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Advances in Ostrowski-Mercer Like Inequalities within Fractal Space

Miguel Vivas-Cortez [1], Muhammad Uzair Awan [2,*], Usama Asif [2], Muhammad Zakria Javed [2] and Hüseyin Budak [3]

[1] Escuela de Ciencias Fsicas y Matemáticas, Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Av. 12 de Octubre 1076, Apartado, Quito 17-01-2184, Ecuador; mjvivas@puce.edu.ec
[2] Department of Mathematics, Government College University, Faisalabad 38000, Pakistan
[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey
* Correspondence: awan.uzair@gmail.com

Abstract: The main idea of the current investigation is to explore some new aspects of Ostrowski's type integral inequalities implementing the generalized Jensen–Mercer inequality established for generalized s-convexity in fractal space. To proceed further with this task, we construct a new generalized integral equality for first-order local differentiable functions, which will serve as an auxiliary result to restore some new bounds for Ostrowski inequality. We establish our desired results by employing the equality, some renowned generalized integral inequalities like Hölder's, power mean, Yang-Hölder's, bounded characteristics of the functions and considering generalized s-convexity characteristics of functions. Also, in support of our main findings, we deliver specific applications to means, and numerical integration and graphical visualization are also presented here.

Keywords: convex; Jensen-Mercer; Ostrowski; fractal; inequalities; means

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1. Introduction

Fractals have been used in many different scientific disciplines since it first came into existence over a hundred years ago. But in the last four decades, its influence has increased many times in mathematics. Mandelbrot has published many books on this topic and introduced the notion of a fractal set, whose Hausdroff dimensions strictly increase the topological dimensions. After this major development, numerous studies have been done regarding this issue. In [1], Yang computed some $\varpi^{\dagger}$-level sets assuming that $\varpi^{\dagger}$ is the dimension of the fractal set.

The theory of convexity has its own very crucial character in the growth of inequalities, and several integral inequalities are restored by considering the notion of convex functions and its generalized forms. Recently, various kinds of novel and innovative convexity have been established. One of them is the generalized convexity defined over the fractal set. After this, many notable and fruitful extensions over fractal sets have been made essentially by making use of non-negative mapping h and parameter $s \in (0, 1]$ and some strong convexities.

Fractional calculus, which deals with non-integer order derivatives and integrals, is one of the key calculus modifications devised recently for a better understanding and depiction of real-world issues. The fractional calculus over the fractal set, frequently referred to as the local fractal calculus, was first developed in 2012 by Yang [1]. The creation of photographs, small-angle scattering theory, the music industry, soil mechanics, etc., are all fields where fractals are beneficial. In non-differentiable problems pertaining to science and engineering nowadays, fractal calculus outperforms wholly and practically. For more details, see [2–4]. Inspired by the pre-mentioned facts, Mo et al. [5] introduced the concept

of generalized convexity over $\mathbb{R}^{\omega^\dagger}$ and have discussed some algebraic properties of this new class and developed the fractal version of well-known integral inequalities.

In 2017 Sarikaya and Budak et al. [6] formulated the Ostrowski-type integral inequalities by considering the class of generalized convexity and local fractional approach. In [7], the authors have explored the notion of generalized s-convexity and established some new integral inequalities. In [8], the authors explored the Simpson's like inequalities through a novel class of convexity named (s, p)-convex functions in the frame of local fractional calculus. Kilicman and Saleh [9] studied some new aspects of generalized convexity and developed some applications for integral inequalities. Chu et al. [10] introduced the conception of exponential convex functions over the fractal set and by implementing this notion developed some new unified bounds of integral inequalities. In 2020, Sanchez and Sanabria [11] reported the new class of strongly convex functions defined over $\mathbb{R}^{\omega^\dagger}$ and discussed some algebraic properties. By using this notion, some strong fractal versions of fundamental inequalities of Jensen's type have been obtained. Luo et al. [12] have reported some weighted Hermite–Hadamard type inequalities associated with h-convexity defined over $\mathbb{R}^{\omega^\dagger}$ and have discussed some special cases to enhance the study with previous results in the literature. In 2020, Sun et al. [13] focused on generalized harmonic convexity to prove some new estimates of trapezium-type inequalities with the help of local identities. Sun et al. [14] have examined the Hermite–Hadamard type inequalities involving the class of generalized harmonic convex mappings defined over the fractal set in association with local fractional calculus. In 2021, Weibing Sun [15] focused on Ostrowski-type inequalities implementing the generalized convex functions through fractal concepts. Razzak et al. [16] have introduced the notion of generalized F convex mapping to explore some new integral inequalities with the help of local fractal calculus. In [17], Kian and Moslehian derived the Jensen–Mercer inequality for operator convexity and some other related inequalities. After this series of work has been done, for further investigation see [18–22]. The first time, Butt and colleagues [23] examined the local fractional variants of Jensen–Mercer inequality and some other inequalities of trapezoidal type in association with the newly proved Jensen–Mercer inequality. In [24], authors investigated the well-known Jensen–Mercer inequality through generalized convexity defined by non-negative mapping h in the context of the fractal domain. In 2017, the authors of [25] studied the Jensen–Mercer inequality from the perspective of harmonic convexity over the fractal domain. In [26], Kalsoom and her colleagues studied Hermite–Hadamard–Fejer-type inequalities involving h-harmonically convex mappings. In [27], the authors developed new counterparts of Simpson's schemes involving generalized p convexity. Erden and Sarikaya [28] investigated the new error estimates for Bullen-type inequalities along with applications. In [29], Wenbing Sun implemented the s-preinvex mappings to acquire new local fractional analogs of Hermite–Hadamard-type inequalities and applications. Du et al. [30] explored certain integral inequalities based on m-convexity convexity over the fractal set and discussed their applications as well. In [31], Yu and his colleagues derived a new variant of improved power-mean inequality in the fractal domain and established new fractional mid-point inequalities and some interesting applications.

2. Preliminaries

Here, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ donate the set of natural numbers, set of integers, set of rational numbers, and set of real numbers, respectively. Through the idea of Yang [1], we recall basic concepts and known results regarding local fractional calculus.

Let us start by the notion of $\omega^\dagger$ type set of element sets: In the following sequel, Mo and Sui [5] introduced a new class of convex functions over the fractal domain, which is defined as

1. $\mathbb{Z}^{\omega^\dagger} := \{\pm 0^{\omega^\dagger}, \pm 1^{\omega^\dagger}, \pm 2^{\omega^\dagger}, ...\} =: \omega^\dagger$ type set of irrational number.
2. $\mathbb{Q}^{\omega^\dagger} := \{v^{\omega^\dagger} = \left(\frac{p_1}{q_1}\right)^{\omega^\dagger} : p_1, q_1 \in \mathbb{Z}, q_2 \neq 0\} =: \omega^\dagger$ type set of irrational number.

3. $\mathbb{Q}'^{\omega^\dagger} := \{v^{\omega^\dagger} \neq \left(\frac{p_1}{q_1}\right)^{\omega^\dagger} : p_1, q_1 \in \mathbb{Z}, q_2 \neq 0\}$=:$\omega^\dagger$ type set of irrational number.
4. $\mathbb{R}^{\omega^\dagger} := \mathbb{Q}^{\omega^\dagger} \bigcup \mathbb{Q}'^{\omega^\dagger} =: \omega^\dagger$ type set of real number.

We also consider two binary operations, the addition '$+$' and multiplication '$*$' on the $\omega^\dagger$-type set $\mathbb{R}^{\omega^\dagger}$ of real numbers as follows.

$$c^{\omega^\dagger} + d^{\omega^\dagger} := (c+d)^{\omega^\dagger} \qquad \& \qquad c^{\omega^\dagger} * d^{\omega^\dagger} = c^{\omega^\dagger} d^{\omega^\dagger} := (cd)^{\omega^\dagger}.$$

and both $c^{\omega^\dagger} + d^{\omega^\dagger}, c^{\omega^\dagger} d^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$.

- Further more one can observe that $(\mathbb{R}^{\omega^\dagger}, +)$ forms commutative group. For any $c^{\omega^\dagger}, d^{\omega^\dagger}, e^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$.
  $c^{\omega^\dagger} + d^{\omega^\dagger} = d^{\omega^\dagger} + c^{\omega^\dagger}$.
  $(c^{\omega^\dagger} + d^{\omega^\dagger}) + e^{\omega^\dagger} = c^{\omega^\dagger} + (d^{\omega^\dagger} + e^{\omega^\dagger})$.
  $0^{\omega^\dagger}$ is the additive identity of $\mathbb{R}^{\omega^\dagger}$, $0^{\omega^\dagger} + c^{\omega^\dagger} = c^{\omega^\dagger} + 0^{\omega^\dagger}$, $\forall c^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$.
  For any $c^{\omega^\dagger}$ then there exist $(-c)^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$ such that $c^{\omega^\dagger} + (-c)^{\omega^\dagger} = 0^{\omega^\dagger}$.
- Also, $(\mathbb{R}^{\omega^\dagger}, *)$ $\{0\}$ forms a commutative group. For any $c^{\omega^\dagger}, d^{\omega^\dagger}, e^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$.
  $c^{\omega^\dagger} d^{\omega^\dagger} = d^{\omega^\dagger} c^{\omega^\dagger}$
  $(c^{\omega^\dagger} d^{\omega^\dagger}) e^{\omega^\dagger} = c^{\omega^\dagger} (d^{\omega^\dagger} e^{\omega^\dagger})$
  $1^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$ then for each $c^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$ such that $1^{\omega^\dagger} c^{\omega^\dagger} = c^{\omega^\dagger} 1^{\omega^\dagger} = c^{\omega^\dagger}$.
  For each $c^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$ $\{0\}$ then there exist $\left(\frac{1}{c}\right)^{\omega^\dagger}$ such that $c^{\omega^\dagger}\left(\frac{1}{c}\right)^{\omega^\dagger} = \left(c\frac{1}{c}\right)^{\omega^\dagger} = 1^{\omega^\dagger}$.

Remark 1. • *$(\mathbb{R}^{\omega^\dagger}, +, *)$ forms field.*

- *If the order $<$ relation is defined on $\mathbb{R}^{\omega^\dagger}$ is defined as follows: $c^{\omega^\dagger} < d^{\omega^\dagger} \Leftrightarrow c < d$ in $\mathbb{R}$. Then, $(\mathbb{R}^{\omega^\dagger}, +, *)$ is an ordered field.*

Now, we have local fractional continuity, which is described as

Definition 1. *A non-differentiable mapping $\Upsilon : \mathbb{R} \to \mathbb{R}^{\omega^\dagger}$, $v \to \Upsilon(v)$ is named as local fractional continuous at v_0, if for any $\epsilon > 0$ then there exist $\delta > 0$ such that*

$$|\Upsilon(v) - \Upsilon(v_0)| < \epsilon^{\omega^\dagger}, \qquad |v - v_0| < \delta.$$

If $\Upsilon(v)$ is local continuous at (w_1, w_2), then $\Upsilon(v) \in C_{\omega^\dagger}(w_1, w_2)$.

Now, we have a look at local differentiability, which is given as

Definition 2. *The local fractional derivative of $\Upsilon(v)$ of order $\omega^\dagger$ at $v = v_0$ is stated as*

$$\Upsilon^{\omega^\dagger}(v) = v_0 D_v^{\omega^\dagger} \Upsilon(v) = \left| \frac{\mathrm{d}^{\omega^\dagger} \Upsilon(v)}{(\mathrm{d}v)^{\omega^\dagger}} \right|_{v=v_0} = \lim_{v \to v_0} \frac{\triangle^{\omega^\dagger}(\Upsilon(v) - \Upsilon(v_0))}{(v - v_0)^{\omega^\dagger}}.$$

where $\triangle^{\omega^\dagger}(\Upsilon(v) - \Upsilon(v_0)) = \Gamma(1 + \omega^\dagger)(\Upsilon(v) - \Upsilon(v_0))$.

Let $\Upsilon^{\omega^\dagger}(v) = D_v^{\omega^\dagger} \Upsilon(v)$. If there exists $\Upsilon^{(l+1)\omega^\dagger}(v) = \overbrace{D_v^{\omega^\dagger} \Upsilon(v).D_v^{\omega^\dagger} \Upsilon(v) \ldots D_v^{\omega^\dagger} \Upsilon(v)}^{(l+1) times}$ for any $v \in [w_1, w_2]$, then it is denoted by $\Upsilon \in D_{(l+1)\omega^\dagger}$, where $k = 1, 2, 3, \ldots$.

Now, we describe the local integration of $\Upsilon(v) \in C_{\omega^\dagger}(w_1, w_2)$.

Definition 3. *Let* $\triangle = \{\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_n\}$ *where* $n \in \mathbb{N}$ *is a partition of* $[w_1, w_2]$ *such that* $\mathfrak{x}_0 < \mathfrak{x}_1 < \mathfrak{x}_2 < \cdots < \mathfrak{x}_n$. *Then, the local fractional integral of* Υ *on* $[w_1, w_2]$ *is defined as*

$$ {}_{w_1}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{x}) = \frac{1}{\Gamma(1+\omega^\dagger)}\int_{w_1}^{b} f(\mathfrak{x})(\mathrm{d}\mathfrak{x})^{\omega^\dagger} = \frac{1}{\Gamma(1+\omega^\dagger)}\lim_{\triangle\mathfrak{x}_i\to 0}\sum_{i=1}^{n}\Upsilon(\mathfrak{x}_i(\triangle\mathfrak{x})^{\omega^\dagger}, $$

where $\triangle\mathfrak{x}_i = \nu_i - \nu_{i-1}$ *for* $i = 1, 2, 3, \ldots, n$.

From the above expression, it is clear that ${}_{w_1}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{x}) = 0$ *if* $w_1 = w_2$ *and* ${}_{w_1}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{x}) = -{}_{w_2}I^{\omega^\dagger}_{w_1}\Upsilon(\mathfrak{x})$ *when* $w_1 < w_2$. *For any* $\mathfrak{x} \in [w_1, w_2]$, *if there exist* ${}_{w_1}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{x})$, *then it is denoted by* $\Upsilon(v) \in I^{\omega^\dagger}_{\mathfrak{x}}[w_1, w_2]$.

Lemma 1. *The following equalities hold:*

1. *(Local fractional integration is anti-differentiation) If* $\Upsilon(\mathfrak{x}) = r^{\omega^\dagger}(\mathfrak{x}) \in C_{\omega^\dagger}[w_1, w_2]$, *then*

$$ {}_{w_1}I^{\omega^\dagger}_{w_2}g(\mathfrak{x}) = r(w_2) - r(w_1). $$

2. *Local fractional derivative of* $\mathfrak{x}^{l\omega^\dagger}$ *is*

$$ \frac{\mathrm{d}^{\omega^\dagger}\mathfrak{x}^{l\omega^\dagger}}{(\mathrm{d}\mathfrak{x})^{\omega^\dagger}} = \frac{\Gamma(1+l\omega^\dagger)}{\Gamma(1+(l-1)\omega^\dagger)}\mathfrak{x}^{(l-1)\omega^\dagger}. $$

3. *Local fractional integration of* $\mathfrak{x}^{l\omega^\dagger}$ *is*

$$ \frac{1}{\Gamma(1+\omega^\dagger)}\int_{w_1}^{w_2}\mathfrak{x}^{l\omega^\dagger}(\mathrm{d}\mathfrak{x})^{\omega^\dagger} = \frac{\Gamma(1+l\omega^\dagger)}{\Gamma(1+(l+1)\omega^\dagger)}({w_2}^{(l+1)\omega^\dagger} - {w_1}^{(l+1)\omega^\dagger}). $$

In 2014, Mo et al. [5] introduced the concept of generalized convexity over fractal set $\mathbb{R}^{\omega^\dagger}$, which is stated as

Definition 4. *Any mapping* $\Upsilon : [w_1, w_2] \to \mathbb{R}^{\omega^\dagger}$ *is said to be a generalized convexity on fractal set, if*

$$ \Upsilon(\nu w_1 + (1-\nu)w_2) \le \nu^{\omega^\dagger} + (1-\nu)^{\omega^\dagger}\Upsilon(w_2), \tag{1} $$

$\nu \in [w_1, w_2]$ *and* $0 < \omega^\dagger \le 1$.

Also, Υ *is said to be a generalized concave* $\Leftrightarrow -\Upsilon$ *is a generalized convex function.*

If the inequality (2) holds strictly, then it is known as a strictly generalized convex function.

Now, we give the second local derivative test for generalized convexity proved by Mo et al. [5].

Theorem 1. *Let* $\Upsilon \in D_{2\omega^\dagger}(w_1, w_2)$. *Then* Υ *is said to be a generalized convex (generalized concave) function, if and only if,*

$$ \Upsilon^{2\omega^\dagger}(\mathfrak{x}) > 0 \quad (\Upsilon^{2\omega^\dagger} < 0), \quad \mathfrak{x} \in (w_1, w_2). $$

In [7], Mo and Sui explored the notion generalized s-convexity mappings, which are stated as

Definition 5. *Any mapping* $\Upsilon : [w_1, w_2] \to \mathbb{R}^{\omega^\dagger}$ *is said to be generalized s-convexity on a fractal set, if*

$$ \Upsilon(\nu w_1 + (1-\nu)w_2) \le \nu^{s\omega^\dagger}\Upsilon w_1 + (1-\nu)^{s\omega^\dagger}\Upsilon(w_2), \tag{2} $$

$\nu \in [0,1], \quad s \in (0,1]$ *and* $0 < \varpi^{\dagger} \leq 1$.

Our first concerned inequality regarding generalized convex functions is generalized Jensen inequality, which is stated as

Theorem 2. *Let* $\Upsilon : [w_1, w_2] \to \mathbb{R}^{\varpi^{\dagger}} \quad (0 < \varpi^{\dagger} \leq 1)$ *be a generalized convex function. Then, for any* $\mathfrak{x}_i \in [w_1, w_2]$ *and* $\nu_i \in [0,1]$ *with* $\sum_{i=1}^{n} \nu_i = 1$, *then*

$$\Upsilon\left(\sum_{i=1}^{n} \nu_i \mathfrak{x}_i\right) \leq \sum_{i=1}^{n} \nu_i^{\varpi^{\dagger}} \Upsilon(\mathfrak{x}_i),$$

Now, we revisit another consequence of Jensen's inequality for two points, which is described as follows:

Theorem 3. *Let* $\Upsilon : [w_1, w_2] \to \mathbb{R}^{\varpi^{\dagger}} \quad (0 < \varpi^{\dagger} \leq 1)$ *be a generalized convex function, then*

$$\Upsilon\left(\frac{w_1 + w_2}{2}\right) \leq \frac{\Gamma(1+\varpi^{\dagger})}{(w_2 - w_1)^{\varpi^{\dagger}}} {}_{w_1} I_{w_2}^{\varpi^{\dagger}} \Upsilon(\mathfrak{x}) \leq \frac{\Upsilon(w_1) + \Upsilon(w_2)}{2^{\varpi^{\dagger}}}, \tag{3}$$

where $0 < \varpi^{\dagger} \leq 1$. *For more details, see [5].*

Recently, in 2022, Xu and colleagues [24] established a new variant of Jensen–Mercer inequality involving h-convex functions over the fractal set, which is followed as

Theorem 4. *Let* $h : J \to \mathbb{R}$ *be non-negative supermultiplicative mapping,* $h \neq 0$. *Let* $\nu_1, \nu_2, \ldots, \nu_n$ *be a positive real numbers* $(n \geq 2)$ *such that* $W_n = \sum_{k=1}^{n} \nu_k$ *and* $\sum_{k=1}^{n} h\left(\frac{\nu_k}{W_n}\right)$. *If* $\Upsilon : I = [w_1, w_2] \to \mathbb{R}^{\varpi^{\dagger}}$ *is a generalized h-convex mapping, then for any positive finite sequence* $\{\mathfrak{x}_k\} \in I$, *then*

$$\Upsilon\left(w_1 + w_2 - \frac{1}{W_n} \sum_{k=1}^{n} \nu_k \mathfrak{x}_k\right) \leq \Upsilon(w_1) + \Upsilon(w_2) - \sum_{k=1}^{n} h^{\varpi^{\dagger}}\left(\frac{\nu_k}{W_n}\right) \Upsilon(\mathfrak{x}_k).$$

Remark 2 ([24]). *1. If* $h^{\varpi^{\dagger}}(\nu) = \nu^{\varpi^{\dagger}}$, *then it reduces to a Jensen–Mercer inequality for generalized convex mappings, which is proved in [23].*

2. If $h^{\varpi^{\dagger}}(\nu) = \nu^{s\varpi^{\dagger}}$, *then it reduces to Jensen–Mercer inequality for generalized s-convex mappings.*

Motivated by the research work going on, we have organized the current study to investigate Ostrowski's type inequalities via generalized s-convexity in the second sense over fractal space. The novelty of our research is that we will propose some new upper bounds for the remainder in the well-known Ostrowski quadrature rule. We have distributed our study in several parts. Initially, we give a brief introduction of the current research and some essential facts, which are required for further proceedings. In the next part, we will propose new local fractional counterparts of Osrowski–Mercer inequality involving s-convexity of the functions along with Holder's type inequalities. Later, applications and numerical and graphical illustrations of our primary findings will be discussed in detail. We hope the problem's idea and technique will attract interested readers' attention.

3. Main Results

The current portion of the study is specified for the detailed investigation of Ostrowski's type inequalities over fractal settings. The substantial part of this study is to formulate a new identity involving local differentiable mappings.

Lemma 2. *Let* $\Upsilon : I = [w_1, w_2] \to \mathbb{R}^{\omega^\dagger} \quad (0 < \omega^\dagger \leq 1)$ *be function, such that* $\Upsilon \in D_{\omega^\dagger}(I^\circ)$ *where* I° *is the interior of* I *and* $\Upsilon^{\omega^\dagger} \in C_{\omega^\dagger}(w_1, w_2)$*, where* $[u_1, u_2] \subseteq [w_1, w_2]$ *and* $\mathfrak{x} \in [u_1, u_2]$*, then*

$$\begin{aligned}
&(\mathfrak{x}-u_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\Upsilon(u)\right]\\
&=\frac{(\mathfrak{x}-u_1)^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1 \nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu u_1+(1-\nu)\mathfrak{x}))(dt)^{\omega^\dagger}\\
&\quad-\frac{(u_2-\mathfrak{x})^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1 \nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu u_2+(1-\nu)\mathfrak{x}))(dt)^{\omega^\dagger}
\end{aligned}\tag{4}$$

Proof. Consider the right hand side of (4), we have

$$I=(\mathfrak{x}-u_1)^{2\omega^\dagger}I_1-(u_2-\mathfrak{x})^{2\omega^\dagger}I_2,\tag{5}$$

where,

$$\begin{aligned}
I_1&=\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1 \nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu u_1+(1-\nu)\mathfrak{x}))(dt)^{\omega^\dagger}\\
&=\frac{1}{(\mathfrak{x}-u_1)^{\omega^\dagger}}\left[\Upsilon(w_1+\mathfrak{x}-u_1)-\frac{\Gamma(1+\omega^\dagger)}{(\mathfrak{x}-u_1)^{\omega^\dagger}\Gamma(1+\omega^\dagger)}\int_{w_1}^{w_1+\mathfrak{x}-u_1}\Upsilon(u)(du)^{\omega^\dagger}\right]\\
&=\frac{\Upsilon(w_1+\mathfrak{x}-u_1)}{(\mathfrak{x}-u_1)^{\omega^\dagger}}-\frac{\Gamma(1+\omega^\dagger)}{(\mathfrak{x}-u_1)^{2\omega^\dagger}}{}_{w_1}I^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\Upsilon(u),
\end{aligned}\tag{6}$$

similarly, we obtain

$$\begin{aligned}
I_2&=\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1 \nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu u_2+(1-\nu)\mathfrak{x}))(dt)^{\omega^\dagger}\\
&=-\frac{\Upsilon(w_2+\mathfrak{x}-u_2)}{(u_2-\mathfrak{x})^{\omega^\dagger}}-\frac{\Gamma(1+\omega^\dagger)}{(u_2-\mathfrak{x})^{2\omega^\dagger}}{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\Upsilon(u),
\end{aligned}\tag{7}$$

Substituting (6) and (7) in (5), we obtain (4). □

Remark 3. *If we set* $u_1 = w_1$ *and* $u_2 = w_2$ *in Lemma 2, then we obtain Lemma 3 of [15].*

Theorem 5. *Under the assumptions of Lemma 2, If* $|\Upsilon^{\omega^\dagger}|$ *be a generalized s-convex function in second sense, then*

$$\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\Upsilon(u)\right]\right|\\
&\leq(\mathfrak{x}-u_1)^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_1)|+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|\right.\\
&\left.-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_1)|\right]+(u_2-\mathfrak{x})^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_2)|\right.\\
&\left.+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_2)|\right],
\end{aligned}\tag{8}$$

$B_{\omega^\dagger}(.,.)$ *is the well-known local beta function and is defined as*

$$B_{\omega^\dagger}(m_1,m_2)=\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(\nu)^{m_1\omega^\dagger-1}(1-\nu)^{m_2\omega^\dagger-1}(\mathrm{d}\nu)^{\omega^\dagger}.$$

Proof. Applying the modulus property and Jensen–Mercer inequality for generalized *s*-convex function in equality Lemma 2, we have

$$
\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\Upsilon(u)\right]\right|\\
&\le\frac{(\mathfrak{x}-u_1)^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\left|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu u_1+(1-\nu)\mathfrak{x}))\right|(dt)^{\omega^\dagger}\\
&\quad+\frac{(u_2-\mathfrak{x})^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\left|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu u_2+(1-\nu)\mathfrak{x}))\right|(dt)^{\omega^\dagger}\\
&\le\frac{(\mathfrak{x}-u_1)^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}\left[|\Upsilon^{\omega^\dagger}(w_1)|+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\nu^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(u_1)|-(1-\nu)^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|\right](dt)^{\omega^\dagger}\\
&\quad+\frac{(u_2-\mathfrak{x})^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}\left[|\Upsilon^{\omega^\dagger}(w_2)|+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\nu^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(u_2)|-(1-\nu)^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|\right](dt)^{\omega^\dagger}\\
&\le(\mathfrak{x}-u_1)^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_1)|+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|\right.\\
&\left.-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_1)|\right]+(u_2-\mathfrak{x})^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_2)|\right.\\
&\left.+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_2)|\right],
\end{aligned}
$$

which gives the inequality (8). □

Corollary 1. *If we take $s=1$ in (8), then*

$$
\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\Upsilon(u)\right]\right|\\
&\le(\mathfrak{x}-u_1)^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_1)|+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_1)|\right]\\
&\quad+(u_2-\mathfrak{x})^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_2)|+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_2)|\right].
\end{aligned}
$$

Corollary 2. *If we set $u_1=w_1$ and $u_2=w_2$ in (8), then we obtain*

$$
\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\Upsilon(u)\right]\right|\\
&\le(\mathfrak{x}-u_1)^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_1)|+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_1)|\right]\\
&\quad+(u_2-\mathfrak{x})^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_2)|+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|-\frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}|\Upsilon^{\omega^\dagger}(u_2)|\right].
\end{aligned}
$$

Corollary 3. *If we set $u_1=w_1, u_2=w_2$ and $\mathfrak{x}=\frac{w_1+w_2}{2}$ in (8), then we obtain*

$$
\begin{aligned}
&\left|\frac{(w_2-w_1)^{\omega^\dagger}\Upsilon\left(\frac{w_1+w_2}{2}\right)}{\Gamma(1+\omega^\dagger)}-\frac{1}{\Gamma(1+\omega^\dagger)}\int_{w_1}^{b}f(u)(du)^{\omega^\dagger}\right|\\
&\le\frac{1}{\Gamma(1+\omega^\dagger)}\left(\frac{w_2-w_1}{2}\right)^{2\omega^\dagger}\left[\left(\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}\right)|\Upsilon^{\omega^\dagger}(w_1)|\right.\\
&\left.+2^{\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]\left|\Upsilon^{\omega^\dagger}\left(\frac{w_1+w_2}{2}\right)\right|+\left(\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}\right)|\Upsilon^{\omega^\dagger}(w_2)|\right].
\end{aligned}
$$

Theorem 6. *Under the assumptions of Lemma 2, if $|\Upsilon^{\omega^\dagger}|^q$ be a generalized s-convex function in second sense, then*

$$
\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\Upsilon(u)\right]\right|
$$

$$
\begin{aligned}
&\leq \left(\frac{\Gamma(1+\varpi^{\dagger}p)}{\Gamma(1+(p+1)\varpi^{\dagger})}\right)^{\frac{1}{p}}\left[(\mathfrak{x}-\mathfrak{u}_1)^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}|\mathrm{Y}^{\varpi^{\dagger}}(w_1)|^q\right.\right.\\
&\quad+\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}\right)|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{u}_1)|^q\Big)^{\frac{1}{q}}\\
&\quad+(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}|\mathrm{Y}^{\varpi^{\dagger}}(w_2)|^q\right.\\
&\quad\left.\left.+\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}\right)|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{u}_2)|^q\Big)^{\frac{1}{q}}\right],
\end{aligned}
\tag{9}
$$

where $p^{-1}+q^{-1}=1$.

Proof. Applying the modulus property, Hölder's inequality, and generalized convexity property of $|\mathrm{Y}^{\varpi^{\dagger}}|^q$, then

$$
\begin{aligned}
&\left|(\mathfrak{x}-\mathfrak{u}_1)^{\varpi^{\dagger}}\mathrm{Y}(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\varpi^{\dagger}}\mathrm{Y}(w_2+\mathfrak{x}-\mathfrak{u}_2)-\Gamma(1+\varpi^{\dagger})\left[{}_{a}I^{\varpi^{\dagger}}_{w_1+\mathfrak{x}-\mathfrak{u}_1}\mathrm{Y}(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u}_2)}I^{\varpi^{\dagger}}_{w_2}\mathrm{Y}(\mathfrak{u})\right]\right|\\
&\leq\frac{(\mathfrak{x}-\mathfrak{u}_1)^{2\varpi^{\dagger}}}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left|\nu^{\varpi^{\dagger}}\mathrm{Y}^{\varpi^{\dagger}}(w_1+\mathfrak{x}-(\nu\mathfrak{u}_1+(1-\nu)\mathfrak{x}))\right|(dt)^{\varpi^{\dagger}}\\
&\quad+\frac{(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left|\nu^{\varpi^{\dagger}}\mathrm{Y}^{\varpi^{\dagger}}(w_2+\mathfrak{x}-(\nu\mathfrak{u}_2+(1-\nu)\mathfrak{x}))\right|(dt)^{\varpi^{\dagger}}\\
&\leq(\mathfrak{x}-\mathfrak{u}_1)^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\nu^{\varpi^{\dagger}p}(dt)^{\varpi^{\dagger}}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left|\mathrm{Y}^{\varpi^{\dagger}}(w_1+\mathfrak{x}-(\nu\mathfrak{u}_1+(1-\nu)\mathfrak{x}))\right|(dt)^{\varpi^{\dagger}}\right)^{\frac{1}{q}}\\
&\quad+(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\nu^{\varpi^{\dagger}p}(dt)^{\varpi^{\dagger}}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left|\mathrm{Y}^{\varpi^{\dagger}}(w_2+\mathfrak{x}-(\nu\mathfrak{u}_2+(1-\nu)\mathfrak{x}))\right|(dt)^{\varpi^{\dagger}}\right)^{\frac{1}{q}}\\
&\leq(\mathfrak{x}-\mathfrak{u}_1)^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\nu^{\varpi^{\dagger}p}(dt)^{\varpi^{\dagger}}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left[|\mathrm{Y}^{\varpi^{\dagger}}(w_1)|^q+|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q-\nu^{\varpi^{\dagger}s}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{u}_1)|^q-(1-\nu)^{\varpi^{\dagger}s}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q\right](dt)^{\varpi^{\dagger}}\right)^{\frac{1}{q}}\\
&\quad+(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\nu^{\varpi^{\dagger}p}(dt)^{\varpi^{\dagger}}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left[|\mathrm{Y}^{\varpi^{\dagger}}(w_2)|^q+|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q-\nu^{\varpi^{\dagger}s}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{u}_2)|^q-(1-\nu)^{\varpi^{\dagger}s}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q\right](dt)^{\varpi^{\dagger}}\right)^{\frac{1}{q}}\\
&\leq\left(\frac{\Gamma(1+\varpi^{\dagger}p)}{\Gamma(1+(p+1)\varpi^{\dagger})}\right)^{\frac{1}{p}}\left[(\mathfrak{x}-\mathfrak{u}_1)^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}|\mathrm{Y}^{\varpi^{\dagger}}(w_1)|^q\right.\right.\\
&\quad+\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}\right)|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{u}_1)|^q\Big)^{\frac{1}{q}}\\
&\quad+(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}|\mathrm{Y}^{\varpi^{\dagger}}(w_2)|^q\right.\\
&\quad\left.\left.+\left(\frac{1}{\Gamma(\varpi^{\dagger}+1)}-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}\right)|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{x})|^q-\frac{\Gamma(1+\varpi^{\dagger}s)}{\Gamma(1+(s+1)\varpi^{\dagger})}|\mathrm{Y}^{\varpi^{\dagger}}(\mathfrak{u}_2)|^q\Big)^{\frac{1}{q}}\right],
\end{aligned}
$$

which completes the proof. □

Corollary 4. *If we take* $s=1$ *in (9), then*

$$
\left|(\mathfrak{x}-\mathfrak{u}_1)^{\varpi^{\dagger}}\mathrm{Y}(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\varpi^{\dagger}}\mathrm{Y}(w_2+\mathfrak{x}-\mathfrak{u}_2)-\Gamma(1+\varpi^{\dagger})\left[{}_{a}I^{\varpi^{\dagger}}_{w_1+\mathfrak{x}-\mathfrak{u}_1}\mathrm{Y}(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u}_2)}I^{\varpi^{\dagger}}_{w_2}\mathrm{Y}(\mathfrak{u})\right]\right|
$$

$$\leq \left(\frac{\Gamma(1+\omega^{\dagger}p)}{\Gamma(1+(p+1)\omega^{\dagger})}\right)^{\frac{1}{p}}\left[(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^{\dagger}}\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}|\mathrm{Y}^{\omega^{\dagger}}(w_1)|^q\right.\right.$$
$$+\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}-\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}\right)|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{x})|^q-\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{u}_1)|^q\Big)^{\frac{1}{q}}$$
$$+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^{\dagger}}\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}|\mathrm{Y}^{\omega^{\dagger}}(w_2)|^q\right.$$
$$\left.\left.+\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}-\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}\right)|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{x})|^q-\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{u}_2)|^q\Big)^{\frac{1}{q}}\right],$$

Corollary 5. *If we set* $\mathfrak{u}_1=w_1,\mathfrak{u}_2=w_2$ *and* $\mathfrak{x}=\frac{w_1+w_2}{2}$ *in (9), then we obtain*

$$\left|\frac{(w_2-w_1)^{\omega^{\dagger}}\mathrm{Y}\left(\frac{w_1+w_2}{2}\right)}{\Gamma(1+\omega^{\dagger})}-\frac{1}{\Gamma(1+\omega^{\dagger})}\int_{w_1}^{b}f(\mathfrak{u})(d\mathfrak{u})^{\omega^{\dagger}}\right|$$
$$\leq \frac{1}{\Gamma(1+\omega^{\dagger})}\left(\frac{\Gamma(1+\omega^{\dagger}p)}{\Gamma(1+(p+1)\omega^{\dagger})}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{w_2-w_1}{2}\right)^{2\omega^{\dagger}}\left(\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}-\frac{\Gamma(1+\omega^{\dagger}s)}{\Gamma(1+(s+1)\omega^{\dagger})}\right)|\mathrm{Y}^{\omega^{\dagger}}(w_1)|^q\right.\right.$$
$$+\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}-\frac{\Gamma(1+\omega^{\dagger}s)}{\Gamma(1+(s+1)\omega^{\dagger})}\right)\left|\mathrm{Y}^{\omega^{\dagger}}\left(\frac{w_1+w_2}{2}\right)\right|^q\Big)^{\frac{1}{q}}$$
$$+\left(\frac{w_2-w_1}{2}\right)^{2\omega^{\dagger}}\left(\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}-\frac{\Gamma(1+\omega^{\dagger}s)}{\Gamma(1+(s+1)\omega^{\dagger})}\right)|\mathrm{Y}^{\omega^{\dagger}}(w_2)|^q\right.$$
$$\left.\left.+\left(\frac{1}{\Gamma(\omega^{\dagger}+1)}-\frac{\Gamma(1+\omega^{\dagger}s)}{\Gamma(1+(s+1)\omega^{\dagger})}\right)\left|\mathrm{Y}^{\omega^{\dagger}}\left(\frac{w_1+w_2}{2}\right)\right|^q\Big)^{\frac{1}{q}}\right].$$

Remark 4. *If we set* $\mathfrak{u}_1=w_1$ *and* $\mathfrak{u}_2=w_2$ *in inequality , then we obtain special case of inequality (3.16), as proved in [15].*

Theorem 7. *Under the assumptions of Lemma 2, if* $|\mathrm{Y}^{\omega^{\dagger}}|^q$ *be a generalized s-convex function in second sense, then*

$$\left|(\mathfrak{x}-\mathfrak{u}_1)^{\omega^{\dagger}}\mathrm{Y}(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\omega^{\dagger}}\mathrm{Y}(w_2+\mathfrak{x}-\mathfrak{u}_2)-\Gamma(1+\omega^{\dagger})\left[{}_{a}I^{\omega^{\dagger}}_{w_1+\mathfrak{x}-\mathfrak{u}_1}\mathrm{Y}(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u}_2)}I^{\omega^{\dagger}}_{w_2}\mathrm{Y}(\mathfrak{u})\right]\right|$$
$$\leq \left(\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}\right)^{1-\frac{1}{q}}\left((\mathfrak{x}-\mathfrak{u}_1)^{2\omega^{\dagger}}\left[\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}|\mathrm{Y}^{\omega^{\dagger}}(w_1)|^q+\left[\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}-B_{\omega^{\dagger}}(2,s+1)\right]|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{x})|^q\right.\right.$$
$$\left.-\frac{\Gamma(1+(s+1)\omega^{\dagger})}{\Gamma(1+(s+2)\omega^{\dagger})}|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{u}_1)|^q\right]^{\frac{1}{q}}+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^{\dagger}}\left[\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}|\mathrm{Y}^{\omega^{\dagger}}(w_2)|^q\right.$$
$$\left.\left.+\left[\frac{\Gamma(1+\omega^{\dagger})}{\Gamma(1+2\omega^{\dagger})}-B_{\omega^{\dagger}}(2,s+1)\right]|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{x})|^q-\frac{\Gamma(1+(s+1)\omega^{\dagger})}{\Gamma(1+(s+2)\omega^{\dagger})}|\mathrm{Y}^{\omega^{\dagger}}(\mathfrak{u}_2)|^q\right]^{\frac{1}{q}}\right),\tag{10}$$

where $p^{-1}+q^{-1}=1$ *and* $B_{\omega^{\dagger}}(w_1,w_2)=\frac{1}{\Gamma(1+\omega^{\dagger})}\int_0^1\nu^{w_1\alpha}(1-\nu)^{w_2\alpha}$ *is a well known beta function over the fractal set.*

Proof. Applying the power mean inequality and generalized convexity property of the $|\mathrm{Y}^{\omega^{\dagger}}|^q$, then

$$
\begin{aligned}
&\Big|(\mathfrak{x}-\mathfrak{u}_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-\mathfrak{u}_2)\\
&\quad-\Gamma(1+\omega^\dagger)\Big[{}_aI^{\omega^\dagger}_{w_1+w_2-\mathfrak{u}_1}\Upsilon(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u})}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{u})\Big]\Big|\\
&\le\frac{(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\Big|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu\mathfrak{u}_1+(1-\nu)\mathfrak{x}))\Big|(dt)^{\omega^\dagger}\\
&\quad+\frac{(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\Big|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu\mathfrak{u}_2+(1-\nu)\mathfrak{x}))\Big|(dt)^{\omega^\dagger}\\
&\le(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}(dt)^{\omega^\dagger}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\Big|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu\mathfrak{u}_1+(1-\nu)\mathfrak{x}))\Big|(dt)^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}(dt)^{\omega^\dagger}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\Big|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu\mathfrak{u}_2+(1-\nu)\mathfrak{x}))\Big|(dt)^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&\le(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}(dt)^{\omega^\dagger}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}\Big[|\Upsilon^{\omega^\dagger}(w_1)|^q+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\nu^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_1)|^q-(1-\nu)^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q\Big](dt)^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&\quad+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}(dt)^{\omega^\dagger}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\frac{1}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}\Big[|\Upsilon^{\omega^\dagger}(w_2)|^q+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\nu^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_2)|^q-(1-\nu)^{\omega^\dagger s}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q\Big](dt)^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&\le\left(\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}\right)^{1-\frac{1}{q}}\left((\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_1)|^q+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q\right.\right.\\
&\quad\left.-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_1)|^q\right]^{\frac{1}{q}}+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_2)|^q\right.\\
&\quad\left.\left.+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_2)|^q\right]^{\frac{1}{q}}\right),
\end{aligned}
$$

which completes the proof. □

Corollary 6. *If we take $s=1$ in (10), then*

$$
\begin{aligned}
&\Big|(\mathfrak{x}-\mathfrak{u}_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-\mathfrak{u}_2)-\Gamma(1+\omega^\dagger)\Big[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-\mathfrak{u}_1}\Upsilon(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u}_2)}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{u})\Big]\Big|\\
&\le\left(\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}\right)^{1-\frac{1}{q}}\left((\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_1)|^q+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q\right.\right.\\
&\quad\left.-\frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_1)|^q\right]^{\frac{1}{q}}+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}|\Upsilon^{\omega^\dagger}(w_2)|^q\right.\\
&\quad\left.\left.+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}-B_{\omega^\dagger}(2,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_2)|^q\right]^{\frac{1}{q}}\right).
\end{aligned}
$$

Corollary 7. *If we set* $\mathfrak{u}_1 = w_1, \mathfrak{u}_2 = w_2$ *and* $\mathfrak{x} = \frac{w_1+w_2}{2}$ *in (10), then we obtain*

$$\begin{aligned}
&\left|\frac{(w_2-w_1)^{\omega^\dagger}\Upsilon\left(\frac{w_1+w_2}{2}\right)}{\Gamma(1+\omega^\dagger)} - \frac{1}{\Gamma(1+\omega^\dagger)}\int_{w_1}^{b} f(\mathfrak{u})(d\mathfrak{u})^{\omega^\dagger}\right| \\
&\leq \frac{1}{\Gamma(1+\omega^\dagger)}\left(\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)}\right)^{1-\frac{1}{q}}\left(\left(\frac{w_2-w_1}{2}\right)^{2\omega^\dagger}\left[\left(\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)} - \frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}\right)|\Upsilon^{\omega^\dagger}(w_1)|^q\right.\right. \\
&\quad \left.\left.+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)} - B_{\omega^\dagger}(2,s+1)\right]\left|\Upsilon^{\omega^\dagger}\left(\frac{w_1+w_2}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad +\left(\frac{w_2-w_1}{2}\right)^{2\omega^\dagger}\left[\left(\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)} - \frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}\right)|\Upsilon^{\omega^\dagger}(w_2)|^q\right. \\
&\quad \left.\left.\left.+\left[\frac{\Gamma(1+\omega^\dagger)}{\Gamma(1+2\omega^\dagger)} - B_{\omega^\dagger}(2,s+1)\right]\left|\Upsilon^{\omega^\dagger}\left(\frac{w_1+w_2}{2}\right)\right|^q\right]^{\frac{1}{q}}\right).
\end{aligned}$$

Theorem 8. *Assume that all the conditions of Lemma 2 are satisfied, if* $|\Upsilon^{\omega^\dagger}|^q$ *is generalized s-convex on the interval* $[w_1, w_2]$ *for* $p, q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned}
&\left|(\mathfrak{x}-\mathfrak{u}_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-\mathfrak{u}_1) + (\mathfrak{u}_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-\mathfrak{u}_2) - \Gamma(1+\omega^\dagger)\left[{}_a I^{\omega^\dagger}_{w_1+w_2-\mathfrak{u}_1}\Upsilon(\mathfrak{u}) + {}_{(w_2+\mathfrak{x}-\mathfrak{u})} I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{u})\right]\right| \\
&\leq (\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left\{(B_{\omega^\dagger}(p+1,2))^{\frac{1}{p}}\left(\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}|\Upsilon^{\omega^\dagger}(w_1)|^q\right.\right. \\
&+\left[\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)} - B_{\omega^\dagger}(s+1,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q - \left.\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_1)|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{\Gamma(1+(p+1)\omega^\dagger)}{\Gamma(1+(p+2)\omega^\dagger)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}|\Upsilon^{\omega^\dagger}(w_1)|^q\right. \\
&\left.\left.+\left[\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)} - B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q - \frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_1)|^q\right)^{\frac{1}{q}}\right\} \\
&+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}\left\{(B_{\omega^\dagger}(p+1,2))^{\frac{1}{p}}\left(\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}|\Upsilon^{\omega^\dagger}(w_2)|^q\right.\right. \\
&+\left[\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)} - B_{\omega^\dagger}(s+1,2)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q - \left.\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_2)|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{\Gamma(1+(p+1)\omega^\dagger)}{\Gamma(1+(p+2)\omega^\dagger)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}|\Upsilon^{\omega^\dagger}(w_2)|^q\right. \\
&\left.\left.+\left[\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)} - B_{\omega^\dagger}(2,s+1)\right]|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q - \frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_2)|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Employing Lemma 2 and the Holder–Yang's inequality, we find that

$$\begin{aligned}
&\left|(\mathfrak{x}-\mathfrak{u}_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-\mathfrak{u}_2)-\Gamma(1+\omega^\dagger)\left[{}_a I^{\omega^\dagger}_{w_1+\mathfrak{x}-\mathfrak{u}_1}\Upsilon(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u}_2)}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{u})\right]\right|\\
&\leq \frac{(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\left|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu\mathfrak{u}_1+(1-\nu)\mathfrak{x}))\right|(dt)^{\omega^\dagger}\\
&\quad+\frac{(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\left|\nu^{\omega^\dagger}\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu\mathfrak{u}_2+(1-\nu)\mathfrak{x}))\right|(dt)^{\omega^\dagger}\\
&\leq(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left\{\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}|\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu\mathfrak{u}_1+(1-\nu)\mathfrak{x}))|^q(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}|\Upsilon^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu\mathfrak{u}_1+(1-\nu)\mathfrak{x}))|^q(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\right\}\\
&\quad+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}\left\{\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}|\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu\mathfrak{u}_2+(1-\nu)\mathfrak{x}))|^q(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}|\Upsilon^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu\mathfrak{u}_2+(1-\nu)\mathfrak{x}))|^q(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Owing to the generalized s-convexity of the mapping $|\Upsilon^{\omega^\dagger}|^q$, we find that

$$\begin{aligned}
&\left|(\mathfrak{x}-\mathfrak{u}_1)^{\omega^\dagger}\Upsilon(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\omega^\dagger}\Upsilon(w_2+\mathfrak{x}-\mathfrak{u}_2)-\Gamma(1+\omega^\dagger)\left[{}_a I^{\omega^\dagger}_{w_1+\mathfrak{x}-\mathfrak{u}_1}\Upsilon(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u}_2)}I^{\omega^\dagger}_{w_2}\Upsilon(\mathfrak{u})\right]\right|\\
&\leq(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left\{\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}[|\Upsilon^{\omega^\dagger}(w_1)|^q+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\nu^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_1)|^q-(1-\nu)^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q](d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\\
&\quad\left.\times\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}[|\Upsilon^{\omega^\dagger}(w_1)|^q+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\nu^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_1)|^q-(1-\nu)^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q](d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\right\}\\
&\quad+(\mathfrak{u}_2-\mathfrak{x})^{2\omega^\dagger}\left\{\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1(1-\nu)^{\omega^\dagger}[|\Upsilon^{\omega^\dagger}(w_2)|^q+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\nu^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_2)|^q-(1-\nu)^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q](d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}\nu^{p\omega^\dagger}(d(\nu))^{\omega^\dagger}\right)^{\frac{1}{p}}\\
&\quad\left.\times\left(\frac{1}{\Gamma(1+\omega^\dagger)}\int_0^1\nu^{\omega^\dagger}[|\Upsilon^{\omega^\dagger}(w_2)|^q+|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q-\nu^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{u}_2)|^q-(1-\nu)^{s\omega^\dagger}|\Upsilon^{\omega^\dagger}(\mathfrak{x})|^q](d(\nu))^{\omega^\dagger}\right)^{\frac{1}{q}}\right\}\\
&\leq(\mathfrak{x}-\mathfrak{u}_1)^{2\omega^\dagger}\left\{(B_{\omega^\dagger}(p+1,2))^{\frac{1}{p}}\left(\frac{\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}|\Upsilon^{\omega^\dagger}(w_1)|^q\right.\right.
\end{aligned}$$

$$+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(s+1,2)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_1)|^{q}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+(p+1)\varpi^{\dagger})}{\Gamma(1+(p+2)\varpi^{\dagger})}\right)^{\frac{1}{p}}\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}|\Upsilon^{\varpi^{\dagger}}(w_1)|^{q}\right.$$
$$\left.+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,s+1)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_1)|^{q}\Bigg)^{\frac{1}{q}}\right\}$$
$$+(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}\left\{(B_{\varpi^{\dagger}}(p+1,2))^{\frac{1}{p}}\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}|\Upsilon^{\varpi^{\dagger}}(w_2)|^{q}\right.\right.$$
$$+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(s+1,2)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_2)|^{q}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+(p+1)\varpi^{\dagger})}{\Gamma(1+(p+2)\varpi^{\dagger})}\right)^{\frac{1}{p}}\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}|\Upsilon^{\varpi^{\dagger}}(w_2)|^{q}\right.$$
$$\left.\left.+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,s+1)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_2)|^{q}\Bigg)^{\frac{1}{q}}\right\}.$$

In this way, we concluded our required result. □

Corollary 8. *If we take $s=1$ in Theorem 8, then we have the following inequality:*

$$\left|(\mathfrak{x}-\mathfrak{u}_1)^{\varpi^{\dagger}}\Upsilon(w_1+\mathfrak{x}-\mathfrak{u}_1)+(\mathfrak{u}_2-\mathfrak{x})^{\varpi^{\dagger}}\Upsilon(w_2+\mathfrak{x}-\mathfrak{u}_2)-\Gamma(1+\varpi^{\dagger})\left[{}_{a}I^{\varpi^{\dagger}}_{w_1+\mathfrak{x}-\mathfrak{u}_1}\Upsilon(\mathfrak{u})+{}_{(w_2+\mathfrak{x}-\mathfrak{u}_2)}I^{\varpi^{\dagger}}_{w_2}\Upsilon(\mathfrak{u})\right]\right|$$
$$\leq(\mathfrak{x}-\mathfrak{u}_1)^{2\varpi^{\dagger}}\left\{(B_{\varpi^{\dagger}}(p+1,2))^{\frac{1}{p}}\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}|\Upsilon^{\varpi^{\dagger}}(w_1)|^{q}\right.\right.$$
$$+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,2)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+2\varpi^{\dagger})}{\Gamma(1+3\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_1)|^{q}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+(p+1)\varpi^{\dagger})}{\Gamma(1+(p+2)\varpi^{\dagger})}\right)^{\frac{1}{p}}\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}|\Upsilon^{\varpi^{\dagger}}(w_1)|^{q}\right.$$
$$\left.+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,2)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+2\varpi^{\dagger})}{\Gamma(1+3\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_1)|^{q}\Bigg)^{\frac{1}{q}}\right\}$$
$$+(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}\left\{(B_{\varpi^{\dagger}}(p+1,2))^{\frac{1}{p}}\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}|\Upsilon^{\varpi^{\dagger}}(w_2)|^{q}\right.\right.$$
$$+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,2)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+2\varpi^{\dagger})}{\Gamma(1+3\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_2)|^{q}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+(p+1)\varpi^{\dagger})}{\Gamma(1+(p+2)\varpi^{\dagger})}\right)^{\frac{1}{p}}\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}|\Upsilon^{\varpi^{\dagger}}(w_2)|^{q}\right.$$
$$\left.\left.+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,2)\right]|\Upsilon^{\varpi^{\dagger}}(\mathfrak{x})|^{q}-\frac{\Gamma(1+2\varpi^{\dagger})}{\Gamma(1+3\varpi^{\dagger})}|\Upsilon^{\varpi^{\dagger}}(\mathfrak{u}_2)|^{q}\Bigg)^{\frac{1}{q}}\right\}.$$

Corollary 9. *If we set $\mathfrak{u}_1=w_1,\mathfrak{u}_2=w_2$ and $\mathfrak{x}=\frac{w_1+w_2}{2}$ in (8), then we obtain*

$$\left|\frac{(w_2-w_1)^{\varpi^{\dagger}}\Upsilon\left(\frac{w_1+w_2}{2}\right)}{\Gamma(1+\varpi^{\dagger})}-\frac{1}{\Gamma(1+\varpi^{\dagger})}\int_{w_1}^{b}f(\mathfrak{u})(d\mathfrak{u})^{\varpi^{\dagger}}\right|$$
$$\leq\frac{1}{\Gamma(1+\varpi^{\dagger})}\left(\frac{w_2-w_1}{2}\right)^{2\varpi^{\dagger}}(B_{\varpi^{\dagger}}(p+1,2))^{\frac{1}{p}}\left\{\left(\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)|\Upsilon^{\varpi^{\dagger}}(w_1)|^{q}\right.\right.$$

$$
\begin{aligned}
&+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(s+1,2)\right]\left|\Upsilon^{\varpi^{\dagger}}\left(\frac{w_1+w_2}{2}\right)\right|^{q}\Bigg)^{\frac{1}{q}}\\
&+\left(\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)|\Upsilon^{\varpi^{\dagger}}(w_2)|^{q}\right.\\
&\left.+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(s+1,2)\right]\left|\Upsilon^{\varpi^{\dagger}}\left(\frac{w_1+w_2}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\Bigg\}\\
&+\frac{1}{\Gamma(1+\varpi^{\dagger})}\left(\frac{w_2-w_1}{2}\right)^{2\varpi^{\dagger}}\left(\frac{\Gamma(1+(p+1)\varpi^{\dagger})}{\Gamma(1+(p+2)\varpi^{\dagger})}\right)^{\frac{1}{p}}\\
&\times\Bigg\{\left(\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)|\Upsilon^{\varpi^{\dagger}}(w_1)|^{q}\right.\\
&+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,s+1)\right]\left|\Upsilon^{\varpi^{\dagger}}\left(\frac{w_1+w_2}{2}\right)\right|^{q}\Bigg)^{\frac{1}{q}}\\
&+\left(\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)|\Upsilon^{\varpi^{\dagger}}(w_2)|^{q}\right.\\
&\left.+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(2,s+1)\right]\left|\Upsilon^{\varpi^{\dagger}}\left(\frac{w_1+w_2}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Theorem 9. *Assume that all the assumptions of Lemma 2 are satisfied, if* $|\Upsilon^{\varpi^{\dagger}}| \leq M^{\varpi^{\dagger}}$, *where* $M^{\varpi^{\dagger}} \in \mathbb{R}^{\varpi^{\dagger}}, \varpi^{\dagger} \in (0,1]$, *then*

$$
\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\varpi^{\dagger}}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\varpi^{\dagger}}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\varpi^{\dagger})\left[{}_{a}I^{\varpi^{\dagger}}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\varpi^{\dagger}}_{w_2}\Upsilon(u)\right]\right|\\
&\leq\frac{M^{\varpi^{\dagger}}\Gamma(1+\varpi^{\dagger})}{\Gamma(1+2\varpi^{\dagger})}[(\mathfrak{x}-u_1)^{2\varpi^{\dagger}}+(u_2-\mathfrak{x})^{2\varpi^{\dagger}}] \qquad (11)
\end{aligned}
$$

Proof. By taking the absolute value in Lemma 2 and using $|\Upsilon^{\varpi^{\dagger}}| \leq M^{\varpi^{\dagger}}, 0 < \varpi^{\dagger} \leq 1$, we have

$$
\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\varpi^{\dagger}}\Upsilon(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\varpi^{\dagger}}\Upsilon(w_2+\mathfrak{x}-u_2)-\Gamma(1+\varpi^{\dagger})\left[{}_{a}I^{\varpi^{\dagger}}_{w_1+\mathfrak{x}-u_1}\Upsilon(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\varpi^{\dagger}}_{w_2}\Upsilon(u)\right]\right|\\
&\leq\frac{(\mathfrak{x}-u_1)^{2\varpi^{\dagger}}}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left|\nu^{\varpi^{\dagger}}\Upsilon^{\varpi^{\dagger}}(w_1+\mathfrak{x}-(\nu u_1+(1-\nu)\mathfrak{x}))\right|(dt)^{\varpi^{\dagger}}\\
&\quad+\frac{(u_2-\mathfrak{x})^{2\varpi^{\dagger}}}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\left|\nu^{\varpi^{\dagger}}\Upsilon^{\varpi^{\dagger}}(w_2+\mathfrak{x}-(\nu u_2+(1-\nu)\mathfrak{x}))\right|(dt)^{\varpi^{\dagger}}\\
&\leq\frac{M^{\varpi^{\dagger}}(\mathfrak{x}-u_1)^{2\varpi^{\dagger}}}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\nu^{\varpi^{\dagger}}(dt)^{\varpi^{\dagger}}+\frac{M^{\varpi^{\dagger}}(u_2-\mathfrak{x})^{2\varpi^{\dagger}}}{\Gamma(\varpi^{\dagger}+1)}\int_0^1\nu^{\varpi^{\dagger}}(dt)^{\varpi^{\dagger}}\\
&\leq\frac{M^{\varpi^{\dagger}}\Gamma(1+\varpi^{\dagger})}{\Gamma(1+2\varpi^{\dagger})}[(\mathfrak{x}-u_1)^{2\varpi^{\dagger}}+(u_2-\mathfrak{x})^{2\varpi^{\dagger}}],
\end{aligned}
$$

which gives the inequality (11). □

Corollary 10. *If we set* $u_1 = w_1, u_2 = w_2$ *and* $\mathfrak{x} = \frac{w_1+w_2}{2}$ *in (11), then we obtain*

$$
\left|\frac{(w_2-w_1)^{\varpi^{\dagger}}\Upsilon\left(\frac{w_1+w_2}{2}\right)}{\Gamma(1+\varpi^{\dagger})}-\frac{1}{\Gamma(1+\varpi^{\dagger})}\int_{w_1}^{b}f(u)(du)^{\varpi^{\dagger}}\right|\leq\frac{2^{\varpi^{\dagger}}M^{\varpi^{\dagger}}}{\Gamma(1+2\varpi^{\dagger})}\left(\frac{w_2-w_1}{2}\right)^{2\varpi^{\dagger}}.
$$

Theorem 10. *Under the assumption of Lemma 2, if* $|\mathrm{Y}^{\omega^\dagger}|$ *satisfies the generalized s-convexity defined on the interval* $[w_1, w_2]$*, and* $\mathrm{Y}^{\omega^\dagger}$ *is bounded, i.e.,* $||\mathrm{Y}^{\omega^\dagger}||_\infty = sup_{\nu\in[w_1,w_2]}|\mathrm{Y}^{\omega^\dagger}| < \infty$*, then*

$$\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\mathrm{Y}(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\mathrm{Y}(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\mathrm{Y}(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\mathrm{Y}(u)\right]\right|\\
&\leq\left[\frac{2^{\omega^\dagger}\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]||\mathrm{Y}^{\omega^\dagger}||_\infty[(\mathfrak{x}-u_1)^{2\omega^\dagger}+(u_2-\mathfrak{x})^{2\omega^\dagger}],
\end{aligned}$$

Proof. Considering the integral identity derived in Lemma 2 and taking advantage of the property of the modulus, we have

$$\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\mathrm{Y}(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\mathrm{Y}(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\mathrm{Y}(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\mathrm{Y}(u)\right]\right|\\
&\leq\frac{(\mathfrak{x}-u_1)^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\left|\nu^{\omega^\dagger}\mathrm{Y}^{\omega^\dagger}(w_1+\mathfrak{x}-(\nu u_1+(1-\nu)\mathfrak{x}))\right|(dt)^{\omega^\dagger}\\
&\quad+\frac{(u_2-\mathfrak{x})^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\left|\nu^{\omega^\dagger}\mathrm{Y}^{\omega^\dagger}(w_2+\mathfrak{x}-(\nu u_2+(1-\nu)\mathfrak{x}))\right|(dt)^{\omega^\dagger}.
\end{aligned}$$

We employ the generalized s-convexity of the mapping $|\mathrm{Y}^{\omega^\dagger}|$ defined on $[w_1, w_2]$, then

$$\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\mathrm{Y}(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\mathrm{Y}(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\mathrm{Y}(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\mathrm{Y}(u)\right]\right|\\
&\leq\frac{(\mathfrak{x}-u_1)^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}\left[|\mathrm{Y}^{\omega^\dagger}(w_1)|+|\mathrm{Y}^{\omega^\dagger}(\mathfrak{x})|-\nu^{\omega^\dagger s}|\mathrm{Y}^{\omega^\dagger}(u_1)|-(1-\nu)^{\omega^\dagger s}|\mathrm{Y}^{\omega^\dagger}(\mathfrak{x})|\right](dt)^{\omega^\dagger}\\
&\quad+\frac{(u_2-\mathfrak{x})^{2\omega^\dagger}}{\Gamma(\omega^\dagger+1)}\int_0^1\nu^{\omega^\dagger}\left[|\mathrm{Y}^{\omega^\dagger}(w_2)|+|\mathrm{Y}^{\omega^\dagger}(\mathfrak{x})|-\nu^{\omega^\dagger s}|\mathrm{Y}^{\omega^\dagger}(u_2)|-(1-\nu)^{\omega^\dagger s}|\mathrm{Y}^{\omega^\dagger}(\mathfrak{x})|\right](dt)^{\omega^\dagger}\\
&\leq\left[\frac{2\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}-\frac{\Gamma(1+(s+1)\omega^\dagger)}{\Gamma(1+(s+2)\omega^\dagger)}-B_{\omega^\dagger}(2,s+1)\right]||\mathrm{Y}^{\omega^\dagger}||_\infty[(\mathfrak{x}-u_1)^{2\omega^\dagger}+(u_2-\mathfrak{x})^{2\omega^\dagger}],
\end{aligned}$$

which is the desired the result asserted in Theorem 10. □

Corollary 11. *If we take* $s = 1$ *in Theorem 10, then*

$$\begin{aligned}
&\left|(\mathfrak{x}-u_1)^{\omega^\dagger}\mathrm{Y}(w_1+\mathfrak{x}-u_1)+(u_2-\mathfrak{x})^{\omega^\dagger}\mathrm{Y}(w_2+\mathfrak{x}-u_2)-\Gamma(1+\omega^\dagger)\left[{}_aI^{\omega^\dagger}_{w_1+\mathfrak{x}-u_1}\mathrm{Y}(u)+{}_{(w_2+\mathfrak{x}-u_2)}I^{\omega^\dagger}_{w_2}\mathrm{Y}(u)\right]\right|\\
&\leq\left[\frac{2\Gamma(\omega^\dagger+1)}{\Gamma(2\omega^\dagger+1)}-\frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}-B_{\omega^\dagger}(2,2)\right]||\mathrm{Y}^{\omega^\dagger}||_\infty[(\mathfrak{x}-u_1)^{2\omega^\dagger}+(u_2-\mathfrak{x})^{2\omega^\dagger}].
\end{aligned}$$

4. Applications

In this section, we are going to present some applications of our main findings. First, we develop some relation between generalized means by considering some results obtained in the previous section. Further, we examine more applications to numerical integration.

4.1. Generalized Special Means

Here, we recapture the renowned generalized means between two numbers ${w_1}^{\omega^\dagger}, {w_2}^{\omega^\dagger} \in \mathbb{R}^{\omega^\dagger}$.

1. The generalized arithmetic mean:

$$A_{\omega^\dagger}(w_1, w_2) = \frac{{w_1}^{\omega^\dagger}+{w_2}^{\omega^\dagger}}{2^{\omega^\dagger}} = \left(\frac{w_1+w_2}{2}\right)^{\omega^\dagger},$$

2. The generalized Weighted arithmetic mean:

$$ {}_{w}A_{\varpi^{\dagger}}(w_1, w_2; m_1, m_2) = \frac{m_1^{\varpi^{\dagger}} {w_1}^{\varpi^{\dagger}} + m_2^{\varpi^{\dagger}} w_2^{\varpi^{\dagger}}}{(w_1 + w_2)^{\varpi^{\dagger}}}. $$

3. The generalized log-p-mean:

$$ L_{\varpi^{\dagger},p}(w_1, w_2) = \left[\frac{\Gamma(1 + p\varpi^{\dagger})}{\Gamma(1 + (1+p)\varpi^{\dagger})} \frac{{w_2}^{(p+1)\varpi^{\dagger}} - {w_1}^{(p+1)\varpi^{\dagger}}}{(p+1)(w_2 - w_1)^{\varpi^{\dagger}}} \right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}. $$

Proposition 1. *All the assumptions of Theorem 5 are satisfied, then the following relation holds:*

$$ \begin{aligned} &\Big| (u_2 - u_1)^{\omega^{\dagger}} {}_{\omega}A_{\varpi^{\dagger}}((\mathfrak{x} - u_1), (u_2 - \mathfrak{x}), (w_1 + \mathfrak{x} - u_1)^s, (w_2 + \mathfrak{x} - u_2)^s) \\ &\quad - \Gamma(1 + \omega^{\dagger}) \Big[(\mathfrak{x} - u_1) L^s_{s,\varpi^{\dagger}}(w_1 + \mathfrak{x} - u_1, w_1) + (u_2 - \mathfrak{x}) L^s_{s,\varpi^{\dagger}}(w_2, w_2 + \mathfrak{x} - u_2) \Big] \Big| \\ &\leq (\mathfrak{x} - u_1)^{2\omega^{\dagger}} \left[\frac{2^{\omega^{\dagger}} \Gamma(1 + \omega^{\dagger}) \Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + 2\omega^{\dagger}) \Gamma(1 + (s-1)\omega^{\dagger})} A_{\omega^{\dagger}}({w_1}^{s-1}, \mathfrak{x}^{s-1}) \right. \\ &\quad \left. - \frac{\Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + (s-1)\omega^{\dagger})} \left[B_{\omega^{\dagger}}(2, s+1) \mathfrak{x}^{(s-1)\omega^{\dagger}} + \frac{\Gamma(1 + (s+1)\omega^{\dagger})}{\Gamma(1 + (s+2)\omega^{\dagger})} u_1^{(s-1)\omega^{\dagger}} \right] \right] \\ &\quad + (u_2 - \mathfrak{x})^{2\omega^{\dagger}} \left[\frac{2^{\omega^{\dagger}} \Gamma(1 + \omega^{\dagger}) \Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + 2\omega^{\dagger}) \Gamma(1 + (s-1)\omega^{\dagger})} w_{1_{\omega^{\dagger}}}({w_2}^{s-1}, \mathfrak{x}^{s-1}) \right. \\ &\quad \left. - \frac{\Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + (s-1)\omega^{\dagger})} \left[B_{\omega^{\dagger}}(2, s+1) \mathfrak{x}^{(s-1)\omega^{\dagger}} + \frac{\Gamma(1 + (s+1)\omega^{\dagger})}{\Gamma(1 + (s+2)\omega^{\dagger})} u_2^{(s-1)\omega^{\dagger}} \right] \right]. \end{aligned} $$

Proof. The assertion follows directly by substituting $\Upsilon(u) = u^{\omega^{\dagger} s}$ with $0 < \omega^{\dagger}, s \leq 1$ in inequality (8). □

Proposition 2. *All the assumptions of Theorem 6 are satisfied, then the following relation holds:*

$$ \begin{aligned} &\Big| (u_2 - u_1)^{\omega^{\dagger}} {}_{\omega}A_{\varpi^{\dagger}}((\mathfrak{x} - u_1), (u_2 - \mathfrak{x}), (w_1 + \mathfrak{x} - u_1)^s, (w_2 + \mathfrak{x} - u_2)^s) \\ &\quad - \Gamma(1 + \omega^{\dagger}) \Big[(\mathfrak{x} - u_1) L^s_{s,\varpi^{\dagger}}(w_1 + \mathfrak{x} - u_1, w_1) + (u_2 - \mathfrak{x}) L^s_{s,\varpi^{\dagger}}(w_2, w_2 + \mathfrak{x} - u_2) \Big] \Big| \\ &\leq \left(\frac{\Gamma(1 + \omega^{\dagger} p)}{\Gamma(1 + (p+1)\omega^{\dagger})} \right)^{\frac{1}{p}} \left[(\mathfrak{x} - u_1)^{2\omega^{\dagger}} \left(\frac{1}{\Gamma(\omega^{\dagger} + 1)} \left(\frac{\Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + (s-1)\omega^{\dagger})} \right)^q A_{\omega^{\dagger}}({w_1}^{(s-1)q}, \mathfrak{x}^{(s-1)q}) \right. \right. \\ &\quad \left. - \frac{\Gamma(1 + \omega^{\dagger} s)}{\Gamma(1 + (s+1)\omega^{\dagger})} \left(\frac{\Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + (s-1)\omega^{\dagger})} \right)^q A_{\omega^{\dagger}}(\mathfrak{x}^{(s-1)q}, u_1^{(s-1)q}) \right)^{\frac{1}{q}} \\ &\quad + (u_2 - \mathfrak{x})^{2\omega^{\dagger}} \left(\frac{1}{\Gamma(\omega^{\dagger} + 1)} \left(\frac{\Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + (s-1)\omega^{\dagger})} \right)^q A_{\omega^{\dagger}}({w_2}^{(s-1)q}, \mathfrak{x}^{(s-1)q}) \right. \\ &\quad \left. \left. - \frac{\Gamma(1 + \omega^{\dagger} s)}{\Gamma(1 + (s+1)\omega^{\dagger})} \left(\frac{\Gamma(1 + s\omega^{\dagger})}{\Gamma(1 + (s-1)\omega^{\dagger})} \right)^q A_{\omega^{\dagger}}(\mathfrak{x}^{(s-1)q}, u_2^{(s-1)q}) \right)^{\frac{1}{q}} \right]. \end{aligned} $$

Proof. The assertion follows directly by substituting $\Upsilon(u) = u^{\omega^{\dagger} s}$ with $0 < \omega^{\dagger}, s \leq 1$ in inequality (9). □

Proposition 3. *All the assumptions of Theorem 9 are satisfied, then the following relation holds:*

$$ \Big| (u_2 - u_1)^{\omega^{\dagger}} {}_{\omega}A_{\varpi^{\dagger}}((\mathfrak{x} - u_1), (u_2 - \mathfrak{x}), (w_1 + \mathfrak{x} - u_1)^s, (w_2 + \mathfrak{x} - u_2)^s) $$

$$-\Gamma(1+\varpi^{\dagger})\Big[(\mathfrak{x}-\mathfrak{u}_1)L^{s}_{s,\varpi^{\dagger}}(w_1+\mathfrak{x}-\mathfrak{u}_1,w_1)+(\mathfrak{u}_2-\mathfrak{x})L^{s}_{s,\varpi^{\dagger}}(w_2,w_2+\mathfrak{x}-\mathfrak{u}_2)\Big]\Big|$$
$$\leq M^{\varpi^{\dagger}}\left[\frac{2^{\varpi^{\dagger}}\Gamma(1+\varpi^{\dagger})}{\Gamma(1+2\varpi^{\dagger})}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}-B_{\varpi^{\dagger}}(2,s+1)\right][(\mathfrak{x}-\mathfrak{u}_1)^{2\varpi^{\dagger}}+(\mathfrak{u}_2-\mathfrak{x})^{2\varpi^{\dagger}}].$$

Proof. The assertion follows directly by substituting $\Upsilon(\mathfrak{u})=\mathfrak{u}^{\varpi^{\dagger}s}$ with $0<\varpi^{\dagger},s\leq 1$ in Theorem 9. □

4.2. The Quadrature Formula

Here, we present some applications to generalized midpoint rules. If we divide the interval $[w_1,w_2]$ into J subintervals $\mathfrak{x}_i,\mathfrak{x}_{i+1}$ with $i=0,1,\ldots,J-1$, then one finds a partition $\mathcal{P}: w_1=\mathfrak{x}_0<\mathfrak{x}_1<\cdots<\mathfrak{x}_{j-1}<\mathfrak{x}_j=w_2$. Furthermore, we take into account the undermentioned quadrature formula

$$\begin{aligned}{}_{z}I^{\varpi^{\dagger}}_{w_2}\Upsilon(y)&=\frac{1}{\Gamma(1+\varpi^{\dagger})}\int_{w_1}^{w_2}\Upsilon(y)(d(y))^{\varpi^{\dagger}}\\&=T_s(\Upsilon,\gamma)+E_s(\Upsilon,\gamma),\end{aligned}$$

where

$$T_s(\Upsilon,\gamma)=(w_2-w_1)^{\varpi^{\dagger}}\Upsilon\left(\frac{w_1+w_2}{2}\right).$$

Proposition 4. *If all the assumptions of Theorem 8 are held, then*

$$|E_s(\gamma,\gamma)|\leqslant\sum_{i=1}^{n}\frac{1}{\Gamma(1+\varpi^{\dagger})}\left(\frac{\mathfrak{x}_{i+1}-\mathfrak{x}_{\mathfrak{i}}}{2}\right)^{2\varpi^{\dagger}}(B_{\varpi^{\dagger}}(p+1,2))^{\frac{1}{p}}$$
$$\times\left\{\left(\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)\left|\gamma^{\varpi^{\dagger}}(\mathfrak{x}_i)\right|^{q}+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\varpi^{\dagger}}(s+1,2)\right]\left|\gamma^{\dagger}\left(\frac{\mathfrak{x}_i+\mathfrak{x}_{i+1}}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)\left|\gamma^{\varpi^{\dagger}}(\mathfrak{x}_{i+1})\right|^{q}+\left[\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\omega^{\dagger}}(s+1,2)\right]\left|\sigma^{\dagger}\left(\frac{\mathfrak{x}_i+\mathfrak{x}_{i+1}}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right\}$$
$$+\sum_{i=1}^{n}\frac{1}{\Gamma(1+\varpi^{\dagger})}\left(\frac{\mathfrak{x}_{i+1}-\mathfrak{x}_i}{2}\right)^{2\varpi^{\dagger}}\left(\frac{\Gamma(1+(p+1)\varpi^{\dagger})}{\Gamma(1+(p+2)\varpi^{\dagger})}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\left(\frac{\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)\left|\gamma^{\omega^{\dagger}}(\mathfrak{x}_{\mathfrak{i}})\right|^{q}+\left[\frac{\Gamma(\boldsymbol{\varpi}^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-B_{\boldsymbol{\omega}^{\dagger}}(2,s+1)\right]\left|\gamma^{\dagger}\left(\frac{\mathfrak{x}_i+\mathfrak{x}_{i+1}}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{\Gamma(\omega^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}\right)\left|\gamma^{\omega^{\dagger}}(\tilde{r}_{i+1})\right|^{q}+\left[\frac{\Gamma(\omega^{\dagger}+1)}{\Gamma(2\omega^{\dagger}+1)}-B_{\omega^{\dagger}}(2,s+1)\right]\left|\gamma^{\dagger}\left(\frac{\tilde{p}_i+\mathfrak{p}_{i+1}}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion follows directly by taking the sum from $n=0$ to $n-1$ over subinterval $[\mathfrak{x}_i,\mathfrak{x}_{i+1}]$ in Theorem 8. □

Proposition 5. *If all the assumptions of Theorem 10 are held, then*

$$E_s(\Upsilon,\gamma)|\leq\frac{2^{\varpi^{\dagger}}||\Upsilon^{\varpi^{\dagger}}||_{\infty}}{\Gamma(1+\varpi^{\dagger})}\left(\frac{\mathfrak{x}_{i+1}-\mathfrak{x}_i}{2}\right)^{2\varpi^{\dagger}}\left[\frac{2\Gamma(\varpi^{\dagger}+1)}{\Gamma(2\varpi^{\dagger}+1)}-\frac{\Gamma(1+(s+1)\varpi^{\dagger})}{\Gamma(1+(s+2)\varpi^{\dagger})}-B_{\varpi^{\dagger}}(2,s+1)\right].$$

Proof. The assertion follows directly by taking the sum from $n=0$ to $n-1$ over subinterval $[\mathfrak{x}_i,\mathfrak{x}_{i+1}]$ in Theorem 10. □

5. Examples

The following section is organized to increase the impact and visibility of main outcomes by establishing some numerical and graphical simulations.

Example 1. *If all the assumptions of Theorem 5 are satisfied, and we one consider the mapping* $\Upsilon(\mathfrak{u}) = \frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}\left(\frac{\mathfrak{u}^3}{3}\right)^{\omega^\dagger}$ *on* $\mathbb{R}^+$ *the generalized convex functions and also* $\left|\Upsilon^{\omega^\dagger}(\mathfrak{u})\right| = \left(\frac{\mathfrak{u}^2}{3}\right)^{\omega^\dagger}$ *is a generalized convex mapping. By specifying the values of* $w_1 = 0, \mathfrak{u}_1 = 1, \mathfrak{u}_2 = 3, w_2 = 4$, *s = 0.8, then*

$$\begin{aligned}&\left|\frac{(\mathfrak{x}-1)^4}{9} + \frac{(3-\mathfrak{x})(1+\mathfrak{x})}{9} - \frac{(\mathfrak{x}-1)^4 + 4^4 - (1+\mathfrak{x})^4}{36}\right| \\ &\leq (\mathfrak{x}-1)^2\left(\frac{13\mathfrak{x}^2}{138} - \frac{5}{42}\right) + (3-\mathfrak{x})^2\left(\frac{8}{3} + \frac{13\mathfrak{x}^2}{138} - \frac{15}{14}\right).\end{aligned}$$

We regard $\mathfrak{x} \in [1,3]$ *as a variable to plot a graph between the left and right-hand side of inequality (8).*

Figure 1 depicts the comparative analysis between the left and right sides of Theorem 5. Here red and green colors reflect the left and right-hand sides respectively.

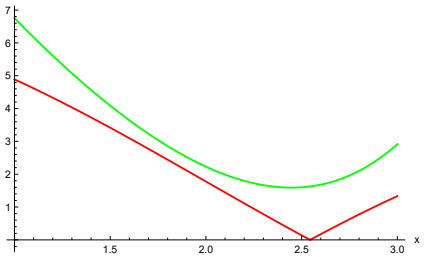

Figure 1. Comparative analysis between both sides of Theorem 5.

Example 2. *If all the assumptions of Theorem 6 are satisfied, and we one consider the mapping* $\Upsilon(\mathfrak{u}) = \frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}\left(\frac{\mathfrak{u}^3}{3}\right)^{\omega^\dagger}$ *on* $\mathbb{R}^+$ *the generalized convex functions and also* $\left|\Upsilon^{\omega^\dagger}(\mathfrak{u})\right| = \left(\frac{\mathfrak{u}^2}{3}\right)^{\omega^\dagger}$ *is a generalized convex mapping. By specifying the values of* $w_1 = 0, \mathfrak{u}_1 = 1, \mathfrak{u}_2 = 3, w_2 = 4$, *s = 1, then*

$$\begin{aligned}&\left|\frac{(\mathfrak{x}-1)^4}{9} + \frac{(3-\mathfrak{x})(1+\mathfrak{x})}{9} - \frac{(\mathfrak{x}-1)^4 + 4^4 - (1+\mathfrak{x})^4}{36}\right| \\ &\leq (\mathfrak{x}-1)^2\left(\frac{\mathfrak{x}^4-1}{54}\right)^{\frac{1}{2}} + (3-\mathfrak{x})^2\left(\frac{256}{27} + \frac{\mathfrak{x}^4}{54} - \frac{3}{2}\right)^{\frac{1}{2}}.\end{aligned}$$

We regard $\mathfrak{x} \in [1,3]$ *as a variable to plot a graph between the left and right-hand side of inequality 9).*

Figure 2 depicts the comparative analysis between the left and right sides of Theorem 5. Here red and green colors reflect the left and right-hand sides respectively.

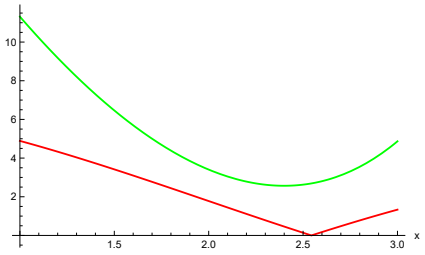

Figure 2. Comparative analysis between the left and right sides of Theorem 6.

Example 3. *If all the assumptions of Theorem 7 are satisfied, and we one consider the mapping* $\Upsilon(\mathfrak{u}) = \frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}\left(\frac{\mathfrak{u}^3}{3}\right)^{\omega^\dagger}$ *on* $\mathbb{R}^+$ *the generalized convex functions and also* $\left|\Upsilon^{\omega^\dagger}(\mathfrak{u})\right| = \left(\frac{\mathfrak{u}^2}{3}\right)^{\omega^\dagger}$ *be a generalized convex mapping. By specifying the values of* $w_1 = 0, \mathfrak{u}_1 = 1, \mathfrak{u}_2 = 3, w_2 = 4$, *s = 1, then*

$$\left|\frac{(\mathfrak{x}-1)^4}{9} + \frac{(3-\mathfrak{x})(1+\mathfrak{x})}{9} - \frac{(\mathfrak{x}-1)^4 + 4^4 - (1+\mathfrak{x})^4}{36}\right|$$
$$\leq (\mathfrak{x}-1)^2\left(\frac{\mathfrak{x}^4-1}{54}\right)^{\frac{1}{2}} + (3-\mathfrak{x})^2\left(\frac{256}{36} + \frac{\mathfrak{x}^4}{54} - \frac{3}{2}\right)^{\frac{1}{2}}.$$

We regard $\mathfrak{x} \in [1,3]$ *as a variable to plot a graph between the left and right-hand side of inequality (10).*

Figure 3 depicts the comparative analysis between the left and right sides of Theorem 7. Here red and green colors reflect the left and right-hand sides respectively.

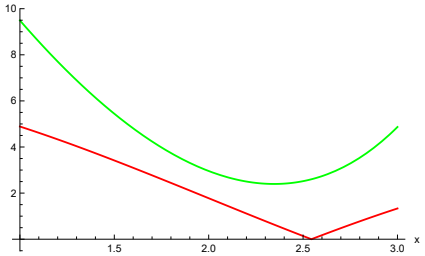

Figure 3. Comparative analysis between the left and right sides of Theorem 7.

Example 4. *If all the assumptions of Theorem 8 are satisfied, and we consider the mapping* $\Upsilon(\mathfrak{u}) = \frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}\left(\frac{\mathfrak{u}^3}{3}\right)^{\omega^\dagger}$ *on* $\mathbb{R}^+$ *the generalized convex functions and also* $\left|\Upsilon^{\omega^\dagger}(\mathfrak{u})\right| = \left(\frac{\mathfrak{u}^2}{3}\right)^{\omega^\dagger}$ *is a generalized convex mapping. By specifying the values of* $w_1 = 0, \mathfrak{u}_1 = 1, \mathfrak{u}_2 = 3, w_2 = 4$, *s = 1, then*

$$\left|\frac{(\mathfrak{x}-1)^4}{9} + \frac{(3-\mathfrak{x})(1+\mathfrak{x})}{9} - \frac{(\mathfrak{x}-1)^4 + 4^4 - (1+\mathfrak{x})^4}{36}\right|$$
$$\leq 0.7886(\mathfrak{x}-1)^2\left(\frac{\mathfrak{x}^4-1}{27}\right)^{\frac{1}{2}} + 0.7886(3-\mathfrak{x})^2\left(\frac{126}{9} + \frac{\mathfrak{x}^4}{27} - 3\right)^{\frac{1}{2}}.$$

We regard $\mathfrak{x} \in [1,3]$ *as a variable to plot a graph between the left- and right-hand side of Theorem 8.*

Figure 4 depicts the comparative analysis between the left and right sides of Theorem 7. Here red and green colors reflect the left and right-hand sides respectively.

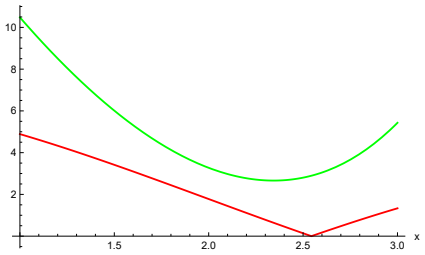

Figure 4. Comparative analysis between the left and right sides of Theorem 8.

Example 5. *If all the assumptions of Theorem 10 are satisfied, and we consider the mapping* $\Upsilon(\mathfrak{u}) = \frac{\Gamma(1+2\omega^\dagger)}{\Gamma(1+3\omega^\dagger)}\left(\frac{\mathfrak{u}^3}{3}\right)^{\omega^\dagger}$ *on* $\mathbb{R}^+$ *the generalized convex functions and also* $\left|\Upsilon^{\omega^\dagger}(\mathfrak{u})\right| = \left(\frac{\mathfrak{u}^2}{3}\right)^{\omega^\dagger}$

be a generalized convex mapping. By specifying the values of $w_1 = 0, \mathfrak{u}_1 = 1, \mathfrak{u}_2 = 3, w_2 = 4$, $s = 1$, *then*

$$\left| \frac{(\mathfrak{x}-1)^4}{9} + \frac{(3-\mathfrak{x})(1+\mathfrak{x})}{9} - \frac{(\mathfrak{x}-1)^4 + 4^4 - (1+\mathfrak{x})^4}{36} \right| \leq \frac{8}{3}\Big((\mathfrak{x}-1)^2 + (3-\mathfrak{x})^2\Big).$$

We regard $\mathfrak{x} \in [1,3]$ *as a variable to plot the graph between the left- and right-hand side of Theorem 10.*

Figure 5 depicts the comparative analysis between the left and right sides of Theorem 7. Here red and green colors reflect the left and right-hand sides respectively.

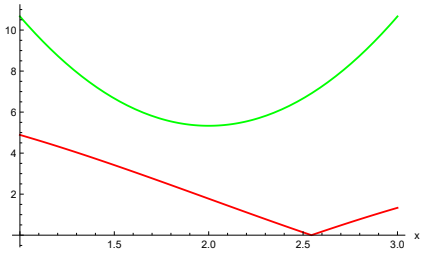

Figure 5. Comparative analysis between the left and right sides of Theorem 10.

6. Conclusions

Numerous techniques have been utilized to formulate the precise upper bounds of quadrature or cubature rules. In this regard, various well-known inequalities have been generalized and modified via different approaches like quantum calculus, fractional calculus, interval analysis, and fractal domains. The current study contains several integral inequalities of Ostrowski's type which have been explored here invoking the local differentiable functions and some classical concepts of inequalities as well. Furthermore, we have supported our primary findings with interesting applications and numerical examples with figures. In the future, we will conclude some new variants of other related inequalities through the implementation of majorization concepts and some generalized local fractional operators in association with generalized Mittag–Leffler functions. Also, this work can be explored by utilizing other classes of convexity. I hope this study will be a major development in the literature and bring curiosity to interested readers.

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