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On q-Hermite–Hadamard Type Inequalities via s-Convexity and (α,m)-Convexity

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Abstract: The purpose of the paper is to present new q-parametrized Hermite–Hadamard-like type integral inequalities for functions whose third quantum derivatives in absolute values are s-convex and (α, m)-convex, respectively. Two new q-integral identities are presented for three time q-differentiable functions. These lemmas are used like basic elements in our proofs, along with several important tools like q-power mean inequality, and q-Holder's inequality. In a special case, a non-trivial example is considered for a specific parameter and this case illustrates the investigated results. We make links between these findings and several previous discoveries from the literature.

Keywords: convex functions; Hermite–Hadamard type inequalities; quantum calculus

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1. Introduction

The q-calculus is used today in a lot of mathematical areas, like combinatorics, orthogonal polynomials, number theory, mechanics and theory of relativity, and basic hypergeometric functions, and this field started with the work of Jackson [1,2]. It has the advantage of substituting the classical derivative with a difference operator in order to more easily manipulate the sets of non-differentiable mappings. In recent times, Q-calculus has gained increasing importance in some fields of science, like geometry function theory, statistics, quantum mechanics, and also, particle physics, cosmology, and economic geology.

Two fundamental publications on quantum-calculus were written by Ernst [3], and Kac and Cheung [4]. Ntouyas and Tariboon in [5,6] analyzed q-derivatives and q-integrals across intervals like $[\varphi_1, \varphi_2] \subset \mathbb{R}$, establishing many q-analogues for the well-known Holder, Ostrowski, Hermite–Hadamard, Gruss, Cauchy–Buniakovski–Schwarz inequalities based on classical convexity. The novel ideas and the fruitful methodologies of these authors attracted researchers, especially those that work in the field of inequalities involving convexity.

The concept of convexity is used for solving problems in mathematics, which have many applications in other domains such as business and industry. Hermite–Hadamard inequality (H-H) [7] is one of the most well-known inequalities from the theory of convexity and have attracted a lot of attention; see the large number of refinements [8–10] and generalizations [11–13], as for example [14–19] without using q-calculus, and in the frame of q-calculus and (p, q)-calculus, see for example, [20–26]. In [14,17], for example, different types of Hermite–Hadanmard inequalities were presented for functions whose second derivatives in absolute value satisfy different types of convexities using in the equality of the corresponding key lemma two integrals from 0 to 1. In this paper, a kind of similar structure will hold, but in the frame of q-calculus. q-Ostrowski's inequalities for q-differentiable convex functions were given in [27]. Also, the techniques involved in such inequalities have enough applications in different areas in which symmetry plays an important role.

Inspired by the recent work of [20,21], respectively, our first purpose is to find new q-Hermite–Hadamard like type inequalities for three time q-differentiable s-convex functions by using a new auxiliary tool, a q-integral identity for the q-left and the q-right derivatives of order three. Our second purpose is to present new q-Hermite-Hadamard like type inequalities with two parameters for twice q-differentiable (α, m)-convex functions, for the q-left and q-right derivatives of order two. The advantage of using one integral instead of two integrals from 0 to 1 in Lemma 1 in the right member of the identity is that in the proofs of the next inequalities in which it is used, we do not have a sum of two modulus; we have only a modulus, which is better for our majorizations. For the second goal, a new lemma has been demonstrated, which is a generalization of the corresponding identity from [28] when the parameter $m = 1$.

The paper has been organized as follows. In Section 2, the basic definitions and properties regarding q-calculus and the H-H integral inequality have been restated. In Section 3, in the first part, we formulate and demonstrate our first main result in Lemma 1. Then, the corresponding inequalities are given in Theorems 6 and 7, and also in Theorems 8 and 9. We list the new q-midpoints, trapezoidal and q-H-H-like type integral inequalities for functions whose third left and right q-derivatives in absolute value are s-convex. Some consequences have been presented, and an example was discussed in detail. It was utilized for the Matlab R2023a software version and also for some calculus, but Mathematica software could be also suitable to use here, for example. In the second part of Section 3, we use Theorem 10 to develop a new two parametrized q-H-H type integral inequality for (α, m)-convex functions. Section 4 contains discussions and conclusions and potential further studies.

2. Preliminaries of Quantum Calculus and Hermite–Hadamard's Inequality

Some well-known type of convexity will be restated for the beginning below.

Definition 1 ([20]). *If the function* $k : [c, d] \subset \mathbb{R} \to \mathbb{R}$ *is convex, then for every* $x, y \in [c, d]$ *and every* $t \in [0, 1]$, *we have*

$$k(tx + (1 - t)y) \le tk(x) + (1 - t)k(y).$$

Definition 2 ([20]). *If the function* $k : [0, d] \to \mathbb{R}$ *is called* (α, m)*-convex, then the following inequality holds*

$$k(tx + m(1 - t)y) \le t^{\alpha} k(x) + m(1 - t^{\alpha})k(y),$$

holds $\forall x,\ y \in [0, d],\ t \in [0, 1],\ (\alpha, m) \in [0, 1]^2$.

In [9], the authors mention the definition of s-convex functions in the first sense, which was stated by Orlicz in [29].

Definition 3 ([9]). *A function* $k : [0, \infty] \to \mathbb{R}$ *is said to be s-convex in the first sense if*

$$k(\alpha x + \beta y) \le \alpha^s k(x) + \beta^s k(y),$$

for all $x, y \in [0, \infty),\ \alpha,\ \beta \ge 0$ *with* $\alpha^s + \beta^s = 1$ *and for some fixed* $s \in (0, 1]$.

From Definitions 2 and 3, we see that if k is s-convex in the first sense, then k is also $(\alpha, 1)$-convex with $\alpha = s$, when $c \ge 0$.

Hudzik and Maligranda in [30] defined the s-convex function in the second sense,

Definition 4 ([30]). *A function* $k : [0, \infty] \to \mathbb{R}$ *is said to be s-convex in the second sense if*

$$k(\alpha x + \beta y) \le \alpha^s k(x) + \beta^s k(y),$$

for all $x, y \in [0, \infty),\ \alpha,\ \beta \ge 0$ *with* $\alpha + \beta = 1$ *and for some fixed* $s \in (0, 1]$.

We start with the famous Hermite–Hadamard inequality [7], which states that a convex function $k : [c, d] \to \mathbb{R}$ satisfies inequality from below:

$$k\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d k(x)dx \leq \frac{k(c)+k(d)}{2}, \tag{1}$$

and if k is a concave function, then the below inequality takes place:

$$k\left(\frac{c+d}{2}\right) \geq \frac{1}{d-c}\int_c^d k(x)dx \geq \frac{k(c)+k(d)}{2}. \tag{2}$$

Here, it will be assumed that $0 < q < 1$, and that $c < d$, with $[c, d]$ being a real interval. Below, we will present several important and previously determined notions, observations and lemmas of the quantum calculus.

First, we start by recalling the definition of right q derivative (from [21]) and then the corresponding definition of the left q derivative of a function [6,21].

Definition 5 ([21]). *The right or q^d-derivative of $k : [c, d] \to \mathbb{R}$ at $x \in [c, d]$ is expressed as:*

$$^{d}D_q k(x) = \frac{k(qx + (1-q)d) - k(x)}{(1-q)(d-x)}, \quad x \neq d.$$

Definition 6 ([6,21]). *The left or q_c-derivative of $k : [c, d] \to \mathbb{R}$ at $x \in [c, d]$ is expressed as*

$$_{c}D_q k(x) = \frac{k(x) - k(qx + (1-q)c)}{(1-q)(x-c)}, \quad x \neq c.$$

The corresponding q-right and q-right integral of the function k were defined for example in [21,31,32] as follows:

Definition 7 ([21,31]). *The right or q^d-integral of $k : [c, d] \to \mathbb{R}$ at $x \in [c, d]$ is defined as:*

$$\int_x^d k(t)^d d_q t = (1-q)(d-x)\sum_{n=0}^{\infty} q^n k(q^n x + (1-q^n)d) = (d-x)\int_0^1 g(td + (1-t)x)^1 d_q t.$$

Definition 8 ([21,32]). *The left or q_c-integral of $k : [c, d] \to \mathbb{R}$ at $x \in [c, d]$ is defined as:*

$$\int_c^x k(t)_c d_q t = (1-q)(x-c)\sum_{n=0}^{\infty} q^n k(q^n x + (1-q^n)c) = (x-c)\int_0^1 k(tx + (1-t)c) d_q t.$$

In [21], the authors also stated the next results, which will be very useful in our calculus.

Definition 9 ([21]). *We have the equality for q_c-integrals*

$$\int_c^d (x-c)^{\alpha}{}_c d_q x = \frac{(d-c)^{\alpha+1}}{[\alpha+1]_q},$$

for $\alpha \in \mathbb{R} - \{-1\}$.

It is also necessary to recall the q-Holder's inequality, here; see [20], Theorem 3.

Theorem 1 ([20]). *Let $x > 0$, $\varphi_1 > 1$. If $\frac{1}{\varphi_1} + \frac{1}{\varphi_2} = 1$, then*

$$\int_0^x |f(x)g(x)| d_q x \leq \left(\int_0^x |f(x)|^{\varphi_1} d_q x \right)^{\frac{1}{\varphi_1}} \left(\int_0^x |g(x)|^{\varphi_2} d_q x \right)^{\frac{1}{\varphi_2}}.$$

Basic properties of q-derivatives and of q-integrals are given, for example, in [3,4,6].

In [20], the authors presented new estimates for q-Hermite–Hadamard inequality for mappings whose second q-derivatives in absolute value are (α, m)-convex in Theorems 4–7, which are demonstrated below. Similar inequalities will be provided in the next section for functions whose third q-derivatives in absolute value are s-convex.

Theorem 2 ([20]). *If $k : [c, d] \subset \mathbb{R} \to \mathbb{R}$ is a twice q^d-differentiable function on (c, d), such that ${}^dD_q^2 k \in C[c, d]$ and integrable on $[c, d]$; then, we have the following inequality, provided that $|{}^dD_q^2 k|$ is (α, m)-convex on $[c, d]$*

$$|\frac{k(c) + qk(md)}{[2]_q} - \frac{1}{md - c} \int_c^{md} k(t) \, {}^{md}d_q t|$$

$$\leq \frac{q^2(md - c)^2}{1 + q} [\frac{[\alpha + 3]_q - q[\alpha + 2]_q}{[\alpha + 2]_q [\alpha + 3]_q} \left| {}^dD_q^2 k(c) \right|$$

$$+ m(\frac{1}{[2]_q [3]_q} - \frac{[\alpha + 3]_q - q[\alpha + 2]_q}{[\alpha + 2]_q [\alpha + 3]_q}) |{}^dD_q^2 k(d)|].$$

Theorem 3 ([20]). *Let $k : [c, d] \subset \mathbb{R} \to \mathbb{R}$ be a twice q^d-differentiable function on (c, d), and ${}^dD_q^2 k \in C[c, d]$ and integrable on $[c, d]$. If $|{}^dD_q^2 k^{\varphi_1}|$, $\varphi_1 \geq 1$ is (α, m)-convex on $[c, d]$ we have the following inequality:*

$$|\frac{k(c) + qk(md)}{[2]_q} - \frac{1}{md - c} \int_c^{md} k(t) \, {}^{md}d_q t|$$

$$\leq \frac{q^2(md - c)^2}{\left([2]_q\right)^{2 - \frac{1}{\varphi_1}} \left([3]_q\right)^{\frac{1}{\varphi_1}}} [(\frac{[\alpha + 3]_q - q[\alpha + 2]_q}{[\alpha + 2]_q [\alpha + 3]_q} \left| {}^dD_q^2 k(c) \right|^{\varphi_1}$$

$$+ m(\frac{1}{[2]_q [3]_q} - \frac{[\alpha + 3]_q - q[\alpha + 2]_q}{[\alpha + 2]_q [\alpha + 3]_q}) |{}^dD_q^2 k(d)|^{\varphi_1}]^{\frac{1}{\varphi_1}}.$$

Theorem 4 ([20]). *Let $k : [c, d] \subset \mathbb{R} \to \mathbb{R}$ be a twice q^d-differentiable function on (c, d), and ${}^dD_q^2 k \in C[c, d]$ and integrable on $[c, d]$. If $|{}^dD_q^2 k^{\varphi_1}|$, is (α, m)-convex on $[c, d]$, for some $\varphi_1 > 1$ and $\frac{1}{\varphi_1} + \frac{1}{\varphi_2} = 1$, then we have*

$$|\frac{k(c) + qk(md)}{1 + q} - \frac{1}{md - c} \int_c^{md} k(t) \, {}^{md}d_q t|$$

$$\leq \frac{q^2(md - c)^2}{1 + q} (u_1)^{\frac{1}{\varphi_2}} \left(\frac{|{}^dD_q^2 k(c)|^{\varphi_1} + m\left([\alpha + 1]_q - 1\right) \left| {}^dD_q^2 k(d) \right|^{\varphi_1}}{[\alpha + 1]_q} \right)^{\frac{1}{\varphi_1}},$$

where $u_1 = (1 - q) \sum_{n=0}^{\infty} (q^n)^{\varphi_2 + 1} (1 - q^{n+1})^{\varphi_2}$.

Theorem 5 ([20]). *By using the assumptions of Theorem 6, the following inequality holds*

$$|\frac{k(c)+qk(md)}{1+q}-\frac{1}{md-c}\int_c^{md}k(t)\ {}^{md}d_qt|$$

$$\leq\frac{q^2(md-c)^2}{1+q}(\frac{1}{[\varphi_2+1]_q})^{\frac{1}{\varphi_2}}\left(u_2|{}^dD_q^2k(c)|^{\varphi_1}+mu_3\left|{}^dD_q^2k(d)\right|^{\varphi_1}\right)^{\frac{1}{\varphi_1}},$$

where $u_1=(1-q)\sum\limits_{n=0}^{\infty}q^{n(\alpha+1)}(1-q^{\alpha n+1})^{\varphi_1}$ *and* $u_3=(1-q)\sum\limits_{n=0}^{\infty}q^n(1-q^{\alpha n}\alpha)(1-q^{\alpha n+1})^{\varphi_1}$.

3. Results

This section is dedicated to main results presented in this paper.

3.1. Hermite–Hadamard Type Inequalities for Functions with the Modulus of Quantum Third Derivatives Being s-Convex

We will provide new q-Hermite–Hadamard-like type inequalities analogue to the same from Theorems 4–7 from [20], which it is then necessary to recall below, for the case of mappings whose third derivatives in absolute values are s-convex. In order to obtain new q-Hermite–Hadamard-like type inequalities for mappings whose third derivatives in absolute values are s-convex, a new quantum identity will be established below.

Lemma 1. *Let* $k:[c,d]\to\mathbb{R}$ *be a three times* q^d*-differentiable function on* $[c,d]$ *and* ${}^dD_q^3k\in C[c,d]$ *and integrable on* $[c,d]$*. Then, the following equality holds:*

$$\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_qt$$

$$=\frac{q^5(d-c)^3}{[2]_q[3]_q}\int_0^1 t^2(1-qt)\ {}^dD_q^3k(tc+(1-t)d)d_qt. \tag{3}$$

Proof. By using Definition 5 and calculus, we obtain

$${}^dD_q^3k(tc+(1-t)d)=\frac{1}{q^3t^3(d-c)^3(1-q)^3}\{k\left(q^3tc+d\left(1-q^3t\right)\right)-[3]_qk\left(q^2tc+d\left(1-q^2t\right)\right)$$

$$+q[3]_qk(qtc+d(1-qt))-q^3k(tc+d(1-t))\}.$$

Here, I denotes the integral, $\int_0^1 t^2(1-qt){}^dD_q^3k(tc+(1-t)d)d_qt$ and we have

$$I=\int_0^1(\frac{k(q^3tc+d(1-q^3t))-[3]_qk(q^2tc+d(1-q^2t))}{q^3(d-c)^3(1-q)^3t}$$

$$+\frac{q[3]_qk(qtc+d(1-qt))-q^3k(tc+d(1-t))}{q^3(d-c)^3(1-q)^3t})d_qt$$

$$-\int_0^1(\frac{k(q^3tc+d(1-q^3t))-[3]_qk(q^2tc+d(1-q^2t))}{q^2(d-c)^3(1-q)^3}$$

$$+\frac{q[3]_qk(qtc+d(1-qt))-q^3k(tc+d(1-t))}{q^2(d-c)^3(1-q)^3})d_qt.$$

If we also denote the first integral of I by A and the second integral of I by B, then we can rewrite the previous equality as $I = A - B$.

By using the Definition 7, and calculus, we successively find

$$A = (1-q)\sum_{n=0}^{\infty}\frac{k\left(q^{n+3}c+d\left(1-q^{n+3}\right)\right)}{(d-c)^3(1-q)^3q^3} - (1-q)\left(1+q+q^2\right)\sum_{n=0}^{\infty}\frac{k\left(q^{n+2}c+d\left(1-q^{n+2}\right)\right)}{(d-c)^3(1-q)^3q^3}$$

$$+(1-q)q\left(1+q+q^2\right)\sum_{n=0}^{\infty}\frac{k\left(q^{n+1}c+d\left(1-q^{n+1}\right)\right)}{(d-c)^3(1-q)^3q^3} - (1-q)q^3\sum_{n=0}^{\infty}\frac{k(q^nc+d(1-q^n))}{(d-c)^3(1-q)^3q^3}$$

$$= \sum_{n=0}^{\infty}\frac{k\left(q^{n+3}c+d\left(1-q^{n+3}\right)\right)}{(d-c)^3(1-q)^2q^3} - \sum_{n=0}^{\infty}\frac{k\left(q^{n+2}c+d\left(1-q^{n+2}\right)\right)}{(d-c)^3(1-q)^2q^3}$$

$$-\left(q+q^2\right)[\sum_{n=0}^{\infty}\frac{k\left(q^{n+2}c+d\left(1-q^{n+2}\right)\right)}{(d-c)^3(1-q)^2q^3} - \sum_{n=0}^{\infty}\frac{k\left(q^{n+1}c+d\left(1-q^{n+1}\right)\right)}{(d-c)^3(1-q)^2q^3}]$$

$$+q^3[\sum_{n=0}^{\infty}\frac{k\left(q^{n+1}c+d\left(1-q^{n+1}\right)\right)}{(d-c)^3(1-q)^2q^3} - \sum_{n=0}^{\infty}\frac{k(q^nc+d(1-q^n))}{(d-c)^3(1-q)^2q^3}]$$

$$= \frac{k(d) - k\left(q^2c+d\left(1-q^2\right)\right)}{(d-c)^3(1-q)^2q^3} - q(1+q)\frac{k(d)-k(qc+d(1-q))}{(d-c)^3(1-q)^2q^3} + q^3\frac{k(d)-k(c)}{(d-c)^3(1-q)^2q^3},$$

or

$$A = \frac{k(d)(1+q)(1-q)^2 - k\left(q^2c+d\left(1-q^2\right)\right) + q(1+q)k(qc+d(1-q)) - q^3k(c)}{(d-c)^3(1-q)^2q^3}.$$

Using the Definition 7 again, but this time for B, we find

$$B = \{\sum_{n=0}^{\infty}\frac{q^{n+3}k\left(q^{n+3}c+d\left(1-q^{n+3}\right)\right)}{q^5(d-c)^3(1-q)^2} - \left(q^2+q+1\right)\sum_{n=0}^{\infty}\frac{q^{n+2}k\left(q^{n+2}c+d\left(1-q^{n+2}\right)\right)}{q^4(d-c)^3(1-q)^2}$$

$$+\left(q^2+q+1\right)\sum_{n=0}^{\infty}\frac{q^{n+1}k\left(q^{n+1}c+d\left(1-q^{n+1}\right)\right)}{q^2(d-c)^3(1-q)^2} - q\sum_{n=0}^{\infty}\frac{q^nk(q^nc+d(1-q^n))}{(d-c)^3(1-q)^2}\}$$

$$= \frac{1}{q^2(d-c)^3(1-q)^2}\{\frac{1}{q^3}[\frac{\int_c^d k(t)^d d_qt}{(d-c)(1-q)} - k(c) - qk(qc+d(1-q)) - q^2k\left(q^2c+d\left(1-q^2\right)\right)]$$

$$-\frac{q^2+q+1}{q^2}[\frac{1}{(d-c)(1-q)}\int_c^d k(t)^d d_qt - k(c) - qk(qc+d(1-q))]$$

$$+\left(q^2+q+1\right)[\frac{1}{(d-c)(1-q)}\int_c^d k(t)^d d_qt - k(c)] - q^3\frac{\int_c^d k(t)^d d_qt}{(d-c)(1-q)}\},$$

or

$$B = \frac{[2]_q[3]_q}{q^5(d-c)^4}\int_c^d k(t)^d d_qt - \frac{k(c)\left(1-q-q^2+q^4+q^5\right)}{q^5(d-c)^3(1-q)^2} - \frac{k(qc+d(1-q))\left(1-q-q^2-q^3\right)}{q^4(d-c)^3(1-q)^2}$$

$$-\frac{k\left(q^2c+d\left(1-q^2\right)\right)}{q^3(d-c)^3(1-q)^2}.$$

Thus, $A-B$ becomes

$$A-B=-\frac{[2]_q[3]_q}{q^5(d-c)^4}\int_c^d k(t)^d d_q t+\frac{k(c)\left(1-q-q^2+q^4+q^5\right)}{q^5(d-c)^3(1-q)^2}$$
$$+\frac{k(qc+d(1-q))\left(1-q-q^2-q^3\right)}{q^4(d-c)^3(1-q)^2}+\frac{k\left(q^2c+d\left(1-q^2\right)\right)}{q^3(d-c)^3(1-q)^2}$$
$$+\frac{k(d)(1+q)(1-q)^2-k\left(q^2c+d\left(1-q^2\right)\right)+q(1+q)k(qc+d(1-q))-q^3k(c)}{(d-c)^3(1-q)^2q^3},$$

or

$$\begin{aligned} A-B &= \tfrac{-[2]_q[3]_q}{q^5(d-c)^4}\textstyle\int_c^d k(t)^d d_q t+\tfrac{1}{q^5(d-c)^3}\left[\tfrac{1-q^2(1+q)}{1-q}k(c)+q^2(1+q)k(d)\right] \\ &+ \tfrac{k(qc+d(1-q))}{(d-c)^3(1-q)q^4}. \end{aligned}$$

But, we also can write the previous expression as

$$A-B=-\frac{[2]_q[3]_q}{q^5(d-c)^4}\int_c^d k(t)^d d_q t+\frac{q^2[2]_q}{q^5(d-c)^3}k(d)+\frac{1-q^2-q^3+q}{(d-c)^3q^5(1-q)}k(c)+\frac{{}^dD_qk(c)}{(d-c)^2q^4},$$

or

$$A-B=-\frac{[2]_q[3]_q}{q^5(d-c)^4}\int_c^d k(t)^d d_q t+\frac{[2]_q}{q^5(d-c)^3}\left(q^2k(d)+[2]_qk(c)\right)+\frac{{}^dD_qk(c)}{(d-c)^2q^4}. \quad (4)$$

Multiplying both sides of (4) by $\frac{(d-c)^3q^5}{[2]_q[3]_q}$, we obtain

$$I\frac{(d-c)^3q^5}{[2]_q[3]_q}=\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}\cdot{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_qt,$$

which is the desired identity. □

Theorem 6. *Let $k:[c,d]\to\mathbb{R}$ be a three times q^d-differentiable function on (c,d) and ${}^dD_q^3k\in C[c,d]$ and integrable on $[c,d]$, $c\geq 0$. If $|{}^dD_q^3k|$ is s-convex in the first sense on $[c,d]$, then the following inequality holds:*

$$\left|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}\cdot{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_q\,t\right| \quad (5)$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\left[|{}^dD_q^3k(c)|\frac{[s+4]_q-q[s+3]_q}{[s+3]_q[s+4]_q}+|{}^dD_q^3k(d)|\left(\frac{1}{[3]_q[4]_q}-\frac{[s+4]_q-q[s+3]_q}{[s+3]_q[s+4]_q}\right)\right].$$

Proof. By using Lemma 1, and the modulus properties, we get

$$\left|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}\cdot{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_q\,t\right|=\left|I\,\frac{(d-c)^3q^5}{[2]_q[3]_q}\right|$$
$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\int_0^1 t^2(1-qt)\left|{}^dD_q^3k(tc+(1-t)d)\right|d_qt.$$

Assuming that $|{}^dD_q^3k|$ is s-convex in the first sense on $[c,d]$, we have

$$|I\,\frac{(d-c)^3q^5}{[2]_q[3]_q}|\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}[|{}^dD_q^3k(c)|\int_0^1 t^{s+2}(1-qt)d_q\,t+|{}^dD_q^3k(d)|\int_0^1 t^2(1-qt)(1-t^s)d_qt],$$

or by calculus

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\;{}^dd_q\,t|$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}[|{}^dD_q^3k(c)|\frac{[s+4]_q-q[s+3]_q}{[s+3]_q[s+4]_q}+|{}^dD_q^3k(d)|\left(\frac{1}{[3]_q[4]_q}-\frac{[s+4]_q-q[s+3]_q}{[s+3]_q[s+4]_q}\right)].$$

□

Remark 1. *If $s=1$, then the inequality (5) from Theorem 6 will be*

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\;{}^dd_q\,t|$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q^2[4]_q[5]_q}[[3]_q|{}^dD_q^3k(c)|+\left([5]_q-q[3]_q\right)|{}^dD_q^3k(d)|]. \tag{6}$$

Theorem 7. *Let $k:[c,d]\to\mathbb{R}$ be a three times q^d-differentiable function on (c,d) and ${}^dD_q^3k\in C[c,d]$ and integrable on $[c,d]$, $c\geq 0$. If $|{}^dD_q^3k|^r$ is s-convex in the first sense on $[c,d]$, $r\geq 1$, then the following inequality is satisfied:*

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\;{}^dd_q\,t| \tag{7}$$

$$\leq\frac{1}{[2]_q[3]_q^{2-\frac{1}{r}}[4]_q^{1-\frac{1}{r}}}\times[|{}^dD_q^3k(c)|^r\frac{[s+4]_q-q[s+3]_q}{[s+3]_q[s+4]_q}$$

$$+|{}^dD_q^3k(d)|^r(\frac{1}{[3]_q[4]_q}-\frac{[s+4]_q-q[s+3]_q}{[s+3]_q[s+4]_q})]^{\frac{1}{r}}.$$

Proof. Here, we will use the modulus properties in Lemma 1 and then a well-known power mean inequality, successively obtaining

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\;{}^dd_q\,t|$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\int_0^1 t^2(1-qt)\left|{}^dD_q^3k(tc+(1-t)d)\right|d_qt$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\left(\int_0^1 t^2(1-qt)d_qt\right)^{1-\frac{1}{r}}\times\left(\int_0^1 t^2(1-qt)\left|{}^dD_q^3k(tc+(1-t)d)\right|^rd_qt\right)^{\frac{1}{r}}.$$

Applying the s-convexity in the first sense of $|{}^dD_q^3k|^r$ on $[c,d]$, we have,

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_q\,t|$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\left(\frac{1}{[3]_q}-\frac{q}{[4]_q}\right)^{1-\frac{1}{r}}\times\left(\int_0^1 t^2(1-qt)[t^s|{}^dD_q^3k(c)|^r+(1-t^s)|{}^dD_q^3k(d)\,|^r]d_qt\right)^{\frac{1}{r}}$$

$$=\frac{(d-c)^3q^5}{[2]_q[3]_q}\left(\frac{1}{[3]_q[4]_q}\right)^{1-\frac{1}{r}}(|{}^dD_q^3k(c)|^r\int_0^1 t^{s+2}(1-qt)d_qt$$

$$+|{}^dD_q^3k(d)|^r\int_0^1 t^2(1-qt)(1-t^s)d_qt)^{\frac{1}{r}}.$$

From here, computing these two integrals, we obtain the inequality (7). □

Theorem 8. *We assume that* $k:[c,d]\to\mathbb{R}$ *is a three times* q^d*-differentiable function on* (c,d) *and* ${}^dD_q^3k\in C[c,d]$ *and integrable on* $[c,d]$, $c\geq 0$. *If* $|{}^dD_q^3k|^{p_1}$ *is s-convex in the first sense on* $[c,d]$, *when* $p_1>1$ *and* $\frac{1}{p_1}+\frac{1}{p_2}=1$, *then we have*

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_q\,t| \tag{8}$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}u_1^{\frac{1}{p_2}}\left(\frac{|{}^dD_q^3k(c)|^{p_1}}{[s+1]_q}+\left|{}^dD_q^3k(d)\right|^{p_1}(1-\frac{1}{[s+1]_q})\right)^{\frac{1}{p_1}}$$

where $u_1=(1-q)\sum_{n=0}^{\infty}(q^n)^{2p_2+1}(1-q(n+1))^{p_2}$.

Proof. Taking into account first the modulus properties, then the auxiliary Lemma 1, it is found by applying the q-Holder's inequality that

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_q\,t|$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\int_0^1 t^2(1-qt)\left|{}^dD_q^3k(tc+(1-t)d)\right|d_qt$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\left(\int_0^1\left(t^2(1-qt)\right)^{p_2}d_qt\right)^{\frac{1}{p_2}}\left(\int_0^1\left|{}^dD_q^3k(tc+(1-t)d)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}.$$

Since $|{}^dD_q^3k|^r$ is s-convex in the first sense on $[c,d]$, we obtain

$$|\frac{q^2k(d)+[2]_qk(c)}{[3]_q}+\frac{q(d-c)}{[2]_q[3]_q}.{}^dD_qk(c)-\frac{1}{d-c}\int_c^d k(t)\ {}^dd_q\,t|$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}\left(\int_0^1\left(t^2(1-qt)\right)^{p_2}d_qt\right)^{\frac{1}{p_2}}[|{}^dD_q^3k(c)|^{p_1}\int_0^1 t^sd_q\,t+|{}^dD_q^3k(d)|^{p_1}\int_0^1(1-t^s)d_qt]^{\frac{1}{p_1}}$$

$$\leq\frac{(d-c)^3q^5}{[2]_q[3]_q}u_1^{\frac{1}{p_2}}[\frac{|{}^dD_q^3k(c)|^{p_1}}{[s+1]_q}+|{}^dD_q^3k(d)|^{p_1}(1-\frac{1}{[s+1]_q})]^{\frac{1}{p_1}},$$

where

$$u_1 = \int_0^1 \left(t^2(1-qt)\right)^{p_2} d_q t = (1-q)\sum_{n=0}^{\infty} q^n \left(q^{2n}\left(1-q^{n+1}\right)\right)^{p_2}$$
$$= (1-q)\sum_{n=0}^{\infty} (q^n)^{2p_2+1}\left(1-q^{n+1}\right)^{p_2}.$$

□

Theorem 9. *Under assumptions of Theorem 8, the following inequality is satisfied*

$$\left|\frac{q^2k(d)+[2]_q k(c)}{[3]_q} + \frac{q(d-c)}{[2]_q[3]_q}.{}^dD_q k(c) - \frac{1}{d-c}\int_c^d k(t)\ {}^dd_q t\right|$$
$$\leq \frac{(d-c)^3q^5}{[2]_q[3]_q}\frac{1}{[2p_2+1]_q^{\frac{1}{p_2}}}\left(|{}^dD_q^3k(c)|^{p_1}u_2 + \left|{}^dD_q^3k(d)\right|^{p_1}u_3\right)^{\frac{1}{p_1}},$$

where

$$u_2 = (1-q)\sum_{n=0}^{\infty} q^{n(s+1)}\left(1-q^{n+1}\right)^{p_1},$$
$$u_3 = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^{n+1}\right)^{p_1}(1-q^{ns}).$$

Proof. Taking into account the modulus properties, Lemma 1 and well-known Holder's inequality, we find

$$\left|\frac{q^2k(d)+[2]_q k(c)}{[3]_q} + \frac{q(d-c)}{[2]_q[3]_q}.{}^dD_q k(c) - \frac{1}{d-c}\int_c^d k(t)\ {}^dd_q t\right|$$
$$\leq \frac{(d-c)^3q^5}{[2]_q[3]_q}\int_0^1 t^2(1-qt)\left|{}^dD_q^3k(tc+(1-t)d)\right|d_q t$$
$$\leq \frac{(d-c)^3q^5}{[2]_q[3]_q}\left(\int_0^1 t^{2p_2}d_q t\right)^{\frac{1}{p_2}}\left(\int_0^1 (1-qt)^{p_1}\left|{}^dD_q^3k(ct+(1-t)d)\right|^{p_1}d_q t\right)^{\frac{1}{p_1}}.$$

Because the function $|{}^dD_q^3 k|^{p_1}$ is s-convex in the first sense on $[c,d]$, we can write

$$\left|\frac{q^2k(d)+[2]_q k(c)}{[3]_q} + \frac{q(d-c)}{[2]_q[3]_q}.{}^dD_q k(c) - \frac{1}{d-c}\int_c^d k(t)\ {}^dd_q t\right|$$
$$\leq \frac{(d-c)^3q^5}{[2]_q[3]_q}\left(\int_0^1 t^{2p_2}d_q t\right)^{\frac{1}{p_2}}\left(|{}^dD_q^3k(c)|^{p_1}\int_0^1(1-qt)^{p_1}t^s d_q t\right.$$
$$\left.+|{}^dD_q^3k(d)|^{p_1}\int_0^1(1-qt)^{p_1}(1-t^s)d_q t\right)^{\frac{1}{p_1}}.$$

It can be seen that

$$\left|\frac{q^2k(d)+[2]_q k(c)}{[3]_q} + \frac{q(d-c)}{[2]_q[3]_q}.{}^dD_q k(c) - \frac{1}{d-c}\int_c^d k(t)\ {}^dd_q t\right|$$
$$\leq \frac{(d-c)^3q^5}{[2]_q[3]_q}\frac{1}{[2p_2+1]_q^{\frac{1}{p_2}}}\left(|{}^dD_q^3k(c)|^{p_1}u_2 + \left|{}^dD_q^3k(d)\right|^{p_1}u_3\right)^{\frac{1}{p_1}},$$

where

$$u_2 = \int_0^1 (1-qt)^{p_1} t^s d_q t = (1-q) \sum_{n=0}^{\infty} q^{n(s+1)} \left(1-q^{n+1}\right)^{p_1},$$

$$u_3 = \int_0^1 (1-qt)^{p_1} (1-t^s) d_q t = (1-q) \sum_{n=0}^{\infty} q^{n} \left(1-q^{n+1}\right)^{p_1} (1-q^{ns}).$$

□

Example 1. *We will take into account the function* $k : [0,1] \to \mathbb{R}$, $k(x) = x^5$, *and* $s = 1$. *The hypothesis of Theorem 6 is satisfied. By Definition 5, we have*

$${}^1D_q^3 k(x) = \frac{\left(q^3x+1-q^3\right)^5 - [3]_q\left(q^2x+1-q^2\right)^5 + q[3]_q(qx+1-q)^5 - q^3x^5}{(1-q)^3(1-x)^3q^3},$$

and from here, the following will be obtained

$${}^1D_q^3 k(0) = q^9+3q^8+6q^7+4q^6-4q^5-14q^4-11q^3+q^2+8q+6\,{}^1D_q^3 k(1) = 10[2]_q[3]_q.$$

Therefore, the right member of (6) is

$$\frac{q^5}{[2]_q[3]_q[4]_q[5]_q}[|{}^1D_q^3 k(0)|+q^3(1+q)|{}^1D_q^3 k(1)|],$$

or

$$\frac{q^5}{[2]_q[3]_q[4]_q[5]_q}\left(q^9+3q^8+6q^7+4q^6-4q^5-14q^4-11q^3+q^2+8q+6+10q^3[2]_q^2\right).$$

On the other hand, according to Definitions 5 and 7, the left member of (6) is

$$|\frac{q^2}{[3]_q} + \frac{q}{[2]_q[3]_q}{}^1D_q k(0) - \int_0^1 t^5\, {}^1d_q t|,$$

or by calculus,

$$|\frac{q\left[q[2]_q+(1-q)^4\right]}{[2]_q[3]_q} - 1 + \frac{5}{[2]_q} - \frac{10}{[3]_q} + \frac{10}{[4]_q} - \frac{5}{[5]_q} + \frac{1}{[6]_q}|.$$

The validity of (5) can be observed in Figure 1. The red line in the graphic is the left member and the blue line in the graphic is the right member of (5) from Theorem 6.

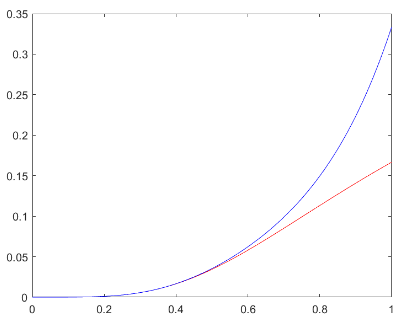

Figure 1. Example for the inequality (6) from Theorem 6.

3.2. Quantum Integral Inequalities for (α, m) Convex Functions

This second result is the main result for the last theorem of this subsection, and is a refinement of the parametric identity given in Lemma 2 from [28] when we consider a second parameter m.

Lemma 2. *Let $k : [cm, d] \to \mathbb{R}$ be a twice q-differentiable function on (cm, d) and $0 < c < d$, $\frac{c}{d} < m \leq 1$. Moreover, if the function, ${}_{mc}D_q^2 k$ and ${}^{md}D_q^2 k$ are continuous and q-integrable over $[cm, d]$, then we have:*

$$ {}_c^d M_q(\lambda, m) = \frac{q^3}{1+q}[(md-c)^2 \int_0^1 t^2 \, {}^{md}D_q^2 k(\lambda tc + m(1-\lambda t)d) d_q t $$

$$ +(d-mc)^2 \int_0^1 t^2 \, {}_{mc}D_q^2 k(\lambda td + m(1-\lambda t)c) d_q t], \tag{9} $$

where

$$ {}_c^d M_q(\lambda, m) = \frac{1}{\lambda^3}[\frac{1}{md-c} \int_{\lambda c + m(1-\lambda)d}^{md} k(t)^{md} d_q t + \frac{1}{d-mc} \int_{mc}^{\lambda d + mc(1-\lambda))} k(t)_{mc} d_q t] - $$

$$ -\frac{1-q-q^2}{(1+q)(1-q)\lambda^2}[k(\lambda c + m(1-\lambda)d) + k(\lambda d + m(1-\lambda)c)] - $$

$$ -\frac{q}{(1+q)(1-q)\lambda^2}[k(\lambda qc + m(1-q\lambda)d) + k(\lambda qd + m(1-q\lambda)c)]. $$

Proof. The demonstration is similar to the proof of Lemma 1. We consider I_1 as the expression $\int_0^1 t^{2md} D_q^2 k(\lambda tc + m(1-\lambda t)d) d_q t$ and I_2 is the expression $\int_0^1 t^2{}_{mc} D_q^2 k(\lambda td + mc(1-\lambda t)) d_q t$, we have ${}_c^d M_q(\lambda, m) = \frac{q^3}{1+q}\left[(md-c)^2 I_1 + (d-mc)^2 I_2\right]$.

By using Definition 5 of the right quantum derivative of k, we will obtain

$$ I_1 = \int_0^1 t^2 \, {}^{md}D_q^2 k(\lambda tc + m(1-\lambda t)d) d_q t $$

$$
\begin{aligned}
&= \int_0^1 \frac{1}{(1-q)^2\lambda^2(md-c)^2 q}\Big[k\big(\lambda tq^2 c + md\big(1-\lambda tq^2\big)\big) - [2]_q k(\lambda tqc + md(1-\lambda tq) \\
&+ qk(\lambda tc + m(1-\lambda t)d)] d_q t \\
&= \frac{1}{(1-q)(md-c)^2\lambda^2 q}[\sum_{n=0}^{\infty} q^n k\big(\lambda q^{n+2} c + md\big(1-\lambda q^{n+2}\big)\big) \\
&- [2]_q \sum_{n=0}^{\infty} q^n k\big(\lambda q^{n+1} c + md\big(1-\lambda q^{n+1}\big)\big) + q\sum_{n=0}^{\infty} q^n k(\lambda q^n c + md(1-\lambda q^n))] \\
&= \frac{1}{(1-q)(md-c)^2\lambda^2 q}[\frac{1}{q^2}(\sum_{n=0}^{\infty} q^n k(\lambda q^n c + md(1-\lambda q^n)) - k(\lambda c + m(1-\lambda)d) \\
&- qk(m(1-\lambda q)d + \lambda qc)) \\
&- [2]_q \frac{\sum_{n=0}^{\infty} q^n k(\lambda q^n c + m(1-\lambda q^n)d) - k(md(1-\lambda) + \lambda c)}{q} \\
&+ q\sum_{n=0}^{\infty} q^n k(\lambda q^n c + md(1-\lambda q^n))].
\end{aligned}
$$

Utilizing Definition 7, of the right q-integral of k, and using calculus, we have,

$$ I_1 = \frac{1+q}{(md-c)^3\lambda^3 q^3} \int_{\lambda c + m(1-\lambda)d}^{md} k(t) \, {}^{md}d_q t - \frac{1-q-q^2}{(1-q)(md-c)^2 q^3 \lambda^2} k(md(1-\lambda) + \lambda c) - $$

$$ -\frac{1}{(1-q)(md-c)^2 q^2 \lambda^2} k(md(1-\lambda q) + \lambda qc). $$

From Definition 6 of the left quantum derivative of k, we have

$$I_2 = \int_0^1 t^2 {}_{mc}D_q^2 k(mc(1-\lambda t)+\lambda td)d_q t$$

$$= \int_0^1 \frac{1}{q(1-q)^2\lambda^2(d-mc)^2}\Big[k\Big(q^2\lambda td + mc\Big(1-q^2\lambda t\Big)\Big) - [2]_q k(q\lambda td + mc(1-q\lambda t)$$

$$+qk(\lambda td + mc(1-\lambda t))]d_q t$$

$$= \frac{1}{(1-q)(d-mc)^2\lambda^2 q}[\sum_{n=0}^{\infty} q^n k\Big(\lambda q^{n+2}d + m\Big(1-\lambda q^{n+2}\Big)c\Big)$$

$$-[2]_q \sum_{n=0}^{\infty} q^n k\Big(\lambda q^{n+1}d + m\Big(1-\lambda q^{n+1}\Big)c\Big) + q\sum_{n=0}^{\infty} q^n k(q^n\lambda d + mc(1-\lambda q^n))]$$

$$= \frac{1}{(1-q)(d-mc)^2\lambda^2 q}[\frac{1}{q^2}(\sum_{n=0}^{\infty} q^n k(q^n\lambda d + mc(1-q^n\lambda)) - k(\lambda d + m(1-\lambda)c)$$

$$-qk(\lambda qd + mc(1-\lambda q))) - [2]_q \frac{\sum_{n=0}^{\infty} q^n k(\lambda q^n d + m(1-\lambda q^n)c) - k(\lambda d + m(1-\lambda)c)}{q}$$

$$+q\sum_{n=0}^{\infty} q^n k(\lambda q^n d + m(1-\lambda q^n)c)].$$

Utilizing Definition 8, of the left quantum integral of k, and then using calculus, we get

$$I_2 = \frac{1+q}{(d-mc)^3\lambda^3 q^3}\int_{mc}^{\lambda d+m(1-\lambda)c} k(t)\ {}_{mc}d_q t - \frac{1-q-q^2}{(1-q)(d-mc)^2\lambda^2 q^3}k(\lambda d + m(1-\lambda)c)$$

$$-\frac{1}{(d-mc)^2(1-q)\lambda^2 q^2}k(q\lambda d + mc(1-q\lambda)).$$

Then, multiplying the expression $(md-c)^2 I_1 + (d-mc)^2 I_2$ by $\frac{q^3}{1+q}$, it follows that

$$\frac{q^3}{1+q}\Big[(dm-c)^2 I_1 + (d-mc)^2 I_2\Big] = \frac{1}{\lambda^3}[\frac{1}{(dm-c)}\int_{\lambda c+m(1-\lambda)d}^{md} k(t)\ {}^{md}d_q t$$

$$\begin{aligned} &+ \tfrac{1}{d-mc}\textstyle\int_{mc}^{\lambda d+m(1-\lambda)c} k(t)\ {}_{mc}d_q\, t] \\ &- \tfrac{1-q-q^2}{(1-q^2)\lambda^2}[k(\lambda c + md(1-\lambda)) + k(\lambda d + mc(1-\lambda))] \\ &- \tfrac{q}{(1-q^2)\lambda^2}[k(\lambda qc + md(1-\lambda q)) + k(\lambda qd + mc(1-\lambda q))], \end{aligned}$$

which completes the proof. □

Remark 2. *Lemma 1 from [28] is a consequence of Lemma 2 when the parameter $m = 1$.*

Theorem 10. *Under conditions of Lemma 2, if $|{}_{mc}D_q^2 k|$ and $|{}^{md}D_q^2 k|$ are $(\alpha,\ m)$ convex functions on $[cm, d]$, then the following will be obtained*

$$\begin{aligned} |{}_c^d M_q(\lambda, m)| \quad &\le \quad \tfrac{q^3}{q+1}\{(dm-c)^2[\tfrac{\lambda^\alpha}{[3+\alpha]_q}|{}^{md}D_q^2 k(c)| + m|{}^{md}D_q^2 k(d)|(\tfrac{1}{[3]_q} - \tfrac{\lambda^\alpha}{[3+\alpha]_q})] \\ &+ \quad (d-mc)^2[\tfrac{\lambda^\alpha}{[\alpha+3]_q}|{}_{mc}D_q^2 k(d)| + m|{}_{mc}D_q^2 k(c)|(\tfrac{1}{[3]_q} - \tfrac{\lambda^\alpha}{[\alpha+3]_q})]\}. \end{aligned}$$

Proof. The properties of the modulus with the (α, m) convexity of $|_{mc}D_q^2 k|$ and $|^{md}D_q^2 k|$ will help us to prove the following inequalities:

$$\begin{aligned} |_c^d M_q(\lambda, m)| \leq & \frac{q^3}{[2]_q}[(dm-c)^2 \int_0^1 t^2 \left|^{md}D_q^2 k(m(1-\lambda t)d + \lambda tc)\right| d_q t \\ + & (d-mc)^2 \int_0^1 t^2 |_{mc}D_q^2 k(m(1-\lambda t)c + \lambda td)| d_q t] \\ \leq & \frac{q^3}{[2]_q}\{(dm-c)^2 \int_0^1 t^2 [t^\alpha \lambda^\alpha |^{md}D_q^2 k(c)| + m(1-\lambda^\alpha t^\alpha)|^{md}D_q^2 k(d)|] d_q t \\ + & (d-mc)^2 \int_0^1 t^2 [t^\alpha \lambda^\alpha |_{mc}D_q^2 k(d)| + m(1-\lambda^\alpha t^\alpha)|_{mc}D_q^2 k(c)|] d_q t\} \\ \leq & \frac{q^3}{[2]_q}\{(dm-c)^2 [\frac{\lambda^\alpha}{[\alpha+3]_q}|^{md}D_q^2 k(c)| + m(\frac{1}{[3]_q} - \frac{\lambda^\alpha}{[\alpha+3]_q})|^{md}D_q^2 k(d)|] \\ + & (d-mc)^2 [\frac{\lambda^\alpha}{[\alpha+3]_q}|_{mc}D_q^2 k(d)| + m(\frac{1}{[3]_q} - \frac{\lambda^\alpha}{[\alpha+3]_q})|_{mc}D_q^2 k(c)|]\} \end{aligned}$$

which completes the proof. □

Remark 3. *It could be given some similar applications to special means as arithmetic mean, harmonic mean, and geometric mean of real numbers by using the established results as in [26].*

4. Discussion and Conclusions

The main findings of this work are intended to prove novel parametrized q-Hermite–Hadamard-like type integral inequalities for mappings whose third left and right q-derivative in absolute value have s-convex and (α, m)-convex, respectively. The main inequalities, quantum Holder's integral inequality and quantum power mean inequality, have been utilized to obtain the new estimated bounds. Two auxiliary quantum lemmas have been used as a basic tool in our proofs in the two subsections of Section 3. Several consequences appear for special choice of λ as a parameter in the first subsection of Section 3, and the corresponding example was analyzed and discussed in detail to illustrate the obtained results, in order to prove the consistency of the conclusions. The obtained results could be useful in certain optimization studies. Convex functions have applications in fields of applied sciences such as numerical analysis, approximation theory, optimization and statistics, which can be useful tools in economy and business.

We used the Matlab R2023a software version, as a tool, for the figure and for some calculus in the given example. The researchers can establish similar inequalities for coordinated convex functions in the future.

We hope that the results will continue to sharpen our understanding of the nature of q-calculus, and as a further study, it could be interesting to extend these findings to recent newly defined kinds of convexities, to (p, q)-calculus, and to q-fractional calculus, which would be several good generalizations.

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