Erzurum 25240, Turkey
[4] Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan; muhammad.samraiz@uos.edu.pk or msamraizuos@gmail.com (M.S.); smjhanda@gmail.com (S.M.)
[5] Department of Mathematics, College of Science and Humanities in Al-Aflaj, Prince Sattam bin Abdulaziz University, Al-Kharj 11912, Saudi Arabia; mf.ali@psau.edu.sa
[6] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Damietta, Egypt
* Correspondence: mjvivas@puce.edu.ec (M.V.-C.); cetin@atauni.edu.tr (Ç.Y.)

Abstract: The Ostrowski inequality expresses bounds on the deviation of a function from its integral mean. The Ostrowski's type inequality is frequently used to investigate errors in numerical quadrature rules and computations. In this work, Ostrowski-type inequality is demonstrated using the generalized fractional integral operators. From an application perspective, we present the bounds of the fractional Hadamard inequalities. The results that are being presented involve a number of fractional inequalities that are already known and have been published.

Keywords: Ostrowski inequality; fractional integrals; convex functions

MSC: 26A33; 26A51; 33E12

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1. Introduction

Numerous mathematical problems, including approximation theory, spectral analysis, statistical analysis, and the concept of distributions, involve the use of integral inequalities. Integral inequalities play a significant role in many branches of science and engineering. Since integral inequalities and their applications are vital to the theory of differential equations and applied mathematics, there has been a significant rise in interest in the study of numerous classical inequalities applied to integral operators associated with various kinds of fractional derivatives in recent years.

The Ostrowski inequality is named after the Polish mathematician Aleksander Ostrowski, who first proved the inequality in 1938. The inequality has since been extended and generalized to various settings, including fractional calculus and multivariate functions.

One important application of the Ostrowski inequality is in the analysis of numerical quadrature rules, which are methods for approximating definite integrals numerically. The inequality can be used to estimate the error in a quadrature rule by providing an upper bound on the deviation of the approximating function from the actual integrand.

The Ostrowski inequality has also been used in the study of differential equations, particularly in the proof of existence and uniqueness theorems for solutions of ordinary and partial differential equations. The inequality provides a tool for estimating the difference between two solutions of a differential equation, which is necessary for proving the convergence and stability properties of numerical methods for solving these equations.

In recent years, the Ostrowski inequality has also been extended to the setting of fractional calculus, which deals with derivatives and integrals of non-integer orders. Fractional versions of the Ostrowski inequality have been used in the study of fractional differential equations and in the development of numerical methods for solving these equations.

Ostrowski [1] introduced the following inequality known as the Ostrowski inequality.

Theorem 1. *Suppose that the mapping $\mathcal{G} : I \to \mathbb{R}$ is differentiable in I^0, where I^0 denotes the interior of I, and let $r, s \in I^0$ with $r < s$. If $|\mathcal{G}'(\tau)| \leq K$ for all $t \in [r, s]$, then we have*

$$| \mathcal{G}(\theta) - \frac{1}{s-1}\int_r^s \mathcal{G}(\tau)d\tau | \leq K(s-r)\left[\frac{1}{4} + \frac{(\theta - \frac{r+s}{2})^2}{(s-r)^2}\right]. \tag{1}$$

The Ostrowski inequality (1) gives the boundedness of the difference between the values of $\mathcal{G}(\theta)$ at an arbitrary value $\theta \in [r, s]$ and its integral mean and $\frac{1}{s-1}\int_r^s \mathcal{G}(\tau)d\tau$ shows the boundedness of $\mathcal{G}'$. From an application point of view, this inequality (1) provides the error bounds of the midpoint and trapezoidal quadrature rules and some of its applications to special means [2].

Using various convex functions, several researchers have recently examined classical inequalities for different kinds of fractional integrals. For instance, using s-convex and h-convex functions, Set [3] and Liu [4] demonstrated the Ostrowski-type inequality for Riemann–Liouville fractional integrals. The Ostrowski-type inequalities for Riemann–Liouville k-fractional integrals were provided by Kermausuor [5] and Lakhal [6] via strongly (a, m)-convex functions and $k - beta$-convex functions.

By using convex and AG-convex functions, Deeb and Awrejcewicz [7] demonstrated the Ostrowski–Trapezoid–Grüss-type difference operator of (q, d)-Hahn type. By using p-convex functions, Gürbüz et al. [8] provided the Ostrowski-type inequalities for Katugampola fractional integrals. For Ψ-Hilfer fractional integrals, Basci and Baleanu [9] developed the Ostrowski-type inequalities. Hermite–Hadamard–Jensen–Mercer fractional inequalities for convex functions were established by Faisal et al. in [10], and related inequalities for $alpha$-type real-valued convex functions were also provided in [11]. The Ostrowski-type inequalities for fractional integrals incorporating an extended modified Mittag–Leffler function were developed by Farid et al. in [12].

The Mittag–Leffler function, on the other hand, is essential for resolving fractional differential equations. In last few decades, various types of fractional integrals have been developed by using Mittag–Leffler functions in the kernel [13–16]. Several inequalities for different fractional integrals comprising the Mittag–Leffler function have been established by researchers [17–19].

The three-parameter M-L function is defined by [20] as follows:

$$\varepsilon^c_{a,b}(z_1) = \sum_{l=0}^{\infty} \frac{(c)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \qquad (a, b, \lambda \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0).$$

The multivariate M-L function is defined by [21] as

$$\begin{aligned} \mathcal{E}^{(\gamma_j)}_{(\sigma_j),\zeta}(z_1, z_2, \ldots, z_j) &= \mathcal{E}^{(\gamma_1,\gamma_2,\ldots,\gamma_j)}_{(\sigma_1,\sigma_2,\ldots,\sigma_j),\zeta}(z_1, z_2, \ldots z_j) \\ &= \sum_{m_1,m_2\ldots m_j=0}^{\infty} \frac{(\gamma_1)_{m_1}(\gamma_2)_{m_2}\ldots(\gamma_j)_{m_j}(z_1)^{m_1}\ldots(z)^{m_j}}{\Gamma(\sigma_1 m_1 + \sigma_2 m_2 + \ldots \sigma_j m_j + \zeta)m_1!\ldots m_j!}, \end{aligned} \tag{2}$$

where $z_i, \sigma_i, \zeta, \gamma_i \in \mathbb{C}$; $i = 1, 2, \ldots, j$, $\Re(\sigma_i) > 0$, $\Re(\zeta) > 0$ and $\Re(\gamma_i) > 0$.

The Riemann–Liouville (R-L) fractional integral (left and right sided) $\mathcal{R}^{\mu}_{r+}$ and $\mathcal{R}^{\mu}_{s-}$ of order $\mu > 0$, for a function $\mathcal{U}$, are, respectively, given in [22–24] by

$$\left(\mathcal{R}^{\mu}_{r+}\mathcal{G}\right)(\theta) = \frac{1}{\Gamma(\mu)}\int_r^{\theta}(\theta-\tau)^{\mu-1}\mathcal{G}(\tau)d\tau, r < \theta \tag{3}$$

and

$$\left(\mathcal{R}^{\mu}_{s-}\mathcal{G}\right)(\theta) = \frac{1}{\Gamma(\mu)}\int_{\theta}^{s}(\tau-\theta)^{\mu-1}\mathcal{G}(\tau)d\tau, s > \tau. \tag{4}$$

The Prabhakar-type fractional integrals are defined in [20] by

$$(\mathcal{R}^{c;\varpi}_{a,b,r+}\mathcal{G})(\theta) = \int_r^{\theta}(\theta-\tau)^{\mu-1}\varepsilon^{c}_{a,b}(\varpi(\theta-\tau)^{a})\mathcal{G}(\tau)d\tau \tag{5}$$

and

$$(\mathcal{R}^{c;\varpi}_{a,b,s-}\mathcal{G})(\theta) = \int_{\theta}^{s}(\tau-\theta)^{\mu-1}\varepsilon^{c}_{a,b}(\varpi(\tau-\theta)^{a})\mathcal{G}(\tau)d\tau. \tag{6}$$

The fractional integral operators having a multivariate M-L function are defined by Saxena et al. [21] as

$$\mathcal{R}^{(c_1,\ldots,c_j),(\varpi_1,\ldots,\varpi_j)}_{(a_1,\ldots,a_j),b,r+}\mathcal{G}(\theta) = \int_r^{\theta}(\theta-\tau)^{\mu-1}\varepsilon^{(c_i)}_{(a_i),b}(\varpi_1(\theta-\tau)^{a_1}\ldots\varpi_j(\theta-\tau)^{a_j})\mathcal{G}(\tau)d\tau, \tag{7}$$

and

$$\mathcal{R}^{(c_1,\ldots,c_j),(\varpi_1,\ldots,\varpi_j)}_{(a_1,\ldots,a_j),b,s-}\mathcal{G}(\theta) = \int_{\theta}^{s}(\tau-\theta)^{\mu-1}\varepsilon^{(c_i)}_{(a_i),b}(\varpi_1(\tau-\theta)^{a_1}\ldots\varpi_j(\tau-\theta)^{a_j})\mathcal{G}(\tau)d\tau, \tag{8}$$

where $\mu, \eta_i, \lambda_i, \gamma_i \in \mathbb{C}$, $\Re(\eta_i) > 0$, $\Re(\mu) > 0$, $\Re(\gamma_i) > 0$ for $i = 1, 2, \cdots, j$.

Remark 1. *If we consider $j = 1$ in (7) and (8), then we obtain the operators defined by (5) and (6), respectively. Similarly, if we take $\varpi = 0$ in (7) and (8), then we have R-L operators (3) and (4), respectively.*

The aim of this article is to establish Ostrowski-type fractional integral inequalities with a multivariate M-L function. From an application point of view, we present some error bounds of the Hadamard-type inequality.

This article is divided into three sections. In the first part, different forms of the M-L function and some fractional operators are given. In the second part, new inequalities and generalizations of the Ostrowski type are obtained using the multivariate M-L function. In Section 3, some applications are presented of Theorem 5 by applying it to the endpoints of the interval $[r, s]$. The conclusion of the overall work is presented in the last section.

2. Main Result

In this section, the Ostrowski-type inequality is shown using generalized fractional integral operators. From an application perspective, we present the bounds of the fractional Hadamard inequalities. Firstly, we recall the following lemma.

Lemma 1. *[21] For the generalized multivariate M-L function defined in (2), the following relation holds true:*

$$
\begin{aligned}
&\left(\frac{d}{dz_1}\right)^m [z_1^{b-1}\varepsilon^{(c_j)}_{(a_j),b;p}(\varpi_1 z_1^{a_1},\cdots,\varpi_j z_1^{a_j}))]\\
=&z_1^{b-m-1}\varepsilon^{(c_j)}_{(a_j),b-m;p}(\varpi_1 z_1^{a_1},\cdots,\varpi_j z_1^{a_j}) \quad (\Re(b-m)>0, m\in\mathbb{N}),
\end{aligned}
$$

where $a_i, b, c_i \in \mathbb{C}; \Re(a_i) > 0, \Re(b) > 0, \Re(p) > 0, p \geq 0$ *for* $i = 1, 2, \cdots, j$.

In particular, for $m = 1$, we have

$$
\begin{aligned}
&\left(\frac{d}{dz_1}\right) [z_1^{b-1}\varepsilon^{(c_j)}_{(a_j),b;p}(\varpi_1 z_1^{a_1},\cdots,\varpi_j z_1^{a_j}))]\\
=&z_1^{b-2}\varepsilon^{(c_j)}_{(a_j),b-m;p}(\varpi_1 z_1^{a_1},\cdots,\varpi_j z_1^{a_j}) \quad (\Re(b-m)>0, m\in\mathbb{N}),
\end{aligned}
\tag{9}
$$

Next, we present Ostrowski-type fractional integral inequalities concerning the multivariate M-L function.

Theorem 2. *Suppose that* $\mathcal{G} : I \to \mathbb{R}$ *is a differentiable mapping in* I^0*, where* I^0 *denotes the interior of* I *and let* $r, s \in I^0$ *with* $r < s$*. If* $|\mathcal{G}'(\tau)| \leq K$ *for all* $t \in [r, s]$*, then for* $\alpha, \beta \geq 1$*,* $a_i, b, c_i \in \mathbb{C}$*, where* $i = 1, 2, \cdots, j$*, we have*

$$
\begin{aligned}
&\Big|\Big((s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j}) + (s-\theta)^{\alpha-1}\\
&\times\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\Big)\mathcal{G}(\theta) - \Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta) + \Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,s-}\mathcal{G}\Big)(\theta)\Big)\Big|\\
\leq& K\Big(\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j}) + (s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\\
&-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(\theta) + \Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,s-}1\Big)(\theta)\Big)\Big).
\end{aligned}
$$

Proof. Let $\theta \in [r, s]$ and $\tau \in [r, \theta]$. Then, for the multivariate M-L function, the following inequality holds:

$$
\begin{aligned}
&(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})\\
\leq&(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j}).
\end{aligned}
\tag{10}
$$

Now, by the given hypothesis $|\mathcal{G}'(\tau)| \leq K$ and (10), we have

$$
\begin{aligned}
&\int_r^{\theta}(K-\mathcal{G}'(\tau))(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau\\
\leq&(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^{\theta}(K-\mathcal{G}'(\tau))d\tau
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
&\int_r^{\theta}(K+\mathcal{G}'(\tau))(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau\\
\leq&(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^{\theta}(K+\mathcal{G}'(\tau))d\tau.
\end{aligned}
\tag{12}
$$

Now, consider first (11)

$$
\begin{aligned}
&K\int_r^\theta(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau\\
&-\int_r^\theta\mathcal{G}'(\tau)(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau\\
&\leq(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^\theta(K-\mathcal{G}'(\tau))d\tau.
\end{aligned}
$$

Integrating by parts yields
and using (9), we have

$$
\begin{aligned}
&(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(\theta)-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta)\\
&\leq K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(\theta)\Big).
\end{aligned}
\tag{13}
$$

Similarly, (12) gives

$$
\begin{aligned}
&\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta)-(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(\theta)\\
&\leq K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(\theta)\Big).
\end{aligned}
\tag{14}
$$

From (13) and (14), we obtain

$$
\begin{aligned}
&\mid(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(\theta)-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta)\mid\\
&\leq K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(\theta)\Big).
\end{aligned}
\tag{15}
$$

Again, let $\theta\in[r,s]$, $\tau\in[\theta,s]$ and $\beta\geq 1$. Then, for the multivariate M-L function, the following inequality holds:

$$
\begin{aligned}
&(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})\\
&\leq(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j}).
\end{aligned}
\tag{16}
$$

Now, by the given hypothesis $|\mathcal{G}'(\tau)|\leq K$ and (16), we have

$$
\begin{aligned}
&\int_\theta^s(K-\mathcal{G}'(\tau))(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})d\tau\\
&\leq(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\int_\theta^s(K-\mathcal{G}'(\tau))d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\theta^s(K+\mathcal{G}'(\tau))(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})d\tau\\
&\leq(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\int_\theta^s(K+\mathcal{G}'(\tau))d\tau.
\end{aligned}
$$

By applying the similar procedure as we did for (11) and (12), we obtain

$$
\begin{aligned}
&\mid(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\mathcal{G}(\theta)-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,s-}\mathcal{G}\Big)(\theta)\mid\\
&\leq K\Big((s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,s-}1\Big)(\theta)\Big).
\end{aligned}
\tag{17}
$$

Inequalities (15) and (17) give the desired result. □

Corollary 1. *If we consider $\alpha=\beta$ in Theorem 2, then we have*

$$
\begin{aligned}
&|\Big((s-\theta)^{\alpha-1}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})+(s-\theta)^{\alpha-1}\\
&\times\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\Big)\mathcal{G}(\theta)-\Big(\mathcal{R}_{(a_i),\alpha-1,r+}^{(c_i),(\varpi_i)}\mathcal{G}\Big)(\theta)+\Big(\mathcal{R}_{(a_i),\alpha-1,s-}^{(c_i),(\varpi_i)}\mathcal{G}\Big)(\theta)\,|\\
&\leq K\Big(\theta-r)^{\alpha}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})+(s-\theta)^{\alpha}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\\
&-\Big(\Big(\mathcal{R}_{(a_i),\alpha,r+}^{(c_i),(\varpi_i)}1\Big)(\theta)+\Big(\mathcal{R}_{(a_i),\alpha,s-}^{(c_i),(\varpi_i)}1\Big)(\theta)\Big)\Big).
\end{aligned}
$$

Remark 2.

i. *Applying Theorem 2 for $j=1$, we obtain a certain new inequality for the fractional integral operator pertaining to the three-parameter M-L function [20].*
ii. *Applying Theorem 2 for $\varpi_i=0$ leads to the result proved by [25].*
iii. *Applying Theorem 2 for $\alpha=1=\beta$ and $\varpi_i=0$ leads to inequality (1).*

Theorem 3. *Suppose that $\mathcal{G}: I\to\mathbb{R}$ is a differentiable mapping in I^0, where I^0 denotes the interior of I, and let $r,s\in I^0$ with $r<s$. If $\mathcal{G}$ is an integrable function and $k<\mathcal{G}'(\tau)\leq K$ for all $t\in[r,s]$, then for $\alpha,\beta\geq 1$, $a_i,b,c_i\in\mathbb{C}$ where $i=1,2,\cdots,j$, we have*

$$
\begin{aligned}
&\Big((\theta-r)^{\alpha-1}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-(s-\theta)^{\beta-1}\\
&\times\varepsilon_{(a_i),\beta}^{(c_i),(\varpi_i)}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\Big)\mathcal{G}(\theta)-\Big(\Big(\mathcal{R}_{(a_i),\alpha-1,r+}^{(c_i),(\varpi_i)}\mathcal{G}\Big)(\theta)-\Big(\mathcal{R}_{(a_i),\beta-1,s-}^{(c_i),(\varpi_i)}\mathcal{G}\Big)(\theta)\Big)\\
&\leq K\Big((\theta-r)^{\alpha}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\\
&+(s-\theta)^{\beta}\varepsilon_{(a_i),\beta}^{(c_i),(\varpi_i)}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\Big(\mathcal{R}_{(a_i),\alpha,r+}^{(c_i),(\varpi_i)}1\Big)(\theta)+\Big(\mathcal{R}_{(a_i),\beta,s-}^{(c_i),(\varpi_i)}1\Big)(\theta)\Big)\Big)
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
&\Big((s-\theta)^{\beta-1}\varepsilon_{(a_i),\beta}^{(c_i),(\varpi_i)}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-(\theta-r)^{\alpha-1}\\
&\times\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\Big)\mathcal{G}(\theta)+\Big(\Big(\mathcal{R}_{(a_i),\alpha-1,r+}^{(c_i),(\varpi_i)}\mathcal{G}\Big)(\theta)-\Big(\mathcal{R}_{(a_i),\beta-1,s-}^{(c_i),(\varpi_i)}\mathcal{G}\Big)(\theta)\Big)\\
&\leq -k\Big((\theta-r)^{\alpha}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\\
&+(s-\theta)^{\beta}\varepsilon_{(a_i),\beta}^{(c_i),(\varpi_i)}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\Big(\mathcal{R}_{(a_i),\alpha,r+}^{(c_i),(\varpi_i)}1\Big)(\theta)+\Big(\mathcal{R}_{(a_i),\beta,s-}^{(c_i),(\varpi_i)}1\Big)(\theta)\Big)\Big).
\end{aligned}
\tag{19}
$$

Proof. By the given boundedness condition of $\mathcal{G}'$ and (10), we have

$$
\begin{aligned}
&\int_r^{\theta}(K-\mathcal{G}'(\tau))(\theta-\tau)^{\alpha-1}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau\\
&\leq(\theta-r)^{\alpha-1}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^{\theta}(K-\mathcal{G}'(\tau))d\tau
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
&\int_r^{\theta}(\mathcal{G}'(\tau)-k)(\theta-\tau)^{\alpha-1}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau\\
&\leq(\theta-r)^{\alpha-1}\varepsilon_{(a_i),\alpha}^{(c_i),(\varpi_i)}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^{\theta}(\mathcal{G}'(\tau)-k)d\tau.
\end{aligned}
\tag{21}
$$

From (20) and (21), we obtain the following inequalities, respectively, by applying integration by parts

$$\begin{aligned}&(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta)\\ \le& K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(\theta)\Big)\end{aligned}\tag{22}$$

and

$$\begin{aligned}&\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta)-(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\\ \le&-k\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(\theta)\Big).\end{aligned}\tag{23}$$

Now, by the given hypothesis $k<|\mathcal{G}'(\tau)|\le K$ and (16), we have

$$\begin{aligned}&\int_\theta^s(K-\mathcal{G}'(\tau))(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})d\tau\\ \le&(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\int_\theta^s(K-\mathcal{G}'(\tau))d\tau\end{aligned}\tag{24}$$

and

$$\begin{aligned}&\int_\theta^s(\mathcal{G}'(\tau)-k)(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})d\tau\\ \le&(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\int_\theta^s(\mathcal{G}'(\tau)-k)d\tau.\end{aligned}\tag{25}$$

From (24) and (25), we obtain the following inequalities, respectively, by applying integrations by parts

$$\begin{aligned}&\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,s-}\mathcal{G}\Big)(\theta)-(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\\ \le& K\Big((s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,s-}1\Big)(\theta)\Big)\end{aligned}\tag{26}$$

and

$$\begin{aligned}&(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,s-}\mathcal{G}\Big)(\theta)\\ \le&-k\Big((s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,s-}1\Big)(\theta)\Big).\end{aligned}\tag{27}$$

The inequalities (22) and (26) give the desired inequality (18). Similarly, the inequalities (23) and (27) give inequality (19). □

Theorem 4. *Under the same hypothesis of Theorem 3, the following inequalities hold:*

$$\begin{aligned}&\Big((\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})+(s-\theta)^{\beta-1}\\ &\times\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\Big)\mathcal{G}(\theta)-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,s-}\mathcal{G}\Big)(\theta)\Big)\\ \le& K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)\Big)\\ &-k\Big((s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,s-}1\Big)(\theta)\Big),\end{aligned}\tag{28}$$

and

$$-\Big((s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-(\theta-r)^{\alpha-1}$$
$$\times\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\Big)\mathcal{G}(\theta)+\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(\theta)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,s-}\mathcal{G}\Big)(\theta)\Big)$$
$$\leq K\Big((s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,s-}1\Big)(\theta)\Big)$$
$$-k\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(\theta)\Big). \tag{29}$$

Proof. The proof of inequality (28) can be proved using (22) and (27). Similarly, one can prove inequality (29) using (23) and (26). □

Theorem 5. *Suppose that $\mathcal{G}: I \to \mathbb{R}$ is a differentiable mapping in I^0, where I^0 denotes the interior of I, and let $r,s \in I^0$ with $r < s$. If $|\mathcal{G}'(\tau)| \leq K$ for all $t \in [r,s]$, then for $\alpha,\beta \geq 1$, $a_i, b, c_i \in \mathbb{C}$, where $i = 1,2,\cdots,j$, we have*

$$\Big|\ (s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\mathcal{G}(s)+(\theta-r)^{\alpha-1}$$
$$\times\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(r)-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,\theta+}\mathcal{G}\Big)(s)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,\theta-}\mathcal{G}\Big)(r)\Big)\ \Big|$$
$$\leq K\Big[(\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})+(s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})$$
$$-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,\theta-}1\Big)(r)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,\theta+}1\Big)(s)\Big)\Big].$$

Proof. Let $\theta \in [r,s]$ and $\tau \in [r,\theta]$. Then, for the multivariate M-L function, the following inequality holds

$$(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})\leq(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j}). \tag{30}$$

Now, by the given hypothesis $|\mathcal{G}'(\tau)| \leq K$ and (10), we have

$$\int_r^\theta (K-\mathcal{G}'(\tau))(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau$$
$$\leq(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^\theta(K-\mathcal{G}'(\tau))d\tau \tag{31}$$

and

$$\int_r^\theta (K+\mathcal{G}'(\tau))(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau$$
$$\leq(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^\theta(K+\mathcal{G}'(\tau))d\tau. \tag{32}$$

Now, consider first (31)

$$K\int_r^\theta(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau$$
$$-\int_r^\theta\mathcal{G}'(\tau)(\theta-\tau)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-\tau)^{a_1},\cdots,\varpi_j(\theta-\tau)^{a_j})d\tau$$
$$\leq(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\int_r^\theta(K-\mathcal{G}'(\tau))d\tau.$$

Integrating by parts yields

$$\begin{aligned}&\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,\theta-}\mathcal{G}\right)(r)-(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(r)\\&\leq K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,\theta-}1\right)(r)\Big).\end{aligned}\tag{33}$$

Similarly, (32) gives

$$\begin{aligned}&(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(r)-\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,\theta-}\mathcal{G}\right)(r)\\&\leq K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,\theta-}1\right)(r)\Big).\end{aligned}\tag{34}$$

From (33) and (34), we obtain the following:

$$\begin{aligned}&\mid(\theta-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(r)-\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,\theta-}\mathcal{G}\right)(r)\mid\\&\leq K\Big((\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})-\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,\theta-}1\right)(r)\Big).\end{aligned}\tag{35}$$

Again, let $\theta\in[r,s]$, $\tau\in[\theta,s]$ and $\beta\geq 1$. Then, for the multivariate M-L function, the following inequality holds

$$\begin{aligned}&(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})\\&\leq(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j}).\end{aligned}\tag{36}$$

Now, by the given hypothesis $|\mathcal{G}'(\tau)|\leq K$ and (36), we have

$$\begin{aligned}&\int_\theta^s(K-\mathcal{G}'(\tau))(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})d\tau\\&\leq(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\int_\theta^s(K-\mathcal{G}'(\tau))d\tau\end{aligned}$$

and

$$\begin{aligned}&\int_\theta^s(K+\mathcal{G}'(\tau))(\tau-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(\tau-\theta)^{a_1},\cdots,\varpi_j(\tau-\theta)^{a_j})d\tau\\&\leq(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\int_\theta^s(K+\mathcal{G}'(\tau))d\tau.\end{aligned}$$

By applying the similar procedure as we did for (31) and (32), we obtain

$$\begin{aligned}&\mid(s-\theta)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\mathcal{G}(s)-\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,\theta+}\mathcal{G}\right)(s)\mid\\&\leq K\Big((s-\theta)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})-\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,\theta+}1\right)(s)\Big).\end{aligned}\tag{37}$$

Inequalities (31) and (37) give the desired result. □

Corollary 2. *If we consider $\alpha=\beta$ in Theorem 5, then we have*

$$\begin{aligned}&\mid(s-\theta)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\mathcal{G}(s)+(\theta-r)^{\alpha-1}\\&\times\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})\mathcal{G}(r)-\Big(\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,\theta+}\mathcal{G}\right)(b)+\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,\theta-}\mathcal{G}\right)(r)\Big)\mid\\&\leq K\Big(\theta-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(\theta-r)^{a_1},\cdots,\varpi_j(\theta-r)^{a_j})+(s-\theta)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-\theta)^{a_1},\cdots,\varpi_j(s-\theta)^{a_j})\\&-\Big(\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,\theta-}1\right)(r)+\left(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,\theta+}1\right)(s)\Big)\Big).\end{aligned}$$

Remark 3.

i. *Applying Theorem 4 for $j = 1$, we obtain a certain new inequality for the fractional integral operator pertaining to the three-parameter M-L function [20].*
ii. *Applying Theorem 4 for $\varpi_i = 0$ leads to the result proved by [25].*
iii. *Applying Theorem 4 for $\alpha = 1 = \beta$ and $\varpi_i = 0$ leads to inequality (1).*

3. Application

In this section, we present some applications of Theorem 5 by applying it to the endpoints of the interval $[r, s]$, and then the summing the resulting inequalities yields the following inequality.

Theorem 6. *Suppose that the assumptions of Theorem 5 are satisfied, then we have*

$$
\begin{aligned}
&\mid (s-r)^{\beta-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-r)^{a_1},\cdots,\varpi_j(s-r)^{a_j})\mathcal{G}(s)+(s-r)^{\alpha-1}\\
&\times\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-r)^{a_1},\cdots,\varpi_j(s-r)^{a_j})\mathcal{G}(r)-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta-1,r+}\mathcal{G}\Big)(s)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,s-}\mathcal{G}\Big)(r)\Big)\mid\\
&\leq K\Big(s-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-r)^{a_1},\cdots,\varpi_j(s-r)^{a_j})+(s-r)^{\beta}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\beta}(\varpi_1(s-r)^{a_1},\cdots,\varpi_j(s-r)^{a_j})\\
&-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,s-}1\Big)(r)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\beta,r+}1\Big)(s)\Big)\Big).
\end{aligned}
$$

Proof. By applying Theorem 5 for $\theta = r$ and $\theta = s$, and then adding the obtained inequalities, we obtain the desired Theorem 6. □

Corollary 3. *The error bound of the Hadamard-type inequality can be achieved by applying Theorem 6 for $\alpha = \beta$ as follows:*

$$
\begin{aligned}
&\mid \Big(\frac{\mathcal{G}(s)+\mathcal{G}(r)}{2}\Big)\Big((s-r)^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-r)^{a_1},\cdots,\varpi_j(s-r)^{a_j})\Big)\\
&-\frac{1}{2}\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,r+}\mathcal{G}\Big)(s)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,s-}\mathcal{G}\Big)(r)\Big)\mid\\
&\leq K\Big(s-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-r)^{a_1},\cdots,\varpi_j(s-r)^{a_j})\\
&-\frac{1}{2}\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,s-}1\Big)(r)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,r+}1\Big)(s)\Big)\Big).
\end{aligned}
$$

Theorem 7. *Suppose that the assumptions of Theorem 5 are satisfied, then we have*

$$
\begin{aligned}
&\mid (s-(\frac{r+s}{2}))^{\alpha-1}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-(\frac{r+s}{2}))^{a_1},\cdots,\varpi_j(s-(\frac{r+s}{2}))^{a_j})\mathcal{G}(s)+((\frac{r+s}{2})-r)^{\alpha-1}\\
&\times\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1((\frac{r+s}{2})-r)^{a_1},\cdots,\varpi_j((\frac{r+s}{2})-r)^{a_j})\mathcal{G}(r)\\
&-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,(\frac{r+s}{2})+}\mathcal{G}\Big)(s)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha-1,(\frac{r+s}{2})-}\mathcal{G}\Big)(r)\Big)\mid\\
&\leq K\Big((\frac{r+s}{2})-r)^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1((\frac{r+s}{2})-r)^{a_1},\cdots,\varpi_j((\frac{r+s}{2})-r)^{a_j})\\
&+(s-(\frac{r+s}{2}))^{\alpha}\varepsilon^{(c_i),(\varpi_i)}_{(a_i),\alpha}(\varpi_1(s-(\frac{r+s}{2}))^{a_1},\cdots,\varpi_j(s-(\frac{r+s}{2}))^{a_j})\\
&-\Big(\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,(\frac{r+s}{2})-}1\Big)(r)+\Big(\mathcal{R}^{(c_i),(\varpi_i)}_{(a_i),\alpha,(\frac{r+s}{2})+}1\Big)(s)\Big)\Big).
\end{aligned}
$$

Proof. By applying Theorem 5 for $\alpha = \beta$ and $\theta = \frac{r+s}{2}$, we obtain the desired inequality of Theorem 7. □

4. Concluding Remarks

In the present article, we established the general form of Ostrowski-type fractional integral inequalities pertaining the multivariate M-L function in the kernel. Some new and existing inequalities are presented in this paper. If we consider $j = 1$ throughout in the paper, then we will have inequalities for the Prabhakar fractional operators containing the three-parameter M-L function. If we consider $\varpi = 0$, then the outcomes of this article will reduce to the Ostrowski-type inequalities for the R-L fractional integral operator [25].

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