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Nonlinear Filter-Based Adaptive Output-Feedback Control for Uncertain Fractional-Order Nonlinear Systems with Unknown External Disturbance

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Abstract: This study is devoted to a nonlinear filter-based adaptive fuzzy output-feedback control scheme for uncertain fractional-order (FO) nonlinear systems with unknown external disturbance. Fuzzy logic systems (FLSs) are applied to estimate unknown nonlinear dynamics, and a new FO fuzzy state observer based on a nonlinear disturbance observer is established for simultaneously estimating the unmeasurable states and mixed disturbance. Then, with the aid of auxiliary functions, a novel FO nonlinear filter is given to approximately replace the virtual control functions, together with the corresponding fractional derivative, which not only erases the inherent complexity explosion problem under the framework of backstepping, but also completely compensates for the effects of the boundary errors induced by the constructed filters compared to the previous FO linear filter method. Under certain assumptions, and in line with the FO stability criterion, the stability of the controlled system is ensured. An FO Chua–Hartley simulation study is presented to verify the validity of the proposed method.

Keywords: fractional-order nonlinear systems; adaptive fuzzy control; nonlinear filter; output-feedback control; disturbance observer

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1. Introduction

As the extension and generalization of integral calculus, fractional calculus, on the one hand, is more applicable for describing systems with memory and history-dependent processes, and can more accurately model and characterize objective phenomena that cannot be described by an integer-order (IO) system [1,2], such as the semi-derivative relationship between heat flow and temperature, and the "trailing" phenomenon of solute transport in porous media, etc. Fractional calculus, on the other hand, has higher design degrees of freedom and can, therefore, exhibit better robustness and transient performance, such as in FO CRONE controllers [3] and $PI^{\lambda}D^{\mu}$ controllers [4]. Consequently, the investigation of FO system control has profound theoretical value and practical significance.

Unfortunately, fractional calculus has weak singularity and global properties, so it is much more difficult to focus on uncertain FO systems than IO systems. The first difficulty is how to define a proper Lyapunov function to prove the stability of feedback control. To solve this problem, two construction methods were formed based on in-depth study of scholars. One is the Lyapunov indirect function construction method put forward in [5,6]; that is, a frequency distribution model is utilized to approximate the FO system to the IO system. This method can truly reflect the internal energy of the FO system. However, with current research progress, the system model considered is relatively conservative, and when there are completely unknown nonlinearities in the system, ideal control results cannot be obtained. Another method is the direct Lyapunov construction method based on Mittag–Leffler stability theory [7], and driven by the unequal relation ${}_0^C D_t^{\alpha} V(x(t)) \leq 2x(t){}_0^C D_t^{\alpha} x(t)$ [8], nonlinear systems with parameter uncertainty investigated in [9].

System structure uncertainties (such as system modeling errors, unmodeled dynamics, etc.) inevitably appear in nonlinear systems. Their presence can degrade the efficiency of the system and even lead to unanticipated outcomes. In order to deal with these problems, some authors have used intelligent approximation tools, such as FLSs [10,11], neural networks [12,13], Takagi–Sugeno fuzzy models [14,15] and machine learning methods [16–18], to border the unknown nonlinear uncertainties over a given compact set. The above intelligent control methods have been widely used in practical control systems.

In addition, many physical systems, such as circuit systems, can be expressed as FO nonlinear systems and are susceptible to electromagnetic perturbations from external environments. Electromagnetic interference is caused by the conduction and electrical radiation generated by devices similar to electromagnetic circles in the process of using current and voltage in the network environment [19–21], which has a destructive effect on electronic components and leads to abnormalities in the internal pulse clock system of electronic components. Consequently, it is necessary to address the disturbance suppression control for FO nonlinear systems with disturbances and modeling errors. When the state is completely known and the uncertain disturbance has an upper bound, some scholars have effectively compensated the influence of the disturbance by introducing a sign function into the controller [22,23]. However, the introduction of a sign function causes the phenomenon of buffeting. Subsequently, some scholars introduced interference compensation adaptive sliding mode control technology to successfully eliminate such problems [24]. However, this method is only suitable for the strict assumption condition; that is, the external disturbance should gradually approach zero increase in time. In order to avoid such problems, some scholars have constructed a fractional disturbance observer under the adaptive backstepping framework [25–28]. Although the above approaches can counteract the effect of disturbance uncertainty on the system, the conclusions should be based on the complete measurement of system state information. In practical control systems, due to technical or economic reasons, the energy measurement or the choice not to measure all state information, means that, in this case, the proposed control scheme becomes ineffective. Thus, it is of critical theoretical and practical importance to design output feedback control such that system status information is no longer required to be directly measurable.

Toward this end, many effective output-feedback control schemes were developed for nonlinear systems (see [29–31] and the references therein). By constructing a linear observer, the authors in [29,30] developed observer-based adaptive fuzzy/neural network output-feedback control schemes for nonlinear systems to erase the restriction condition of the unmeasurable states in the known nonlinear functions case. In order to model the nonlinear system with unmeasurable states more accurately, the authors in [31] first proposed an adaptive fuzzy output-feedback control scheme for a nonlinear system by constructing a fuzzy nonlinear state observer. Further, the output feedback control method is widely used in practical engineering systems, like active suspension systems [32], unmanned aerial vehicles [33], spacecraft attitude systems [34], etc. It is worth noting that the above-mentioned output-feedback results are only applicable to integer-order nonlinear systems. Since the fractional-order system is fundamentally changed based on the integer-order system, and compared with a traditional integer-order controller, the parameters in the fractional-order case not only have more degrees of freedom, but also lead to the Newton–Leibniz formula, and the derivative rule of a compound function in an integer-order operation cannot be used directly. Consequently, it is extremely difficult to extend the integer-order method to a fractional-order method directly. Therefore, the question arises: Is it feasible to present an output feedback control strategy based on a disturbance observer for FO nonlinear systems with unknown disturbance and modeling error uncertainty? This question prompted our research.

On the other hand, on account of the repetitive derivative of the virtual controller, the computing burden of the traditional backstepping recursive method will inevitably increase, which is the so-called issue of “complexity explosion”. To reduce the effects of this issue, a filter-based adaptive backstepping algorithm was proposed in [35] for IO nonlinear systems by replacing the virtual controller with an introduced filter signal. In terms of this technique, great progress has been made for IO nonlinear systems, and numerous filter-based adaptive backstepping strategies have been developed for heterogeneous IO nonlinear systems [36–39]. However, the introduced strategies described in [35–39] can only be employed to IO nonlinear systems. Further, in [40–42], the above results were extended to FO ones, with positive progress achieved, in which the results were utilized on the basis of linear filter design methods, where the effects caused by boundary error were not effectively suppressed, which inevitably increased the bounds of the error signals. Although work has been presented in [43] to concurrently erase the issues of "complexity explosion" and the effects of boundary layer errors by constructing nonlinear filter-based backstepping control schemes, the commonality of these works is that the states of the systems are assumed to be known or the system is considered to be in an ideal state without external disturbances. To the best of our knowledge, there is still no mature adaptive backstepping procedure for FO nonlinear systems with unknown external disturbances and unmeasurable states, which further motivates our investigation.

Stimulated by the aforesaid motivations, this work focuses on an adaptive backstepping output-feedback control problem based on nonlinear filters and a disturbance observer for FO nonlinear systems subject to unknown external disturbances and unmeasurable states. The innovative aspects of this paper are mainly reflected in two aspects:

(1) An adaptive fuzzy output-feedback control-strategy-based disturbance-observer for strict-feedback FO nonlinear systems with unknown external disturbances is achieved for the first time. It should be noted that the authors in [25–28] have considered a related topic. However, the references [25–28] are based on the complete measurement of system state information.

(2) A novel FO nonlinear filter based on an auxiliary function is constructed to approximately replace the virtual control functions together with the corresponding fractional derivative, which not only erases the issue of complexity explosion, but also completely compensates for the effects of the boundary errors induced by the constructed filters. Although the authors of [40–42] considered adaptive control based on a filter signal for FO nonlinear systems, these results were obtained on the basis of a linear filter, and cannot directly compensate for the aforementioned effects.

Notations: In this paper, some specific notations are employed. R^i means i-dimensional Euclidean space; $\|\cdot\|$ represents the Euclidean norm of a vector or matrix; N^+ is a positive integer.

2. Preliminaries and Problem Formulations

2.1. Preliminaries

Some standard definitions and Lemmas are presented.

Definition 1 ([44])**.** *Let $F : [t_0, +\infty) \to R$ be a continuously differentiable function, then the Caputo FO derivative of F with order α satisfies:*

$$ {}_0^C D_t^\alpha F(t) = \frac{1}{\Gamma(w-\alpha)} \int_0^t \frac{F^{(w)}(\tau)}{(t-\tau)^{\alpha+1-w}} d\tau, \tag{1}$$

with ${}_0^C D_t^\alpha$ being the fractional-integral of order α, $\alpha \in [w-1, w)$ with $w \in N^+$, when $\alpha = w$, ${}_0^C D_t^\alpha$ converts to the traditional IO differential operator. $\Gamma(\star) = \int_0^{+\infty} \tau^{\star-1} e^{-\tau} d\tau$ represents Euler's Gamma function complying with $\Gamma(1) = 1$.

Remark 1. *In the course of the development of fractional calculus theory, many definitions, properties and theorems have been established on the basis of the Riemann–Liouville (R-L) definition. It is worth noting that the R–L definition must specify the fractional derivative value of the unknown solution at the initial time to ensure the uniqueness of the solution. Although the R–L definition* *is mathematically rigorous, the fractional derivative does not have a good physical or geometric explanation. On the contrary, the Caputo derivative enables utilization of the initial values of the IO derivatives with physical meaning. If the R–L operator is applied in the system's model, when the initial condition is zero, the R–L operator and the Caputo operator are equivalent. Regarding the non-zero case, the physical meaning of the fractional derivative in the R–L definition is not very clear. Therefore, the Caputo FO derivative will be employed.*

Lemma 1 ([23,45])**.** *If the α-order derivative of a continuous function $V(t) : [0, \infty) \longrightarrow R$ satisfies*

$$ {}_{0}^{C}D_{t}^{\alpha}V(t) \leq -\kappa V(t) + \mu, \tag{2} $$

with $\alpha \in (0, 1]$, $\kappa, \mu > 0$, then one obtains

$$ V(t) \leq V(0)E_{\alpha}(-\kappa t^{\alpha}) + \tfrac{\mu\bar{\varrho}}{\kappa}, t > 0, \tag{3} $$

with $\bar{\varrho} = \max\{1, \lambda\}$.

Lemma 2 ([44])**.** *Fractional differential operators of Caputo type satisfy the linear relation, i.e.,*

$$ {}_{0}^{C}D_{t}^{\alpha}(ah_1(t) + bh_2(t)) = a{}_{0}^{C}D_{t}^{\alpha}h_1(t) + b{}_{0}^{C}D_{t}^{\alpha}h_2(t) \tag{4} $$

where $a > 0$, $b > 0$, $0 < \alpha < 1$ and $h_1(t), h_2(t)$ are continuously differentiable functions.

Lemma 3 ([46])**.** *For any $x, y \in R^n, \epsilon > 0, p > 1, q < 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, one has*

$$ xy \leq \frac{\epsilon^p}{p} \mid x \mid^p + \frac{1}{q\epsilon^q} \mid y \mid^q \tag{5} $$

Lemma 4 ([8])**.** *For all $t \geq t_0$, a smooth function $\iota(t) \in R$ satisfies:*

$$ \tfrac{1}{2}{}_{0}^{C}D_{t}^{\alpha}(\iota^T(t)\iota(t)) \leq \iota^T(t){}_{0}^{C}D_{t}^{\alpha}(\iota(t)). \tag{6} $$

Lemma 5 ([47,48])**.** *Consider $\varpi > 0$ and $\kappa \in R$, it holds:*

$$ 0 \leq |\kappa| - \frac{\kappa^2}{\sqrt{\kappa^2 + \varpi^2}} < \varpi. \tag{7} $$

2.2. System Descriptions and Control Objective

Consider a class of FO nonlinear systems expressed as

$$ \begin{cases} {}_{0}^{C}D_{t}^{\alpha}x_i = x_{i+1} + f_i(\bar{x}_i) + d_i(t), i = 1, \ldots, n-1 \\ {}_{0}^{C}D_{t}^{\alpha}x_n = u + f_n(\bar{x}_n) + d_n(t) \\ y = x_1 \end{cases} \tag{8} $$

where $\bar{x}_n = [x_1, x_2, \ldots, x_n]^T \in R^n$ represents the system state, $y \in R$ represents the output of the system which is measured, $u \in R$ represents the control input. $f_i(\cdot) \in R, i = 1, \ldots, n$ are unknown nonlinear dynamics, $d_i(t) \in R, i = 1, \ldots, n$ represent unknown bounded external disturbances, and $x_2, x_3, \ldots, x_n$ are assumed to be unmeasured.

Control Objective: Design an adaptive fuzzy output-feedback controller, such that all the closed-loop signals are bounded and the system output y can track the reference signal y_d well.

Some assumptions are given below.

Assumption 1 ([40,49])**.** *The given reference signals y_d, ${}_{0}^{C}D_{t}^{\alpha}y_d$ and ${}_{0}^{C}D_{t}^{2\alpha}y_d$ are smooth, bounded and usable.*

Assumption 2 ([50,51])**.** *A constant $q > 0$ exists, which guarantees $V(0) \leq q$ is always satisfied.*

Remark 2. *Assumption 1 and Assumption 2 are standard assumptions to ensure the tracking requirement and avoid the issue of complexity explosion, respectively.*

3. Nonlinear Filter-Based Adaptive Fuzzy Output-Feedback Control Design

An FO fuzzy state observer will be introduced to learn the unmeasurable states. Furthermore, to counteract the influence of unknown external disturbances, a disturbance observer is proposed, and an adaptive output feedback controller is presented with an FO nonlinear filter. Finally, a strict stability analysis is provided in view of the FO Lyapunov stability criterion.

3.1. Fractional-Order Fuzzy Observer Design

Since the system states $x_2,\ x_3,\ \dots,\ x_n$ in (8) are unmeasured, an FO fuzzy state observer will be introduced. In addition, according to the system (8), FLSs $\bar{\xi}_i^T\bar{\psi}_i(\bar{x}_i)$ are presented to learn the unknown nonlinear dynamics $a_i f_i(\bar{x}_i) \in R, i = 1, \dots, n, a_i > 0$, i.e.,

$$a_i f_i(\bar{x}_i) = \bar{\xi}_i^{*T}\bar{\psi}_i(\bar{x}_i) + \bar{\varepsilon}_i \tag{9}$$

where $\bar{\psi}_i(\bar{x}_i)$ is the fuzzy basis functions, $\bar{\varepsilon}_i$ is the fuzzy minimum approximation error, and $\bar{\xi}_i^*$ denotes the optimal weight. A more specific introduction to FLSs is provided in [52–54].

According to (9), (8) is rewritten as :

$$\begin{cases} {}_0^CD_t^\alpha x_i = x_{i+1} + \frac{1}{a_i}\bar{\xi}_i^{*T}\bar{\psi}_i(\bar{x}_i) + D_i, i = 1, \dots, n-1 \\ {}_0^CD_t^\alpha x_n = u + \frac{1}{a_n}\bar{\xi}_n^{*T}\bar{\psi}_n(\bar{x}_n) + D_n \\ y = x_1 \end{cases} \tag{10}$$

where $D_k = \bar{\varepsilon}_k + d_k, k = 1, \dots, n$.

To move forward a single step, an FO fuzzy nonlinear state observer can be constructed

$$\begin{cases} {}_0^CD_t^\alpha \hat{x}_i = \hat{x}_{i+1} + \frac{1}{a_i}\bar{\xi}_i^T\bar{\psi}_i(\hat{\bar{x}}_i) + \hat{D}_i + r_i(y-\hat{y}), i = 1, \dots, n-1 \\ {}_0^CD_t^\alpha \hat{x}_n = u + \frac{1}{a_n}\bar{\xi}_n^T\bar{\psi}_n(\hat{\bar{x}}_n) + \hat{D}_n + r_n(y-\hat{y}) \\ y = x_1 \end{cases} \tag{11}$$

where $r_i, i = 1, \dots, n$ denote the observer gains, and $\bar{\xi}_i$ denotes the estimate of $\bar{\xi}_i^*$. Let $\hat{D}_i$ be the disturbance observer for the hybrid disturbances D_i, which is defined later in the backstepping control design.

Let $e = [e_1, e_2, \dots, e_n]^T$ with $e_k = x_k - \hat{x}_k (k = 1, , \dots, n)$ be the observer error. In light of (10) and (11), one has

$$\begin{cases} {}_0^CD_t^\alpha e_i = e_{i+1} - r_i e_1 + \tilde{D}_i + \frac{1}{a_i}(\bar{\xi}_i^{*T}\bar{\psi}_i(\bar{x}_i) - \bar{\xi}_i^T\bar{\psi}_i(\hat{\bar{x}}_i)) \\ i = 1, \dots, n-1 \\ {}_0^CD_t^\alpha e_n = -r_n e_1 + \tilde{D}_n + \frac{1}{a_n}(\bar{\xi}_n^{*T}\bar{\psi}_n(\bar{x}_n) - \bar{\xi}_n^T\bar{\psi}_n(\hat{\bar{x}}_n)) \end{cases} \tag{12}$$

with $\tilde{D}_i = D_i - \hat{D}_i$. Define

$$B_j = [\underbrace{0 \dots 1}_{j} \dots 0]_{n\times 1}^T,\ A = \begin{bmatrix} -r_1 & & \\ \vdots & I_{n-1} & \\ -r_n & & 0 \end{bmatrix}_{n\times n},$$

then (13) is rearranged as:

$$ {}_0^CD_t^\alpha e = Ae + \sum_{j=1}^{n} B_j(\frac{1}{a_j}(\bar{\xi}_j^{*T}\bar{\psi}_j(\bar{x}_j) - \bar{\xi}_j^T\bar{\psi}_j(\hat{\bar{x}}_j)) + \tilde{D}_j) \tag{13}$$

Choose r_i to ensure the matrix A is Hurwitz. Hence, if any matrix $Q = Q^T > 0$ is proposed, there is a matrix $P = P^T > 0$, which holds

$$A^T P + PA = -Q \tag{14}$$

For verifying the validity of (11), define the Lyapunov function

$$V_0 = e^T Pe \tag{15}$$

The time differentiation of (15) is

$${}_0^C D_t^\alpha V_0 \leq -e^T Qe + 2e^T P \sum_{j=1}^{n} B_j(\bar{\xi}_j^{*T} \bar{\psi}_j(\bar{x}_j) - \bar{\xi}_j^T \bar{\psi}_j(\hat{\bar{x}}_j) + \tilde{D}_j) \tag{16}$$

Along with Lemma 3, Lemma 4 and the fact that $0 < \bar{\psi}_j^T(\cdot)\bar{\psi}_j(\cdot) \leq 1$, one can obtain

$$2e^T P \sum_{j=1}^{n} B_j \tilde{D}_j \leq \frac{n\|P\|^2}{b_{02}} \|e\|^2 + b_{02} \textstyle\sum_{j=1}^n \tilde{D}_j^2 \tag{17}$$

$$\begin{aligned} &2e^T P \sum_{j=1}^{n} B_j(\bar{\xi}_j^{*T} \bar{\psi}_j(\bar{x}_j) - \bar{\xi}_j^T \bar{\psi}_j(\hat{\bar{x}}_j)) \\ &= 2e^T P \sum_{j=1}^{n} B_j(\bar{\xi}_j^{*T} \bar{\psi}_j(\bar{x}_j) - \bar{\xi}_j^{*T} \bar{\psi}_j(\hat{\bar{x}}_j) + \tilde{\bar{\xi}}_j^{*T} \bar{\psi}_j(\hat{\bar{x}}_j) + \tilde{\bar{\xi}}_j^T \bar{\psi}_j(\hat{\bar{x}}_j)) \\ &\leq \frac{2n}{b_{01}} \|e\|^2 + b_{01} \|P\|^2 \textstyle\sum_{j=1}^n (\|\bar{\xi}_j^*\|^2 + \tilde{\bar{\xi}}_j^T \tilde{\bar{\xi}}_j) \end{aligned} \tag{18}$$

where $b_{01} > 0$ and $b_{02} > 0$.

Substituting (18) and (19) into (17) gives

$${}_0^C D_t^\alpha V_0 \leq -H_0 \|e\|^2 + b_{01} \|P\|^2 \sum_{j=1}^{n} \tilde{\bar{\xi}}_j^T \tilde{\bar{\xi}}_j + b_{02} \sum_{j=1}^{n} \tilde{D}_j^2 + \bar{\mu}_0 \tag{19}$$

where $H_0 = \lambda_{\min}(Q) - \frac{2n}{b_{01}} - \frac{n\|P\|^2}{b_{02}}$, and $\bar{\mu}_0 = b_{01} \|P\|^2 \sum_{j=1}^n \|\bar{\xi}_j^*\|^2$.

3.2. Fractional-Order Nonlinear Filter Design

For handling the issue of the inherent complexity explosion under the framework of backstepping, an FO nonlinear filter is presented by the following form:

$$\bar{\varsigma}_i {}_0^C D_t^\alpha \bar{\theta}_i = -\frac{\bar{\varsigma}_i \hat{\bar{M}}_i^2 \bar{\vartheta}_i}{\sqrt{\hat{\bar{M}}_i^2 + \bar{\varpi}^2}} - \bar{\vartheta}_i - \bar{\varsigma}_i \bar{S}_i,\ \bar{\theta}_i(0) = \bar{\tau}_i(0), i = 1, 2, \ldots, n-1, \tag{20}$$

where $\bar{\varsigma}_i > 0$, $\bar{\theta}_i$ denotes the filter signal that can approximately replace the virtual control signal $\bar{\tau}_i$, while $\bar{\vartheta}_i = \bar{\theta}_i - \bar{\tau}_i$ denotes the i-th boundary layer error. $\hat{M}_i$ are the estimates of M_i, while M_i are the unknown upper bounds of ${}_0^C D_t^\alpha \bar{\tau}_i$.

The FO adaptive law for $\hat{\bar{M}}_i$ is given by

$${}_0^C D_t^\alpha \hat{\bar{M}}_i = \zeta_{i1} |\vartheta_i| - \bar{\zeta}_{i1} \hat{\bar{M}}_i \tag{21}$$

where $\zeta_{i1} > 0$ and $\bar{\zeta}_{i1} > 0$.

The effectiveness and performance analysis of the constructed novel nonlinear filter will be addressed in Section 3.4.

Remark 3. *In comparison to the traditional linear filter design scheme widely reported in [35–39], the benefit of the proposed nonlinear filter constructed in this paper is manifested in* $-\frac{\bar{\varsigma}_i \hat{\bar{M}}_i^2 \bar{\vartheta}_i}{\sqrt{\hat{\bar{M}}_i^2 + \bar{\varpi}^2}}$, *which is utilized to compensate for the upper bound of the derivative of the virtual controller. Further, by combining the adaptation law for $\hat{\bar{M}}_j$, the effects of the boundary layer errors $\bar{\vartheta}_i$ can be completely compensated.*

3.3. Disturbance Observer-Based Controller Design

In view of the observer design and the nonlinear filter design given above, this section will present the overall backstepping control design by constructing a disturbance observer.

In order to achieve the subsequent design, the following assumptions need to be satisfied:

Assumption 3 ([25])**.** *There are unknown constants $\bar{d}_i$ and ε_i^* satisfying $|{}_0^C D_t^\alpha d_i(t)| \leq d_i^*$ and $\left|{}_0^C D_t^\alpha \varepsilon_i\right| \leq \bar{\varepsilon}_i$.*

Remark 4. *The assumption for an unknown external disturbance is widely referred to in the literature. While the FO derivative of the fuzzy approximation error ε_i is a continuous function relying on the variables $x_1, x_2, \ldots, x_i$, Assumption 3 is rational.*

From Lemma 2 and Assumption 3, it yields $|{}_0^C D_t^\alpha D_i(t)| = |{}_0^C D_t^\alpha d_i(t) +{}_0^C D_t^\alpha \varepsilon_i| \leq d_i^* + \bar{\varepsilon}_i$, and the unknown upper bound of ${}_0^C D_t^\alpha D_i(t)$ is $\bar{D}_i$.

Firstly, construct the changes in the coordinates as:

$$\bar{S}_1 = x_1 - y_d, \bar{S}_i = \hat{x}_i - \bar{\theta}_{i-1}, \bar{\vartheta}_{i-1} = \bar{\theta}_{i-1} - \bar{\tau}_{i-1}, i = 2, \ldots, n \tag{22}$$

where $\bar{S}_i$ denotes an error surface.

From Equations (10), (11) and (22), the ${}_0^C D_t^\alpha \bar{S}_1$ satisfies:

$$\begin{aligned}
{}_0^C D_t^\alpha \bar{S}_1 &= {}_0^C D_t^\alpha x_1 - {}_0^C D_t^\alpha y_d \\
&= \hat{x}_2 + e_2 + \tfrac{1}{a_1}\bar{\xi}_1^{*T}(\bar{\psi}_1(x_1) - \bar{\psi}_1(\hat{x}_1)) \\
&\quad + \tfrac{1}{a_1}\tilde{\bar{\xi}}_1^T \bar{\psi}_1(\hat{x}_1) + \tfrac{1}{a_1}\bar{\xi}_1^T \bar{\psi}_1(\hat{x}_1) + D_1 -{}_0^C D_t^\alpha y_d \\
{}_0^C D_t^\alpha \bar{S}_i &= \hat{x}_{i+1} + \tfrac{1}{a_i}\bar{\xi}_i^T \bar{\psi}_i(\hat{\bar{x}}_i) + \hat{D}_i + r_i(y - \hat{y}) - {}_0^C D_t^\alpha \bar{\theta}_{i-1} \\
{}_0^C D_t^\alpha \bar{\vartheta}_{i-1} &= {}_0^C D_t^\alpha \bar{\theta}_{i-1} - {}_0^C D_t^\alpha \bar{\tau}_{i-1}, i = 2, \ldots, n
\end{aligned} \tag{23}$$

Design the virtual control laws $\bar{\tau}_i, i = 1, 2, \ldots, n-1$ and the actual control input u

$$\begin{aligned}
\bar{\tau}_1 &= -c_1\bar{S}_1 - \tfrac{1}{a_1}\bar{\xi}_1^T \bar{\psi}_1(\hat{x}_1) - \hat{D}_1 + {}_0^C D_t^\alpha y_d - (\tfrac{1}{2b_{11}} + \tfrac{b_{14}+b_{15}+b_{16}(1+a_1)}{2})\bar{S}_1 \\
\bar{\tau}_i &= -c_i\bar{S}_i - r_i(y - \hat{y}) - \bar{S}_{i-1} - \tfrac{1}{a_i}\bar{\xi}_i^T \bar{\psi}_i(\hat{\bar{x}}_i) - \hat{D}_i - \tfrac{b_{i4}}{2a_i^2}\bar{S}_i + {}_0^C D_t^\alpha \bar{\theta}_{i-1} \\
u &= -c_n\bar{S}_n - \bar{S}_{n-1} - r_n(y - \hat{y}) - \tfrac{1}{a_n}\bar{\xi}_n^T \bar{\psi}_n(\hat{\bar{x}}_n) - \hat{D}_n - \tfrac{b_{n4}}{2a_n^2}\bar{S}_n + {}_0^C D_t^\alpha \bar{\theta}_{n-1}
\end{aligned} \tag{24}$$

where $c_i > 0$, $b_{1k} > 0$ and $b_{j4} > 0$, $k = 4, 5, 6$, $j = 2, 3, \ldots, n$.

In view of (20) and (24), one has

$$ {}_0^C D_t^\alpha \bar{\vartheta}_i = {}_0^C D_t^\alpha \bar{\theta}_i -{}_0^C D_t^\alpha \bar{\tau}_i = -\bar{S}_i - \frac{\bar{\vartheta}_i}{\bar{\zeta}_i} - \frac{\hat{\bar{M}}_i^2 \bar{\vartheta}_i}{\sqrt{\hat{\bar{M}}_i^2 + \bar{\omega}^2}} + \bar{N}_i(\cdot) \tag{25}$$

where $\bar{N}_i(\cdot)$ denotes a smooth function of $\bar{S}_1, \ldots, \bar{S}_i$, $\hat{D}_1, \ldots, \hat{D}_i$, $\hat{\bar{M}}_1, \ldots, \hat{\bar{M}}_{i-1}$, $\bar{\xi}_1, \ldots, \bar{\xi}_i$, $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_i$, y_d, ${}_0^C D_t^\alpha y_d$, ${}_0^C D_t^\alpha({}_0^C D_t^\alpha y_d)$. It is reasonable that an unknown constant $\bar{M}_i > 0$, in a proposed compact set, is such that $|\bar{N}_i(\cdot)| \leq \bar{M}_i$.

Design the FO adaptive law for ξ_i as

$$ {}_0^C D_t^\alpha \xi_i = \frac{\eta_i}{a_i}\bar{S}_i \bar{\psi}_i(\hat{\bar{x}}_i) - \bar{\eta}_i \bar{\xi}_i \tag{26}$$

where $\eta_i > 0$ and $\bar{\eta}_i > 0$.

For the purpose of designing a disturbance observer $\hat{D}_i, i = 1, 2, \ldots, n$ to estimate the hybrid disturbance D_i, by introducing auxiliary variables $\varphi_i = D_i - a_i x_i$, the $\alpha - th$ order derivative of φ_i becomes

$$\begin{aligned} {}_0^C D_t^\alpha \varphi_i &= {}_0^C D_t^\alpha D_i - a_i {}_0^C D_t^\alpha x_i \\ &= {}_0^C D_t^\alpha D_i - a_i(x_{i+1} + \tfrac{1}{a_i}\bar{\xi}_i^{*T}\bar{\psi}_i(\hat{\bar{x}}_i) + \varphi_i + a_i x_i) \end{aligned} \tag{27}$$

Further, design the disturbance observer as:

$$\begin{aligned} &\hat{D}_i = \hat{\varphi}_i + a_i \hat{x}_i, \\ &{}_0^C D_t^\alpha \hat{\varphi}_i = -a_i(\hat{x}_{i+1} + \tfrac{1}{a_i}\hat{\bar{\xi}}_i^T \bar{\psi}_i(\hat{\bar{x}}_i) + \hat{\varphi}_i + a_i \hat{x}_i) \end{aligned} \tag{28}$$

where $x_{n+1} = \hat{x}_{n+1} = u$.

Let $\tilde{\varphi}_i = \varphi_i - \hat{\varphi}_i$ be the disturbance observer error, i.e.,

$$\tilde{\varphi}_i = \tilde{D}_i - a_i e_i \tag{29}$$

Invoking (28)–(30) yields

$$\begin{aligned} {}_0^C D_t^\alpha \tilde{\varphi}_i &= {}_0^C D_t^\alpha D_i \\ &-a_i[e_{i+1} + \tfrac{1}{a_i}\bar{\xi}_i^{*T}(\bar{\psi}_i(\bar{x}_i) - \bar{\psi}_i(\hat{\bar{x}}_i)) + \tfrac{1}{a_i}\tilde{\bar{\xi}}_i^T \bar{\psi}_i(\hat{\bar{x}}_i) + \tilde{\varphi}_i + a_i e_i] \end{aligned} \tag{30}$$

3.4. Stability Analysis

Consequently, the control design has been achieved, and the specific results will be summarized as:

Theorem 1. *Consider FO nonlinear systems (8) subject to unknown external disturbances and unmeasurable states. If the Assumptions 1–3 are satisfied, and a constant $q > 0$ exists , which guarantees $V(0) \le q$, then, by constructing a fractional fuzzy state observer (11), control functions (24), FO nonlinear filter (20), parameter updating laws (21) and (26), and a disturbance observer (28), all the closed-loop signals are bounded, and the system output y can track the given reference signal y_d well.*

Proof. Construct the Lyapunov function candidate:

$$\begin{aligned} &V_1 = V_0 + \tfrac{1}{2}\bar{S}_1^2 + \tfrac{1}{2}\tilde{\varphi}_1^2 + \tfrac{1}{2\eta_1}\tilde{\bar{\xi}}_1^T\tilde{\bar{\xi}}_1, \\ &V_i = V_{i-1} + \tfrac{1}{2}\bar{S}_i^2 + \tfrac{1}{2}\tilde{\varphi}_i^2 + \tfrac{1}{2\eta_i}\tilde{\bar{\xi}}_i^T\tilde{\bar{\xi}}_i + \tfrac{1}{2}\bar{\vartheta}_{i-1}^2 + \tfrac{1}{2\zeta_{i-1,1}}\tilde{\bar{M}}_{i-1}^2, i = 2, 3, \ldots, n \end{aligned} \tag{31}$$

By utilizing Lemma 4, (10), (11), (23), (30) and (31), one can get

$$\begin{aligned} {}_0^C D_t^\alpha V_1 &\le {}_0^C D_t^\alpha V_0 + \bar{S}_1 {}_0^C D_t^\alpha \bar{S}_1 + \tilde{\varphi}_1 {}_0^C D_t^\alpha \tilde{\varphi}_1 - \tfrac{1}{\eta_1}\tilde{\bar{\xi}}_1^T {}_0^C D_t^\alpha \hat{\bar{\xi}}_1 \\ &\le {}_0^C D_t^\alpha V_0 + \bar{S}_1(\bar{S}_2 + \bar{\vartheta}_1) + \tfrac{1}{a_1}\bar{S}_1\bar{\xi}_1^{*T}(\bar{\psi}_1(x_1) - \bar{\psi}_1(\hat{x}_1)) \\ &+\bar{S}_1(e_2 + \tilde{\varphi}_1 + a_1 e_1) - c_1\bar{S}_1^2 - (\tfrac{b_{14}+b_{15}+b_{16}(1+a_1)}{2})\bar{S}_1^2 \\ &+\tfrac{\bar{\eta}_1}{\eta_1}\tilde{\bar{\xi}}_1^T\bar{\xi}_1 - a_1\varphi_1^2 + \tilde{\varphi}_1\{{}_0^C D_t^\alpha D_1 - a_1[e_2 \\ &+\tfrac{1}{a_1}\bar{\xi}_1^{*T}(\bar{\psi}_1(x_1) - \bar{\psi}_1(\hat{x}_1)) + \tfrac{1}{a_1}\tilde{\bar{\xi}}_1^T\bar{\psi}_1(\hat{x}_1) + a_1 e_1]\} \end{aligned} \tag{32}$$

$$\begin{aligned}{}_0^C D_t^\alpha V_i \leq& {}_0^C D_t^\alpha V_{i-1} + \bar{S}_i(\bar{S}_{i+1} + \bar{\vartheta}_i) - \tfrac{1}{a_i}\bar{S}_i\tilde{\bar{\xi}}_i^T\bar{\psi}_i(\hat{x}_i) - \tfrac{b_{i4}}{2a_i^2}\bar{S}_i^2 \\ &+ \tfrac{\bar{\eta}_i}{\eta_i}\tilde{\bar{\xi}}_i^T\bar{\xi}_i + \tfrac{\bar{\zeta}_{i-1}}{\zeta_{i-1}}\tilde{\bar{M}}_{i-1}\hat{\bar{M}}_{i-1} - c_i\bar{S}_i^2 - \tilde{\bar{M}}_{i-1}|\vartheta_{i-1}| - a_i\tilde{\varphi}_i^2 \\ &+ \bar{\vartheta}_{i-1}[\bar{N}_{i-1}(\cdot) - \tfrac{\bar{\vartheta}_{i-1}}{\bar{\zeta}_{i-1}} - \tfrac{\hat{\bar{M}}_{i-1}^2\bar{\vartheta}_{i-1}}{\sqrt{\hat{\bar{M}}_{i-1}^2 + \bar{\varpi}^2}} - \bar{S}_{i-1}] + \tilde{\varphi}_i\{{}_0^C D_t^\alpha D_i \\ &- a_i[e_{i+1} + \tfrac{1}{a_i}\bar{\xi}_i^{*T}(\bar{\psi}_i(\bar{x}_i) - \bar{\psi}_i(\hat{x}_i)) + \tfrac{1}{a_i}\tilde{\bar{\xi}}_i^T\bar{\psi}_i(\hat{x}_i) + a_ie_i]\}\end{aligned} \quad (33)$$

where $\bar{S}_{n+1} + \bar{\vartheta}_n = 0$. □

On account of Lemma 3, Lemma 5 and the property $0 < \bar{\psi}_i^T(\hat{x}_i)\bar{\psi}_i(\hat{x}_i) \leq 1$, some terms in Formulas (32) and (33) satisfy the following inequalities

$$\begin{aligned}&\bar{S}_1(e_2 + \tilde{\varphi}_1 + a_1e_1) \\ &\qquad \leq \tfrac{b_{15}+b_{16}(1+a_1)}{2}\bar{S}_1^2 + \tfrac{1}{2b_{15}}\tilde{\varphi}_1^2 + \tfrac{1+a_1}{2b_{16}}\|e\|^2\end{aligned} \quad (34)$$

$$\tfrac{1}{a_1}\bar{S}_1\bar{\xi}_1^{*T}(\bar{\psi}_1(x_1) - \bar{\psi}_1(\hat{x}_1)) \leq \tfrac{b_{14}}{2}\bar{S}_1^2 + \tfrac{1}{2b_{14}}\|\bar{\xi}_1^*\|^2 \quad (35)$$

$$-\tfrac{1}{a_j}\bar{S}_j\tilde{\bar{\xi}}_j^T\bar{\psi}_j(\hat{x}_j) \leq \tfrac{b_{j4}}{2a_j^2}S_j^2 + \tfrac{1}{2b_{j4}}\tilde{\bar{\xi}}_j^T\tilde{\bar{\xi}}_j \quad (36)$$

$$\begin{aligned}&\tilde{\varphi}_i\{{}_0^C D_t^\alpha D_i - a_k[e_{k+1} + \tfrac{1}{a_k}\bar{\xi}_k^{*T}(\bar{\psi}_k(\bar{x}_k) - \bar{\psi}_k(\hat{x}_k)) + \tfrac{1}{a_k}\tilde{\bar{\xi}}_k^T\bar{\psi}_k(\hat{x}_k) + a_ke_k]\} \\ &\quad \leq (\tfrac{1}{2b_{k1}} + b_{k2} + b_{k3})\varphi_k^2 + \tfrac{b_{k1}}{2}\bar{D}_k^2 + \tfrac{a_k^2+a_k^4}{2b_{k2}}\|e\|^2 + \tfrac{1}{2b_{k3}}\|\bar{\xi}_k^*\|^2 + \tfrac{1}{2b_{k3}}\tilde{\bar{\xi}}_k^T\tilde{\bar{\xi}}_k\end{aligned} \quad (37)$$

$$\begin{aligned}&\tilde{\varphi}_n\{{}_0^C D_t^\alpha D_n - a_n[\tfrac{1}{a_n}\bar{\xi}_n^{*T}(\bar{\psi}_n(\bar{x}_n) - \bar{\psi}_n(\hat{x}_n)) + \tfrac{1}{a_n}\tilde{\bar{\xi}}_n^T\bar{\psi}_n(\hat{x}_n) + a_ne_n]\} \\ &\quad \leq (\tfrac{1}{2b_{n1}} + \tfrac{b_{n2}}{2} + b_{n3})\varphi_n^2 + \tfrac{b_{n1}}{2}\bar{D}_n^2 + \tfrac{a_n^4}{2b_{n2}}\|e\|^2 + \tfrac{1}{2b_{n3}}\|\bar{\xi}_n^*\|^2 + \tfrac{1}{2b_{n3}}\tilde{\bar{\xi}}_n^T\tilde{\bar{\xi}}_n\end{aligned} \quad (38)$$

$$\begin{aligned}&\bar{\vartheta}_{i-1}\bar{N}_{i-1} \leq |\bar{\vartheta}_{i-1}|\bar{M}_{i-1} \\ &\qquad \leq \tfrac{\hat{\bar{M}}_{i-1}^2\bar{\vartheta}_{i-1}^2}{\sqrt{\hat{\bar{M}}_{i-1}^2\bar{\vartheta}_{i-1}^2 + \bar{\varpi}^2}} + \bar{\varpi} + |\bar{\vartheta}_{i-1}|\tilde{\bar{M}}_{i-1},\ i = 2, 3, \ldots, n\end{aligned} \quad (39)$$

with $b_{i2}, b_{i3}, i = 1, 2, \ldots, n$ being design parameters, $k = 1, 2, \ldots, n-1$, $j = 2, 3, \ldots, n$.

Subsequently, one can induce

$$\begin{aligned}{}_0^C D_t^\alpha V_n \leq& -H_1\|e\|^2 + b_{02}\textstyle\sum_{i=1}^n \bar{D}_i^2 + \bar{\mu}_1 - \sum_{i=1}^n c_j\bar{S}_j^2 - \sum_{k=1}^{n-1}\tfrac{1}{\bar{\zeta}_k}\bar{\vartheta}_k^2 \\ &- \textstyle\sum_{k=2}^{n-1}[a_k - (\tfrac{1}{2b_{k1}} + b_{k2} + b_{k3})]\tilde{\varphi}_k^2 - [a_1 - (\tfrac{1}{2b_{11}} \\ &+ b_{12} + b_{13}) + \tfrac{1}{2b_{15}}]\tilde{\varphi}_1^2 - [a_n - (\tfrac{1}{2b_{n1}} + \tfrac{a_n^2b_{n2}}{2} + b_{n3})]\tilde{\varphi}_n^2 \\ &+ \textstyle\sum_{i=1}^n(b_{01}\|P\|^2 + \tfrac{1}{2b_{i3}} + \tfrac{1}{2b_{i4}})\tilde{\bar{\xi}}_i^T\tilde{\bar{\xi}}_i + \tfrac{\bar{\eta}_i}{\eta_i}\tilde{\bar{\xi}}_i^T\bar{\xi}_i + \tfrac{\bar{\zeta}_{i-1}}{\zeta_{i-1}}\tilde{\bar{M}}_{i-1}\hat{\bar{M}}_{i-1}\end{aligned} \quad (40)$$

where $H_1 = H_0 - (1 + a_1)/(2b_{16}) - \sum_{j=1}^{n-1}(a_j^2 + a_j^4)/(2b_{j2}) - a_n^4/2b_{n2}$ and $\bar{\mu}_1 = b_{j1}\bar{D}_j^2/2 + \bar{\mu}_0 + \sum_{j=1}^n(\|\bar{\xi}_j^*\|^2/(2b_{n3}) + \|\bar{\xi}_1^*\|^2/(2b_{14})$.

By invoking Young's inequality, some terms in Formula (40) satisfy

$$\begin{aligned}b_{02}\textstyle\sum_{i=1}^n \bar{D}_i^2 &= b_{02}\textstyle\sum_{i=1}^n(\tilde{\varphi}_i + a_ie_i)^2 \\ &\leq 2b_{02}\textstyle\sum_{i=1}^n \tilde{\varphi}_i^2 + 2b_{02}\sum_{i=1}^n a_i^2\|e\|^2\end{aligned} \quad (41)$$

$$\tfrac{\bar{\eta}_i}{\eta_i}\tilde{\bar{\xi}}_i^T\bar{\xi}_i \leq -\tfrac{\bar{\eta}_i}{2\eta_i}\tilde{\bar{\xi}}_i^T\tilde{\bar{\xi}}_i + \tfrac{\bar{\eta}_i}{2\eta_i}\|\bar{\xi}_i^*\|^2 \quad (42)$$

$$\tfrac{\bar{\zeta}_{i-1}}{\zeta_{i-1}}\tilde{\bar{M}}_{i-1}\hat{\bar{M}}_{i-1} \leq \tfrac{\bar{\zeta}_{i-1}}{2\zeta_{i-1}}\tilde{\bar{M}}_{i-1}^2 + \tfrac{\bar{\zeta}_{i-1}}{2\zeta_{i-1}}\bar{M}_{i-1}^2 \quad (43)$$

Selecting the appropriate parameters to make $\bar{a}_1 = a_1 - (\frac{1}{2b_{11}} + b_{12} + b_{13} + \frac{1}{2b_{15}}) - 2b_{02} > 0$, $\bar{a}_k = a_k - (\frac{1}{2b_{k1}} + b_{k2} + b_{k3}) - 2b_{02} > 0, k = 2, 3, \ldots, n-1$, $\bar{a}_n = a_n - (\frac{1}{2b_{n1}} + \frac{b_{n2}}{2} + b_{n3}) - 2b_{02} > 0$, $H = H_1 - 2b_{02}\sum_{i=1}^{n} a_i^2 > 0$, $bb_i = \frac{\bar{\eta}_i}{2\eta_i} - (b_{01}\|P\|^2 + \frac{1}{2b_{i3}} + \frac{1}{2b_{i4}}) > 0, i = 1, 2, \ldots, n$.

Denote $aa_i = \min\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n\}$, $\bar{c} = \min\{H\lambda_{\min}(P), 2c_i, 2\eta_i bb_i, 2aa_i, \bar{\zeta}_{i-1}\}$, $\bar{\mu} = \bar{\mu}_1 + \sum_{i=1}^{n}(\frac{\bar{\eta}_i}{2\eta_i}\|\bar{\xi}_i^*\|^2 + \frac{\bar{\zeta}_{i-1}}{2\zeta_{i-1}}\bar{M}_{i-1}^2)$, (41) can be rewritten as

$$ {}_0^C D_t^{\alpha} V_n \le -\bar{c}V_n + \bar{\mu} \tag{44}$$

In terms of Lemma 1, it yields

$$ V_n(t) \le E_{\alpha,1}(-\bar{c}t^{\alpha})V_n(0) + \frac{\bar{\mu}\bar{\varrho}}{\bar{c}} \le \frac{V_n(0)\lambda}{1+\bar{c}t^{\alpha}} + \frac{\bar{\mu}\bar{\varrho}}{\bar{c}} \tag{45}$$

According to (45) and the definition of $V_n(t)$, the output tracking error and viewing measure error satisfy $|S_1| \le \sqrt{\frac{2V_n(0)\lambda}{1+\bar{c}t^{\alpha}} + \frac{2\bar{\mu}\bar{\varrho}}{\bar{c}}}$, $\|e\| \le \sqrt{\frac{V_n(0)\lambda}{(1+\bar{c}t^{\alpha})\lambda_{\min}(P)} + \frac{\bar{\mu}\bar{\varrho}}{\bar{c}\lambda_{\min}(P)}}$. Then, $\lim_{t\to\infty}|S_1| = \sqrt{\frac{2\bar{\mu}\bar{\varrho}}{\bar{c}}}$, $\lim_{t\to\infty}\|e\| = \sqrt{\frac{\bar{\mu}\bar{\varrho}}{\bar{c}\lambda_{\min}(P)}}$, the tracking error and observation error approach a small neighborhood of the origin by selecting the appropriate parameter. The rest can be performed in the same manner; consequently, one can deduce that all the closed-loop signals are bounded. Thus, Theorem 1 is proved.

To better show the way of tuning the observer gains and the controller parameters, the guideline is summarized as follows:

(1) Construct the IF-THEN rules, select fuzzy membership functions, and generate the FLS (9).
(2) Choose the observer gains $r_1, r_2, \ldots, r_n$ such that A is Hurwitz.
(3) Select the matrix $Q > 0$, and, by solving (14), the symmetric matrix $P > 0$ is acquired.
(4) Choose appropriate parameters to ensure $\bar{a}_1 > 0$, $\bar{a}_k > 0, k = 2, 3, \ldots, n-1$, $\bar{a}_n > 0$, $H > 0$, $bb_i > 0, i = 1, 2, \ldots, n$, and construct the FO state observer (11), the virtual controller and the actual controller (24), the parameter adaptation law (26), the disturbance observer (28), and the FO nonlinear filter (20), respectively.

Remark 5. *It is clear that, if the parameters $\bar{\eta}_i$ and $\bar{\zeta}_{i-1}$ are fixed, for $i = 1, 2, \ldots, n$, by increasing c_i, H and aa_i, might result in small $\bar{\mu}$ and large $\bar{c}$. Thus, this will lead to a smaller neighborhood. In fact, the size of aa_i depends on the selection of parameters $\bar{a}_i$. Take $\bar{a}_1$ as an example. A different value of $\bar{a}_i$ may have a direct impact on aa_i, and then affect the neighborhood of the tracking error and observer error. The smaller $\bar{a}_i$ is, the greater the tracking error, not vice versa. The same is true for the other parameters. In addition, note that if the modification parameters are too small, parameter drifting may occur to a large extent, while a smaller tracking error will lead to a larger control gain. Hence, the design parameters should be suitably chosen to trade-off the transient performance and the control action in real-life applications.*

Remark 6. *It should be noted that the authors in [21] considered the disturbance rejection issue for fractional-order nonlinear systems. However, unlike this paper, reference [21] is based on complete measurement of the system state information. In fact, due to technical or economic reasons, the energy measurement or the choice not to measure all state information are relevant. In addition, the disturbance signal considered in [21] needs to satisfy the so-called matching condition, i.e., the disturbance signal appears only in the last equation of the system, rather than the non-matching case considered in this paper. Therefore, the approach taken in this paper is difficult and challenging.*

4. Simulation Study

In this part, an FO Chua–Hartley system [55] is introduced to verify the validity of the given control strategy:

$$\begin{cases} {}_0^C D_t^\alpha x_1 = x_2 + \frac{10}{7}(x_1 - x_1^3) + d_1(t) \\ {}_0^C D_t^\alpha x_2 = x_3 + 10x_1 - x_2 + d_2(t) \\ {}_0^C D_t^\alpha x_3 = -\frac{100}{7}x_2 + u + d_3(t) \\ y = x_1, \end{cases} \tag{46}$$

where $\alpha = 0.98$, when $u = d_1(t) = d_2(t) = d_3(t) = 0$, and the system status information is fully available, the system dynamic is depicted in Figure 1.

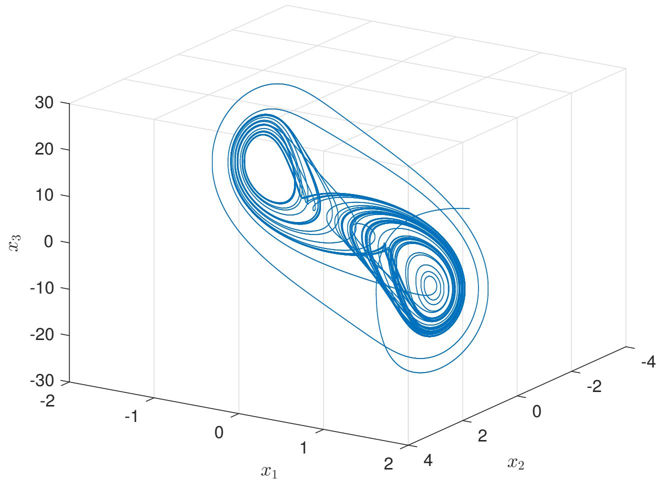

Figure 1. Dynamical behavior of the FO Chua's system.

Denote $d_1(t) = \sin(t)$, $d_2(t) = \sin(0.5t)$, $d_3(t) = \sin(2t)$. The reference signal is chosen as $y_d = \cos(2t)$.

Remark 7. *Since $y_d = cos(2t)$, according to fractional order calculation rules, the α-th order and 2α-th order derivative of the reference signal y_d are both bounded. The details are as follows: ${}_0^C D_t^\alpha y_d = 2^\alpha cos(2t + \frac{\alpha\pi}{2})$ and ${}_0^C D_t^{2\alpha} y_d = 2^{2\alpha} cos(2t + \alpha\pi)$. It is obvious that y_d and its α-th order and 2α-th order derivative are both bounded and smooth, which also reflects the plausibility of Assumption 1.*

The parameters of the state observer, disturbance observer, nonlinear filters, adaptive laws and control laws are selected as $c_i = 2, i = 1,2,3$, $a_1 = 1$,$a_2 = a_3 = 2$, $r_1 = 5$, $r_2 = 30$, $r_3 = 60$, $\varsigma_1 = \varsigma_2 = 1/15$, $\zeta_{11} = \bar{\zeta}_{11} = \zeta_{21} = \bar{\zeta}_{21} = 1$, $\eta_i = 0.001$, $\bar{\eta}_i = 1$, $b_{01} = 20$, $b_{02} = 1/4$, $b_{i1} = 20$, $b_{i2} = 1/6$, $b_{i3} = 1/20$, $b_{14} = 10$, $b_{15} = 3$, $b_{16} = 2$, $b_{24} = 1$, $b_{34} = 1$.

The fuzzy membership functions are: $\mu_{F_h^l}(x_h) = exp(-\frac{x_h+3-l}{4})$, $l = 1,2,\ldots,5$. The initial conditions are selected as $x_1(0) = 0.8$, $x_2(0) = -2$, $x_3(0) = 1$; the others are chosen as zeros.

The simulation results are shown in Figures 2–6. Figure 2 shows the system states x_i and their estimates $\hat{x}_i, i = 1,2,3$. The parameters $\bar{\xi}_i$ and $\hat{M}_k$ for $k = 1,2$, $i = 1,2,3$ are shown in Figures 3 and 4, respectively; The trajectories of the output y and reference signal y_d are plotted in Figure 5; Figure 6 depicts the control input u.

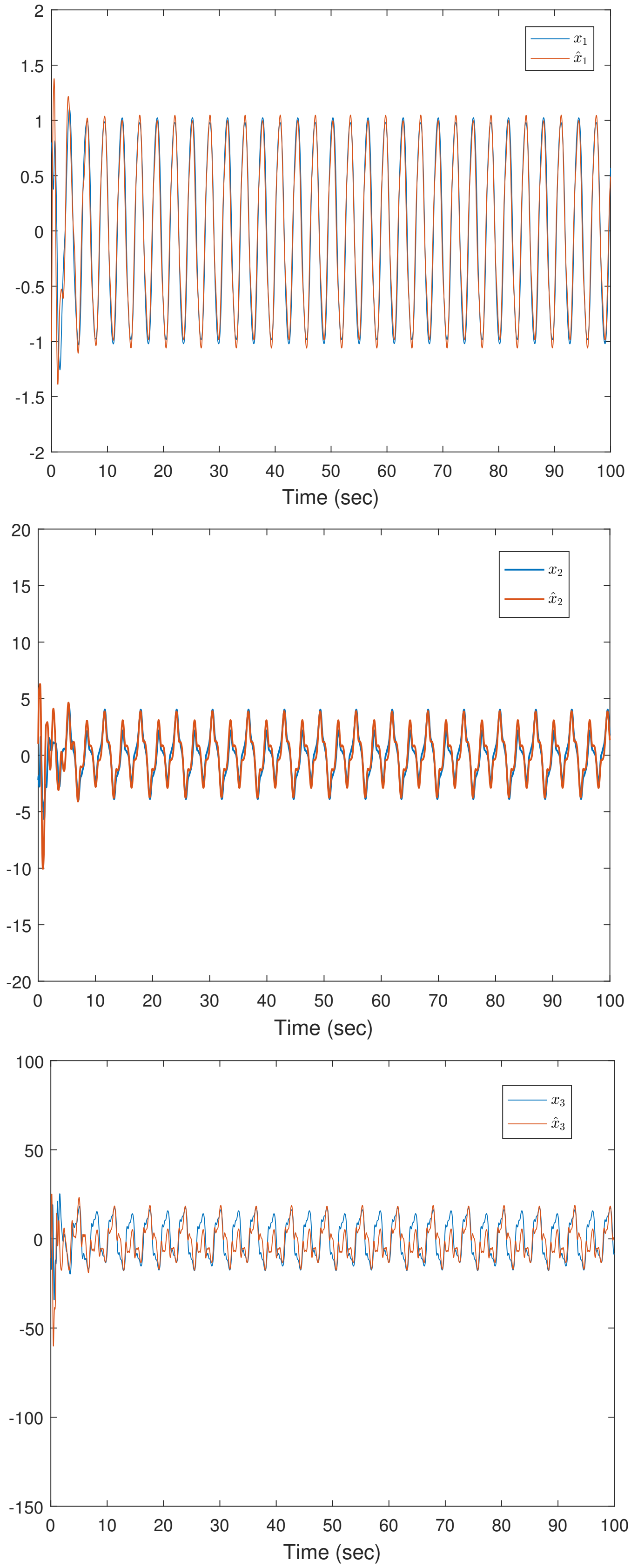

Figure 2. The system states x_i and their estimates $\hat{x}_i, i = 1, 2, 3$.

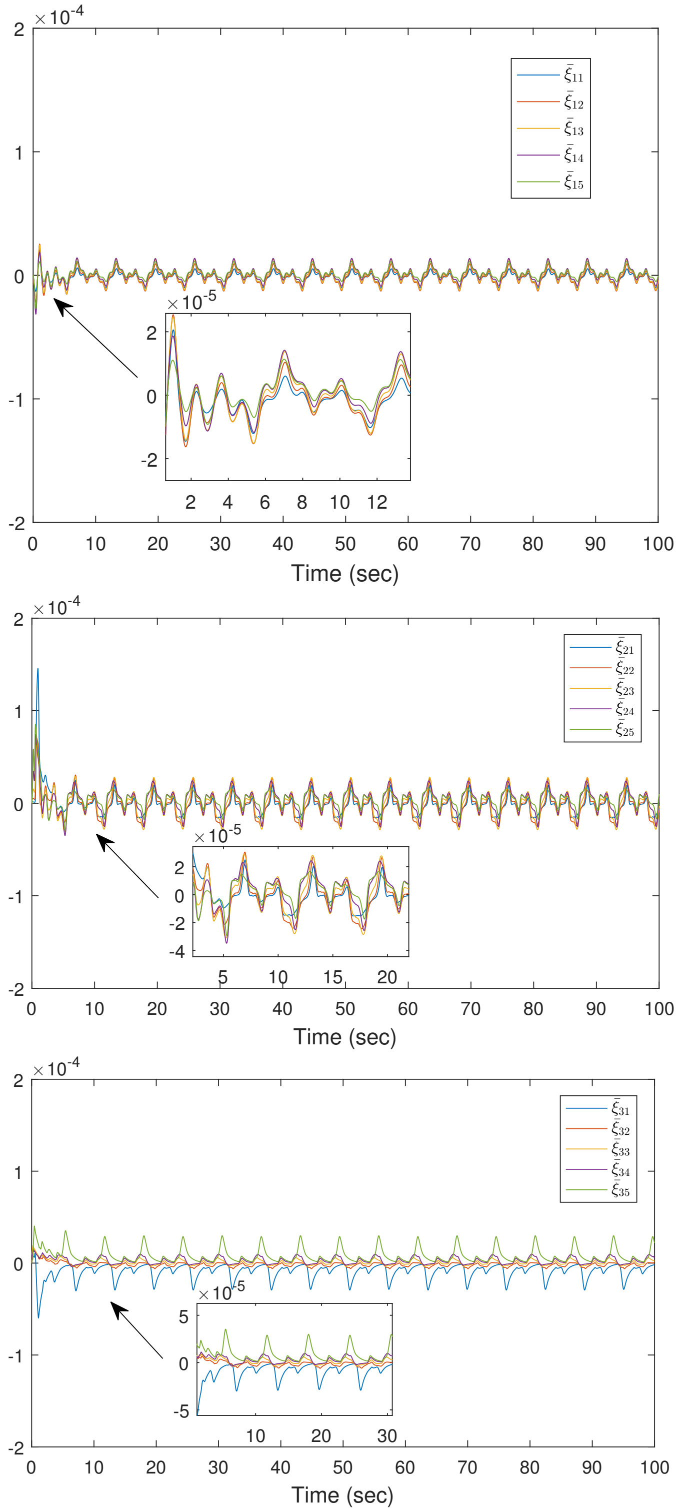

Figure 3. The parameters ξ_i, $i = 1, 2, 3$.

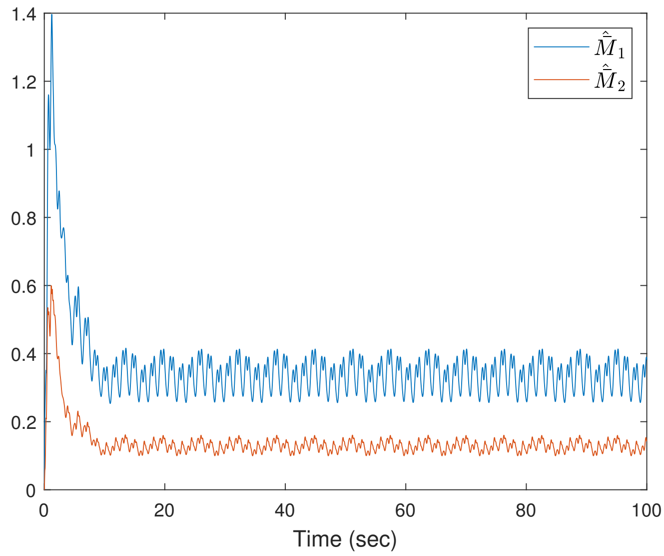

Figure 4. The parameters $\hat{\hat{M}}_k$, $k = 1, 2$.

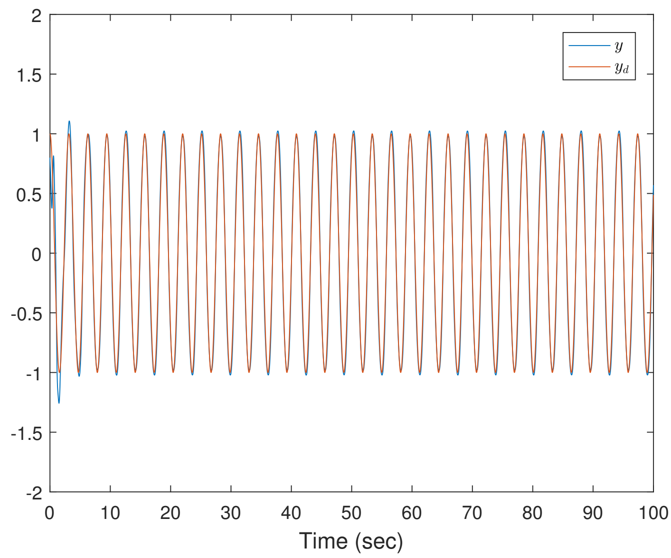

Figure 5. The trajectories of the output y and reference signal y_d.

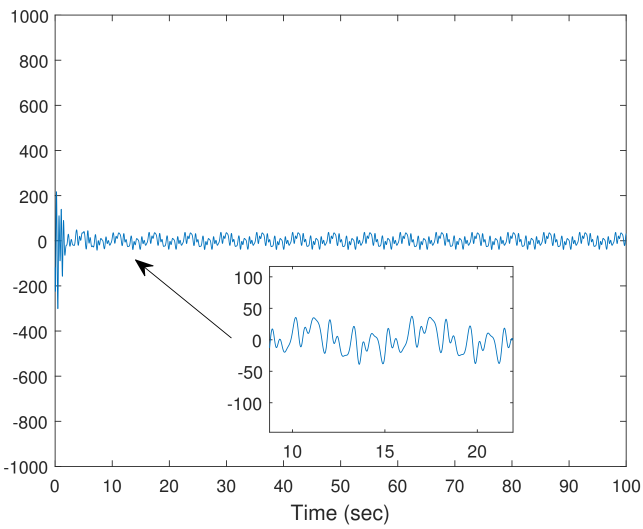

Figure 6. The control input u.

From Figures 2–6, it is obvious that the developed control algorithm not only achieves the boundedness of all the closed-loop signals, but also in a way such that the tracking error and the observer error approach to small neighborhoods of the origin. This further confirms Theorem 1.

5. Conclusions

In this work, a nonlinear filter-based adaptive fuzzy output-feedback control strategy was presented for uncertain FO nonlinear systems subject to unknown external disturbance. By borrowing the universal approximation principle of FLS, the unknown nonlinear dynamics existing in the systems have been effectively approximated. Then, a new FO fuzzy state observer based on a nonlinear disturbance observer was developed to handle the issues of immeasurable states and unknown compounded disturbance simultaneously. Further, by introducing auxiliary functions that conform to specific inequality relations, a novel FO nonlinear filter based on a smooth auxiliary function was defined to not only erase the issue of complexity explosion, but also to completely compensate for the effects of the boundary errors induced by the constructed filters compared with the existing FO linear filter method. What is more, by combining a backsteppng algorithm, an adaptive fuzzy output-feedback control strategy was presented. On account of the FO stability criterion, the boundedness of all the closed-loop signals is achieved, and the tracking error and observer error approach to small neighborhoods of the origin. Simulation results based on the FO Chua–Hartley system further demonstrate the validity of the presented scheme. In addition, the FO system is an extension of the IO system in traditional control theory. It can more accurately describe non-rigid dynamic systems, such as thermodynamic systems, flexible systems, etc., and is extensively applied to vehicle suspension systems, fractal and chaos, battery management systems, robots and other fields. Consequently, future research will consider an FO control strategy for large-scale cases.

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