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The Uniqueness and Iterative Properties of Positive Solution for a Coupled Singular Tempered Fractional System with Different Characteristics

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Abstract: In this paper, we focus on the uniqueness and iterative properties of positive solution for a coupled p-Laplacian system of singular tempered fractional equations with differential order and characteristics. Firstly, the system is converted to an integral equation, and then, a coupled iterative technique and some suitable growth conditions are proposed; furthermore, some elaborate results about the uniqueness and iterative properties of positive solutions of the system are established, which include the uniqueness, the convergence analysis, the asymptotic behavior, and error estimation, as well as the convergence rate of the positive solution. The interesting points of this paper are that the order of the system of equations is different and the nonlinear terms of the system possess the opposite monotonicity and allow for stronger singularities at space variables.

Keywords: iterative solutions; uniqueness; double iterative techniques; convergence analysis; iterative properties

Citation: Chen, P.; Zhang, X.; Wang, Y.; Wu, Y. The Uniqueness and Iterative Properties of Positive Solution for a Coupled Singular Tempered Fractional System with Different Characteristics. *Fractal Fract.* **2024**, *8*, 636. https://doi.org/10.3390/fractalfract8110636

Academic Editor: Ivanka Stamova

Received: 26 September 2024
Revised: 25 October 2024
Accepted: 26 October 2024
Published: 28 October 2024

1. Introduction

In this paper, we consider the following coupled p-Laplacian system of singular tempered fractional equations with different characteristics

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p({}_0^R\mathbb{D}_t^{\beta,\lambda}u(t))\right) = f_1(t,v(t)), \\ {}_0^R\mathbb{D}_t^{\jmath,\lambda}\left(\varphi_p({}_0^R\mathbb{D}_t^{\ell,\lambda}v(t))\right) = f_2(t,u(t)), \\ u(0) = v(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta,\lambda}u(0) = {}_0^R\mathbb{D}_t^{\ell,\lambda}v(0) = 0, \\ u(1) = \int_0^1 e^{-\lambda(1-t)}u(t)dt, \quad v(1) = \int_0^1 e^{-\lambda(1-t)}v(t)dt. \end{cases} \tag{1}$$

where $\alpha, \jmath \in (0,1)$ and $\beta, \ell \in (1,2)$ are real constants, which represent the order of tempered fractional derivatives, λ is a positive constant, $\varphi_p(t) = |t|^{p-2}t$ denotes the p-Laplacian operator with conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1, p > 1$, ${}_0^R\mathbb{D}_t^{\varsigma,\lambda}(\varsigma = \jmath, \ell, \alpha, \beta)$ are tempered fractional derivatives, which have the following relationship with the Riemann–Liouville fractional derivative ${}_0^R\mathbf{D}_t^{\varsigma}$

$$ {}_0^R\mathbb{D}_t^{\varsigma,\lambda}x(t) = e^{-\lambda t}{}_0^R\mathbf{D}_t^{\varsigma}\left(e^{\lambda t}x(t)\right), \tag{2}$$

where ${}_0^R\mathbf{D}_t{}^{\varsigma}x(t) = \frac{d^n}{dt^n}\Big(\mathbf{I}_t^{n-\varsigma}x(t)\Big)$ and

$$\mathbf{I}_t^{n-\varsigma}x(t) = \int_0^1 (t-s)^{\varsigma-1}x(s)ds, \tag{3}$$

is a Riemann–Liouville fractional integral; see [1].

System (1) comes from the dynamics phenomenon for modeling a particle's random walk in Brownian motion, where the transition has a feature of a semi-heavy tail from a power law to a Gaussian. Let $g(x,t)$ be the particle jump density function. To more accurately describe the long-range dependence property that the particle diffusion possesses, Sokolov and Klafter [2] replaced Einstein's classical diffusion equation [3] by the following fractional variant:

$$\partial_t^J g(x,t) = \partial_x^\ell g(x,t), \tag{4}$$

where the space fractional derivative $\partial_x^\ell (0<\ell<2)$ describes particle jumps $P[J>x] \approx x^{-\ell}$ and the time fractional derivative $\partial_t^J (0<J\le 1)$ represents the waiting time $P[W>t] \approx t^{-J}$ between particle jumps. Obviously, ∂_x^ℓ and ∂_t^J all obey a heavy-tailed power law. However, in anomalous diffusion, the delivery speed of a particle's random walk in Brownian motion is faster than that expected in traditional diffusion, which means the decay of the particle abides by a power law in moderate time, but follows an exponential rule on long time scales, i.e., the particle's Brownian motion in anomalous diffusion possesses a feature of a semi-heavy tail. In order to temper the feature of the semi-heavy tail, Sabzikar et al. [4] introduced an exponential factor into the particle jump density by a Fourier transform

$$\mathcal{F}[\partial_{\pm,x}^{\ell,\lambda} g](x,t) = B_{\pm}^{\ell,\lambda}(x)\Big[e^{-[pB_+^{\ell,\lambda}(x)+(1-p)B_-^{\ell,\lambda}(x)]\mu t}\Big], \ \ 0\le p\le 1,$$

where

$$B_{\pm}^{\ell,\lambda}(x) = \begin{cases} (\lambda \pm xi)^\ell - \lambda^\ell, & 0<\ell<1, \\ (\lambda \pm xi)^\ell - \lambda^\ell - \pm \ell\lambda^{\ell-1}xi, & 1<\ell<2, \end{cases}$$

and derived a tempered anomalous diffusion equation:

$$\partial_t g(x,t) = (-1)^k C_T\{p\partial_{+,x}^{\ell,\lambda} + (1-p)\partial_{-,x}^{\ell,\lambda}\}g(x,t), \ell \in (k-1,k), k=1,2, \tag{5}$$

Thus, the tempered anomalous diffusion Equation (5) offers an exponential decay advantage over the fractional diffusion model (4) at long time scales, and has more applications in the tempered Lévy flight diffusion [5], in geophysics [6,7], and in finance [8,9].

In recent years, many new concepts of fractional derivatives and integral operators have been proposed, either to model natural phenomena where the existing fractional integral or derivative operators are inadequate, or to derive some wonderful mathematic properties that the traditional derivatives do not possess, for example, in the study of dynamic systems model for bioprocesses [10], eco-economical processes [11], fractional Kelvin–Voigt models [12], fractional Fourier transforms [13], fractional Jeffreys fluid in a porous medium [14], fractional temperature fields [15], mathematical properties for fractional problems [16–26], and so on. In a single equation, the author [27] established iterative solutions for a class of fractional nonlocal equations subject to integral conditions:

$$\begin{cases} {}_0^R\mathbf{D}_t{}^{\alpha}x(t) + f(t,x(t)) = 0, \quad 0<t<1, \\ x(0) = x'(0) = 0, \ {}_0^R\mathbf{D}_t{}^{\beta}x(1) = \int_0^1 {}_0^R\mathbf{D}_t{}^{\beta}x(t)dA(t), \end{cases} \tag{6}$$

where $2<\alpha\le 3, 0<\beta\le 1$, $\int_0^1 {}_0^R\mathbf{D}_t{}^{\beta}x(t)dA(t)$ denotes a Riemann–Stieltjes integral, and the nonlinearity $f(\cdot,\cdot)$ is continuous and increasing on the second variable in the local interval. By two iterative sequences with known initial values, it was proven that Equation (6) has two nontrivial solutions. However, the author neither obtained the positive solution of the problem nor established the uniqueness and iterative properties of the solution. In a recent work [28], Wu et al. considered the unique positive solution for a p-Laplacian fractional differential equation in the sense of Riemann–Liouville fractional derivatives:

$$\begin{cases} -{}_0^R\mathbf{D}_t{}^{\alpha}\left(\varphi_p\left(-{}_0^R\mathbf{D}_t{}^{\gamma}x\right)\right)(t)=f(t,x(t)), \quad 0<t<1, \\ x(0)=0,\ {}_0^R\mathbf{D}_t{}^{\gamma}x(0)={}_0^R\mathbf{D}_t{}^{\gamma}x(1)=0,\ x(1)=\int_0^1 x(t)d\chi(t), \end{cases} \tag{7}$$

where $\gamma,\alpha \in (1,2]$, $\int_0^1 z(t)d\chi(t)$ is a Riemann–Stieltjes integral and χ is a function of bounded variation. By using an iterative technique, the uniqueness and iterative properties of the positive solution were established in the case where the nonlinearity was decreasing in the space variable. Recently, by employing the monotone iterative technique, Zhao et al. [29] studied the existence of iterative positive solutions for a coupled system of fractional differential equations,

$$\begin{cases} {}_0^R\mathbf{D}_t{}^{\alpha_1}x(t)+f_1(t,x(t),y(t),{}_0^R\mathbf{D}_t{}^{\gamma_1}x(t),{}_0^R\mathbf{D}_t{}^{\gamma_2}y(t)), \quad 0<t<1, \\ {}_0^R\mathbf{D}_t{}^{\alpha_2}y(t)+f_2(t,x(t),y(t),{}_0^R\mathbf{D}_t{}^{\gamma_1}x(t),{}_0^R\mathbf{D}_t{}^{\gamma_2}y(t)), \quad 0<t<1, \end{cases} \tag{8}$$

subject to multipoint mixed boundary conditions:

$$\begin{cases} x(0)=x'(0)=0,\ x(1)=\sum_{i=1}^{m} a_i\mathbf{I}_t^{\beta_1}y(\xi_i)+\sum_{j=1}^{n} b_j y(\eta_i), \\ y(0)=y'(0)=0,\ y(1)=\sum_{i=1}^{m} c_i\mathbf{I}_t^{\beta_2}x(\xi_i)+\sum_{j=1}^{n} b_j x(\eta_i), \end{cases} \tag{9}$$

where $\alpha_i \in (2,3]$, $\beta_i \in (1,2]$, $0<\gamma_i<\alpha_i-2$, $i=1,2$ and $a_i,c_i>0, b_j\geq 0$, $0<\xi_i,\eta_j<1$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$. Under the condition that the nonlinearities $f_i(t,x_1,x_2,x_3,x_4)$ $(i=1,2)$ are all increasing in $x_j \in [0,l]$ for $j=1,2,3,4$, the iterative positive solutions and iterative sequence were established. In addition, to make our paper more self-integrated and readers conveniently understand our work, here we also recall some theories and methods related to this paper such as space theory [30–35], iterative techniques [36–38], regular theory [39–41], operator theory [42–49], upper–lower solution methods [50–52], the method of moving spheres [53], critical point theory [20,54–61], fixed point theorem [62,63], and many numerical techniques [64–66], etc.

By reviewing the above work, we find that the nonlinear terms of the equations of the systems in existing work are almost all required to have the same characteristic: from an analytical point of view, there is no essential difference to studying a single equation, [27–29]. However, in system (1), f_1, f_2 have opposite monotonicity, i.e., $f_1(\cdot,\cdot)$ is increasing, and $f_2(\cdot,\cdot)$ is decreasing on the second variable, respectively. No work has been performed on this type of system of equations with different characteristics as far as we know. The research of this paper is motivated by this source; to fill this gap, we propose a new double iterative technique and introduce some new growth conditions to overcome the difficulty of systems of equations with different characteristics, in particular, we not only establish the uniqueness of the iterative positive solution for system (1), but also derive the explicit iterative properties of the positive solution, which include the iterative sequence, the convergence analysis, error estimation, and the asymptotic behavior of the positive solution of system (1). The new contributions of this paper are listed as follows:

- The nonlinear terms of the system of equations possess opposite monotonicity, the method in [29] cannot be applied to solve the system (1).
- To overcome the obstruction of opposite monotonicity, a new double iterative technique is introduced.
- The nonlinear term of the system of equations can have stronger singularity in space variables.

- Some new growth conditions are introduced which generalize and improve the conditions and results of [28,67].
- Different from [29], the assumption of the upper and lower solutions is not required in our work.
- The explicit iterative properties of the solution for (1) are established, which makes up for the lack of work on it [29].

This paper is structured as follows. In Section 2, some preliminaries and lemmas are given. The main results are stated in Section 3. In Section 4, we give an example to illustrate our main results.

2. Preliminaries and Lemmas

In this sections, we firstly gather some properties of the Riemann–Liouville fractional derivative and integral for the subsequent studies.

Lemma 1 ([1]). *The Riemann–Liouville fractional derivative and integral have the following properties:*
(1) *Let* $z(t) \in L^1[0,1] \cap C[0,1]$, *then*

$$\boldsymbol{I}_t^{\hbar}{}_0^R\boldsymbol{D}_t^{\hbar} x(t) = x(t) + \sum_{i=1}^{n} b_i t^{\hbar - i},$$

where b_i ($i = 1,2,3,\cdots,n, n = [\hbar]+1$) *are real numbers, and* $[\cdot]$ *denotes the integer function.*
(2) *If* $z(t) \in L^1(0,1), \jmath > \ell > 0$, *then*

$${}_0^R\boldsymbol{D}_t^{\ell}\boldsymbol{I}_t^{\jmath} z(t) = \boldsymbol{I}_t^{\jmath-\ell} z(t), \quad \boldsymbol{I}_t^{\jmath}\boldsymbol{I}_t^{\ell} z(t) = \boldsymbol{I}_t^{\jmath+\ell} z(t), \quad {}_0^R\boldsymbol{D}_t^{\ell}\boldsymbol{I}_t^{\ell} z(t) = z(t);$$

(3) *Let* $\jmath > 0, \ell > 0$, *then*

$${}_0^R\boldsymbol{D}_t^{\jmath} t^{\ell-1} = \frac{\Gamma(\ell)}{\Gamma(\ell-\jmath)} t^{\ell-\jmath-1}.$$

Lemma 2 (see [50]). *If* $h(t)$ *is a positive continuous function in* $[0,1]$, *then the linear tempered fractional equation*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\beta,\lambda} x(t) = h(t), \\ x(0) = 0, x(1) = \int_0^1 e^{-\lambda(1-t)} x(t)dt, \end{cases} \tag{10}$$

has a unique positive solution

$$x(t) = \int_0^1 H_1(t,s)h(s)ds, \tag{11}$$

where $H_1(t,s)$ *is the Green function of* (10)

$$H_1(t,s) = \begin{cases} \dfrac{[\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}t^{\beta-1} - \beta(\beta-1)e^{\lambda s}(t-s)^{\beta-1}]e^{-\lambda t}}{(\beta-1)\Gamma(\beta+1)}, 0 \le s \le t \le 1; \\ \dfrac{\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)} t^{\beta-1} e^{-\lambda t}, 0 \le t \le s \le 1. \end{cases} \tag{12}$$

For

$$\begin{cases} {}_0^R\mathbb{D}_t^{\jmath,\lambda} x(t) = h(t), \\ x(0) = 0, x(1) = \int_0^1 e^{-\lambda(1-t)} x(t)dt, \end{cases}$$

we write the Green function as

$$H_2(t,s) = \begin{cases} \dfrac{\left[\ell(1-s)^{\ell-1}(\ell-1+s)e^{\lambda s}t^{\ell-1} - \ell(\ell-1)e^{\lambda s}(t-s)^{\ell-1}\right]e^{-\lambda t}}{(\ell-1)\Gamma(\ell+1)}, 0 \le s \le t \le 1; \\ \dfrac{\ell(1-s)^{\ell-1}(\ell-1+s)e^{\lambda s}}{(\ell-1)\Gamma(\ell+1)}t^{\ell-1}e^{-\lambda t}, 0 \le t \le s \le 1. \end{cases} \tag{13}$$

According to [50], $H_1(t,s)$ and $H_2(t,s)$ satisfy the following properties:

(a) $H_i(t,s)$ is a non-negative and continuous function for $(t,s) \in [0,1] \times [0,1]$.

(b) For all $t,s \in [0,1]$, $H_i(t,s), i = 1,2$ satisfy

$$A_1(s)e^{-\lambda t}t^{\beta-1} \le H_1(t,s) \le A_2(s)e^{-\lambda t}t^{\beta-1},$$

$$A_3(s)e^{-\lambda t}t^{\ell-1} \le H_2(t,s) \le A_4(s)e^{-\lambda t}t^{\ell-1},$$

where

$$A_1(s) = \frac{\beta s(1-s)^{\beta-1}e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}, \quad A_2(s) = \frac{\beta(\beta-1+s)(1-s)^{\beta-1}e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)},$$

$$A_3(s) = \frac{\ell s(1-s)^{\ell-1}e^{\lambda s}}{(\ell-1)\Gamma(\ell+1)}, \quad A_4(s) = \frac{\ell(\ell-1+s)(1-s)^{\ell-1}e^{\lambda s}}{(\ell-1)\Gamma(\ell+1)}.$$

Lemma 3. *Let $g(t) \in L^1(0,1)$, then the tempered fractional equation*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p({}_0^R\mathbb{D}_t^{\beta,\lambda}x(t))\right) = g(t), \\ x(0) = 0, {}_0^R\mathbb{D}_t^{\beta,\lambda}x(0) = 0, x(1) = \int_0^1 e^{-\lambda(1-t)}x(t)dt, \end{cases} \tag{14}$$

has one unique solution:

$$x(t) = \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}g(\tau)d\tau\right)^{q-1}ds. \tag{15}$$

Proof. Let $\rho = {}_0^R\mathbb{D}_t^{\beta,\lambda}x(t), y = \varphi_p(\rho)$, we firstly consider the following initial value problem:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}y(t) = g(t), t \in [0,1], \\ y(0) = 0. \end{cases} \tag{16}$$

It follows from (2) and Lemma 1 that

$$e^{\lambda t}y(t) = {}_0\mathbf{I}_t^{\alpha}e^{\lambda t}g(t) - b_1t^{\alpha-1},$$

that is,

$$e^{\lambda t}y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds - b_1t^{\alpha-1}.$$

Since $y(0) = 0, \alpha \in (0,1]$, one obtains $b_1 = 0$, and then,

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds.$$

On the other hand, by $y = \varphi_p(\rho)$, one obtains

$$y(t) = \varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}x(t)\right) = \int_0^t \frac{(t-s)^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds.$$

Consequently, Equation (14) can be converted to the following form:

$$\begin{cases} {}_{0}^{R}\mathbb{D}_{t}^{\beta,\lambda} x(t) = \varphi_p^{-1}\left(\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda t}}{\Gamma(\alpha)} g(s)e^{\lambda s} ds\right) \\ x(0) = 0, x(1) = \int_0^1 e^{-\lambda(1-t)} x(t) dt. \end{cases} \tag{17}$$

Notice that $g(s) \geq 0,\ \ s \in [0,1]$, then we have $\varphi_p(t) = t^{p-1}$, and due to

$$\varphi_p^{-1}\left(\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda t}}{\Gamma(\alpha)} g(s)e^{\lambda s} ds\right) = \left(\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda t}}{\Gamma(\alpha)} g(s)e^{\lambda s} ds\right)^{q-1},$$

then from Lemma 2, we know that Equation (14) has a unique solution that can be expressed by

$$x(t) = \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} g(\tau) e^{\lambda\tau} d\tau\right)^{q-1} ds.$$

The proof is completed. □

Thus, it follows from Lemmas 2 and 3 that system (1) is equivalent to the following system of integral equations:

$$\begin{cases} u(t) = \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} e^{\lambda\tau} f_1(\tau, v(\tau)) d\tau\right)^{q-1} ds, \quad t \in [0,1], \\ v(t) = \int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\jmath)} (s-\tau)^{\jmath-1} e^{\lambda\tau} f_2(\tau, u(\tau)) d\tau\right)^{q-1} ds, \quad t \in [0,1]. \end{cases} \tag{18}$$

To simplify the above system, we define an operator Q as follows,

$$(Qu)(t) = v(t) = \int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\jmath)} (s-\tau)^{\jmath-1} e^{\lambda\tau} f_2(\tau, u(\tau)) d\tau\right)^{q-1} ds, t \in [0,1], \tag{19}$$

and then, substituting (19) into the first equation of (18), we derive

$$u(t) = \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} e^{\lambda\tau} f_1(\tau, (Qu)(\tau)) d\tau\right)^{q-1} ds, t \in [0,1]. \tag{20}$$

Thus, if $u(t)$ is a solution of the integral Equation (20), then $(u,(Qu)) = (u,v)$ is a solution of system (1). So in order to find the solution of system (1), we only need to focus on the study of Equation (20).

Now, define a Banach space $E = C[0,1]$ with $||u|| = \max_{t\in[0,1]} |u(t)|$, and then, define a cone P in E:

$$P = \{u \in C[0,1] : u(t) \geq 0, t \in [0,1]\}.$$

Obviously, P is a normal cone with normality constant 1. Next, define a nonlinear operator $T : E \to E$ by

$$(Tu)(t) = \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} e^{\lambda\tau} f_1(\tau, (Qu)(\tau)) d\tau\right)^{q-1} ds, \quad t \in [0,1].$$

Thus, the fixed point of the operator T is a solution of Equation (20). Based on the above discussion, we state the main results and proceed to the proof of the main results in the next section.

3. Main Results

For the following studies, define two positive constants a, b satisfying

$$ab \in (0,1), \quad ab(q-1)^2 \in (0,1),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

To derive our main results, we introduce a double iterative process with the help of the following hypothesis and growth conditions:

($\mathbf{P}_1$) $f_1 \in C((0,1) \times [0,+\infty), [0,+\infty))$ is increasing on the second variable, $f_2 \in ((0,1) \times (0,+\infty), [0,+\infty))$ is decreasing on the second variable.

($\mathbf{P}_2$) There exists a constant $\sigma > 0$ such that

$$\begin{aligned} 0 &< \int_0^1 e^{\frac{\lambda t}{\sigma}} f_2^{\frac{1}{\sigma}}(t, e^{-\lambda t} t^{\beta-1}) dt < +\infty, \\ 0 &< \int_0^1 e^{\frac{\lambda t}{\sigma}} f_1^{\frac{1}{\sigma}}(t, e^{-\lambda t} t^{\ell-1}) dt < +\infty. \end{aligned} \tag{21}$$

($\mathbf{P}_3$) For the above a, b and any $0 < \delta < 1$, there exist two real functions $\phi, \psi : (0,1) \to [0,+\infty)$ with $\phi(\delta) > \delta^b$ and $\psi(\delta) < \delta^{-a}$ such that

$$f_1(t, \delta v) \geq \phi(\delta) f_1(t, v), \;\; f_2(t, \delta u) \leq \psi(\delta) f_2(t, u). \tag{22}$$

Remark 1. *If* ($\mathbf{P}_3$) *holds, for* $\delta \geq 1$*, by a simple proof, we have*

$$f_1(t, \delta v) \leq \phi^{-1}(\delta^{-1}) f_1(t, v), \;\; f_2(t, \delta u) \geq \psi^{-1}(\delta^{-1}) f_2(t, u). \tag{23}$$

Remark 2. *In [28], the following condition was employed to study the convergence analysis and error estimation for the unique solution of a single p-Laplacian fractional differential equation with singular decreasing nonlinearity,*

($\mathbf{F_1}$) $f \in C((0,1) \times (0,+\infty), [0,+\infty))$*, and* $f(t,u)$ *is decreasing in* u *and, for any* $\delta \in (0,1)$*, there exists a constant* $0 < a < \frac{1}{p-1}$ *such that, for any* $(t,u) \in (0,1) \times (0,+\infty)$*,*

$$f(t, \delta u) \leq \delta^{-a} f(t, u). \tag{24}$$

In recent work [67], Zhang et al. used the following condition to establish the convergence analysis of the unique solution for a Dirichlet problem of the general k-Hessian equation in a ball:

($\mathbf{F_2}$) $f : [0,+\infty) \to (0,+\infty)$ *is continuous and nondecreasing and, for any* $\delta \in (0,1)$*, there exists a constant* $0 < b < 1$ *such that, for any* $v \in [0,+\infty)$*,*

$$f(\delta v) \geq \delta^b f(v).$$

Clearly ($\mathbf{P}_3$) *generalizes and improves the conditions* ($\mathbf{F_1}$) *and* ($\mathbf{F_2}$) *and includes* ($\mathbf{F_1}$) *and* ($\mathbf{F_2}$) *as special cases.*

To facilitate further study, we define a subset of P,

$$K = \Big\{ u \in P : \text{there exists a number } \; 0 < L < 1 \; \text{ such that} \\ L e^{-\lambda t} t^{\beta-1} \leq u(t) \leq L^{-1} e^{-\lambda t} t^{\beta-1}, \;\; t \in [0,1] \Big\},$$

and state our main results as follows.

Theorem 1. *Assume that* ($\mathbf{P_1}$), ($\mathbf{P_2}$)*, and* ($\mathbf{P_3}$) *hold, then the following conclusions hold:*

($\mathbf{C_1}$) *Uniqueness: The tempered fractional system (1) has one unique positive solution* $(u^*(t), (Qu^*)(t))$ *in* $K \times K$.

($\mathbf{C_2}$) *Iterative sequence: For any initial value* $\nu_0 \in K$*, construct an iterative sequence as*

$$\nu_i(t) = \int_0^1 H_1(t,s) \left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} e^{\lambda \tau} f_1(\tau, (Q\nu_{i-1})(\tau)) d\tau \right)^{q-1} ds, t \in [0,1],$$

$$(Q\nu_{i-1})(t) = \int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\jmath)}(s-\tau)^{\jmath-1}e^{\lambda\tau}f_2(\tau,\nu_{i-1}(\tau))d\tau\right)^{q-1} ds, t \in [0,1].$$

Then,

$$\lim_{i\to+\infty}\nu_i(t) = u^*(t), \quad \lim_{i\to+\infty}(Q\nu_{i-1})(t) = (Qu^*)(t)$$

uniformly hold for $t \in [0,1]$.

($\mathbf{C_3}$) *Error estimation and convergence rate: There is an error estimation between* $u^*(t)$ *and the ith iterative value* $\nu_i(t)$,

$$||\nu_i - u^*|| \leq 2\left(1-\epsilon^{[ab(q-1)^2]^{2i}}\right)\epsilon^{-\frac{1}{2}},$$

where $\epsilon \in (0,1)$, *and the following convergence rate holds:*

$$||\nu_i - u^*|| = o\left(1-\epsilon^{[ab(q-1)^2]^{2i}}\right).$$

($\mathbf{C_4}$) *Asymptotic behavior: The unique solution* $(u^*(t),(Qu^*)(t))$ *of the tempered fractional system* (1) *has the following asymptotic behavior:*

There exist two positive constants $\varpi, \omega \in (0,1)$ *such that for any* $t \in [0,1]$,

$$\varpi e^{-\lambda t}t^{\beta-1} \leq u^*(t) \leq \varpi^{-1}e^{-\lambda t}t^{\beta-1},$$

$$\omega e^{-\lambda t}t^{\ell-1} \leq (Qu^*)(t) \leq \omega^{-1}e^{-\lambda t}t^{\ell-1}.$$

Proof. Let us firstly prove that $T: K \to K$ is a completely continuous operator. In fact, for any $u \in K$, according to the definition of K, there exists a constant $L_1 \in (0,1)$ such that

$$L_1 e^{-\lambda t}t^{\beta-1} \leq u(t) \leq L_1^{-1}e^{-\lambda t}t^{\beta-1}. \tag{25}$$

It follows from ($\mathbf{P_1}$)–($\mathbf{P_3}$), (25), and the Hölder inequality that

$$\begin{aligned}
(Qu)(t) &= \int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_2(\tau,u(\tau))d\tau\right)^{q-1} ds \\
&\leq e^{-\lambda t}t^{\ell-1}\int_0^1 A_4(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\jmath)}(s-\tau)^{\jmath-1}e^{\lambda\tau}f_2(\tau,L_1e^{-\lambda\tau}\tau^{\beta-1}\right)d\tau^{q-1} ds \\
&\leq e^{-\lambda t}t^{\ell-1}\int_0^1 A_4(s)\left(\frac{e^{-\lambda s}}{\Gamma(\jmath)}\right)^{q-1}\left(\int_0^s (s-\tau)^{\frac{\alpha-1}{1-\sigma}}d\tau\right)^{(1-\sigma)(q-1)} \\
&\times\left(\int_0^s e^{\frac{\lambda\tau}{\sigma}}f_2^{\frac{1}{\sigma}}(\tau,L_1e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} ds \\
&\leq \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)} e^{-\lambda t}t^{\ell-1}\int_0^1 \frac{\ell(1-s)^{\ell-1}(\ell-1+s)e^{\lambda s}}{(\ell-1)\Gamma(\ell+1)}\left(\frac{e^{-\lambda s}}{\Gamma(\jmath)}\right)^{q-1} \\
&\times s^{(\alpha-\sigma)(q-1)}ds\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_2^{\frac{1}{\sigma}}(\tau,L_1e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} \\
&\leq \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)} e^{-\lambda t}t^{\ell-1}\frac{\ell^2e^{\lambda}}{(\ell-1)\Gamma(\ell+1)}\left(\frac{1}{\Gamma(\jmath)}\right)^{q-1} \\
&\times\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_2^{\frac{1}{\sigma}}(\tau,L_1e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} \\
&\leq \left(\frac{(1-\sigma)^{(1-\sigma)}L_1^{-a}}{(\alpha-\sigma)^{(1-\sigma)}\Gamma(\jmath)}\right)^{q-1}\frac{\ell^2e^{\lambda}}{(\ell-1)\Gamma(\ell+1)} \\
&\times\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_2^{\frac{1}{\sigma}}(\tau,e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} e^{-\lambda t}t^{\ell-1}.
\end{aligned} \tag{26}$$

By a similar method, the following inequality is still valid:

$$\begin{aligned}(Qu)(t) &= \int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_2(\tau,u(\tau))d\tau\right)^{q-1}ds\\ &\geq e^{-\lambda t}t^{\ell-1}\int_0^1 A_3(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\jmath)}(s-\tau)^{\jmath-1}e^{\lambda\tau}f_2(\tau,u(t))d\tau\right)^{q-1}ds\\ &\geq e^{-\lambda t}t^{\ell-1}\int_0^1 A_3(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\jmath)}(s-\tau)^{\jmath-1}e^{\lambda\tau}f_2(\tau,L_1^{-1}e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds\\ &\geq e^{-\lambda t}t^{\ell-1}\left(\frac{L_1^a}{\Gamma(\jmath)}\right)^{(q-1)}\int_0^1 A_3(s)\left(e^{-\lambda s}\right)^{q-1}\left(\int_0^s (s-\tau)^{\jmath-1}e^{\lambda\tau}f_2(\tau,e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds.\end{aligned} \tag{27}$$

Take

$$L_1^* = \min\left\{\frac{1}{2}, \left(\frac{L_1^a}{\Gamma(\jmath)}\right)^{(q-1)}\int_0^1 A_3(s)\left(e^{-\lambda s}\right)^{q-1}\left(\int_0^s (s-\tau)^{\jmath-1}e^{\lambda\tau}f_2(\tau,e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds,\right.$$
$$\left.\left(\left(\frac{(1-\sigma)^{(1-\sigma)}L_1^{-a}}{(\alpha-\sigma)^{(1-\sigma)}\Gamma(\jmath)}\right)^{q-1}\frac{\ell^2 e^{\lambda}}{(\ell-1)\Gamma(\ell+1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_2^{\frac{1}{\sigma}}(\tau,e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)}\right)^{-1}\right\},$$

then we have

$$L_1^* e^{-\lambda t}t^{\ell-1} \leq (Qu)(t) \leq L_1^{*-1}e^{-\lambda t}t^{\ell-1}, \quad t\in[0,1].$$

Now, by $(\mathbf{P_2})$, $(\mathbf{P_3})$, and the monotonicity of f_1, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,(Qu)(\tau))d\tau\right)^{q-1}ds\\ &\leq e^{-\lambda t}t^{\beta-1}\int_0^1 A_2(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,L_1^{*-1}e^{-\lambda\tau}\tau^{\ell-1}\right)^{q-1}ds\\ &\leq e^{-\lambda t}t^{\beta-1}\int_0^1 A_2(s)\left(\frac{e^{-\lambda s}}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^s (s-\tau)^{\frac{\alpha-1}{1-\sigma}}d\tau\right)^{(1-\sigma)(q-1)}\\ &\times\left(\int_0^s e^{\frac{\lambda\tau}{\sigma}}f_1^{\frac{1}{\sigma}}(\tau,L_1^{*-1}e^{\lambda\tau}\tau^{\ell-1})d\tau\right)^{\sigma(q-1)}ds\\ &\leq \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\frac{\beta^2 e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}\left(\frac{1}{\Gamma(\alpha)}\right)^{q-1}\\ &\times\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_1^{\frac{1}{\sigma}}(\tau,L_1^{*-1}e^{-\lambda\tau}\tau^{\ell-1})d\tau\right)^{\sigma(q-1)}e^{-\lambda t}t^{\beta-1}\\ &= \left(\frac{(1-\sigma)^{(1-\sigma)}L_1^{*-b}}{(\alpha-\sigma)^{(1-\sigma)}\Gamma(\alpha)}\right)^{q-1}\frac{\beta^2 e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}\\ &\times\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_1^{\frac{1}{\sigma}}(\tau,e^{-\lambda\tau}\tau^{\ell-1})d\tau\right)^{\sigma(q-1)}e^{-\lambda t}t^{\beta-1}\\ &< +\infty.\end{aligned} \tag{28}$$

Therefore, the operator T is uniformly bounded.

On the other hand, notice that $H_1(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, thus for any $\varepsilon>0$, there exists $\zeta>0$ such that for any $0\le t_1<t_2\le 1$ and $|t_1-t_2|<\zeta$, we have

$$\begin{aligned}&|H(t_1,s)-H(t_2,s)|\\&<\left[\left(\frac{(1-\sigma)^{(1-\sigma)}L_1^{*-b}}{(\alpha-\sigma)^{(1-\sigma)}\Gamma(\alpha)}\right)^{q-1}\frac{\beta^2e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_1^{\frac{1}{\sigma}}(\tau,e^{-\lambda\tau}\tau^{\ell-1})d\tau\right)^{\sigma(q-1)}\right]^{-1}\varepsilon.\end{aligned}$$

Consequently, for any $u\in K$ and $|t_1-t_2|<\zeta$, it follows from $(\mathbf{P_2})$, $(\mathbf{P_3})$, (26), and (28) that

$$\begin{aligned}&|(Tu)(t_1)-(Tu)(t_2)|\\&\le\int_0^1|H(t_1,s)-H(t_2,s)|\left(\int_0^s\frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,(Qu)(\tau))d\tau\right)^{q-1}ds\\&\le\int_0^1|H(t_1,s)-H(t_2,s)|ds\\&\quad\times\left(\frac{(1-\sigma)^{(1-\sigma)}L_1^{*-b}}{(\alpha-\sigma)^{(1-\sigma)}\Gamma(\alpha)}\right)^{q-1}\frac{\beta^2e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_1^{\frac{1}{\sigma}}(\tau,e^{-\lambda\tau}\tau^{\ell-1})d\tau\right)^{\sigma(q-1)}\\&<\varepsilon,\end{aligned}\tag{29}$$

which implies that $T(K)$ is equicontinuous.

Next, we show that $T(K)\subset K$. From $(\mathbf{P_2})$, $(\mathbf{P_3})$, (26), and (27), we have

$$\begin{aligned}(Tu)(t)&=\int_0^1H_1(t,s)\left(\int_0^s\frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,(Qu)(\tau))d\tau\right)^{q-1}ds\\&\ge e^{-\lambda t}t^{\beta-1}\int_0^1A_1(s)\left(\int_0^s\frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,L_1^*e^{-\lambda\tau}\tau^{\ell-1}\right)^{q-1}ds\\&\ge e^{-\lambda t}t^{\beta-1}L_1^{*b(q-1)}\int_0^1A_1(s)\left(\int_0^s\frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,e^{-\lambda\tau}\tau^{\ell-1}\right)^{q-1}ds\\&\ge\left(\frac{L_1^{*b}}{\Gamma(\alpha)}\right)^{q-1}\int_0^1A_1(s)\left(e^{-\lambda s}\right)^{q-1}\left(\int_0^s(s-\tau)^{\alpha-1}f_1(\tau,e^{\lambda\tau}\tau^{\ell-1})d\tau\right)^{q-1}ds\,e^{-\lambda t}t^{\beta-1}.\end{aligned}\tag{30}$$

Take

$$\begin{aligned}L_T=\min\Bigg\{&\frac{1}{2},\left(\frac{L_1^{*b}}{\Gamma(\alpha)}\right)^{q-1}\int_0^1A_1(s)\left(e^{-\lambda s}\right)^{q-1}\left(\int_0^s(s-\tau)^{\alpha-1}f_1(\tau,e^{\lambda\tau}\tau^{\ell-1})d\tau\right)^{q-1}ds,\\&\left(\left(\frac{(1-\sigma)^{(1-\sigma)}L_1^{*-b}}{(\alpha-\sigma)^{(1-\sigma)}\Gamma(\alpha)}\right)^{q-1}\frac{\beta^2e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f_1^{\frac{1}{\sigma}}(\tau,e^{-\lambda\tau}\tau^{\ell-1})d\tau\right)^{\sigma(q-1)}\right)^{-1}\Bigg\},\end{aligned}$$

it follows from (28) and (30) that

$$L_Te^{-\lambda t}t^{\beta-1}\le(Tu)(t)\le L_T^{-1}e^{-\lambda t}t^{\beta-1},$$

which implies that $T(K)\subset K$. Obviously, T is a continuous operator; thus, according to the Arzela–Ascoli theorem, $T:K\to K$ is a completely continuous operator.

Next, let $\mu(t)=e^{-\lambda t}t^{\beta-1}$. Obviously $\mu\in K$, and then, $T\mu(t)\in K$, thus there is a constant $L_T^*\in(0,1)$ such that

$$L_T^*\mu(t)\le T\mu(t)\le L_T^{*-1}\mu(t).\tag{31}$$

Fixing $0 < \gamma < 1$, since $ab(q-1)^2 \in (0,1)$, one has

$$\lim_{\vartheta \to +\infty} \gamma^{\vartheta[-ab(q-1)^2+1]} = 0,$$

which implies that there is a sufficiently large real number $\vartheta > 1$ such that

$$\gamma^{\vartheta[-ab(q-1)^2+1]} \leq L_T^*. \tag{32}$$

In the following, we select $u_0(t) = \gamma^{\vartheta}\mu(t)$ as an initial value; clearly,

$$\begin{aligned}(Qu_0)(t) &= \int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_2(\tau,\gamma^{\vartheta}\mu(\tau))d\tau\right)^{q-1} ds \\ &\leq \gamma^{-a\vartheta(q-1)}\int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_2(\tau,\mu(\tau))d\tau\right)^{q-1} ds \\ &= \gamma^{-a\vartheta(q-1)}(Q\mu)(t).\end{aligned} \tag{33}$$

Let $u_1(t) = (Tu_0)(t)$, and construct an iterative sequence as follows:

$$u_i(t) = (Tu_{i-1})(t) = \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,(Qu_{i-1})(\tau))d\tau\right)^{q-1} ds, \ \ i = 1,2,3,\cdots. \tag{34}$$

Since the operator T is decreasing in u, by (32)–(34), we have

$$u_0(t) \leq \mu(t),$$

and

$$u_1(t) = Tu_0(t) \geq T\mu(t) \geq L_T^*\mu(t) \geq \gamma^{\vartheta[-ab(q-1)^2+1]}\mu(t) = \gamma^{-\vartheta(q-1)^2ab}\gamma^{\vartheta}\mu(t) \geq u_0(t).$$

Consequently,

$$u_2(t) = (Tu_1)(t) \leq (Tu_0)(t) = u_1(t).$$

As a result, by (28), (32), and (33), one obtains

$$\begin{aligned}u_1(t) = (Tu_0)(t) &= \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,(Qu_0)(\tau))d\tau\right)^{q-1} ds \\ &\leq \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,\gamma^{-\vartheta a(q-1)}Q\mu(\tau))d\tau\right)^{q-1} ds \\ &\leq \frac{1}{[\phi(\gamma^{\vartheta a(q-1)})]^{q-1}}\int_0^1 H(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,Q\mu(\tau))d\tau\right)^{q-1} ds \\ &\leq \gamma^{-\vartheta ab(q-1)^2}T\mu(t) \leq \gamma^{-\vartheta ab(q-1)^2}L_T^{*\,-1}\mu(t) \leq \gamma^{-\vartheta}\gamma^{\vartheta[-ab(q-1)^2+1]}L_T^{*\,-1}\mu(t) \\ &\leq \gamma^{-\vartheta}\mu(t).\end{aligned} \tag{35}$$

It follows from (35) that

$$\begin{aligned}u_2(t) &= Tu_1(t) \geq T\left(\gamma^{-\vartheta}\mu(t)\right) \\ &= \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,Q(\gamma^{-\vartheta}\mu(\tau)))d\tau\right)^{q-1} ds \\ &\geq \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f_1(\tau,\gamma^{\vartheta a(q-1)}Q\mu(\tau))d\tau\right)^{q-1} ds \\ &\geq \gamma^{\vartheta ab(q-1)^2}T\mu(t) \geq \gamma^{\vartheta ab(q-1)^2}L_T^*\mu(t) \geq \gamma^{\vartheta}\mu(t) = u_0(t),\end{aligned} \tag{36}$$

which implies that

$$u_0 \leq u_2 \leq u_1.$$

Thus, according to induction, we have

$$u_0 \leq u_2 \leq, \ldots \leq u_{2i} \leq \ldots \leq u_{2i+1} \leq \ldots \leq u_3 \leq u_1. \tag{37}$$

On the other hand, for any fixed $\xi \in (0,1)$, from $(\mathbf{P_3})$, one has

$$T(\xi u) \leq \xi^{-ab(q-1)^2}(Tu), \quad T^2(\xi u) \geq \xi^{[(q-1)^2 ab]^2} T^2 u. \tag{38}$$

Notice that the operator T^2 is nondecreasing in u; from (35), (37), and (38), one obtains

$$\begin{aligned}
&u_{2i}(t) = Tu_{2i-1}(t) = T^{2i}u_0(t) = T^{2i}(\gamma^{\vartheta}\mu(t)) = T^{2i}(\gamma^{2\vartheta}\gamma^{-\vartheta}\mu(t)) \\
&\geq T^{2i}(\gamma^{2\vartheta}u_1(t)) \geq T^{2i-2}\left(\left(\gamma^{2\vartheta}\right)^{[ab(q-1)^2]^2} T^2 u_1(t)\right) \\
&\geq T^{2i-4}\left(\left(\gamma^{2\vartheta}\right)^{[ab(q-1)^2]^4} T^4 u_1(t)\right) \geq \ldots \geq \left(\gamma^{2\vartheta}\right)^{[ab(q-1)^2]^{2i}} T^{2i} u_1(t) \\
&= \left(\gamma^{2\vartheta}\right)^{[ab(q-1)^2]^{2i}} T^{2i+1} u_0(t) = \left(\gamma^{2\vartheta}\right)^{[ab(q-1)^2]^{2i}} u_{2i+1}(t),
\end{aligned} \tag{39}$$

that is,

$$u_{2i}(t) \geq \gamma^{2\vartheta[ab(q-1)^2]^{2i}} u_{2i+1}(t).$$

Consequently, for any $i, j \in \mathbb{N}$, we have

$$\begin{aligned}
&0 \leq u_{2(i+j)}(t) - u_{2i}(t) \leq u_{2i+1}(t) - u_{2i}(t) \\
&\leq (1 - \gamma^{2\vartheta[ab(q-1)^2]^{2i}})u_{2i+1}(t) \leq (1 - \gamma^{2\vartheta[ab(q-1)^2]^{2i}})u_1(t) \\
&\leq (1 - \gamma^{2\vartheta[ab(q-1)^2]^{2i}})\gamma^{-\vartheta}\mu(t)
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&0 \leq u_{2i+1}(t) - u_{2(i+j)+1}(t) \leq u_{2i+1}(t) - u_{2i}(t) \\
&\leq (1 - \gamma^{2\vartheta[ab(q-1)^2]^{2i}})\gamma^{-\vartheta}\mu(t).
\end{aligned} \tag{41}$$

Thus, for any $j \in \mathbb{N}$, by the normality of P and (40) and (41), we obtain

$$||u_{i+j}(t) - u_i(t)|| \leq (1 - \gamma^{2\vartheta[ab(q-1)^2]^{2i}})\gamma^{-\vartheta} \to 0, \quad i \to +\infty,$$

which indicates that $\{u_i\}_{i \geq 1}$ is a Cauchy sequence in K, and then, there exists $u^* \in K$ such that

$$u_i \to u^*, \quad i \to \infty,$$

and (37) implies that

$$u_{2i} \leq u^* \leq u_{2i+1}.$$

Thus, it follows from the fact that T is a decreasing operator that

$$u_{2i+2}(t) = Tu_{2i+1}(t) \leq Tu^* \leq Tu_{2i}(t) = u_{2i+1}(t).$$

Taking the limit on both sides of the above inequality, we have

$$u^* = Tu^*,$$

which implies that $u^*(t)$ is a positive solution of (19) and $(u^*(t), (Qu^*)(t))$ is a positive solution of the tempered fractional system (1).

In the following, we assert that $(u^*(t), (Qu^*)(t))$ is also a unique solution of the tempered fractional system (1) in $K \times K$. In fact, if $(\nu(t), (Q\nu(t)))$ is another solution of the tempered fractional system (1), let

$$\theta_* = \sup\{\theta > 0 | \nu \geq \theta_* u^*\}.$$

Obviously, $\theta_* \in (0, +\infty)$. Now, we show $\theta_* \geq 1$. If not, one has $0 < \theta_* < 1$, which leads to

$$\nu(t) = T\nu(t) = T^2\nu \geq T^2(\theta_* u^*(t)) \geq \theta_*^{[ab(q-1)^2]^2} T^2 u^*(t) = \theta_*^{[ab(q-1)^2]^2} u^*(t).$$

From the definition of θ, one obtains $\theta_* \geq \theta_*^{[ab(q-1)^2]^2}$. However, since $ab(q-1)^2 \in (0,1)$ and $\theta_* \in (0,1)$, we have $\theta_* < \theta_*^{[ab(q-1)^2]^2}$, which is a contradiction. Thus, $\theta_* \geq 1$, which yields $\nu \geq u^*$. Follows the same strategy, we also have $\nu \leq u^*$, thus $u^* = \nu$. So $u^*(t)$ is the unique solution of Equation (19), and thus, the solution of system (1) in $K \times K$ is unique.

Finally, we focus on the iterative properties of the unique solution of system (1). Take any $\nu_0 \in K$ as the initial value and construct an iterative sequence:

$$\begin{aligned} \nu_i(t) &= T\nu_{i-1}(t) \\ &= \int_0^1 H_1(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau} f_1(\tau, (Q\nu_{i-1})(\tau))d\tau\right)^{q-1} ds, \quad t \in [0,1], \quad i = 1,2,3,\cdots, \end{aligned} \tag{42}$$

and

$$\begin{aligned} &(Q\nu_{i-1})(t) \\ &= \int_0^1 H_2(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau} f_2(\tau, \nu_{i-1}(\tau))d\tau\right)^{q-1} ds, \quad t \in [0,1], \quad i = 1,2,3,\cdots. \end{aligned} \tag{43}$$

Since $T(K) \subset K$ and $\nu_1 = T\nu_0 \in K$, there exist two constants $L_0^*, L_1^* \in (0,1)$ such that

$$L_0^* \mu(t) \leq \nu_0(t) \leq L_0^{*-1}\mu(t), \quad L_1^*\mu(t) \leq \nu_1(t) \leq L_1^{*-1}\mu(t).$$

As

$$\lim_{\vartheta \to +\infty} \gamma^{\vartheta[-ab(q-1)^2+1]} = 0,$$

take a large enough constant $\vartheta > 1$ such that

$$\gamma^{\vartheta[-ab(q-1)^2+1]} \leq \min\{L_0^*, L_1^*, L_T^*\}.$$

For this ϑ, still let $u_0(t) = \gamma^\vartheta \mu(t)$; then, the iterative process of (33)–(41) is valid. Thus, we have

$$u_0(t) = \gamma^\vartheta\mu(t) \leq \gamma^{-\vartheta ab(q-1)^2}\gamma^\vartheta\mu(t) \leq L_0^*\mu(t) \leq \nu_0(t),$$
$$u_0(t) = \gamma^\vartheta\mu(t) \leq \gamma^{-\vartheta ab(q-1)^2}\gamma^\vartheta\mu(t) \leq L_1^*\mu(t) \leq \nu_1(t),$$

which yield

$$\begin{aligned} &\nu_1(t) = T\nu_0(t) \leq Tu_0(t) = u_1(t), \\ &u_0(t) \leq \nu_1(t) \leq u_1(t), \\ &u_2(t) \leq \nu_2(t) \leq u_1(t), \end{aligned} \tag{44}$$

and the continuous iteration

$$\begin{aligned} &u_{2i}(t) \leq \nu_{2i+1}(t) \leq u_{2i+1}(t), \\ &u_{2i+2}(t) \leq \nu_{2i+2}(t) \leq u_{2i+1}(t). \end{aligned} \tag{45}$$

Taking the limit on both sides of (45), one obtains $\nu_i \to u^*$ and $(Q\nu_i) \to (Qu^*)$ as $i \to \infty$. Meanwhile, according to (40), (41), and (45), one obtains

$$\begin{aligned} &||\nu_{2i+1}(t) - u^*(t)|| \le ||\nu_{2i+1}(t) - u_{2i}(t)|| + ||u_{2i}(t) - u^*(t)|| \\ &\le ||u_{2i+1}(t) - u_{2i}(t)|| + ||u_{2i}(t) - u^*(t)|| \\ &\le ||u_{2i+1}(t) - u_{2i}(t)|| + ||u_{2i+1}(t) - u_{2i}(t)|| \\ &\le 2(1 - \gamma^{2\vartheta[ab(q-1)^2]^{2i}})\gamma^{-\vartheta} \\ &= 2\Big(1 - \epsilon^{[ab(q-1)^2]^{2i}}\Big)\epsilon^{-\frac{1}{2}}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} &||\nu_{2i+1}(t) - u^*(t)|| \le ||\nu_{2i+2}(t) - u_{2i+2}(t)|| + ||u_{2i+2}(t) - u^*(t)|| \\ &\le ||u_{2i+1}(t) - u_{2i+2}(t)|| + ||u_{2i+2}(t) - u^*(t)|| \\ &\le ||u_{2i+1}(t) - u_{2i+2}(t)|| + ||u_{2i+1}(t) - u_{2i+2}(t)|| \\ &\le 2(1 - \gamma^{2\vartheta[ab(q-1)^2]^{2i}})\gamma^{-\vartheta} \\ &= 2\Big(1 - \epsilon^{[ab(q-1)^2]^{2i}}\Big)\epsilon^{-\frac{1}{2}}. \end{aligned} \tag{47}$$

It follows from (46) and (47) that

$$||\nu_i(t) - u^*(t)|| \le 2\Big(1 - \epsilon^{[ab(q-1)^2]^{2i}}\Big)\epsilon^{-\frac{1}{2}},$$

where $\epsilon = \gamma^{2\vartheta} \in (0,1)$ is a constant determined by $\mu(t)$ and ν_0. Moreover, there is a accurate convergence rate:

$$||\nu_i - u^*|| = o\Big(1 - \epsilon^{[ab(q-1)^2]^{2i}}\Big).$$

Finally, it follows from $u^* \in K$, (26)–(28), and (30) that there exist two constants $\varpi, \omega \in (0,1)$ such that for any $t \in [0,1]$,

$$\varpi e^{-\lambda t}t^{\beta-1} \le u^*(t) \le \varpi^{-1}e^{-\lambda t}t^{\beta-1},$$

$$\omega e^{-\lambda t}t^{\ell-1} \le (Qu^*)(t) \le \omega^{-1}e^{-\lambda t}t^{\ell-1}.$$

□

Consequently, by the above proofs, we can conclude that the above conclusions are valid.

4. Numerical Results

Example 1. *In system* (1), *take*

$$\alpha = \frac{1}{2}, \beta = \frac{3}{2}, \lambda = 2, p = \frac{3}{2}, \jmath = \frac{1}{3}, \ell = \frac{4}{3}$$

and

$$f_1(t,v) = (1 + t^{\frac{1}{3}})^3(1 + v^{\frac{1}{2}}),\ f_2(t,u) = t^{-\frac{1}{3}}(1 - t^{\frac{1}{2}})^{\frac{1}{2}}u^{-\frac{1}{6}},$$

then we have

$$H_1(t,s) = \begin{cases} \dfrac{\Big[\frac{3}{2}(1-s)^{\frac{1}{2}}(\frac{1}{2}+s)t^{\frac{1}{2}} - \frac{3}{4}(t-s)^{\frac{1}{2}}\Big]e^{-2t}e^{2s}}{\frac{1}{2}\Gamma(\frac{5}{2})}, 0 \le s \le t \le 1; \\ \dfrac{\frac{3}{2}(1-s)^{\frac{1}{2}}(\frac{1}{2}+s)t^{\frac{1}{2}}e^{-2t}e^{2s}}{\frac{1}{2}\Gamma(\frac{5}{2})}, 0 \le t \le s \le 1, \end{cases}$$

and

$$H_2(t,s)=\begin{cases}\dfrac{\left[\frac{4}{3}(1-s)^{\frac{1}{3}}(\frac{1}{3}+s)t^{\frac{1}{3}}-\frac{4}{9}(t-s)^{\frac{1}{3}}\right]e^{-2t}e^{2s}}{\frac{1}{3}\Gamma(\frac{7}{3})}, 0\le s\le t\le 1;\\ \dfrac{\frac{4}{3}(1-s)^{\frac{1}{3}}(\frac{1}{3}+s)t^{\frac{1}{3}}e^{-2t}e^{2s}}{\frac{1}{3}\Gamma(\frac{7}{3})}, 0\le t\le s\le 1.\end{cases}$$

Consider the following coupled p-Laplacian system of singular tempered fractional equations with different characteristics:

$$\begin{cases}{}_0^R\mathbb{D}_t^{\frac{1}{2},2}\Big(\varphi_{\frac{3}{2}}({}_0^R\mathbb{D}_t^{\frac{3}{2},2}u(t))\Big)=(1+t^{\frac{1}{3}})^3(1+v^{\frac{1}{2}}),\\ {}_0^R\mathbb{D}_t^{\frac{1}{3},2}\Big(\varphi_{\frac{3}{2}}({}_0^R\mathbb{D}_t^{\frac{4}{3},2}v(t))\Big)=t^{-\frac{1}{3}}(1-t^{\frac{1}{2}})^{\frac{1}{2}}u^{-\frac{1}{6}},\\ u(0)=v(0)=0,\\ {}_0^R\mathbb{D}_t^{\frac{3}{2},2}u(0)={}_0^R\mathbb{D}_t^{\frac{4}{3},2}v(0)=0,\\ u(1)=\displaystyle\int_0^1 e^{-2(1-t)}u(t)dt,\quad v(1)=\int_0^1 e^{-2(1-t)}v(t)dt.\end{cases}\tag{48}$$

For system (48), the following conclusions hold:

(**i**) *The singular tempered fractional system (48) has one unique positive solution $(u^*,(Qu^*))$ in $K\times K$.*

(**ii**) *For any initial value $\nu_0\in K$, construct the iterative sequences*

$$\begin{aligned}\nu_i(t)&=\\ &\int_0^t\frac{\left[\frac{3}{2}(1-s)^{\frac{1}{2}}(\frac{1}{2}+s)t^{\frac{1}{2}}-\frac{3}{4}(t-s)^{\frac{1}{2}}\right]e^{-2t}}{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}\left(\int_0^s(s-\tau)^{-\frac{1}{2}}e^{2\tau}(1+\tau^{\frac{1}{3}})^3(1+((Q\nu_{i-1}))^{\frac{1}{2}}(\tau))d\tau\right)^{-\frac{1}{2}}ds\\ &+\int_t^1\frac{\frac{3}{2}(1-s)^{\frac{1}{2}}(\frac{1}{2}+s)t^{\frac{1}{2}}e^{-2t}}{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}\left(\int_0^s(s-\tau)^{-\frac{1}{2}}e^{2\tau}(1+\tau^{\frac{1}{3}})^3(1+((Q\nu_{i-1}))^{\frac{1}{2}}(\tau))d\tau\right)^{-\frac{1}{2}}ds,\\ (Q\nu_{i-1})(t)&=\\ &\int_0^t\frac{\left[\frac{4}{3}(1-s)^{\frac{1}{3}}(\frac{1}{3}+s)t^{\frac{1}{3}}-\frac{4}{9}(t-s)^{\frac{1}{3}}\right]e^{-2t}}{\Gamma(\frac{4}{3})\Gamma(\frac{7}{3})}\left(\int_0^s(s-\tau)^{-\frac{2}{3}}e^{2\tau}\tau^{-\frac{1}{3}}(1-\tau^{\frac{1}{2}})^{\frac{1}{2}}\nu_{i-1}^{-\frac{1}{6}}(\tau)d\tau\right)^{-\frac{1}{2}}ds\\ &+\int_t^1\frac{\frac{4}{3}(1-s)^{\frac{1}{3}}(\frac{1}{3}+s)t^{\frac{1}{3}}e^{-2t}}{\Gamma(\frac{4}{3})\Gamma(\frac{7}{3})}\left(\int_0^s(s-\tau)^{-\frac{2}{3}}e^{2\tau}\tau^{-\frac{1}{3}}(1-\tau^{\frac{1}{2}})^{\frac{1}{2}}\nu_{i-1}^{-\frac{1}{6}}(\tau)d\tau\right)^{-\frac{1}{2}}ds.\end{aligned}$$

Then, we have that

$$\lim_{i\to+\infty}\nu_i(t)=u^*(t),\quad \lim_{i\to+\infty}(Q\nu_{i-1})(t)=(Qu^*)(t)$$

uniformly holds for $t\in[0,1]$.

(**iii**) *The error between $u^*(t)$ and the iterative value $\nu_i(t)$ can be expressed by*

$$||\nu_i-u^*||\le 2\Big(1-\epsilon^{(\frac{1}{9})^i}\Big)\epsilon^{-\frac{1}{2}},$$

where $\epsilon\in(0,1)$, and the convergence rate is

$$||\nu_i-u^*||=o\Big(1-\epsilon^{(\frac{1}{9})^i}\Big).$$

(**iv**) *There exist two constants $\varpi,\omega\in(0,1)$ such that*

$$\varpi e^{-2t}t^{\frac{1}{2}}\le u^*(t)\le\varpi^{-1}e^{-2t}t^{\frac{1}{2}}, t\in[0,1],$$

$$\omega e^{-2t}t^{\frac{1}{3}} \leq (Qu^*)(t) \leq \omega^{-1}e^{-2t}t^{\frac{1}{3}}, t \in [0,1].$$

Proof. Take $a = \frac{1}{2}, b = \frac{1}{3}$, $\phi(\delta) = 1 + \delta^{\frac{1}{2}}$, $\psi(\delta) = \delta^{-\frac{1}{6}}$, then $\phi(\delta) > \delta^b, \psi(\delta) < \delta^{-a}$, and $0 < ab(q-1)^2 = \frac{2}{3} < 1$. Thus, $(\mathbf{P_1})$ and $(\mathbf{P_3})$ are satisfied.

Now, we verify $(\mathbf{P_2})$. Taking $\sigma = 2$, we have

$$0 < \int_0^1 e^t f_1^{\frac{1}{2}}(t, e^{-2t}t^{\frac{1}{3}})dt = \int_0^1 e^t(1 + t^{\frac{1}{3}})^{\frac{3}{2}}(1 + e^{-t}t^{\frac{1}{6}})^{\frac{1}{2}}dt \leq 4e < +\infty,$$

$$0 < \int_0^1 e^t f_2^{\frac{1}{2}}(t, e^{-2t}t^{\frac{1}{2}})dt = \int_0^1 e^{\frac{7t}{6}}t^{-\frac{5}{24}}(1 - t^{\frac{1}{2}})^{\frac{1}{4}}dt \leq \frac{24}{19}e^{\frac{7}{6}} < +\infty.$$

Thus, the condition $(\mathbf{P_2})$ holds; according to Theorem 1, the above conclusions hold. □

Example 2. *In system* (1), *take*

$$\alpha = \frac{1}{4}, \beta = \frac{5}{4}, \lambda = 3, p = 4, \jmath = \frac{1}{5}, \ell = \frac{6}{5}$$

and

$$f_1(t,v) = t^{-\frac{1}{12}}(1 - t^{\frac{1}{4}})^{\frac{1}{3}}v^{\frac{2}{3}},\ f_2(t,u) = t^{-\frac{1}{10}}(1 - t^{\frac{1}{3}})^{\frac{1}{2}}u^{-\frac{1}{5}},$$

then we have

$$H_1(t,s) = \begin{cases} \dfrac{\left[\frac{5}{4}(1-s)^{\frac{1}{4}}(\frac{1}{4}+s)t^{\frac{1}{4}} - \frac{5}{16}(t-s)^{\frac{1}{4}}\right]e^{-3t}e^{3s}}{\frac{1}{4}\Gamma(\frac{9}{4})}, 0 \leq s \leq t \leq 1; \\ \dfrac{\frac{5}{4}(1-s)^{\frac{1}{4}}(\frac{1}{4}+s)t^{\frac{1}{4}}e^{-3t}e^{3s}}{\frac{1}{4}\Gamma(\frac{9}{4})}, 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$H_2(t,s) = \begin{cases} \dfrac{\left[\frac{6}{5}(1-s)^{\frac{1}{5}}(\frac{1}{5}+s)t^{\frac{1}{5}} - \frac{6}{25}(t-s)^{\frac{1}{5}}\right]e^{-3t}e^{3s}}{\frac{1}{5}\Gamma(\frac{11}{5})}, 0 \leq s \leq t \leq 1; \\ \dfrac{\frac{6}{5}(1-s)^{\frac{1}{5}}(\frac{1}{5}+s)t^{\frac{1}{5}}e^{-3t}e^{3s}}{\frac{1}{5}\Gamma(\frac{11}{5})}, 0 \leq t \leq s \leq 1. \end{cases}$$

Consider the following coupled p-Laplacian system of singular tempered fractional equations with different characteristics:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\frac{1}{4},3}\left(\varphi_3({}_0^R\mathbb{D}_t^{\frac{5}{4},3}u(t))\right) = t^{-\frac{1}{12}}(1 - t^{\frac{1}{4}})^{\frac{1}{3}}v^{\frac{2}{3}}, \\ {}_0^R\mathbb{D}_t^{\frac{1}{5},3}\left(\varphi_3({}_0^R\mathbb{D}_t^{\frac{6}{5},3}v(t))\right) = t^{-\frac{1}{10}}(1 - t^{\frac{1}{3}})^{\frac{1}{2}}u^{-\frac{1}{5}}, \\ u(0) = v(0) = 0, \\ {}_0^R\mathbb{D}_t^{\frac{5}{4},3}u(0) = {}_0^R\mathbb{D}_t^{\frac{6}{5},3}v(0) = 0, \\ u(1) = \int_0^1 e^{-2(1-t)}u(t)dt, \quad v(1) = \int_0^1 e^{-2(1-t)}v(t)dt. \end{cases} \tag{49}$$

For system (49), *the following conclusions hold:*

(**i**) *The singular tempered fractional system* (49) *has one unique positive solution* $(u^*, (Qu^*))$

in $K \times K$.

(**ii**) *For any initial value* $\nu_0 \in K$, *construct the iterative sequences*

$$\nu_i(t) = \int_0^t \frac{\left[5(1-s)^{\frac{1}{4}}(\frac{1}{4}+s)t^{\frac{1}{4}} - \frac{5}{4}(t-s)^{\frac{1}{4}}\right]e^{-3t}e^{\frac{3}{2}s}}{\Gamma(\frac{9}{4})(\Gamma(\frac{1}{4})^{\frac{1}{2}}}$$
$$\times \left(\int_0^s (s-\tau)^{-\frac{3}{4}} e^{3\tau}\tau^{-\frac{1}{12}}(1-\tau^{\frac{1}{4}})^{\frac{1}{3}}(Q\nu_{i-1})^{\frac{2}{3}}(\tau)d\tau\right)^{\frac{1}{2}} ds$$
$$+ \int_t^1 \frac{5(1-s)^{\frac{1}{4}}(\frac{1}{4}+s)t^{\frac{1}{4}}e^{-3t}e^{\frac{3}{2}s}}{\Gamma(\frac{9}{4})(\Gamma(\frac{1}{4})^{\frac{1}{2}}} \left(\int_0^s (s-\tau)^{-\frac{3}{4}} e^{3\tau}\tau^{-\frac{1}{12}}(1-\tau^{\frac{1}{4}})^{\frac{1}{3}}(Q\nu_{i-1})^{\frac{2}{3}}(\tau)d\tau\right)^{\frac{1}{2}} ds,$$

$$(Q\nu_{i-1})(t) = \int_0^t \frac{\left[6(1-s)^{\frac{1}{5}}(\frac{1}{5}+s)t^{\frac{1}{5}} - \frac{6}{5}(t-s)^{\frac{1}{5}}\right]e^{-3t}e^{\frac{3}{2}s}}{\Gamma(\frac{11}{5})(\Gamma(\frac{1}{5}))^{\frac{1}{2}}}$$
$$\times \left(\int_0^s (s-\tau)^{-\frac{2}{3}} e^{2\tau}\tau^{-\frac{1}{10}}(1-\tau^{\frac{1}{3}})^{\frac{1}{2}}\nu_{i-1}^{-\frac{1}{5}}(\tau)d\tau\right)^{\frac{1}{2}} ds$$
$$+ \int_t^1 \frac{6(1-s)^{\frac{1}{5}}(\frac{1}{5}+s)t^{\frac{1}{5}}e^{-3t}e^{\frac{3}{2}s}}{\Gamma(\frac{11}{5})(\Gamma(\frac{1}{5}))^{\frac{1}{2}}} \left(\int_0^s (s-\tau)^{-\frac{2}{3}} e^{2\tau}\tau^{-\frac{1}{10}}(1-\tau^{\frac{1}{3}})^{\frac{1}{2}}\nu_{i-1}^{-\frac{1}{5}}(\tau)d\tau\right)^{\frac{1}{2}} ds.$$

Then, we have that

$$\lim_{i\to+\infty} \nu_i(t) = u^*(t), \quad \lim_{i\to+\infty} (Q\nu_{i-1})(t) = (Qu^*)(t)$$

uniformly holds for $t \in [0,1]$.

(**iii**) *The error between* $u^*(t)$ *and the iterative value* $\nu_i(t)$ *can be expressed by*

$$||\nu_i - u^*|| \leq 2\left(1 - \epsilon^{(\frac{4}{27^2})^i}\right)\epsilon^{-\frac{1}{2}},$$

where $\epsilon \in (0,1)$, *and the convergence rate is*

$$||\nu_i - u^*|| = o\left(1 - \epsilon^{(\frac{4}{27^2})^i}\right).$$

(**iv**) *There exist two constants* $\varpi, \omega \in (0,1)$ *such that*

$$\varpi e^{-3t}t^{\frac{1}{4}} \leq u^*(t) \leq \varpi^{-1}e^{-3t}t^{\frac{1}{4}}, t \in [0,1],$$

$$\omega e^{-3t}t^{\frac{1}{5}} \leq (Qu^*)(t) \leq \omega^{-1}e^{-3t}t^{\frac{1}{5}}, t \in [0,1].$$

Proof. Take $a = 2, b = \frac{1}{3}$, $\phi(\delta) = (1+\delta)^{\frac{1}{3}}$, $\psi(\delta) = (1+\delta)^{-\frac{1}{3}}$, then

$$\phi(\delta) = (1+\delta)^{\frac{1}{3}} > \delta^{\frac{1}{3}}, \ \psi(\delta) = (1+\delta)^{-\frac{1}{3}} < \delta^{-2},$$

and $0 < ab(q-1)^2 = \frac{2}{27} < 1$. Thus, ($\mathbf{P_1}$) and ($\mathbf{P_3}$) are satisfied.

Now, take $\sigma = \frac{1}{6}$, we have

$$0 < \int_0^1 e^{18t} f_1^6(t, e^{-3t}t^{\frac{1}{5}})dt = \int_0^1 e^{18t}\left(t^{-\frac{1}{12}}(1-t^{\frac{1}{4}})^{\frac{1}{3}}\left(e^{-3t}t^{\frac{1}{5}}\right)^{\frac{2}{3}}\right)^6 dt \leq \frac{30}{39}e^6 < +\infty,$$

$$0 < \int_0^1 e^{18t} f_2^6(t, e^{-3t}t^{\frac{1}{4}})dt = \int_0^1 \left(t^{-\frac{1}{10}}(1-t^{\frac{1}{3}})^{\frac{1}{2}}\left(e^{-3t}t^{\frac{1}{4}}\right)^{-\frac{1}{5}}\right)^6 dt \leq \frac{5}{108}(e^{\frac{108}{5}} - 1) < +\infty.$$

Thus, the condition $(\mathbf{P_2})$ holds; according to Theorem 1, the above conclusions hold. □

5. Conclusions

Tempered fractional system can describe some dynamics phenomena arising from a particle's random walk in Brownian motion, where the transition has a feature of a semi-heavy tail from a power law to a Gaussian. In this paper, by proposing a coupled iterative technique and offering some suitable growth conditions, we establish the uniqueness of the positive solution for a coupled p-Laplacian system of singular tempered fractional equations with differential order and characteristics. Moreover, some elaborate iterative properties of positive solutions of the system are given, such as the convergence analysis, the asymptotic behavior, error estimation as well as the convergence rate of the positive solution. The interesting points of this paper are that the order of the system of equations is different and the nonlinear terms of the system possess the opposite monotonicity and permit stronger singularities at space variables. Finally, we shall also address that in this paper we only consider the coupled case of a system by coupling techniques; if the nonlinearities of the system rely on both time variables and space variables, then further study will become more challenging and interesting.

Author Contributions: Investigation, P.C., Y.W. (Ying Wang) and Y.W. (Yonghong Wu); writing—original draft preparation, P.C., Y.W. (Ying Wang), Y.W. (Yonghong Wu) and X.Z.; Writing—review & editing, X.Z.; Supervision, X.Z. The study was carried out in collaboration among all authors. All authors have read and agreed to the published version of the manuscript.

Funding: The authors are supported financially by the Natural Science Foundation of Shandong Province of China (ZR2022AM015) and an ARC Discovery Project Grant.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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