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Existence, Uniqueness, and Stability of Solutions for Nabla Fractional Difference Equations

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Abstract: In this paper, we study a class of nabla fractional difference equations with multipoint summation boundary conditions. We obtain the exact expression of the corresponding Green's function and deduce some of its properties. Then, we impose some sufficient conditions in order to ensure existence and uniqueness results. Also, we establish some conditions under which the solution to the considered problem is generalized Ulam–Hyers–Rassias stable. In the end, some examples are included in order to illustrate our main results.

Keywords: nabla fractional difference; summation conditions; existence results; uniqueness result; stability

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1. Introduction

The notion of the fractional derivative [1,2] dates back to the works of Euler, but the idea of fractional difference is recent. Discrete fractional calculus is an integrated theory of sums and differences of an arbitrary order [3,4]. Two perspectives may be found in the literature on fractional differences: the Δ point of view, also known as the delta fractional difference, and the ∇ perspective, also known as the nabla fractional difference. We limit ourselves to the second method in this article.

The notion of nabla fractional difference can be traced back to the work of Gray and Zhang [5], and Miller and Ross [6]. In this line, Atici and Eloe [7] developed the Riemann–Liouville nabla fractional difference; initiated the study of nabla fractional initial value problem; and established exponential law, product rule, and nabla Laplace transform.

Since then, the non-local character of nabla fractional differences has attracted a lot of attention regarding the theory and applications of nabla fractional calculus. It is an ideal tool for simulating non-local phenomena in time or space. There is a long-term memory effect in the nabla fractional difference of a function as it holds information about this function at previous times. Many natural systems, including those with non-local effects, are better described by nabla fractional difference equations than by integer-order difference equations. A strong theory of nabla fractional calculus for discrete-variable, real-valued functions was developed as a consequence of the contributions of multiple mathematicians. We refer to a recent monograph [4] and its sources for a thorough introduction to the development of nabla fractional calculus.

During the past decade, interest in analyzing discrete fractional boundary value problems increased. To name a few works, we refer to [8–24]. Recently, Ulam–Hyers-type stability [25–28] has palyed an important role in many applied problems in biology and economics. However, it is not a common result in discrete fractional calculus and there are only a few papers in this direction [13,29–32]. To the best of our knowledge, both the existence results and the Ulam–Hyers–Rassias stability results for multipoint nabla fractional problems had been obtained before.

Motivated by these developments, in this work, we study existence, uniqueness, and stability of solutions for the following nonlinear nabla fractional difference problem with multipoint summation boundary conditions

$$-\left(\nabla^{\nu}_{\rho(a)}u\right)(t) = f(t,u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \tag{1}$$

$$u(a) = A, \quad u(b) + \mu \sum_{s=a+1}^{b-1} u(s) = B, \tag{2}$$

where a, b, A, $B \in \mathbb{R}$; $\mu > 0$; $u : \mathbb{N}^{b}_{a} \to \mathbb{R}$; $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to the second argument; $1 < \nu < 2$ and $\nabla^{\nu}_{\rho(a)}u$ denotes the ν^{th}-order Riemann–Liouville nabla fractional difference of u based at $\rho(a) = a - 1$.

Our interest in the above problem also comes from the fact that the mathematical models of many real-world phenomena can be represented by multi-point boundary value problems. Such models have a large number of applications in numerous areas of science and engineering, such as electric power networks, electric railway systems, elasticity, thermodynamics, telecommunication lines, and wave propagation. For more details, we refer to [33] and the references therein. As mentioned above, there are no results for the solutions of nabla fractional boundary value problems with multi-point boundary conditions, and our work seems to be the first one in this direction. The present paper is organized as follows. In Section 2, we recall some preliminaries on nabla fractional calculus, Ulam–Hyers stability, and fixed-point theory. In Section 3, we construct the Green's function associated with (1) and (2). We also derive a few of their essential properties. Then, in Section 4, we impose some sufficient conditions in order to deduce the existence and uniqueness of solutions to (1) and (2) using various fixed-point theorems. In Section 5, we state and prove the Ulam–Hyers stability results for (1) and (2). Finally, we provide an example in Section 6 to illustrate our main results.

2. Preliminaries

First, we provide some definitions and fundamental facts of nabla fractional calculus [4], which we are going to use later. Denote by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}^{b}_{a} = \{a, a+1, a+2, \ldots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. The backward jump operator $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$ is defined by

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.$$

For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t + r)}{\Gamma(t)}.$$

Also, $t^{\overline{r}} = 0$, if $t \in \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$, such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$.

Let $\mu \in \mathbb{R} \setminus \{\ldots, -3, -2, -1\}$. The μ^{th}-order nabla fractional Taylor monomial is defined as

$$H_{\mu}(t, a) = \frac{(t - a)^{\overline{\mu}}}{\Gamma(\mu + 1)},$$

provided the right-hand side exists. Note that for all $\mu \in \{\ldots, -3, -2, -1\}$ and $t \in \mathbb{N}_a$, we have $H_{\mu}(a, a) = H_{\mu}(t, a) = 0$.

Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based on a is defined as

$$(\nabla^{-N}_{a}u)(t) = \sum_{s=a+1}^{t} H_{N-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a.$$

Moreover, $\left(\nabla_a^{-N}u\right)(a) = 0$ and $\left(\nabla_a^{-0}u\right)(t) = u(t)$ for all $t \in \mathbb{N}_{a+1}$.

Definition 1 ([4]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is defined as*

$$\left(\nabla_a^{-\nu}u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

with $\left(\nabla_a^{-\nu}u\right)(a) = 0$.

Definition 2 ([4]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and $N \in \mathbb{N}_1$, such that $N-1 < \nu \le N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by*

$$\left(\nabla_a^{\nu}u\right)(t) = \left(\nabla^N\left(\nabla_a^{-(N-\nu)}u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Similar to the definitions given in [27], we introduce the definitions of Ulam stability for nabla fractional difference equations.

Consider the following two inequalities:

$$\left|\left(\nabla_{\rho(a)}^{\nu}u\right)(t) + f(t, u(t))\right| \le \epsilon, \quad t \in \mathbb{N}_{a+2}^{b}, \tag{3}$$

$$\left|\left(\nabla_{\rho(a)}^{\nu}u\right)(t) + f(t, u(t))\right| \le \epsilon\psi(t), \quad t \in \mathbb{N}_{a+2}^{b}, \tag{4}$$

where $\psi : \mathbb{N}_{a+2}^{b} \to \mathbb{R}^{+}$.

Definition 3 ([27]). *Problem (1) and (2) is said to be Ulam–Hyers stable if there exists a real number $d_f > 0$, such that for each $\epsilon > 0$ and for every solution $u_2 : \mathbb{N}_a^b \to \mathbb{R}$ of (2) and (3), there exists a solution $u_1 : \mathbb{N}_a^b \to \mathbb{R}$ of (1) and (2) with*

$$|u_1(t) - u_2(t)| \le \epsilon d_f, \quad t \in \mathbb{N}_a^b.$$

Moreover, (1) and (2) is said to be generalized Ulam–Hyers stable if

$$|u_1(t) - u_2(t)| \le \phi_f(\varepsilon), \quad t \in \mathbb{N}_a^b,$$

where $\phi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\phi_f(0) = 0$.

Definition 4 ([27]). *Problem (1) and (2) is said to be Ulam–Hyers–Rassias stable with respect to the function $\psi : \mathbb{N}_a^b \to \mathbb{R}^+$ if there exists a real number $d_{f,\psi} > 0$, such that for each $\epsilon > 0$ and for every solution $u_2 : \mathbb{N}_a^b \to \mathbb{R}$ of (2)–(4), there exists a solution $u_1 : \mathbb{N}_a^b \to \mathbb{R}$ of (1) and (2) with*

$$|u_1(t) - u_2(t)| \le \epsilon\psi(t)d_{f,\psi}, \quad t \in \mathbb{N}_a^b.$$

Moreover, (1) and (2) is said to be generalized Ulam–Hyers–Rassias stable with respect to the function $\Psi : \mathbb{N}_a^b \to \mathbb{R}^+$ if there exists a real number $d_{f,\Psi} > 0$, such that for every solution $u_2 : \mathbb{N}_a^b \to \mathbb{R}$ of (2)–(4), there exists a solution $u_1 : \mathbb{N}_a^b \to \mathbb{R}$ of (1) and (2) with

$$|u_1(t) - u_2(t)| \le \Psi(t)d_{f,\Psi}, \quad t \in \mathbb{N}_a^b.$$

Finally, we provide the statements of Brouwer and Banach fixed-point theorems as follows:

Theorem 1 ([34]). *(Brouwer Fixed-Point Theorem) Let K be a nonempty compact convex subset of a finite dimensional normed space $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$, and let T be a continuous mapping of K into itself. Then, T has a fixed point in K.*

Theorem 2 ([34])**.** *(Banach Fixed-Point Theorem) Let K be a closed subset of a Banach space $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$, and let $T : K \to K$ be a contraction mapping. Then, T has a unique fixed point in K.*

3. Green's Function

First, our aim is to obtain the exact expression of the Green's function corresponding to the linear problem

$$-\left(\nabla^{\nu}_{\rho(a)} u\right)(t) = h(t), \quad t \in \mathbb{N}^{b}_{a+2}, \tag{5}$$

$$u(a) = 0, \quad u(b) + \mu \sum_{s=a+1}^{b-1} u(s) = 0, \quad \mu > 0, \tag{6}$$

with $h : \mathbb{N}^{b}_{a+2} \to \mathbb{R}$ and $1 < \nu < 2$. Denote:

$$\begin{aligned} \phi(r) &= H_{\nu-1}(b, \rho(r)) + \mu H_{\nu}(b, r), \quad r \in \mathbb{N}^{b}_{a}, \\ \Lambda &= H_{\nu-1}(b, a) + \mu H_{\nu}(b-1, a). \end{aligned}$$

Theorem 3. *Assume $\Lambda \neq 0$. The linear problem (5) and (6) has a unique solution*

$$u(t) = \sum_{s=a+2}^{b} G(t,s) h(s), \quad t \in \mathbb{N}^{b}_{a}, \tag{7}$$

where

$$G(t,s) = \begin{cases} G_1(t,s), & t \in \mathbb{N}^{\rho(s)}_{a}, \\ G_2(t,s), & t \in \mathbb{N}^{b}_{s}, \end{cases} \tag{8}$$

with

$$G_1(t,s) = \frac{H_{\nu-1}(t,a)\phi(s)}{\Lambda},$$

and

$$G_2(t,s) = G_1(t,s) - H_{\nu-1}(t, \rho(s)).$$

Proof. Applying $\nabla^{-\nu}_{a+1}$ on both sides of the nabla problem (5), we obtain that the general solution is given by

$$u(t) = C_1 H_{\nu-1}(t, \rho(a)) + C_2 H_{\nu-2}(t, \rho(a)) - \sum_{s=a+2}^{t} H_{\nu-1}(t, \rho(s)) h(s), \quad t \in \mathbb{N}^{b}_{a}, \tag{9}$$

where C_1 and C_2 are arbitrary constants. As $H_{\nu-1}(a, \rho(a)) = H_{\nu-2}(a, \rho(a)) = 1$, the first condition $u(a) = 0$, shows that

$$C_1 + C_2 = 0. \tag{10}$$

From the second boundary condition, we obtain

$$\begin{aligned} &C_1 H_{\nu-1}(b, \rho(a)) + C_2 H_{\nu-2}(b, \rho(a)) - \sum_{s=a+2}^{b} H_{\nu-1}(b, \rho(s)) h(s) \\ &\quad + \mu \sum_{s=a+1}^{b-1} \left[C_1 H_{\nu-1}(s, \rho(a)) + C_2 H_{\nu-2}(s, \rho(a)) - \sum_{r=a+2}^{s} H_{\nu-1}(s, \rho(r)) h(r) \right] = 0. \end{aligned}$$

Replacing C_2 with $-C_1$ in the above equation, we deduce

$$C_1 = \frac{\sum_{s=a+2}^{b} H_{\nu-1}(b, \rho(s)) h(s) + \mu \sum_{s=a+1}^{b-1} \sum_{r=a+2}^{s} H_{\nu-1}(s, \rho(r)) h(r)}{H_{\nu-1}(b, \rho(a)) - H_{\nu-2}(b, \rho(a)) + \mu \sum_{s=a+1}^{b-1} [H_{\nu-1}(s, \rho(a)) - H_{\nu-2}(s, \rho(a))]}. \tag{11}$$

Note that
$$H_{\nu-1}(b,\rho(a)) - H_{\nu-2}(b,\rho(a)) = H_{\nu-1}(b,a).$$
Now, we will show by induction that for every $b \geq a+2$, the following equality holds
$$H_\nu(b-1,a) = \sum_{s=a+1}^{b-1} H_{\nu-1}(s,a).$$
For $b = a+2$, it is necessary to check that $H_{\nu-1}(a+1,a) = H_\nu(a+1,a) = 1$.

Suppose that our claim holds for some $b = k$, i.e,
$$H_\nu(k-1,a) = H_{\nu-1}(a+1,a) + H_{\nu-1}(a+2,a) + ... + H_{\nu-1}(k-1,a).$$
In order to show that the equation holds for $b = k+1$, one needs to check that
$$H_\nu(k,a) = H_\nu(k-1,a) + H_{\nu-1}(k,a),$$
which clearly holds.

As a result,
$$\begin{aligned}
& H_{\nu-1}(b,\rho(a)) - H_{\nu-2}(b,\rho(a)) + \mu \sum_{s=a+1}^{b-1} [H_{\nu-1}(s,\rho(a)) - H_{\nu-2}(s,\rho(a))] \\
= \; & H_{\nu-1}(b,a) + \mu \sum_{s=a+1}^{b-1} H_{\nu-1}(s,a) \\
= \; & H_{\nu-1}(b,a) + \mu H_\nu(b-1,a) \\
= \; & \Lambda.
\end{aligned}$$
Moreover, as $H_\nu(b,b) = 0$, we have
$$\begin{aligned}
& \sum_{s=a+2}^{b} H_{\nu-1}(b,\rho(s))h(s) + \mu \sum_{s=a+1}^{b-1} \sum_{r=a+2}^{s} H_{\nu-1}(s,\rho(r))h(r) \\
= & \sum_{s=a+2}^{b} H_{\nu-1}(b,\rho(s))h(s) + \mu \sum_{s=a+2}^{b-1} \sum_{r=a+2}^{s} H_{\nu-1}(s,\rho(r))h(r) \\
= & \sum_{s=a+2}^{b} H_{\nu-1}(b,\rho(s))h(s) + \mu \sum_{r=a+2}^{b-1} \left[\sum_{s=r}^{b-1} H_{\nu-1}(s,\rho(r)) \right] h(r) \\
= & \sum_{s=a+2}^{b} H_{\nu-1}(b,\rho(s))h(s) + \mu \sum_{r=a+2}^{b-1} H_\nu(b,r)h(r) \\
= & \sum_{s=a+2}^{b} H_{\nu-1}(b,\rho(s))h(s) + \mu \sum_{s=a+2}^{b} H_\nu(b,s)h(s) \\
= & \sum_{s=a+2}^{b} \phi(s)h(s).
\end{aligned}$$
From (10) and (11), we deduce that
$$C_1 = \frac{1}{\Lambda} \sum_{s=a+2}^{b} \phi(s)h(s), \tag{12}$$
and
$$C_2 = -\frac{1}{\Lambda} \sum_{s=a+2}^{b} \phi(s)h(s). \tag{13}$$
Substituting (12) and (13) in (9), we obtain (7). □

Lemma 1. *The Green's function defined in* (8) *satisfies the identity*

$$\sum_{s=a+2}^{b}|G(t,s)| \leq \frac{H_{\nu-1}(b,a)}{\Lambda}[H_{\nu}(b,a+1)+\mu H_{\nu+1}(b,a+2)]+H_{\nu}(b,a\ +1)=\overline{G},$$

for $t\in\mathbb{N}_a^b$.

Proof. Clearly, $\Lambda>0$. Also, for each $s\in\mathbb{N}_a^b$,

$$\phi(s)=H_{\nu-1}(b,\rho(s))+\mu H_{\nu}(b,s)>0.$$

Moreover, since

$$H_{\nu-1}(t,a)\geq 0, \quad t\in\mathbb{N}_a^b,$$

one can check that

$$\begin{aligned}
\sum_{s=a+2}^{b}|G(t,s)| &= \sum_{s=a+2}^{t}|G_2(t,s)|+\sum_{s=t+1}^{b}|G_1(t,s)| \\
&= \sum_{s=a+2}^{t}|G_1(t,s)-H_{\nu-1}(t,\rho(s))|+\sum_{s=t+1}^{b}|G_1(t,s)| \\
&\leq \sum_{s=a+2}^{b}|G_1(t,s)|+\sum_{s=a+2}^{t}H_{\nu-1}(t,\rho(s)) \\
&= \sum_{s=a+2}^{b}\left|\frac{H_{\nu-1}(t,a)\phi(s)}{\Lambda}\right|+H_{\nu}(t,a\ +1) \\
&= \frac{H_{\nu-1}(t,a)}{\Lambda}\sum_{s=a+2}^{b}[H_{\nu-1}(b,\rho(s))+\mu H_{\nu}(b,s)]+H_{\nu}(t,a\ +1) \\
&= \frac{H_{\nu-1}(t,a)}{\Lambda}[H_{\nu}(b,a\ +1)+\mu H_{\nu+1}(b,a\ +2)]+H_{\nu}(t,a\ +1).
\end{aligned}$$

Finally, using $t\in\mathbb{N}_a^b$,

$$H_{\nu-1}(t,a)\leq H_{\nu-1}(b,a) \quad \text{and} \quad H_{\nu}(t,a\ +1)\leq H_{\nu}(b,a\ +1),$$

the proof is complete. □

Now, our aim is to obtain the exact expression of the unique solution of the following nabla problem:

$$-\left(\nabla_{\rho(a)}^{\nu}u\right)(t)=h(t), \quad t\in\mathbb{N}_{a+2}^b, \tag{14}$$

$$u(a)=A, \quad u(b)+\mu\sum_{s=a+1}^{b-1}u(s)=B, \quad \mu>0. \tag{15}$$

First, we establish the following result.

Lemma 2. *Assume* $\Lambda\neq 0$. *The unique solution of the nabla fractional problem*

$$-\left(\nabla_{\rho(a)}^{\nu}v\right)(t)=0, \quad t\in\mathbb{N}_{a+2}^b, \tag{16}$$

$$v(a)=A, \quad v(b)+\mu\sum_{s=a+1}^{b-1}v(s)=B, \quad \mu>0,$$

is

$$v(t) = \frac{1}{\Lambda}\Bigg[(A\mu + B)H_{\nu-1}(t,a) - \frac{A\mu H_{\nu-2}(t,\rho(a))H_{\nu}(b,a)}{(\nu-1)} + A\left(\frac{b-t}{\nu-1}\right)[H_{\nu-2}(t,\rho(a))H_{\nu-2}(b,\rho(a)) + \mu H_{\nu-2}(t,\rho(a))H_{\nu-1}(b,a)]\Bigg], \quad (17)$$

for $t \in \mathbb{N}_a^b$.

Proof. Using similar arguments as before, the general solution of (16) is

$$v(t) = C_1 H_{\nu-1}(t,\rho(a)) + C_2 H_{\nu-2}(t,\rho(a)), \quad t \in \mathbb{N}_a^b. \quad (18)$$

The condition $v(a) = A$, implies that

$$C_1 + C_2 = A. \quad (19)$$

Using the condition $v(b) + \mu \sum_{s=a+1}^{b-1} v(s) = B$ in (18), we obtain

$$C_1 H_{\nu-1}(b,\rho(a)) + C_2 H_{\nu-2}(b,\rho(a)) + \mu \sum_{s=a+1}^{b-1} [C_1 H_{\nu-1}(s,\rho(a)) + C_2 H_{\nu-2}(s,\rho(a))] = B. \quad (20)$$

It is necessary to check that

$$\begin{aligned} \sum_{s=a+1}^{b-1} H_{\nu-1}(s,\rho(a)) &= \sum_{s=a}^{b-1} H_{\nu-1}(s,\rho(a)) - H_{\nu-1}(a,\rho(a)) \\ &= H_{\nu}(b-1,\rho(a)) - 1 \\ &= H_{\nu}(b,a) - 1. \end{aligned}$$

Similarly,

$$\sum_{s=a+1}^{b-1} H_{\nu-2}(s,\rho(a)) = H_{\nu-1}(b,a) - 1.$$

Then, from (20), we obtain

$$C_1[H_{\nu-1}(b,\rho(a)) + \mu(H_{\nu}(b,a) - 1)] + C_2[H_{\nu-2}(b,\rho(a)) + \mu(H_{\nu-1}(b,a) - 1)] = B. \quad (21)$$

Solving (19) and (21), we obtain. Thus,

$$C_1 = \frac{-AH_{\nu-2}(b,\rho(a)) - A\mu H_{\nu-1}(b,a) + A\mu + B}{\Lambda}, \quad (22)$$

and

$$C_2 = \frac{AH_{\nu-1}(b,\rho(a)) + A\mu H_{\nu}(b,a) - A\mu - B}{\Lambda}. \quad (23)$$

Substituting (22) and (23) in (18), we obtain (17). □

From the above results, it follows that the unique solution of the boundary problem (14) and (15) has the following representation.

Theorem 4. *Assume $\Lambda \neq 0$ and $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$. The unique solution of the nabla fractional problem (14) and (15) is*

$$u(t) = v(t) + \sum_{s=a+2}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{24}$$

where $G(t,s)$ is given in (8) and v is given in (17).

Remark 1. *For $t \in \mathbb{N}_a^b$, we have*

$$\begin{aligned} v(t) &= \frac{1}{\Lambda}\Big[(A\mu + B)H_{\nu-1}(t,a) - \frac{A\mu H_{\nu-2}(t,\rho(a))H_\nu(b,a)}{(\nu-1)} \\ &\quad + A\left(\frac{b-t}{\nu-1}\right)[H_{\nu-2}(t,\rho(a))H_{\nu-2}(b,\rho(a)) + \mu H_{\nu-2}(t,\rho(a))H_{\nu-1}(b,a)]\Big] \\ &= \frac{1}{\Lambda}\Big[(A\mu + B)H_{\nu-1}(t,a) + A\left(\frac{b-t}{\nu-1}\right)H_{\nu-2}(t,\rho(a))H_{\nu-2}(b,\rho(a)) \\ &\quad + \frac{A\mu H_{\nu-2}(t,\rho(a))}{(\nu-1)}[(b-t)H_{\nu-1}(b,a) - H_\nu(b,a)]\Big]. \end{aligned}$$

As $H_\nu(b,a) > 0$, and for $t \in \mathbb{N}_a^b$, one can verify that $H_{\nu-1}(t,a) \geq 0$ and $H_{\nu-2}(t,\rho(a)) > 0$, then,

$$\begin{aligned} |v(t)| \leq \frac{1}{\Lambda}\Big[&(|A|\mu + |B|)H_{\nu-1}(t,a) + |A|\left(\frac{b-t}{\nu-1}\right)H_{\nu-2}(t,\rho(a))H_{\nu-2}(b,\rho(a)) \\ &+ \frac{|A|\mu H_{\nu-2}(t,\rho(a))}{(\nu-1)}[(b-t)H_{\nu-1}(b,a) + H_\nu(b,a)]\Big]. \end{aligned}$$

Denote

$$\begin{aligned} \overline{V} = \frac{1}{\Lambda}\Big[&(|A|\mu + |B|)H_{\nu-1}(b,a) + |A|\left(\frac{b-a}{\nu-1}\right)H_{\nu-2}(b,\rho(a)) \\ &+ \frac{|A|\mu}{(\nu-1)}[(b-a)H_{\nu-1}(b,a) + H_\nu(b,a)]\Big]. \end{aligned}$$

It is also clear that for $t \in \mathbb{N}_a^b$, we have

$$H_{\nu-1}(t,a) \leq H_{\nu-1}(b,a), \quad H_{\nu-2}(t,\rho(a)) \leq H_{\nu-2}(a,\rho(a)) = 1, \quad (b-t) \leq (b-a).$$

Thus,

$$|v(t)| \leq \overline{V}, \quad t \in \mathbb{N}_a^b.$$

4. Existence and Uniqueness Results

Now, let X be a Banach space equipped with the standard norm $\|y\| = \max\Big\{|y(t)| : t \in \mathbb{N}_a^b\Big\}$. Set the compact, convex subset

$$K = \{u \in X : \|u\| \leq 2\overline{V}\}$$

of X and the operator $T : X \to X$ by

$$Tu(t) = \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) + v(t), \quad t \in \mathbb{N}_a^b.$$

Now, we are in the position to establish our existence result based on Theorem 1.

Theorem 5. *Let*

$$F = \max\left\{|f(t,u)| : (t,u) \in \mathbb{N}_a^b \times \left[-2\overline{V}, 2\overline{V}\right]\right\} > 0.$$

If $\overline{G}F \leq \overline{V}$, then the nonlinear problem (1) *and* (2) *has at least one solution in K.*

Proof. For any $t \in \mathbb{N}_a^b$ and $u \in K$, we have

$$\begin{aligned} |Tu(t)| &= \left|\sum_{s=a+2}^{b} G(t,s)f(s,u(s)) + v(t)\right| \\ &\leq \sum_{s=a+2}^{b} |G(t,s)||f(s,u(s))| + |v(t)| \\ &\leq F\sum_{s=a+2}^{b} |G(t,s)| + \overline{V} \\ &\leq 2\overline{V}, \end{aligned}$$

which means that $T : K \to K$. Continuity of f on $\mathbb{R}$ implies its uniform continuity on $\left[-2\overline{V}, 2\overline{V}\right]$. Then, one can choose $\delta > 0$, such that for all $t \in \mathbb{N}_a^b$ and for all $u_1, u_2 \in \left[-2\overline{V}, 2\overline{V}\right]$ with $|(t,u_1) - (t,u_2)| < \delta$, we have

$$|f(t,u_1) - f(t,u_2)| < \epsilon\overline{G}^{-1}.$$

Hence, for all $t \in \mathbb{N}_a^b$,

$$\begin{aligned} |Tu_1(t) - Tu_2(t)| &= \left|\sum_{s=a+2}^{b} G(t,s)f(s,u_1(s)) - \sum_{s=a+2}^{b} G(t,s)f(s,u_2(s))\right| \\ &\leq \sum_{s=a+2}^{b} |G(t,s)||f(s,u_1(s)) - f(s,u_2(s))| \\ &< \epsilon\overline{G}^{-1}\sum_{s=a+2}^{b} |G(t,s)| = \epsilon, \end{aligned}$$

which shows us that T has at least one fixed point in K. □

Now, we provide the following uniqueness results based on Theorem 2.

Theorem 6. *Where f is Lipschitz continuous on its second variable with a constant $k > 0$. Moreover, if $k < \overline{G}^{-1}$, then the nonlinear problem* (1) *and* (2) *has a unique solution.*

Proof. For all $t \in \mathbb{N}_a^b$ and $u_1, u_2 \in X$, we have

$$\begin{aligned} \|Tu_1 - Tu_2\| &\leq \max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^{b} |G(t,s)||f(s,u_1(s)) - f(s,u_2(s))| \\ &\leq k\max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^{b} |G(t,s)||u_1(s) - u_2(s)| \\ &\leq k\overline{G}\|u_1 - u_2\|. \end{aligned}$$

As $k\overline{G} < 1$, T is a contraction on X, which means that T has a unique fixed point $u \in X$. □

5. Stability Analysis

Here, we will establish two stability results as follows.

Theorem 7. *Let f is Lipschitz continuous on its second variable with a constant $k > 0$. Moreover, if $k < \overline{G}^{-1}$, then the nonlinear problem (1) and (2) is generalized Ulam–Hyers stable.*

Proof. Let u_1 be a solution of (1) and (2) and u_2 is a solution of (2) and (3). From (3) and (24), for $t \in \mathbb{N}_a^b$, it follows that

$$\left| u_2(t) - \left(v(t) + \sum_{s=a+2}^{b} G(t,s) f(s, u_2(s)) \right) \right| \leq \varepsilon.$$

Moreover, for $t \in \mathbb{N}_a^b$, we have

$$\begin{aligned} |u_2(t) - u_1(t)| &= \left| u_2(t) - \left(v(t) + \sum_{s=a+2}^{b} G(t,s) f(s, u_1(s)) \right) \right| \\ &\leq \left| u_2(t) - \left(v(t) + \sum_{s=a+2}^{b} G(t,s) f(s, u_2(s)) \right) \right| \\ &\quad + \left| \sum_{s=a+2}^{b} G(t,s) f(s, u_2(s)) - \sum_{s=a+2}^{b} G(t,s) f(s, u_1(s)) \right| \\ &\leq \epsilon + k\overline{G} |u_2(t) - u_1(t)|, \end{aligned}$$

implying that

$$|u_2(t) - u_1(t)| \leq \frac{\epsilon}{1 - k\overline{G}} = \epsilon d_f.$$

Clearly, from Definition 3, the solution of problem (1) is Ulam–Hyers stable. Moreover, as one can choose $\phi_f(\varepsilon) = \frac{\epsilon}{1-k\overline{G}}$ with $\phi_f(0) = 0$, the solution of problem (1) and (2) is generalized Ulam–Hyers stable. □

Theorem 8. *Let f be Lipschitz continuous on its second variable with a constant $k > 0$. Moreover, if $k < \overline{G}^{-1}$, then the nonlinear problem (1) and (2) is Ulam–Hyers–Rassias stable with respect to the function $\psi : \mathbb{N}_a^b \to \mathbb{R}^+$ and, consequently, it is generalized Ulam–Hyers–Rassias stable.*

Proof. Let u_1 be a solution to (1) and (2) and u_2 is a solution of (2) and (3). From (4) and (24), for $t \in \mathbb{N}_a^b$, it follows that

$$\left| u_2(t) - \left(v(t) + \sum_{s=a+2}^{b} G(t,s) f(s, u_2(s)) \right) \right| \leq \epsilon\psi(t).$$

Furthermore, for $t \in \mathbb{N}_a^b$, we have

$$\begin{aligned} |u_2(t) - u_1(t)| &= \left| u_2(t) - \left(v(t) + \sum_{s=a+2}^{b} G(t,s) f(s, u_1(s)) \right) \right| \\ &\leq \left| u_2(t) - \left(v(t) + \sum_{s=a+2}^{b} G(t,s) f(s, u_2(s)) \right) \right| \\ &\quad + \left| \sum_{s=a+2}^{b} G(t,s) f(s, u_2(s)) - \sum_{s=a+2}^{b} G(t,s) f(s, u_1(s)) \right| \\ &\leq \epsilon\psi(t) + k\overline{G} |u_2(t) - u_1(t)|, \end{aligned}$$

implying that

$$|u_2(t) - u_1(t)| \leq \frac{\epsilon\psi(t)}{1 - k\overline{G}} = \epsilon\psi(t) d_{f,\psi}.$$

As a result, from Definition 4, the solution of problem (1) and (2) is Ulam–Hyers–Rassias stable with respect to the function ψ and, consequently, choosing $\Psi(t) = \epsilon\psi(t)$, it is generalized Ulam–Hyers–Rassias stable. □

6. Examples

In the end, we provide an example to show the applicability of our main results.

Example 1. *Consider* (1) *and* (2) *with* $a = 0$, $b = 5$, $\nu = 1.5$, $\mu = \frac{1}{2}$, $A = B = 1$, *and* $f(t,\xi) = \frac{1}{35}(t+\xi)$ *for all* $(t,\xi) \in \mathbb{N}_2^5 \times \mathbb{R}$. *Then,*

$$\Lambda = H_{0.5}(5,0) + \frac{H_{1.5}(4,0)}{2} = 5.7422,$$

$$\overline{V} = \frac{1}{\Lambda}\left[(1.5)H_{0.5}(5,0) + \left(\frac{5}{0.5}\right)H_{-0.5}(5,-1) + 5H_{0.5}(5,0) + H_{1.5}(4,0)\right] = 4.4728,$$

$$\overline{G} = \frac{H_{0.5}(5,0)}{\Lambda}\left[H_{1.5}(5,1) + \frac{H_{2.5}(5,2)}{2}\right] + H_{1.5}(5,1) = 11.0628,$$

and

$$F = \max\left\{\left|\frac{1}{35}(t+\xi)\right| : t \in \mathbb{N}_2^5, \quad |\xi| \le 8.9456\right\} = 0.3984.$$

As $\overline{G}F = 4.4074 \le \overline{V}$, *by Theorem 5, the nonlinear problem* (1) *and* (2) *has at least one solution in* $u \in K$.

Moreover, f *is Lipschitz continuous on its second variable with a constant* $k = \frac{1}{35}$. *Since* $k\overline{G} = 0.3161 < 1$, *from Theorem 6, the nonlinear problem* (1) *and* (2) *has a unique solution* $u \in X$. *And, from Theorem 7, the nonlinear problem* (1) *and* (2) *is generalized Ulam–Hyers stable.*

7. Conclusions

In this work, we study a completely new for the literature problem (1) and (2). We were able to construct the Green's function related to the linear problem and to deduce some of its properties. Then, using various fixed-point theorems, under some suitable conditions, we obtained the existence and uniqueness of solutions to (1) and (2). In the end, we proved that these solutions are Ulam–Hyers stable. We point out that, to the best of knowledge, this is the first paper that deals with the existence and stability results for nabla fractional difference equations with summation boundary conditions. Our results can be used in future works as a base for researchers to obtain the existence and multiplicity of positive solutions via some topological methods.

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