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# Sampling Theorems Associated with Offset Linear Canonical Transform by Polar Coordinates

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**Abstract:** The sampling theorem for the offset linear canonical transform (OLCT) of bandlimited functions in polar coordinates is an important signal analysis tool in many fields of signal processing and optics. This paper investigates two sampling theorems for interpolating bandlimited and highest frequency bandlimited functions in the OLCT and offset linear canonical Hankel transform (OLCHT) domains by polar coordinates. Based on the classical Stark's interpolation formulas, we derive the sampling theorems for bandlimited functions in the OLCT and OLCHT domains, respectively. The first interpolation formula is concise and applicable. Due to the consistency of the OLCHT order, the second interpolation formula is superior to the first interpolation formula in computational complexity.

**Keywords:** offset linear canonical transform; offset linear canonical Hankel transform; sampling theorems; polar coordinates

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## 1. Introduction

The offset linear canonical transform (OLCT) is a time-shifted and frequency-modulated version of the linear canonical transform (LCT) and is a powerful tool in signal processing and optics [1–3]. OLCT, also known as special affine Fourier transform (FT) [4] or inhomogeneous regular transform [5], is a class of linear integral transforms with six parameters $(a, b, c, d, \tau, \eta)$. Because it adds two parameters of time shift and frequency modulation on the basis of the LCT [6–8], OLCT has greater degrees of freedom and flexibility in applications such as sampling and time-frequency analysis [3,9]. Therefore, an in-depth study of the theoretical problems of the OLCT, such as sampling and filtering [10–13], can further enrich the linear canonical theory and even the theoretical system of signal processing based on linear transformation.

The sampling theorem converts analog signals into digital signals [14], which plays a fundamental role in signal processing. In recent years, the sampling theorem of angular periodic functions in polar coordinates has had a wide range of application prospects in the fields of computed tomography (CT) and magnetic resonance imaging (MRI) and has attracted much attention from scholars [15–18]. According to the existing research results, a large number of interpolation formulas for angular periodic functions with different bandwidth constraints have appeared in the literature [19–23] from samples on uniform or nonuniform polar lattices. The most famous of these is Stark's work [20], which derived the uniform sampling theorem for bandlimited functions in the two-dimensional FT and Hankel transform (HT) domains in polar coordinates. Generalizations on the basis of this result and [24] provide the azimuthal jitter sampling theorem for bandlimited functions in the FT and HT domains.

Due to wider applicability, people extend the sampling theorem from the traditional FT to the LCT. Thus far, the sampling theory of the LCT in polar coordinates has been well developed [25–27]. Note that the interpolation formulations can only produce perfect reconstructions if $f(r,\theta)$ is bandlimited in the LCT or LCHT domain. However, in practical situations, especially in medical diagnosis, most of $f(r,\theta)$ are non-bandlimited functions in the LCT or linear canonical Hankel transform (LCHT) domain. Due to the higher degree of freedom of the OLCT, the above functions can be bandwidth-limited in the OLCT and OLCHT domains. Therefore, it is more efficient to explore the generalization of the sampling theorem in the LCT and LCHT domains in polar coordinates than the OLCT and OLCHT domains, respectively. The high theoretical value is also a supplement and improvement to the linear canonical theoretical system.

For the above reasons, the sampling theory of the OLCT in polar coordinates is a challenging problem, and more rigorous mathematical logic is required to develop this theory. As the OLCT has been applied in polar coordinates for a relatively short period of time, the theoretical system based on OLCT is not yet perfect, and its sampling and other related theories require further investigation. Therefore, the purpose of this paper is to study two kinds of sampling theorems for interpolating angular periodic functions and the highest frequency bandlimited functions with different bandwidth constraints at the radius and azimuth in the OLCT and OLCHT domains by polar coordinates. The main mathematical idea is to first interpolate the bandwidth-limited radius of the function in the OLCT or OLCHT domain, and then interpolate within the function's bandwidth-limited range to the highest frequency. Due to the consistency of the order of the OLCHT, the interpolation formula in the OLCHT domain is superior to the interpolation formula in the OLCT domain in terms of computational complexity.

The paper is organized as follows. Section 2 presents our previous research work on polar coordinates. Section 3 gives the definitions of bandlimited functions in the OLCT or OLCHT domain and the related results. Section 4 derives the sampling theorem based on bandlimited functions for the OLCT in polar coordinates. Section 5 derives the sampling theorem based on bandlimited functions in the OLCHT domain. Section 6 discusses the potential application of sampling theorems for the OLCT and OLCHT. Section 7 draws conclusions.

## 2. Preliminaries

In a recent work [28], we introduced knowledge related to the OLCT and OLCHT in polar coordinates. In order to facilitate an in-depth study of the integral transformation of the OLCT, we provide some mathematical definitions in polar coordinates.

### *2.1. Offset Linear Canonical Transform in Polar Coordinates*

**Assumption 1.** *Suppose function $f(r,\theta)$ satisfies the Dirichlet condition, is angularly periodic in $2\pi$, and has a Fourier series expansion*

$$f(r,\theta) = \sum_{n=-\infty}^{+\infty} f_n(r)e^{in\theta}. \tag{1}$$

**Definition 1.** *The two-dimensional FT of function $f(r,\theta)$ in polar coordinates is defined by* [20,24,29]

$$F(\rho,\phi) = \mathcal{F}[f](\rho,\phi) = \frac{1}{2\pi}\int_0^{+\infty}\int_{-\pi}^{\pi} f(r,\theta)e^{-ir\rho\cos(\theta-\phi)}r\mathrm{d}r\mathrm{d}\theta. \tag{2}$$

**Definition 2.** *Let parameters $A = (a,b;c,d)$, $\boldsymbol{\tau} = (\tau_1,\tau_2)$, and $\boldsymbol{\eta} = (\eta_1,\eta_2)$ satisfy $a$, $b$, $c$, $d$, $\tau_1$, $\tau_2$, $\eta_1$, $\eta_2 \in \mathbb{R}$ and $\det(A) = 1$. The OLCT of parameters $A$, $\boldsymbol{\tau}$, and $\boldsymbol{\eta}$ of $f(r,\theta)$ in polar coordinates is defined by* [28]

$$F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = O_L^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f](\rho,\phi) = \int_0^{+\infty}\int_{-\pi}^{\pi} f(r,\theta)P_{A,\boldsymbol{\tau},\boldsymbol{\eta}}(r,\theta;\rho,\phi)r\mathrm{d}r\mathrm{d}\theta, \tag{3}$$

*where*

$$P_{A,\boldsymbol{\tau},\boldsymbol{\eta}} = K_{A,\boldsymbol{\tau},\boldsymbol{\eta}}\, e^{i\left[\frac{a}{2b}r^2 - \frac{r\rho}{b}\cos(\theta-\phi) + \frac{d}{2b}\rho^2 + \frac{r|\boldsymbol{\tau}|}{b}\sin(\theta+\varphi_1) - \frac{\rho|d\boldsymbol{\tau}-b\boldsymbol{\eta}|}{b}\sin(\phi+\varphi_2)\right]}, \tag{4}$$

*is the kernel function, and*

$$K_{A,\boldsymbol{\tau},\boldsymbol{\eta}} = \frac{1}{2\pi b} e^{i\frac{d|\boldsymbol{\tau}|^2}{b}}, \tag{5}$$

*where* $|\boldsymbol{\tau}|^2 = \tau_1^2 + \tau_2^2$, $|\boldsymbol{\eta}|^2 = \eta_1^2 + \eta_2^2$, $\tan\varphi_1 = \frac{\tau_1}{\tau_2}$, $\tan\varphi_2 = \frac{d\tau_1 - b\eta_1}{d\tau_2 - b\eta_2}$, $\tau_2 \neq 0$, *and* $d\tau_2 - b\eta_2 \neq 0$. *If* $A = (0,1;-1,0)$, $\boldsymbol{\tau} = \mathbf{0}$, *and* $\boldsymbol{\eta} = \mathbf{0}$, *the OLCT reduces to the FT in polar coordinates.*

It is easy to know that if $b = 0$, the OLCT of the signal reduces to a time-scaled version multiplied by a linear chirp [1]. Without loss of generality, we assume $b > 0$ in the following sections.

**Remark 1.** *It follows that there is a relation between the FT and OLCT in polar coordinates*

$$F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = \frac{\ell_1}{b} e^{i\left[\frac{d}{2b}\rho^2 - \frac{\rho|d\boldsymbol{\tau}-b\boldsymbol{\eta}|}{b}\sin(\phi+\varphi_2)\right]} F[\tilde{f}]\left(\frac{\rho}{b},\phi\right), \tag{6}$$

*where* $\tilde{f}(r,\theta) = e^{i\left[\frac{a}{2b}r^2 + \frac{r|\boldsymbol{\tau}|}{b}\sin(\theta+\varphi_1)\right]} f(r,\theta)$, $\varphi_1$ *and* $\varphi_2$ *are given by (4), and*

$$\ell_1 = e^{i\frac{d|\boldsymbol{\tau}|^2}{b}}. \tag{7}$$

The inversion formula of the OLCT with parameters $A^{-1}$, $\boldsymbol{\xi}$, and $\boldsymbol{\gamma}$ in polar coordinates takes

$$f(r,\theta) = O_L^{A^{-1},\boldsymbol{\xi},\boldsymbol{\gamma}}[F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}](r,\theta), \tag{8}$$

where $A^{-1} = (d,-b;-c,a)$, $\boldsymbol{\xi} = b\boldsymbol{\eta} - d\boldsymbol{\tau}$, and $\boldsymbol{\gamma} = c\boldsymbol{\tau} - a\boldsymbol{\eta}$.

### 2.2. Offset Linear Canonical Hankel Transform in Polar Coordinates

**Definition 3.** *The vth-order Hankel transform (HT) of* $f(r)$ *in polar coordinates is defined by* [20,24,29]

$$H_v[f](\rho) = \int_0^{+\infty} f(r) J_v(\rho r) r\mathrm{d}r, \tag{9}$$

*where* $J_v$ *is the vth-order Bessel function of the first kind, and the corresponding inversion formula is*

$$f(r) = H_v[H_v[f]](r) = \int_0^{+\infty} H_v[f](\rho) J_v(r\rho)\rho\mathrm{d}\rho. \tag{10}$$

**Definition 4.** *The vth-order OLCHT of* $f(r)$ *with the parameters matrix* $A$, $\boldsymbol{\tau}$, *and* $\boldsymbol{\eta}$ *of* $f(r)$ *in polar coordinates is defined by* [28]

$$H_v^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f](\rho) = \frac{i^v \ell_1 e^{im(\varphi_1-\varphi_2)}}{b} \lambda_1 e^{i\frac{d}{2b}\rho^2} \int_0^{+\infty} \lambda_2 e^{i\frac{a}{2b}r^2} f(r) J_v\left(\frac{r\rho}{b}\right) r\mathrm{d}r, \tag{11}$$

*where* $J_v$ *is the vth-order Bessel function of the first kind and order* $v \geq -\frac{1}{2}$, $\varphi_1$ *and* $\varphi_2$ *are given by (4),* $\ell_1$ *is given by (7), and*

$$\lambda_1 = \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\rho|d\boldsymbol{\tau}-b\boldsymbol{\eta}|}{b}\right), \quad \lambda_2 = \sum_{m=-\infty}^{+\infty} J_m\left(\frac{r|\boldsymbol{\tau}|}{b}\right), \tag{12}$$

*where* $|\boldsymbol{\tau}|^2 = \tau_1^2 + \tau_2^2$ *and* $|\boldsymbol{\eta}|^2 = \eta_1^2 + \eta_2^2$*. If* $A = (0, 1; -1, 0)$*,* $\boldsymbol{\tau} = \mathbf{0}$*, and* $\boldsymbol{\eta} = \mathbf{0}$*, the OLCHT reduces to the HT.*

**Remark 2.** *The relationship between the HT and OLCHT is as follows*

$$H_v^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f](\rho) = \frac{i^v \lambda_1 \ell_1 e^{im(\varphi_1 - \varphi_2)}}{b} e^{i\frac{d}{2b}\rho^2} H_v[\tilde{f}]\left(\frac{\rho}{b}\right), \tag{13}$$

*where* $\varphi_1$ *and* $\varphi_2$ *are given by (4),* $\ell_1$ *is given by (7),* $\lambda_1$ *and* $\lambda_2$ *are given by (12), and*

$$\tilde{f}(r) = \lambda_2 e^{i\frac{a}{2b}r^2} f(r). \tag{14}$$

The inversion formula of $v$th-order OLCHT with parameters $A$, $\boldsymbol{\tau}$, and $\boldsymbol{\eta}$ in polar coordinates takes

$$\begin{aligned} f(r) &= H_v^{-A^{-1},-\xi,-\gamma}\left[H_v^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f]\right](r) \\ &= i^v \frac{\ell_2 e^{im(\varphi_2 - \varphi_1)}}{b} \lambda_2 e^{-i\frac{a}{2b}r^2} \int_0^{+\infty} \lambda_1 e^{-i\frac{d}{2b}\rho^2} H_v^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f](\rho) J_v\left(\frac{\rho r}{b}\right)\rho \mathrm{d}\rho, \end{aligned} \tag{15}$$

where $\lambda_1$ and $\lambda_2$ are given by (12), and $\ell_2 = e^{-i\frac{a|b\boldsymbol{\eta} - d\boldsymbol{\tau}|^2}{b}}$.

## 3. Bandlimited Functions in the OLCT and OLCHT Domains

Based on the above basic mathematical knowledge, we next study $\Omega$ bandlimited functions $f(r,\theta)$ and related conclusions in the OLCT and OLCHT domains.

### *3.1. Relationship between the OLCT and OLCHT in Polar Coordinates*

To facilitate the proof of the sampling theorem below, we give the definitions of $\Omega_{FT}$ bandlimited functions $f(r,\theta)$ in the FT domain [20,24] .

**Definition 5.** *Let* $f(r,\theta)$ *satisfy Assumption 1, then it is* $\Omega_{FT}-$*bandlimited in the FT domain to the highest frequency* $\omega_m = \frac{K}{2\pi}$ *if its Fourier expansion takes* [20,24]

$$f(r,\theta) = \sum_{n=-K}^{K} f_n(r) e^{in\theta}. \tag{16}$$

**Definition 6.** *Let* $f(r,\theta)$ *satisfy Assumption 1 and* $b > 0$*, then it is* $\Omega-$*bandlimited in the OLCT domain with the parameters* $A$*,* $\boldsymbol{\tau}$*, and* $\boldsymbol{\eta}$*, if* $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = 0$ *for* $\rho \geq \Omega$*, where* $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi)$ *is the OLCT of* $f(r,\theta)$ *in polar coordinates.*

**Definition 7.** *Let* $f(r) \in L^2(\mathbb{R})$ *and* $b > 0$*.* $f(r)$ *is* $\Omega-$*bandlimited isotropic function in the OLCHT domain with the parameters* $A$*,* $\boldsymbol{\tau}$*, and* $\boldsymbol{\eta}$*, if* $H_v^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f](\rho) = 0$ *for* $\rho \geq \Omega$*, where* $H_v^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f](\rho)$ *is the* $v$*th-order OLCHT of* $f(r)$ *in polar coordinates.*

**Lemma 1.** *Let* $f(r,\theta)$ *satisfy Assumption 1 and* $b > 0$*. Then, the Fourier series expansion of the OLCT of* $f(r,\theta)$ *has a form*

$$F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = \sum_{n=-\infty}^{+\infty} H_{2n}^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f_n](\rho) e^{in\phi}, \tag{17}$$

*where* $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi)$ *is the OLCT of* $f(r,\theta)$ *in polar coordinates,* $H_{2n}^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f](\rho)$ *is the* $2n$*th-order OLCHT of* $f(r)$*.*

**Proof.** By (3) and (16), we obtain

$$\begin{aligned} F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) &= \sum_{n=-\infty}^{+\infty} f_n(r)e^{in\theta}\int_0^{+\infty}\int_{-\pi}^{\pi} P_{A,\boldsymbol{\tau},\boldsymbol{\eta}}(r,\theta;\rho,\phi)r\mathrm{d}r\mathrm{d}\theta \\ &= \sum_{n=-\infty}^{+\infty} e^{in\theta}\int_0^{+\infty}\int_{-\pi}^{\pi} \frac{\ell_1}{2\pi b}e^{i\left[\frac{a}{2b}r^2-\frac{r\rho}{b}\cos(\theta-\phi)+\frac{d}{2b}\rho^2\right]} \\ &\times e^{i\left[\frac{r|\boldsymbol{\tau}|}{b}\sin(\theta+\varphi_1)-\frac{\rho|d\boldsymbol{\tau}-b\boldsymbol{\eta}|}{b}\sin(\phi+\varphi_2)\right]} f_n(r)r\mathrm{d}r\mathrm{d}\theta, \end{aligned} \tag{18}$$

In view of the exponent expansion formula [30] (p. 973), we obtain

$$e^{i\frac{r|\boldsymbol{\tau}|}{b}\sin(\theta+\varphi_1)} = \sum_{m=-\infty}^{+\infty} J_m\left(\frac{r|\boldsymbol{\tau}|}{b}\right)e^{im(\theta+\varphi_1)}, \tag{19}$$

$$e^{-i\frac{\rho|d\boldsymbol{\tau}-b\boldsymbol{\eta}|}{b}\sin(\phi+\varphi_2)} = \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\rho|d\boldsymbol{\tau}-b\boldsymbol{\eta}|}{b}\right)e^{-im(\phi+\varphi_2)}. \tag{20}$$

From (19) and (20), we have

$$\begin{aligned} F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) &= \sum_{n=-\infty}^{+\infty} e^{in\theta}\int_0^{+\infty}\int_{-\pi}^{\pi} \frac{\ell_1}{2\pi b}e^{i\left[\frac{a}{2b}r^2-\frac{r\rho}{b}\cos(\theta-\phi)+\frac{d}{2b}\rho^2\right]} f_n(r)r\mathrm{d}r\mathrm{d}\theta \\ &\times \lambda_1\lambda_2\, e^{im(\theta-\phi)}e^{im(\varphi_1-\varphi_2)} \\ &= \sum_{n=-\infty}^{+\infty} e^{in\theta}\int_0^{+\infty}\int_{-\pi}^{\pi} \frac{\ell_1}{2\pi b}e^{i\left[\frac{a}{2b}r^2-\frac{r\rho}{b}\sin(\frac{\pi}{2}+\theta-\phi)+\frac{d}{2b}\rho^2\right]} f_n(r)\mathrm{d}r\mathrm{d}\theta \\ &\times \lambda_1\lambda_2\, e^{im(\theta-\phi)}e^{im(\varphi_1-\varphi_2)}, \end{aligned} \tag{21}$$

where

$$\lambda_1 = \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\rho|d\boldsymbol{\tau}-b\boldsymbol{\eta}|}{b}\right), \quad \lambda_2 = \sum_{m=-\infty}^{+\infty} J_m\left(\frac{r|\boldsymbol{\tau}|}{b}\right). \tag{22}$$

It follows from a celebrated formula [27]

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i(n\theta-z\sin\theta)}\mathrm{d}\theta, \quad e^{i\pi} = -1, \tag{23}$$

that

$$\begin{aligned} F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) &= \sum_{n=-\infty}^{+\infty} e^{in\theta}\int_0^{+\infty}\int_{-\pi}^{\pi} \frac{\ell_1}{2\pi b}e^{i\left[\frac{a}{2b}r^2-\frac{r\rho}{b}\sin(\frac{\pi}{2}+\theta-\phi)+\frac{d}{2b}\rho^2\right]} f_n(r)r\mathrm{d}r\mathrm{d}\theta \\ &\times \lambda_1\lambda_2\, e^{im(\theta-\phi)}e^{im(\varphi_1-\varphi_2)} \\ &= \sum_{n=-\infty}^{+\infty} \int_0^{+\infty} \frac{\ell_1}{b}e^{i\left[\frac{a}{2b}r^2+\frac{d}{2b}\rho^2\right]}\lambda_1\lambda_2\, e^{in\theta+im(\theta-\phi)}e^{im(\varphi_1-\varphi_2)} \\ &\times \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\left[(m+n)(\frac{\pi}{2}+\theta-\phi)-\frac{r\rho}{b}\sin(\frac{\pi}{2}+\theta-\phi)\right]}\mathrm{d}\theta\right\} f_n(r)r\mathrm{d}r \\ &\times e^{-i\left[(m+n)(\frac{\pi}{2}+\theta-\phi)\right]} \\ &= \sum_{n=-\infty}^{+\infty} \int_0^{+\infty} \frac{\ell_1}{b}e^{i\left[\frac{a}{2b}r^2+\frac{d}{2b}\rho^2\right]}\lambda_1\lambda_2\, J_{m+n}(\frac{r\rho}{b}) f_n(r)r\mathrm{d}r \\ &\times e^{im(\varphi_1-\varphi_2)-i\frac{\pi}{2}(m+n)+in\phi} \\ &= \sum_{n=-\infty}^{+\infty} \frac{i^{m+n}\ell_1 e^{im(\varphi_1-\varphi_2)}}{b}\lambda_1 e^{i\frac{d}{2b}\rho^2}\int_0^{+\infty}\lambda_2 e^{i\frac{a}{2b}r^2} f_n(r)J_{m+n}(\frac{r\rho}{b})e^{in\phi}r\mathrm{d}r. \end{aligned} \tag{24}$$

From (11), we have

$$F^{A,\tau,\eta}(\rho,\phi) = \sum_{n=-\infty}^{+\infty} H_{m+n}^{A,\tau,\eta}[f_n](\rho)e^{in\phi}. \tag{25}$$

Let $m = n$, and we obtain

$$F^{A,\tau,\eta}(\rho,\phi) = \sum_{n=-\infty}^{+\infty} H_{2n}^{A,\tau,\eta}[f_n](\rho)e^{in\phi}. \tag{26}$$

which completes the proof. □

**Remark 3.** *Lemma 1 summarizes that the nth coefficient of the Fourier series of the OLCT of $f(r,\theta)$ is the 2nth order OLCHT of the nth coefficient of the Fourier series of $f(r,\theta)$. If $A = (a,b;c,d) \in \mathbb{R}^{2\times 2}$, $\boldsymbol{\tau} = \mathbf{0}$, and $\boldsymbol{\eta} = \mathbf{0}$, the Lemma 1 degenerates into the relation of the LCT* [27].

**Remark 4.** *When $n \to \infty$, (17) in Lemma 1 and Theorem 1 of* [28] *are essentially the same.*

**Lemma 2.** *Let $f(r,\theta)$ be $\Omega-$bandlimited in the OLCT domain with parameters $A$, $\boldsymbol{\tau}$, and $\boldsymbol{\eta}$ satisfying Assumption 1 and $b > 0$. Then all of the coefficients of the Fourier series of the OLCT $F^{A,\tau,\eta}(\rho)$ are zero outside a circle of radius $\rho = \Omega$, i.e.,*

$$H_n^{A,\tau,\eta}[f_n](\rho) = 0, \quad \text{for} \quad \rho \geq \Omega, \tag{27}$$

*where $n = 0, \pm 1, \pm 2, \cdots$.*

**Proof.** From (17) in Lemma 1, and if $F^{A,\tau,\eta}(\rho,\phi)$ is a periodic function of $\phi$, we can use the Parseval formula [31]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F^{A,\tau,\eta}(\rho,\phi)|^2 \mathrm{d}\phi = \sum_{n=-\infty}^{+\infty} |H_{2n}^{A,\tau,\eta}[f_n](\rho)|^2. \tag{28}$$

However, if $F^{A,\tau,\eta}(\rho,\phi) = 0$ for $\rho \geq \Omega$, then the left-hand side of (28) gives

$$\sum_{n=-\infty}^{+\infty} |H_{2n}^{A,\tau,\eta}[f_n](\rho)|^2 = 0, \quad \text{for} \quad \rho \geq \Omega. \tag{29}$$

Here, (29) implies that

$$H_n^{A}[f_n](\rho) = 0, \quad \text{for} \quad \rho \geq \Omega.$$

which completes the proof. □

*3.2. Sampling of Bandlimited Isotropic Functions in the OLCHT Domain*

**Definition 8.** *Let $f(r,\theta)$ satisfy Assumption 1 and $b > 0$. Then, it is $\Omega-$bandlimited in the OLCHT domain, if all of the coefficients of its Fourier series are $\Omega-$bandlimited isotropic in the OLCHT domain with the parameters $A$, $\boldsymbol{\tau}$, and $\boldsymbol{\eta}$, i.e.,*

$$H_v^{A,\tau,\eta}[f_n](\rho) = 0, \quad \text{for} \quad \rho \geq \Omega,$$

*where $n = 0, \pm 1, \pm 2, \cdots$.*

**Lemma 3.** *Let $f(r) \in L^2(\mathbb{R})$ be $\Omega-$bandlimited isotropic in the OLCHT domain with the parameters $A, \boldsymbol{\tau}, \boldsymbol{\eta}$, and $b > 0$, then the function $f(r)$ can be reconstructed at sampling point $\alpha_{vj} \in \mathbb{R}$ by*

$$f(r) = (-1)^v \varsigma\, e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty}\sum_{j=1}^{\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2\Omega}{b}\right) J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right) e^{i\frac{a}{2b}\alpha_{vj}^2} f(\alpha_{vj})\vartheta_{vj}(r), \tag{30}$$

*where* $\mu_1 = |\boldsymbol{\tau}|$, $\mu_2 = |d\boldsymbol{\tau} - b\boldsymbol{\eta}|$, *and*

$$\vartheta_{vj}(r) = \frac{2b[\mu_2 + \alpha_{vj}] J_v\left(\frac{\Omega r}{b}\right)}{\Omega J_{v+1}(z_{vj})\left[\alpha_{vj}^2 - r^2 + 2\mu_2(\alpha_{vj} - r)\right]},$$

$\vartheta_{vj}(r)$ *denotes the vjth interpolating function with the sample at* $\alpha_{vj}$, $\alpha_{vj} = \frac{bz_{vj}}{\Omega}$, $z_{vj} \in \mathbb{R}$ *is the jth zero of* $J_v(z)$, $\varsigma = e^{i\left(\frac{d|\boldsymbol{\tau}|^2 - a|b\boldsymbol{\eta} - d\boldsymbol{\tau}|^2}{b}\right)}$.

**Proof.** From (13), because $f(r)$ is $\Omega-$bandlimited isotropic in the OLCHT domain, $\tilde{f}(r)$ is a $\frac{\Omega}{b}-$bandlimited isotropic function, such that

$$H_v[\tilde{f}](\rho) = 0 \quad \text{for} \quad \rho \geq \frac{\Omega}{b}. \tag{31}$$

From (31), $H_v[\tilde{f}](\rho)$ can be expanded into a Fourier–Bessel series according to [20]

$$H_v[\tilde{f}](\rho) = \begin{cases} \sum\limits_{j=1}^{\infty} \varepsilon_j J_v(\alpha_{vj}\rho), & 0 < \rho < \frac{\Omega}{b} \\ 0, & \rho \geq \frac{\Omega}{b} \end{cases} \tag{32}$$

where

$$\varepsilon_j = \frac{2b^2}{\Omega^2 J_{v+1}^2(z_{vj})} \int_0^{\frac{\Omega}{b}} H_v[\tilde{f}](\rho) J_v(\alpha_{vj}\rho)\rho \mathrm{d}\rho = \frac{2b^2 \tilde{f}(\alpha_{vj})}{\Omega^2 J_{v+1}^2(\frac{\alpha_{vj}\Omega}{b})}. \tag{33}$$

Therefore, from (13) and (32), we have

$$H_v^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[\tilde{f}](\rho) = \begin{cases} \frac{i^v \lambda_1 \ell_1 e^{im(\varphi_1 - \varphi_2)}}{b} e^{i\frac{d}{2b}\rho^2} \sum\limits_{j=1}^{\infty} \varepsilon_j J_v(\frac{\alpha_{vj}\rho}{b}), & 0 < \rho < \Omega \\ 0, & \rho \geq \Omega \end{cases} \tag{34}$$

where $\ell_1$ is defined as (7), $\lambda_1$ is given by (12).
According to (15), the inverse $v$th-order OLCHT of (34) enables us to write

$$f(r) = (-1)^v \frac{\ell_1 \ell_2}{b^2} e^{-i\frac{a}{2b}r^2} \int_0^{\Omega} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) \rho \mathrm{d}\rho \sum_{j=1}^{\infty} \varepsilon_j \underbrace{J_m^2\left(\frac{\mu_2\rho}{b}\right) J_v(\frac{\alpha_{vj}\rho}{b}) J_v\left(\frac{r\rho}{b}\right)}_{\xi}, \tag{35}$$

where $\mu_1 = |\boldsymbol{\tau}|$ and $\mu_2 = |d\boldsymbol{\tau} - b\boldsymbol{\eta}|$.

From (23), we have

$$J_m\left(\frac{\mu_2\rho}{b}\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\left(m\theta - \frac{\mu_2\rho}{b}\sin\theta\right)} \mathrm{d}\theta, \tag{36}$$

$$J_v\left(\frac{\alpha_{vj}\rho}{b}\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\left(v\theta - \frac{\alpha_{vj}\rho}{b}\sin\theta\right)} \mathrm{d}\theta. \tag{37}$$

Multiplying (36) by (37), we obtain

$$
\begin{aligned}
J_m\left(\frac{\mu_2\rho}{b}\right)J_v\left(\frac{\alpha_{vj}\rho}{b}\right) &= \frac{1}{2\pi}\int_0^{2\pi} e^{i\left(v\theta-\frac{\alpha_{vj}\rho}{b}\sin\theta\right)}e^{i\left(m\theta-\frac{\mu_2\rho}{b}\sin\theta\right)}\mathrm{d}\theta\cdot\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\left[(m+v)\theta-\left(\frac{\mu_2\rho+\alpha_{vj}\rho}{b}\right)\sin\theta\right]}\mathrm{d}\theta \\
&= J_{m+v}\left(\frac{\mu_2\rho+\alpha_{vj}\rho}{b}\right).
\end{aligned} \tag{38}
$$

Similarly

$$
J_m\left(\frac{\mu_2\rho}{b}\right)J_v\left(\frac{r\rho}{b}\right) = J_{m+v}\left(\frac{\mu_2\rho+r\rho}{b}\right). \tag{39}
$$

Using (38) and (39), so

$$
\xi = J_{m+v}\left(\frac{\mu_2\rho+\alpha_{vj}\rho}{b}\right)J_{m+v}\left(\frac{\mu_2\rho+r\rho}{b}\right). \tag{40}
$$

It then follows from a well-known equation [25,32]

$$
\begin{aligned}
&\int_0^{\Omega} J_{m+v}\left(\frac{\mu_2\rho+\alpha_{vj}\rho}{b}\right)J_{m+v}\left(\frac{\mu_2\rho+r\rho}{b}\right)\rho\mathrm{d}\rho \\
&= \frac{b\Omega(\mu_2+\alpha_{vj})}{\alpha_{vj}^2-r^2+2\mu_2(\alpha_{vj}-r)}J_{m+v}\left(\frac{\mu_2\Omega+r\Omega}{b}\right)J_{m+v+1}\left(\frac{\mu_2\Omega+\alpha_{vj}\Omega}{b}\right).
\end{aligned} \tag{41}
$$

Using the relation (40), we obtain

$$
J_{m+v+1}\left(\frac{\mu_2\Omega+\alpha_{vj}\Omega}{b}\right)J_{m+v}\left(\Omega\frac{\mu_2\Omega+r\Omega}{b}\right) = J_m^2\left(\frac{\mu_2\Omega}{b}\right)J_{v+1}\left(\frac{\alpha_{vj}\Omega}{b}\right)J_v\left(\frac{r\Omega}{b}\right). \tag{42}
$$

Applying (14), (35), (41), and (42), thus

$$
\begin{aligned}
f(r) = (-1)^v\varsigma e^{-i\frac{a}{2b}r^2}\sum_{m=-\infty}^{+\infty}J_m\left(\frac{\mu_1 r}{b}\right)J_m^2\left(\frac{\mu_2\Omega}{b}\right)\sum_{j=1}^{\infty}J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right) \\
\times e^{i\frac{a}{2b}\alpha_{vj}^2}f(\alpha_{vj})\frac{2b[\mu_2+\alpha_{vj}]J_v\left(\frac{r\Omega}{b}\right)}{\Omega J_{v+1}(\frac{\alpha_{vj}\Omega}{b})\left[\alpha_{vj}^2-r^2+2\mu_2(\alpha_{vj}-r)\right]},
\end{aligned} \tag{43}
$$

where $\varsigma = \ell_1\ell_2 = e^{i\left(\frac{d|\boldsymbol{\tau}|^2-a|b\boldsymbol{\eta}-d\boldsymbol{\tau}|^2}{b}\right)}$. □

## 4. Sampling Theorems in the OLCT Domain

For simplicity, we denote by $\mathscr{H}_{OLCT}$ the space of all functions that are $\Omega-$bandlimited in the OLCT domain and angularly bandlimited to the highest frequency $\omega_p = \frac{K}{2\pi}$, and by $\mathscr{H}_{OLCHT}$ the space of all functions that are $\Omega-$bandlimited in the OLHCT domain and angularly bandlimited to the highest frequency $\omega_p = \frac{K}{2\pi}$.

**Lemma 4.** *Let $f(r,\theta)$ be $\Omega-$bandlimited in the OLCT domain with parameters $A$, $\boldsymbol{\tau}$, and $\boldsymbol{\eta}$ satisfying Assumption 1 and $b > 0$. Then, the nth Fourier coefficients $f_n(r)$ can be reconstructed at sampling point $\alpha_{nj} \in \mathbb{R}$ by*

$$f_n(r) = (-1)^n \varsigma\, e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2 \Omega}{b}\right) \times \sum_{j=1}^{\infty} J_m\left(\frac{|\tau|\alpha_{nj}}{b}\right) e^{i\frac{a}{2b}\alpha_{nj}^2} f_n(\alpha_{nj}) \vartheta_{nj}(r), \tag{44}$$

*where* $\alpha_{nj} = \frac{bz_{nj}}{\Omega}$, $z_{nj} \in \mathbb{R}$ *is the jth zero of* $J_n(z)$, *and*

$$\vartheta_{nj}(r) = \frac{2b[\mu_2 + \alpha_{nj}] J_n\left(\frac{\Omega r}{b}\right)}{\Omega J_{n+1}(z_{nj}) \left[\alpha_{nj}^2 - r^2 + 2\mu_2(\alpha_{nj} - r)\right]},$$

*here,* $\varsigma$, $\mu_1$, *and* $\mu_2$ *are the same as those stated.*

**Proof.** Let $v = n$ in Lemma 3, and we obtain

$$f(r) = (-1)^n \varsigma\, e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2 \Omega}{b}\right) \times \sum_{j=1}^{\infty} J_m\left(\frac{|\tau|\alpha_{nj}}{b}\right) e^{i\frac{a}{2b}\alpha_{nj}^2} f(\alpha_{nj}) \vartheta_{nj}(r), \tag{45}$$

Following from Lemma 2, we can directly obtain

$$f_n(r) = (-1)^n \varsigma\, e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2 \Omega}{b}\right) \times \sum_{j=1}^{\infty} J_m\left(\frac{|\tau|\alpha_{nj}}{b}\right) e^{i\frac{a}{2b}\alpha_{nj}^2} f_n(\alpha_{nj}) \vartheta_{nj}(r). \tag{46}$$

which completes the proof. □

The sampling theorem for the FT in polar coordinates is mentioned in [19,20,24]. Let us review the classical Stark's interpolation formula [19,20].

**Lemma 5.** *Let* $f(r,\theta)$ *be* $\Omega_{FT}-$*bandlimited in the FT domain to the highest frequency* $\omega_p = \frac{K}{2\pi}$, *satisfying Assumption 1, and* $b > 0$. *Then, it can be uniform reconstruction at azimuthal sampling point* $\left(r, \frac{2\pi l}{2K+1}\right) \in \mathbb{R}^2, l = 0, 1, \cdots, 2K \in \mathbb{N}$ *by* [19,20]

$$f(r,\theta) = \sum_{l=0}^{2K} f\left(r, \frac{2\pi l}{2K+1}\right) o_l(\theta), \tag{47}$$

*where*

$$o_l(\theta) = o\left(\theta - \frac{2\pi l}{2K+1}\right) = \frac{\sin\left[\frac{2K+1}{2}\left(\theta - \frac{2\pi l}{2K+1}\right)\right]}{(2K+1)\sin\left[\frac{1}{2}\left(\theta - \frac{2\pi l}{2K+1}\right)\right]}, \tag{48}$$

*denotes the lth interpolating function in azimuth with the sample at* $\frac{2\pi l}{2K+1}$.

Given that the OLCT is a generalized version of the LCT in polar coordinates, it is of great significance and value to study the sampling theorem in the field of the OLCT. The following theorem is obtained by combining Lemmas 4 and 5.

**Theorem 1.** *Let $f(r,\theta) \in \mathscr{H}_{OLCT}$ satisfy Assumption 1 and $b > 0$. Then, it can be reconstructed at the normalized zeros $\alpha_{nj} \in \mathbb{R}$ and at the uniformly spaced points $\frac{2\pi l}{2K+1} \in \mathbb{R}$ by*

$$f(r,\theta) = \frac{(-1)^n \varsigma}{2K+1} e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} \sum_{n=-K}^{K} \sum_{j=1}^{\infty} \sum_{l=0}^{2K} e^{i\frac{a}{2b}\alpha_{nj}^2} f\left(\alpha_{nj}, \frac{2\pi l}{2K+1}\right) \times J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2 \Omega}{b}\right) J_m\left(\frac{|\tau|\alpha_{nj}}{b}\right) \vartheta_{nj}(r) e^{in\left(\theta - \frac{2\pi l}{2K+1}\right)}. \tag{49}$$

*where $\varsigma$, $\mu_1$, $\mu_2$, $\alpha_{nj}$, and $\vartheta_{nj}(r)$ are the same as those stated.*

**Proof.** By (16), we obtain

$$f_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r,\theta) e^{-in\theta} \mathrm{d}\theta, \tag{50}$$

and

$$f_n(\alpha_{nj}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha_{nj},\theta) e^{-in\theta} \mathrm{d}\theta. \tag{51}$$

From (47), it follows that

$$f_n(\alpha_{nj}) = \frac{1}{2\pi} \sum_{l=0}^{2K} f\left(\alpha_{nj}, \frac{2\pi l}{2K+1}\right) \int_{-\pi}^{\pi} o_l(\theta) e^{-in\theta} \mathrm{d}\theta, \ -K \le n \le K. \tag{52}$$

Following from [19,20], we obtain

$$\int_{-\pi}^{\pi} o_l(\theta) e^{-in\theta} \mathrm{d}\theta = \frac{2\pi}{2K+1} e^{-in\frac{2\pi l}{2K+1}}, \ -K \le n \le K. \tag{53}$$

It follows from (52) that

$$f_n(\alpha_{nj}) = \frac{1}{2K+1} \sum_{l=0}^{2K} f\left(\alpha_{nj}, \frac{2\pi l}{2K+1}\right) e^{-in\frac{2\pi l}{2K+1}}, \ -K \le n \le K. \tag{54}$$

By substituting this result into (44), we obtain

$$f_n(r) = \frac{(-1)^n \varsigma}{2K+1} e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2 \Omega}{b}\right) \sum_{j=1}^{\infty} J_m\left(\frac{|\tau|\alpha_{nj}}{b}\right) \times e^{i\frac{a}{2b}\alpha_{nj}^2} \vartheta_{nj}(r) \sum_{l=0}^{2K} f\left(\alpha_{nj}, \frac{2\pi l}{2K+1}\right) e^{-in\frac{2\pi l}{2K+1}}, \tag{55}$$

for all $-K \le n \le K$.

Hence,

$$f(r,\theta) = \sum_{n=-K}^{K} f_n(r) e^{in\theta} \frac{(-1)^n \varsigma}{2K+1} e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2 \Omega}{b}\right) \times \sum_{n=-K}^{K} \sum_{j=1}^{\infty} J_m\left(\frac{|\tau|\alpha_{nj}}{b}\right) e^{i\frac{a}{2b}\alpha_{nj}^2} \vartheta_{nj}(r) \sum_{l=0}^{2K} f\left(\alpha_{nj}, \frac{2\pi l}{2K+1}\right) e^{in\left(\theta - \frac{2\pi l}{2K+1}\right)}. \tag{56}$$

which completes the proof. □

**Remark 5.** *The sampling points $\alpha_{nj}$ are usually referred to as the scaled jth zero of $J_v(z)$, where $\alpha_{vj} = \frac{bz_{vj}}{\Omega}$, $z_{vj}$ is the jth zero of $J_v(z)$. According to (49) in Theorem 1, it is clear that the required number of samples is*

$$[(2K+1)N]^2,$$

*where the number of normalized zeros takes* $(2K+1)N(N \to \infty)$.

**Remark 6.** *When* $A = (0,1;-1,0)$, $\boldsymbol{\tau} = \mathbf{0}$, *and* $\boldsymbol{\eta} = \mathbf{0}$, *Theorem 1 reduces the classical interpolation formula of the FT in polar coordinates* [20]. *When* $A = (a,b;c,d) \in \mathbb{R}^{2\times 2}$, $\boldsymbol{\tau} = \mathbf{0}$, *and* $\boldsymbol{\eta} = \mathbf{0}$, *Theorem 1 reduces the sampling theorem of the LCT in polar coordinates* [27]. *As evidenced in* [33], *not all* $f(r,\theta)$ *are bandlimited in practical applications. Consequently, our research results can be used to deal with non-bandlimited functions in the FT or LCT domains.*

**Corollary 1.** *Let* $f(r,\theta) \in \mathscr{H}_{OLCT}$ *satisfy Assumption 1 and* $b > 0$. *Then, the OLCT* $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi)$ *of* $f(r,\theta)$ *can be reconstructed at the normalized zeros* $\alpha_{nj} \in \mathbb{R}$ *and at the uniformly spaced points* $\frac{2\pi l}{2K+1} \in \mathbb{R}$ *by*

$$\begin{aligned} F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = \frac{(-1)^n \varsigma}{2K+1} e^{i\frac{d}{2b}\rho^2} \sum_{m=-\infty}^{+\infty} \sum_{n=-K}^{K} \sum_{j=1}^{\infty} \sum_{l=0}^{2K} e^{-i\frac{d}{2b}\alpha_{nj}^2} J_m\left(\frac{\mu_2\rho}{b}\right) J_m^2\left(\frac{\mu_1\Omega}{b}\right) \\ \times J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{nj}}{b}\right) F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}\left(\alpha_{nj}, \frac{2\pi l}{2K+1}\right) \vartheta_{nj}(\rho) e^{in\left(\phi - \frac{2\pi l}{2K+1}\right)}, \end{aligned} \tag{57}$$

*where* $\varsigma$, $\mu_1$, $\mu_2$, $\alpha_{nj}$, *and* $\vartheta_{nj}(\rho)$ *are the same as those stated.*

**Proof.** Because of the inversion formula of the OLCT, we obtain

$$O_L^{A^{-1},\boldsymbol{\xi},\boldsymbol{\gamma}}[F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}](r,\theta) = 0 \quad \text{for} \quad \rho \geq \Omega,$$

which implies that $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}} \in \mathscr{H}_{OLCT}$ with $A^{-1}$, $\boldsymbol{\xi}$, and $\boldsymbol{\gamma}$.

Following from Remark 4, we can directly obtain

$$F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = \sum_{n=-\infty}^{+\infty} H_n^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f_n](\rho) e^{in\phi}. \tag{58}$$

According to (58), it implies that $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}$ satisfies Assumption 1, and

$$F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = \sum_{n=-K}^{K} H_{-n}^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f_{-n}](\rho) e^{in\phi}. \tag{59}$$

By using Theorem 1, we obtain

$$\begin{aligned} F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = \frac{(-1)^n \varsigma}{2K+1} e^{i\frac{d}{2b}\rho^2} \sum_{m=-\infty}^{+\infty} \sum_{n=-K}^{K} \sum_{j=1}^{\infty} \sum_{l=0}^{2K} e^{-i\frac{d}{2b}\alpha_{nj}^2} J_m\left(\frac{\mu_2\rho}{b}\right) J_m^2\left(\frac{\mu_1\Omega}{b}\right) \\ \times J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{nj}}{b}\right) F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}\left(\alpha_{nj}, \frac{2\pi l}{2K+1}\right) \vartheta_{nj}(\rho) e^{in\left(\phi - \frac{2\pi l}{2K+1}\right)}. \end{aligned} \tag{60}$$

which completes the proof. □

**Remark 7.** *When* $A = (0,1;-1,0)$, $\boldsymbol{\tau} = \mathbf{0}$, *and* $\boldsymbol{\eta} = \mathbf{0}$, *Corollary 1 becomes the interpolation formula of the LCT* [27] *(Corollary 2).*

## 5. Sampling Theorem in the OLCHT Domain

Inspired by the classical interpolation formula [19,20], this section mainly studies the sampling theorem for $f(r,\theta) \in \mathscr{H}_{OLCHT}$ from samples at the normalized zeros $\alpha_{nj}$ in radius and at the uniformly spaced points $\frac{2\pi l}{2K+1}$ in azimuth in the OLCHT domain in polar coordinates.

**Lemma 6.** *Let $f(r,\theta)$ be $\Omega-$bandlimited in the OLCHT domain with parameters $A$, $\boldsymbol{\tau}$, and $\boldsymbol{\eta}$ satisfying Assumption 1 and $b>0$. Then, the nth Fourier coefficients $f_n(r)$ can be reconstructed at sampling point $\alpha_{vj}\in\mathbb{R}$ by*

$$f_n(r) = (-1)^v \varsigma\, e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2\Omega}{b}\right) \times \sum_{j=1}^{\infty} J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right) e^{i\frac{a}{2b}\alpha_{vj}^2} f_n(\alpha_{vj})\vartheta_{vj}(r), \tag{61}$$

*where $\varsigma, \mu_1, \mu_2, \alpha_{vj}$, and $\vartheta_{nj}(r)$ are the same as those stated.*

**Proof.** Replacing $f(r)$ in Lemma 3 with $f_n(r)$, we can directly obtain

$$f_n(r) = (-1)^v \varsigma\, e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2\Omega}{b}\right) \times \sum_{j=1}^{\infty} J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right) e^{i\frac{a}{2b}\alpha_{vj}^2} f_n(\alpha_{vj})\vartheta_{vj}(r). \tag{62}$$

which completes the proof. □

According to Lemmas 5 and 6, an interpolation formula is obtained in the OLCHT domain. This interpolation formula is essentially different from Theorem 1 due to the consistency of the OLCHT order, where the sampling points are normalized zeros of the Bessel function on radius. Theorem 2 better reduces the number of normalized zeros.

**Theorem 2.** *Let $f(r,\theta)\in\mathscr{H}_{OLCHT}$ satisfy Assumption 1 and $b>0$. Then, it can be reconstructed at the normalized zeros $\alpha_{vj}\in\mathbb{R}$ and at the uniformly spaced points $\frac{2\pi l}{2K+1}\in\mathbb{R}$ by*

$$f(r,\theta) = (-1)^v \varsigma e^{-i\frac{a}{2b}r^2} \sum_{j=1}^{\infty}\sum_{l=0}^{2K}\sum_{m=-\infty}^{+\infty} e^{i\frac{a}{2b}\alpha_{vj}^2} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2\Omega}{b}\right) \times J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right)\vartheta_{vj}(r) f\left(\alpha_{vj}, \frac{2\pi l}{2K+1}\right) o_l(\theta), \tag{63}$$

*where $\varsigma, \mu_1, \mu_2, \alpha_{vj}, \vartheta_{vj}(r)$, and $o_l(\theta)$ are the same as those stated.*

**Proof.** Replacing $\alpha_{nj}$ in (54) with $\alpha_{vj}$, we have

$$f_n(\alpha_{vj}) = \frac{1}{2K+1}\sum_{l=0}^{2K} f\left(\alpha_{vj}, \frac{2\pi l}{2K+1}\right) e^{-in\frac{2\pi l}{2K+1}},\ -K\leq n\leq K. \tag{64}$$

Using (61), we obtain

$$f_n(r) = \frac{(-1)^v\varsigma}{2K+1} e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2\Omega}{b}\right) \sum_{j=1}^{\infty} J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right) \times e^{i\frac{a}{2b}\alpha_{vj}^2}\vartheta_{vj}(r)\sum_{l=0}^{2K} f\left(\alpha_{vj}, \frac{2\pi l}{2K+1}\right) e^{-in\frac{2\pi l}{2K+1}}, \tag{65}$$

for all $-K\leq n\leq K$.

By the following triangle sum formula [20,25]

$$(2K+1)o_l(\theta) = \sum_{n=-K}^{K} e^{in\left(\theta - \frac{2\pi l}{2K+1}\right)}. \tag{66}$$

Applying (16) and (66), we obtain

$$
\begin{aligned}
f(r,\theta) &= \sum_{n=-K}^{K} f_n(r)e^{in\theta} \\
&= (-1)^v \varsigma e^{-i\frac{a}{2b}r^2} \sum_{m=-\infty}^{+\infty} J_m\left(\frac{\mu_1 r}{b}\right) J_m^2\left(\frac{\mu_2 \Omega}{b}\right) e^{i\frac{a}{2b}\alpha_{vj}^2} \\
&\times \sum_{j=1}^{\infty}\sum_{l=0}^{2K} J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right)\vartheta_{vj}(r) f\left(\alpha_{vj}, \frac{2\pi l}{2K+1}\right) o_l(\theta).
\end{aligned} \tag{67}
$$

which completes the proof. □

**Remark 8.** *The sampling points $\alpha_{nj}$ are usually referred to as the scaled jth zero of $J_v(z)$, where $\alpha_{vj} = \frac{bz_{vj}}{\Omega}$, $z_{vj}$ is the jth zero of $J_v(z)$. According to (63) in Theorem 2, it is clear that the required number of samples is*

$$(2K+1)N^2,$$

*where the number of normalized zeros takes $N(N \to \infty)$.*

**Remark 9.** *When $A = (0,1;-1,0)$, $\boldsymbol{\tau} = \mathbf{0}$, and $\boldsymbol{\eta} = \mathbf{0}$, Theorem 2 reduces the classical reconstruction formula of the HT [20]. When $A = (a,b;c,d) \in \mathbb{R}^{2\times 2}$, $\boldsymbol{\tau} = \mathbf{0}$, and $\boldsymbol{\eta} = \mathbf{0}$, Theorem 2 reduces the sampling theorem of the LCT [27].*

**Remark 10.** *It is emphasized here that the interpolation Formula (63) is essentially different from (49) in Theorem 1 because the transform domain in which the reconstructed object is located is different. By comparing (49) and (63), it is obvious that the second interpolation formula is better than in the first interpolation formula in terms of computational complexity.*

**Corollary 2.** *Let $f(r,\theta) \in \mathscr{H}_{OLHCT}$ satisfy Assumption 1 and $b > 0$. Then, the OLCT $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi)$ of $f(r,\theta)$ can be reconstructed at the normalized zeros $\alpha_{nj} \in \mathbb{R}$ and at the uniformly spaced points $\frac{2\pi l}{2K+1} \in \mathbb{R}$ by*

$$
\begin{aligned}
F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) &= (-1)^v \varsigma e^{i\frac{d}{2b}\rho^2} \sum_{j=1}^{\infty}\sum_{l=0}^{2K}\sum_{m=-\infty}^{+\infty} e^{-i\frac{a}{2b}\alpha_{vj}^2} J_m\left(\frac{\mu_2\rho}{b}\right) J_m^2\left(\frac{\mu_1\Omega}{b}\right) \\
&\times J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right)\vartheta_{vj}(\rho) F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}\left(\alpha_{vj}, \frac{2\pi l}{2K+1}\right) o_l(\phi),
\end{aligned} \tag{68}
$$

*where $\varsigma, \mu_1, \mu_2, \alpha_{vj}, \vartheta_{vj}(\rho)$, and $o_l(\phi)$ are the same as those stated.*

**Proof.** According to (58), it implies that $F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}$ satisfies Assumption 1, and

$$F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) = \sum_{n=-K}^{K} H_{-n}^{A,\boldsymbol{\tau},\boldsymbol{\eta}}[f_{-n}](\rho)e^{in\phi}. \tag{69}$$

By using Theorem 2, we have

$$
\begin{aligned}
F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}(\rho,\phi) &= (-1)^v \varsigma e^{i\frac{d}{2b}\rho^2} \sum_{j=1}^{\infty}\sum_{l=0}^{2K}\sum_{m=-\infty}^{+\infty} e^{-i\frac{a}{2b}\alpha_{vj}^2} J_m\left(\frac{\mu_2\rho}{b}\right) J_m^2\left(\frac{\mu_1\Omega}{b}\right) \\
&\times J_m\left(\frac{|\boldsymbol{\tau}|\alpha_{vj}}{b}\right)\vartheta_{vj}(\rho) F^{A,\boldsymbol{\tau},\boldsymbol{\eta}}\left(\alpha_{vj}, \frac{2\pi l}{2K+1}\right) o_l(\phi).
\end{aligned} \tag{70}
$$

which completes the proof. □

**Remark 11.** *When $A = (a,b;c,d) \in \mathbb{R}^{2\times 2}$, $\boldsymbol{\tau} = \mathbf{0}$, and $\boldsymbol{\eta} = \mathbf{0}$, Theorem 2 becomes the classical result [27] (Corollary 3).*

**Remark 12.** *The difference between Theorem 2 and Corollary 2 is the reconstructed function. Theorem 2 is the original function, while Corollary 2 is the OLCT version in polar coordinates, which leads to the conclusions being applicable to different fields of application.*

## 6. Potential Application

Two-dimensional sampling is a general technique applicable to various fields such as medical imaging, astronomy, radar, and crystallography. There exist numerous diverse sampling methods in these domains, among which polar coordinate sampling proves to be an effective approach. In this paper, we propose two new sampling theorems for the OLCT and OLCHT in polar coordinates. They can serve as a theoretical foundation for applications in the fields of CT and image reconstruction.

On the one hand, the results in Theorems 1 and 2 show that it is feasible to reconstruct a bandlimited (or space-limited) image from uniformly spaced samples. The interpolation function is a Bessel function, and the sample points are proportional to the zeros of the Bessel function. Bessel function subroutines are available in most scientific program libraries. Even if these are not readily accessible, expressions based on polynomial approximations can be employed.

On the other hand, CT image reconstruction based on the OLCT in polar coordinates also has an application basis. Reference [28] presents a numerical experiment on the utilization of the OLCT in CT image reconstruction, which requires the use of two-dimensional interpolation. This paper primarily focuses on the theoretical proof of sampling theorems for the OLCT and OLCHT, and practical applications will be presented in another article.

## 7. Conclusions

This paper studies the sampling theorems of bandlimited functions in the OLCT and OLCHT domains in polar coordinates, that is, interpolating uniform samples in radius and interpolating the highest frequency range samples in azimuth, where the sampling points are normalized zeros of the Bessel function on radius. The first interpolation formula is a generalization of the FT and LCT domains, which is more general. The second interpolation formula is superior to the first interpolation formula in terms of computational complexity due to the consistency of the OLCHT order.

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