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Article

Leader-Following Output Feedback H_∞ Consensus of Fractional-Order Multi-Agent Systems with Input Saturation

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Abstract: This paper investigates the leader-following H_∞ consensus of fractional-order multi-agent systems (FOMASs) under input saturation via the output feedback. Based on the bounded real lemma for FOSs, the sufficient conditions of H_∞ consensus for FOMASs are provided in $\alpha \in (0,1)$ and $[1,2)$, respectively. Furthermore, the iterative linear matrix inequalities (ILMIs) approaches are applied for solving quadratic matrix inequalities (QMIs). The ILMI algorithms show a method to derive initial values and transform QMIs into LMIs. Mathematical tools are employed to transform the input saturation issue into optimal solutions of LMIs for estimating stable regions. The ILMI algorithms avoid the conditional constraints on matrix variables during the LMIs' construction and reduce conservatism. The approach does not disassemble the entire MASs by transformations to the Laplacian matrix, instead adopting a holistic analytical perspective to obtain gain matrices. Finally, numerical examples are conducted to validate the efficiency of the approach.

Keywords: fractional-order multi-agent systems; H_∞ control; static output feedback; iterative linear matrix inequality

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1. Introduction

In recent decades, multi-agent systems (MASs) have been the subject of extensive research, resulting in substantial achievements in the field. MASs are utilized in many practical domains, including traffic management [1] and power systems [2,3]. These applications rely significantly on the consensus of MASs, which involves the states of agents converging to a desired state based on information from neighbouring agents [4]. Extensive studies have been conducted on the consensus for MASs with integer-order differential models. However, integer-order models are inadequate for accurately representing some non-classical phenomena in various physical systems.

With the advancement of fractional calculus theory, fractional-order systems (FOSs) and fractional-order MASs (FOMASs) have emerged as a significant direction. Many achievements have been made in integer-order systems [5–7], but the utilization of fractional derivatives allows for a more comprehensive understanding of the characteristics of materials and systems exhibiting power-law, nonlocal, or long-term memory. FOSs offer enhanced capabilities for modelling and analysing complex systems, e.g., electrical systems [8,9], economic systems [10], motion models [11], and biological models [12,13]. The stability of control systems is a fundamental problem, certainly including those involving FOSs. It is not possible to derive the stability criteria of FOSs directly from those of integer-order systems. Fortunately, in [14], the authors establish LMI-based stability criteria and design a method for robust feedback state stabilization control for commensurate FOSs with $\alpha \in (0,1)$ and $\alpha \in [1,2)$. In [15], a necessary and sufficient condition for unified LMI formulation is provided to ensure the stability of FOSs within $\alpha \in (0,2)$. The aforementioned study makes a contribution to the consensus problem of FOMASs. In leader-following consensus, the state of followers needs to be consistent with the leader. In [16], the leader-following consensus of FOMASs with time delay is studied. Using the Lyapunov method, some consensus criteria are proposed to guarantee the consensus. Focusing on singular systems, the authors in [17] investigate the observer-based leader-following consensus problem of singular FOMASs. Then, the paper provides the corresponding control protocol and the calculation method of gain matrices. In [18], the distributed fixed-time leader-following consensus for FOMASs with a dynamic virtual leader under external disturbances is investigated, and a sliding-mode control protocol is designed. Meanwhile, singular-perturbation FOMASs are modelled and studied in [19], which provides a sufficient condition for the leader-following consensus. Nevertheless, the leader-following consensus of FOMASs in these results is based on state feedback. In most cases, only the measurement output of the agents is available, which indicates that the above method has certain limitations.

In comparison to state feedback, control via output feedback is a challenging problem, largely due to the effect of measurement matrices. Furthermore, the consensus of FOMASs via output feedback is a highly intricate issue. The stability criteria of FOSs with $\alpha \in (0,1)$ comprise two matrix variables, whereas in $\alpha \in [1,2)$, the matrix variable is related to the order number. In [20–23], static output feedback control is employed for FOSs, and a matrix exchange condition is utilized to integrate the matrix variable with gain matrices. This approach utilizes singular value decomposition (SVD) of the measurement matrices. Nevertheless, this approach imposes constraints on the form of matrix variables and tends to be conservative. The authors in [24] also adopt some strong assumptions to obtain a feasible solution, which brings conservatism. Similarly, in the output feedback consensus of FOMASs, limitations often exist. In [20], sufficient conditions are provided for the leader-following consensus of singular FOMASs with $\alpha \in (0,2)$. The authors in [25] perform an analogous study, and the SVD method is always indispensable. To complicate matters, disturbances persist in actual systems, and the output information transmitted between the agents contains the disturbances.

For FOMASs, the H_∞ control method provides substantial benefits in addressing system uncertainty, robustness optimization, and other pertinent issues. Ref. [26] derives the bounded real lemma for FOSs and establishes the foundation for H_∞ control. In [27], a proposal is made to extend the application of the H_∞ control method from integer-order systems to FOSs. Robust fault-tolerant H_∞ control for FOSs with actuator faults and uncertainties is addressed in [28] through the design of an output feedback controller. For singular FOSs, a state feedback control strategy is presented that guarantees the prescribed H_∞ performance in [29]. These works form the foundation of the H_∞ consensus of FOMASs. In [30], the admissible consensus of fuzzy singular FOMASs is considered, a sufficient condition for a system achieving admissible consensus while satisfying H_∞ performance. Ref. [31] investigates the H_∞ consensus problem for discrete-time FOMASs. However, the research remains limited. The output feedback H_∞ consensus remains a challenging field. Meanwhile, the control input saturation is a common feature of practical engineering systems, due to physical limitations [32–34]. This renders the output feedback consensus for FOMASs a complex process.

The discussion provides the impetus for the ILMI algorithms towards the leader-following H_∞ consensus of FOMASs with input saturation via output feedback. The contributions are as follows:

(i) Based on the real bound lemma of FOSs, sufficient conditions for leader-following output feedback H_∞ consensus of FOMASs in $\alpha \in (0,1)$ and $[1,2)$ are provided. The Laplacian matrix precludes the direct design of controllers. In traditional methods, transformation and decomposition of the error system are invariably required, hindering the investigation of its robustness. The proposed method adopts a holistic analytical perspective to the entire error system, which differs from the decomposition of error systems using traditional methods in [17,19,20] .

(ii) For solving the QMIs, the ILMI algorithms are provided, which propose a calculation method for initial values. Based on the stability region of FOSs, the iterative conditions are designed to guarantee the consensus condition of FOMASs. This paper delves deeper into the issue of the input saturation, which is reframed as an LMI-based optimisation problem. The ILMI algorithms circumvent the necessity for matrix exchange conditions from the SVD method. Compared to the work in [20,24,25], no strong assumptions are required for feasible solutions, and this reduces the conservatism.

Notations: Given a matrix A, $\mathrm{sym}(A)$ denotes $A^T + A$, where A^T is the transpose of A, and $\mathrm{her}(A) = A^* + A$, where A^* is the conjugate transpose of matrix A. $A > 0$ and $A < 0$ represent that A is positive definite and negative definite, respectively. Then, $A \geqslant 0$ denotes that A is positive semi-definite and the $\star$ symbol stands for the symmetric part. The Kronecker product is represented by $\otimes$. $\arg(\cdot)$ denotes the argument of complex numbers, and spec(A) indicates the spectrum of matrix A. $\mathbb{R}^{n\times m}$ stands for sets of $n \times m$ real matrices. $\mathrm{diag}(\cdot)$ represents a diagonal matrix. With $\alpha \in (0,2)$, denote $a = \sin\left(\alpha\frac{\pi}{2}\right)$, $b = \cos\left(\alpha\frac{\pi}{2}\right)$.

2. Problem Formulation and Preliminaries

An MAS consists of $\mathcal{N}$ followers represented by the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, \mathcal{N}\}$ represents a set of $\mathcal{N}$ followers, and $\mathcal{E} = \{(i,j) \colon i, j \in \mathcal{V}\}$ is the edge set. The adjacency matrix is $\mathcal{A} = [a_{ij}]_{\mathcal{N}\times\mathcal{N}}$, and if $(i,j) \in \mathcal{E}$, $a_{ij} > 0$, otherwise a_{ij}=0.

Furthermore, the communication graph between the leader and followers is denoted as $\widetilde{\mathcal{G}}$. $\mathrm{diag}(h_1, h_2, \cdots, h_{\mathcal{N}})$ represents the communication. If follower i receives the information from the leader, then $h_i > 0$, otherwise $h_i = 0$. The Laplacian matrix is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, where

$$l_{ij} = \begin{cases} -a_{ij}, i = j, \\ \sum_{j=1}^{\mathcal{N}} a_{ij}, i \neq j. \end{cases}$$

Consider the FOMAS under actuator saturation, and the $\mathcal{N}$ followers are described by

$$\begin{cases} D^\alpha x_i(t) = Ax_i(t) + B\mathrm{sat}(u_i(t)) + D_1 w_i(t), \\ z_i(t) = Cx_i(t) + D_2 w_i(t), \\ y_i(t) = C_y x_i(t) + D_3 w_i(t), \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{n_u}$, $y_i(t) \in \mathbb{R}^m$, $z_i(t) \in \mathbb{R}^{n_z}$, and $w_i(t) \in \mathbb{R}^{n_w}$ denote the state, control input, measured output, controlled output, and disturbances, respectively; $A\in \mathbb{R}^{n\times n}$, $B\in \mathbb{R}^{n\times n_u}$, $D_1\in \mathbb{R}^{n\times n_w}$, $D_2\in \mathbb{R}^{n_z\times n_w}$, $D_3\in \mathbb{R}^{m\times n_w}$, $C\in \mathbb{R}^{n_z\times n}$, and $C_y\in \mathbb{R}^{m\times n}$ are known system matrices; D^α represents the Caputo fractional derivative of $f(t)$ as

$$D^\alpha f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} = \int_0^t \frac{f^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha+1-\lceil\alpha\rceil}} d\tau,$$

and $\Gamma(\cdot)$ is the Euler gamma function; the saturation function $\mathrm{sat}(\cdot)$ for $u(t)$ is denoted as

$$\begin{aligned} &\mathrm{sat}(u) = [\mathrm{sat}(u_1) \ \cdots \ \mathrm{sat}(u_{n_u})]^T, \\ &\mathrm{sat}(u_s) = \mathrm{sign}(u_s)\min\{|u_s|, 1\}, s = 1, \cdots, n_u. \end{aligned}$$

The leader is described by

$$\begin{cases} D^\alpha x_0(t) = Ax_0(t), \\ z_0(t) = Cx_0(t), \\ y_0(t) = C_y x_0(t), \end{cases} \tag{2}$$

where $x_0(t) \in \mathbb{R}^n$ is the state; $z_0(t) \in \mathbb{R}^{n_z}$ and $y_0(t) \in \mathbb{R}^m$ are the control output and the measured output, respectively.

Assumption 1. *The pair (A, B) is controllable and the pair (A, C_y) is observable.*

Definition 1 ([35])**.** *The leader-following consensus of the* FOMAS *in (1) and (2) is achieved if*

$$\lim_{t\to\infty} \|x_i(t) - x_0(t)\| = 0,\ i = 1, 2, \cdots, \mathcal{N}.$$

For achieving consensus of the FOMAS in (1) and (2), a distributed consensus protocol is carried out by

$$u_i(t) = K\left(\sum_{j=1}^{\mathcal{N}} a_{ij}\big(y_i(t) - y_j(t)\big) + h_i\big(y_i(t) - y_0(t)\big)\right), i = 1, 2, \cdots, \mathcal{N}.$$

If the follower i and the leader are directly connected, then $h_i > 0$, otherwise $h_i = 0$.

The initial values are considered as $x_i(0) \in \mathbb{R}^n$. Denote the state trajectory as $T(x_i(0), t)$, and the domain of attraction is

$$\mathbb{D} = \left\{x_i(0) : \lim_{t\to\infty} T(x_i(0), t) = 0\right\}.$$

For $F \in \mathbb{R}^{n_u \times m}$, let

$$\mathcal{L}(F) = \left\{x(t) \in \mathbb{R}^n : \left|f_s C_y x(t)\right| \leqslant 1, s = 1, \cdots, n_u\right\},$$

where f_s denotes the s-th row of the matrix F; $\mathcal{L}(F)$ denotes the region where $FC_y x(t)$ is not saturated.

Lemma 1 ([17,36,37])**.** *Denote K, $F \in \mathbb{R}^{n_u \times m}$, then for any $y(t) \in \mathcal{L}(F)$, there is*

$$\mathrm{sat}(Ky(t)) \in \mathrm{co}\left\{\Gamma_l K y(t) + \Gamma_l^- F y(t),\ l = 1, \cdots, 2^{n_u}\right\},$$

where co$\{\cdot\}$ *indicates a convex hull; Γ_l are diagonal matrices, whose diagonal elements are 0 or 1; Γ_l^- are set as $\Gamma_l^- = 1 - \Gamma_l$. Further, the input saturation is written as*

$$\mathrm{sat}(Ky(t)) = \sum_{l=1}^{2^{n_u}} \eta_l \left(\Gamma_l K + \Gamma_l^- F\right) y(t), \tag{3}$$

where $\eta_l \geqslant 0$ and $\sum_{l=1}^{2^{n_u}} \eta_l = 1$.

Set $x_{ei} = x_i - x_0$, $z_{ei} = z_i - z_0$. With the consensus protocol, the error system of FOMAS in (1) and (2) is written as

$$\begin{cases} D^\alpha x_e(t) = \big(I_\mathcal{N} \otimes A + (\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes C_y)\big)x_e(t) + \big(I_\mathcal{N} \otimes D_1 + (\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes D_3)\big)w(t), \\ z_e(t) = (I_\mathcal{N} \otimes C)x_e(t) + (I_\mathcal{N} \otimes D_2)w(t), \end{cases} \tag{4}$$

where $x_e(t) = \left[x_{e1}^T(t) \ \cdots \ x_{e\mathcal{N}}^T(t)\right]^T$, $z_e(t) = \left[z_{e1}^T(t) \ \cdots \ z_{e\mathcal{N}}^T(t)\right]^T$, $w(t) = \left[w_1^T(t) \ \cdots \ w_\mathcal{N}^T(t)\right]^T$, $\tilde{K} = I_\mathcal{N} \otimes \sum_{l=1}^{2^{n_u}} \eta_l(\Gamma_l K + \Gamma_l^- F)$, $\mathcal{H} = \mathcal{L} + \mathrm{diag}(h_1, h_2, \cdots, h_\mathcal{N})$. Its transfer function $G_{wz}(s)$ is $(I_\mathcal{N} \otimes C)\big(s^\alpha I - (I_\mathcal{N} \otimes A + (\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes C_y))\big)^{-1}(\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes D_3)\big) + I_\mathcal{N} \otimes D_2$.

Lemma 2 ([38]). *Consider the FOS as*

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + B\text{sat}(u(t)), \\ y(t) = C_y x(t), \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, and $y(t) \in \mathbb{R}^{n_u}$; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, and $C_y \in \mathbb{R}^{m\times n}$. With $u(t) = 0$, the system in (5) reduces to

$$D^{\alpha}x(t) = Ax(t). \tag{6}$$

The system in (6) is stable if and only if $|\arg(\text{spec}(A))| > \alpha\frac{\pi}{2}$.

Remark 1. *Based on Lemma 2, if the eigenvalues of A are in the region of the eigenvalues shown in Figure 1, it is obvious that the system in (6) is stable. Meanwhile, the system in (6) is also stable even if the eigenvalues of A are moved by $\frac{\tau}{2}$ units in the positive direction of the x-axis, where $\tau > 0$. Thus, the presence of eigenvalues of A in a specific region of Figure 1 serves as a sufficient condition for the system's stability.*

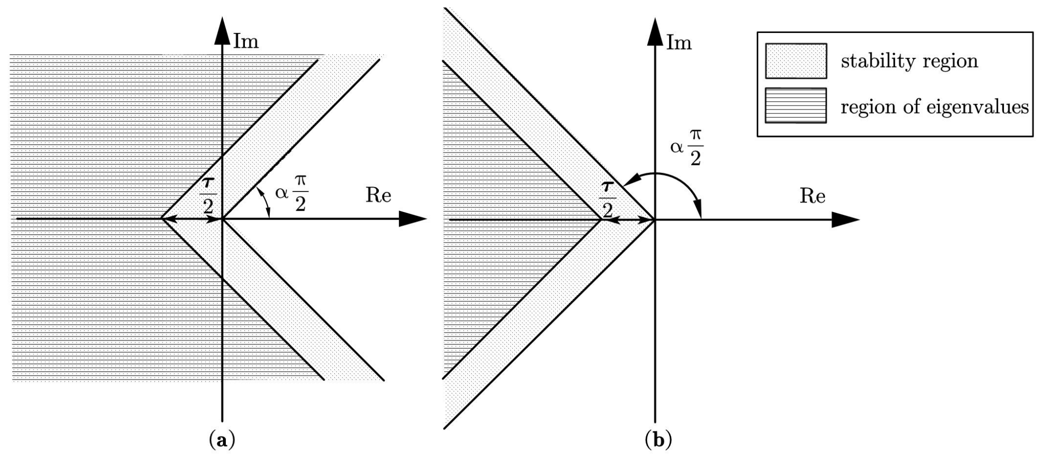

Figure 1. Stability region of the system in (6) and region of spec(A): (**a**) $\alpha \in (0,1)$ and (**b**) $\alpha \in [1,2)$.

Lemma 3 ([39]). *Via $u(t) = KC_y x(t)$, if the system in (5) satisfies the stability condition in Lemma 2, i.e., $\left|\arg\left(\text{spec}\left(A + BKC_y\right)\right)\right| > \alpha\frac{\pi}{2}$, then the system in (5) is asymptotically stable for $x(0) \in \mathcal{B}_{\sigma} = \left\{x(t) \in \mathbb{R}^n : x^T x \leqslant \sigma\right\} \subset \mathcal{L}(F)$.*

Definition 2 ([29]). *Define the H_∞ norm of transfer function $G(s)$ as $\|G(s)\|_\infty = \sup_{\text{Re}(s)\geqslant 0} \sigma_{\max}(G(s))$, where $\sigma_{\max}(\cdot)$ is the maximum singular value of a matrix.*

Lemma 4 ([26]). *Consider the FOS as*

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + D_1 w(t), \\ z(t) = Cx(t) + D_2 w(t), \end{cases} \tag{7}$$

with the transfer function $G_{wz}(s) = C(s^{\alpha}I - A)^{-1}D_1 + D_2$, where $x(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^{n_w}$. Given a scalar $\gamma > 0$, the system in (7) is stable and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed if

(1) For the case $\alpha \in (0,1)$ there exist $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{8}$$

$$\begin{bmatrix} \mathrm{sym}(A(aX+bY)) & (aX+bY)^T C^T & D_1 \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{9}$$

(2) For the case $\alpha \in [1,2)$ there exists $X \in \mathbb{R}^{n\times n} > 0$ such that

$$\begin{bmatrix} \mathrm{her}(rAX) & \bar{r}C^T X & D_1 \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0, \tag{10}$$

where $r = e^{\mathrm{j}(1-\alpha)\frac{\pi}{2}} = a + \mathrm{j}b$.

Lemma 5 ([40]). *$X + Y\mathrm{j} > 0$ is equivalent to (8).*

3. Main Results

This section introduces the iterative algorithms for solving the H_∞ consensus of FOMASs via output feedback.

3.1. ILMI Algorithm for Output Feedback H_∞ Consensus with $\alpha \in (0,1)$

In this subsection, it is essential to reference the following lemmas, which form the basis of the ILMI algorithms. To facilitate the derivation process, the matrices are expressed as follows:

$$P_\alpha = \begin{bmatrix} aX+bY & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}, \tilde{A} = \begin{bmatrix} I_\mathcal{N} \otimes A & I_\mathcal{N} \otimes D_1 & 0 \\ 0 & -\frac{1}{2}\gamma I & 0 \\ I_\mathcal{N} \otimes C & I_\mathcal{N} \otimes D_2 & -\frac{1}{2}\gamma I \end{bmatrix}, \tilde{B} = \begin{bmatrix} \mathcal{H} \otimes B \\ 0 \\ 0 \end{bmatrix}, \tag{11}$$
$$\tilde{C} = \begin{bmatrix} I_\mathcal{N} \otimes C_y & I_\mathcal{N} \otimes D_3 & 0 \end{bmatrix}.$$

Theorem 1. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed in error system (4), if there exist $X \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n}$, $K \in \mathbb{R}^{n_u\times m}$, and $F \in \mathbb{R}^{n_u\times m}$ such that (8) and*

$$(\tilde{A} + \tilde{B}\tilde{K}\tilde{C})^T P_\alpha + P_\alpha^T(\tilde{A} + \tilde{B}\tilde{K}\tilde{C}) < 0, \tag{12}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11) and $\tilde{K}$ is from (4).

Proof. In the case of $\alpha \in (0,1)$ of Lemma 4, consider matrices $X_0 > 0$, $Y_0 = -Y_0^T$ such that

$$\begin{bmatrix} \mathrm{sym}(A(aX_0+bY_0)) & (aX_0+bY_0)^T C^T & D_1 \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{13}$$

Then, introduce a congruence transformation. Pre- and post-multiplying (13) by

$$\begin{bmatrix} (aX_0+bY_0)^{-T} & 0 & 0 \\ \star & 0 & I \\ \star & \star & 0 \end{bmatrix},$$

and its transpose, one obtains

$$\begin{bmatrix} \mathrm{sym}(A^T(aX_0+bY_0)^{-1}) & (aX_0+bY_0)^{-T} D_1 & C^T \\ \star & -\gamma I & D_2^T \\ \star & \star & -\gamma I \end{bmatrix} < 0.$$

Meanwhile, there exist matrices $X = X^T$ and $Y = -Y^T$ that satisfy

$$(aX_0 + bY_0)^{-1} = aX + bY. \tag{14}$$

To verify (14), set X and Y as

$$X = \frac{(aX_0 + bY_0)^{-1} + (aX_0 - bY_0)^{-1}}{2a}, \; Y = \frac{(aX_0 + bY_0)^{-1} - (aX_0 - bY_0)^{-1}}{2b},$$

which also satisfy $X = X^T$ and $Y = -Y^T$. It is easy to see that $(aX_0 + bY_0)(aX + bY) = I$, which proves (14). Then, (13) is transformed into

$$\begin{bmatrix} \text{sym}(A^T(aX+bY)) & (aX+bY)^T D_1 & C^T \\ \star & -\gamma I & D_2^T \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{15}$$

And the positive definite property of $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$ is proved by pre- and post-multiplying it by $I_2 \otimes (aX_0 + bY_0)$.

Under Assumption 1, the consensus is achieved if and only if the error system in (4) is stable. Apply (15) in the consensus problem of the FOMAS via output feedback. Then, the consensus of the FOMAS in (1) and (2) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed in error system (4) with $\alpha \in (0,1)$, if there exist X, Y, K, and F such that (8) and

$$\begin{bmatrix} \text{sym}({A_{\tilde{K}}}^T(aX+bY)) & (aX+bY)^T D_{\tilde{K}} & (I_{\mathcal{N}} \otimes C_y)^T \\ \star & -\gamma I & (I_{\mathcal{N}} \otimes D_2)^T \\ \star & \star & -\gamma I \end{bmatrix} < 0, \tag{16}$$

where

$$A_{\tilde{K}} = I_{\mathcal{N}} \otimes A + (\mathcal{H} \otimes B)\tilde{K}(I_{\mathcal{N}} \otimes C_y), D_{\tilde{K}} = I_{\mathcal{N}} \otimes D_1 + (\mathcal{H} \otimes B)\tilde{K}(I_{\mathcal{N}} \otimes D_3). \tag{17}$$

Then, using basic matrix operations, one simplifies inequality (16) into (12). □

Theorem 2. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed in error system (4) if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $K \in \mathbb{R}^{n_u \times m}$, and $F \in \mathbb{R}^{n_u \times m}$ such that (8) and*

$$\text{sym}(\tilde{A}^T P_\alpha) - P_\alpha^T \tilde{B}\tilde{B}^T P_\alpha + (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T(\tilde{B}^T P_\alpha + \tilde{K}\tilde{C}) < 0, \tag{18}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11) and $\tilde{K}$ is from (4).

Proof. With matrix transformation, (18) is rewritten as

$$(\tilde{A} + \tilde{B}\tilde{K}\tilde{C})P_\alpha + P_\alpha^T(\tilde{A} + \tilde{B}\tilde{K}\tilde{C})^T + \tilde{C}^T\tilde{K}^T\tilde{K}\tilde{C} < 0.$$

As $\tilde{C}^T\tilde{K}^T\tilde{K}\tilde{C} \geqslant 0$ is obvious, it is readily apparent that (12) holds. Based on Lemma 2, the H_∞ consensus is achieved without input saturation. □

Theorem 3. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed in error system (4) if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $K \in \mathbb{R}^{n_u \times m}$, $F \in \mathbb{R}^{n_u \times m}$, and S with appropriate dimensions such that (8) and*

$$\begin{bmatrix} \text{sym}(\tilde{A}^T P_\alpha - S^T\tilde{B}\tilde{B}^T P_\alpha) + S^T\tilde{B}\tilde{B}^T S & (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T \\ \star & -I \end{bmatrix} < 0, \tag{19}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11) and $\tilde{K}$ is from (4).

Proof. For matrices S and P_α, $(S - P_\alpha)^T \tilde{B}\tilde{B}^T (S - P_\alpha) \geqslant 0$ always holds, and this inequality is written as

$$S^T \tilde{B}\tilde{B}^T P_\alpha + P_\alpha^T \tilde{B}\tilde{B}^T S - S^T \tilde{B}\tilde{B}^T S - P_\alpha^T \tilde{B}\tilde{B}^T P_\alpha \leqslant 0. \tag{20}$$

Using the Schur complement on (19), one obtains

$$\mathrm{sym}(\tilde{A}^T P_\alpha - S^T \tilde{B}\tilde{B}^T P_\alpha) + S^T \tilde{B}\tilde{B}^T S + (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C}) < 0. \tag{21}$$

Combining (20) and (21) yields (18). With the sufficient condition from Lemma 2, this completes the proof. □

Theorem 4. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved for any initial value $x_i(0) \in B_\varepsilon \subset \mathcal{L}(F)$ and $\|G_{wz}(s)\|_\infty < \gamma$ and is guaranteed in error system (4) if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $K \in \mathbb{R}^{n_u \times m}$, $F \in \mathbb{R}^{n_u \times m}$, and S with appropriate dimensions and a constant $\varepsilon > 0$ such that (8) and*

$$\begin{bmatrix} \mathrm{sym}(\tilde{A}^T P_\alpha - S^T \tilde{B}\tilde{B}^T P_\alpha) + S^T \tilde{B}\tilde{B}^T S & (\tilde{B}^T P_\alpha + I_{\mathcal{N}} \otimes (\Gamma_l K + \Gamma_l^- F)\tilde{C})^T \\ \star & -I \end{bmatrix} < 0,\ l = 1, \ldots, 2^{n_u}, \tag{22}$$

$$\begin{bmatrix} \frac{1}{\varepsilon} I & C_y^T f_s^T \\ f_s C_y & 1 \end{bmatrix} \geqslant 0,\ s = 1, \cdots, n_u. \tag{23}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11).

Proof. Since $h_l \geqslant 0$, (22) holds, then (19) is guaranteed. Based on Lemma 3, if (19), (8), and (23) hold, the FOMAS in (1) and (2) achieves consensus without input saturation.

Then, consider that input is limited. According to the statement in Lemma 3, the FOMAS achieves consensus for $x(0) \in \mathcal{B}_\sigma \subset \mathcal{L}(F)$. From (23), using the Schur complement, one obtains

$$\frac{1}{\varepsilon} I \geqslant C_y^T f_s^T f_s C_y,\ s = 1, \cdots, n_u.$$

So $1 \geqslant \frac{1}{\varepsilon} x_i(t)^T x_i(t) \geqslant x_i(t)^T C_y^T f_s^T f_s C_y x_i(t)$ holds, for $s = 1, 2, \cdots, n_u$, $x_i(t) \in \mathcal{B}_\varepsilon$, and $\mathcal{B}_\varepsilon \subset \mathcal{L}(F)$. Therefore, $\mathcal{L}(F)$ is estimated by B_ε. □

According to Lemma 4, the stable region is approximated by utilizing optimization theory:

Maximize ε
X, Y, K, F, subject to (8), (22), and (23).

It is evident that the LMIs from (22) display nonlinear characteristics, particularly due to the incorporation of product terms. Nevertheless, if the matrix S is predetermined and holds constant, the QMIs (22) are simplified into LMIs, which inherently possess convexity. This transformation allows the LMIs to contain feasible solutions for the gain matrices K and F. Consequently, this paper presents an iterative algorithm aimed at solving (22) through the following steps.

Prior to commencing the ILMI algorithm, it is essential to define the following notation:

$$P_{\alpha p} = \begin{bmatrix} aX_p + bY_p & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}, \text{ and } \tilde{P}_{\alpha p} = \begin{bmatrix} aX_p & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}. \tag{24}$$

Remark 2. *For simplification, the inequality is still not presented in detail.* P_α*,* $\tilde{A}$*,* $\tilde{B}$*, and* $\tilde{C}$ *are shown in (11), and* $P_{\alpha p}$*,* $\tilde{P}_{\alpha p}$ *are in (24). p only denotes the number of iterations.*

Remark 3. *To verify the necessity of step 1, one considers inequality (20), where the equality holds with* $S = P_\alpha$*. S is close to* P_α*, then the solutions of (19) are also close to those of (18). For obtaining the appropriate initial value* S_1*, consider the following equation based on (18):*

$$\tilde{A}^T P_\alpha + P_\alpha^T \tilde{A} - P_\alpha^T \tilde{B}\tilde{B}^T P_\alpha + \tilde{Q} = 0,$$

where $\tilde{Q}$ *represents the product term* $(\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T(\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})$*. However, if* $\tilde{Q}$ *is set to a certain positive definite matrix directly, the solution of* P_α *is difficult to obtain and may not be formally consistent with that in (11). For convenient calculation, setting* $Y = 0$ *yields*

$$\begin{bmatrix} \mathrm{sym}(a(I_\mathcal{N} \otimes A)^T X) - a^2 X(\mathcal{H} \otimes B)(\mathcal{H} \otimes B)^T X & aX(I_\mathcal{N} \otimes D_1) & (I_\mathcal{N} \otimes C)^T \\ \star & -\gamma I & (I_\mathcal{N} \otimes D_2)^T \\ \star & \star & -\gamma I \end{bmatrix}$$

$$+ \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{bmatrix} = 0,$$

where $Q_{11} = Q_{\alpha 01}$ *is the sub-block of* $\tilde{Q}$*. After setting* Q_{11} *and obtaining the solution of X, it is obvious that the results of other sub-blocks are obtained and* $\tilde{Q}$ *is automatically generated. Therefore, it is only necessary to set* Q_{11} *in Algorithm 1.*

Algorithm 1 ILMI algorithm for $\alpha \in (0, 1)$.

Step 1: Set $p = 1$ and $Y_{\Delta,p} = 0$. Select $Q_{\alpha 01} \in \mathbb{R}^{n \times n} > 0$ and solve the Riccati equation:

$$a(I_\mathcal{N} \otimes A)^T X_{\Delta,p} + aX_{\Delta,p}(I_\mathcal{N} \otimes A) - a^2 X_{\Delta,p}(\mathcal{H} \otimes B)(\mathcal{H} \otimes B)^T X_{\Delta,p} + Q_{\alpha 01} = 0. \quad (25)$$

Step 2: Set

$$S_p = \begin{bmatrix} aX_{\Delta,p} + bY_{\Delta,p} & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}.$$

Maximize τ_p subject to the following LMIs:

$$\begin{bmatrix} \mathrm{sym}\left(\tilde{A}^T P_{\alpha p} - S_p^T \tilde{B}\tilde{B}^T P_{\alpha p}\right) + S_p^T \tilde{B}\tilde{B}^T S_p + \tau_p \tilde{P}_{\alpha p} & \left(\tilde{B}^T P_{\alpha p} + I_\mathcal{N} \otimes (\Gamma_l K + \Gamma_l^- F)\tilde{C}\right)^T \\ \star & -I \end{bmatrix} < 0, \quad (26)$$
$$l = 1, \cdots, 2^{n_u},$$

$$\begin{bmatrix} X_p & Y_p \\ -Y_p & X_p \end{bmatrix} > 0. \quad (27)$$

Step 3: Denote $\tau_{\max,p}$ as the maximum value of τ_p. If $\tau_{\max,p} \geqslant 0$ holds, go to step 7.
Step 4: Minimize $\mathrm{trace}(X_p)$ subject to the LMIs (26) and (27) with $\tau_{\max,p}$ until the minimized trace $X_{\min,p}$ and the corresponding $Y_{\min,p}$ are obtained.
Step 5: Give a small tolerance $\delta > 0$. If $\| X_{\Delta,p} - X_{\min,p} \| < \delta$ holds, then go to step 6; else, set $p = p + 1$, $X_{\Delta,p} = X_{\min,p-1}$, $Y_{\Delta,p} = Y_{\min,p-1}$, and return to step 2.
Step 6: The leader-following consensus of the FOMAS in (1) and (2) may not be achieved by static output feedback, stop.
Step 7: Maximize ε subject to (8), (22), and (23) with S_p and obtain gain matrices K and F for stabilizing the system, stop.

Remark 4. *Considering (26) from step 2, the system matrix is substituted with* $I_{\mathcal{N}} \otimes A + \frac{\tau}{2} I$*, whose eigenvalues are translated* $\frac{\tau}{2}$ *units towards the x-axis relative to the eigenvalues of* $I_{\mathcal{N}} \otimes A$ *with* $\tau > 0$*.*

In step 3, if $\tau_{\max,p} \geqslant 0$ holds, with $\tau = \tau_{\max,p}$, (26) guarantees that the eigenvalues of $I_{\mathcal{N}} \otimes A$ are still in the stability region after they move forward by $\frac{\tau_{\max,p}}{2}$ units towards the x-axis, e.g., in Figure 1. And with $\tau = \tau_{\max,p}$, (26) is seen as a sufficient condition for the consensus.

Remark 5. *In step 4,* Y_p *satisfies* $Y_p^T = -Y_p$ *and its trace is 0. Thus, the minimization problem only involves the trace of* X_p*.*

Remark 6. *After multiple iterations,* S_p *is obtained to ensure (8) and (22) have feasible solutions. To obtain a large domain of attraction* $\mathcal{L}(F)$ *for the initial value* $x_i(0)$*, thus in step 7, an optimization problem is solved subject to (8), (22), and (23) to acquire gain matrices.*

3.2. ILMI Algorithm for Output Feedback Consensus with $\alpha \in [1, 2)$

In the section, the case of $\alpha \in [1, 2)$ is considered. Complex numbers exist in (10) of Lemma 4 and bring difficulties to the solution. Thus, similar to the case in $\alpha \in (0, 1)$, the following lemmas are given for obtaining the gain matrices. Then, the parameter matrices are set as

$$\tilde{X}_a = \begin{bmatrix} aX & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}, \tilde{X}_b = \begin{bmatrix} bX & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}, \check{X} = \begin{bmatrix} \tilde{X}_a & \tilde{X}_b \\ -\tilde{X}_b & \tilde{X}_a \end{bmatrix}, \tag{28}$$

$$\check{A} = I_2 \otimes \tilde{A},\ \check{B} = I_2 \otimes \tilde{B},\ \check{C} = I_2 \otimes \tilde{C},\ \check{K} = I_2 \otimes \tilde{K} = I_{2\mathcal{N}} \otimes \sum_{l=1}^{2^{n_u}} \eta_l (\Gamma_l K + \Gamma_l^- F),$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11).

Theorem 5. *With* $\alpha \in [1, 2)$ *and a given scalar* $\gamma > 0$*, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved and* $\|G_{wz}(s)\|_\infty < \gamma$ *is guaranteed in error system (4) if there exist* $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n} > 0$*,* $K \in \mathbb{R}^{n_u \times m}$*, and* $F \in \mathbb{R}^{n_u \times m}$ *such that*

$$(\check{A} + \check{B}\check{K}\check{C})^T \check{X} + \check{X}^T (\check{A} + \check{B}\check{K}\check{C}) < 0, \tag{29}$$

where $\check{A}$*,* $\check{B}$*,* $\check{K}$*,* $\check{C}$*, and* $\check{X}$ *are shown in (28).*

Proof. Based on the proof from (13) to (15), the system in (7) is stable and $\|G_{wz}(s)\|_\infty < \gamma$ holds if there exists $X > 0$ such that

$$\begin{bmatrix} \text{her}(rA^T X) & \bar{r} X D_1 & C^T \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{30}$$

Considering the FOMAS in (1) and (2) without input saturation, one obtains

$$\text{her}\left(\begin{bmatrix} A_{\tilde{K}} & D_{\tilde{K}} & 0 \\ 0 & -\frac{1}{2}\gamma I & 0 \\ I_{\mathcal{N}} \otimes C & I_{\mathcal{N}} \otimes D_2 & -\frac{1}{2}\gamma I \end{bmatrix} \begin{bmatrix} (a + b\mathrm{j})X & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix} \right) < 0, \tag{31}$$

where $A_{\tilde{K}}$ and $D_{\tilde{K}}$ are shown in (17).

Based on Lemma 5, an equivalent condition of (31) is given as

$$\begin{bmatrix} \mathrm{sym}\big((\tilde{A}+\tilde{B}\tilde{K}\tilde{C})^T\tilde{X}_a\big) & (\tilde{A}+\tilde{B}\tilde{K}\tilde{C})^T\tilde{X}_b-\tilde{X}_b^T(\tilde{A}+\tilde{B}\tilde{K}\tilde{C}) \\ \star & \mathrm{sym}\big((\tilde{A}+\tilde{B}\tilde{K}\tilde{C})^T\tilde{X}_a\big) \end{bmatrix} < 0. \quad (32)$$

Simplify (32) and this completes the proof. □

Due to the similarity in the proof process with Lemmas 2–4, proofs are not provided for the following lemmas.

Theorem 6. *With $\alpha \in [1,2)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed in error system (4) if there exist $X \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n} > 0$, $K \in \mathbb{R}^{n_u\times m}$, and $F \in \mathbb{R}^{n_u\times m}$ such that*

$$\check{A}^T\check{X}+\check{X}^T\check{A}-\check{X}^T\check{B}\check{B}^T\check{X}+(\check{B}^T\check{X}+\check{K}\check{C})^T(\check{B}^T\check{X}+\check{K}\check{C})<0, \quad (33)$$

where $\check{A}$, $\check{B}$, $\check{K}$, $\check{C}$, and $\check{X}$ are shown in (28).

Theorem 7. *With $\alpha \in [1,2)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed in error system (4) if there exist $X \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n} > 0$, $K \in \mathbb{R}^{n_u\times m}$, $F \in \mathbb{R}^{n_u\times m}$, and S with appropriate dimensions such that*

$$\begin{bmatrix} \check{A}^T\check{X}+\check{X}^T\check{A}-S^T\check{B}\check{B}^T\check{X}-\check{X}^T\check{B}\check{B}^TS+S^T\check{B}\check{B}^TS & (\check{B}^T\check{X}+\check{K}\check{C})^T \\ \star & -I \end{bmatrix} < 0, \quad (34)$$

where $\check{A}$, $\check{B}$, $\check{K}$, $\check{C}$, and $\check{X}$ are shown in (28).

Theorem 8. *With $\alpha \in [1,2)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (1) and (2) without input saturation is achieved for any $x_i(0) \in B_\varepsilon \subset \mathcal{L}(F)$ and $\|G_{wz}(s)\|_\infty < \gamma$ and is guaranteed in error system (4) if there exist $X \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n} > 0$, $K \in \mathbb{R}^{n_u\times m}$, $F \in \mathbb{R}^{n_u\times m}$, S with appropriate dimensions, and a constant $\varepsilon > 0$ such that (23) and*

$$\begin{bmatrix} \check{A}^T\check{X}+\check{X}^T\check{A}-S^T\check{B}\check{B}^T\check{X}-\check{X}^T\check{B}\check{B}^TS+S^T\check{B}\check{B}^TS & (\check{B}^T\check{X}+\check{K}_l\check{C})^T \\ \star & -I \end{bmatrix} < 0, \quad (35)$$
$$l = 1,\ldots,2^{n_u},$$

where $\check{A}$, $\check{B}$, $\check{C}$, and $\check{X}$ are shown in (28) and $\check{K}_l = I_{2\mathcal{N}} \otimes (\Gamma_l K + \Gamma_l^- F)$.

Similarly, as with the issue pertaining to $\alpha \in (0,1)$, the domain of attraction is estimated by the following method:

$$\text{Maximize } \varepsilon$$
$$X, Y, K, F, \text{subject to } X > 0,\ (23),\ \text{and } (35).$$

In addition, (35) are not LMIs and difficult to solve with the LMI toolbox in MATLAB R2022a. Accordingly, the following iterative algorithm is provided for obtain matrices K and F. To simplify writing, one denotes

$$\check{P}_{\alpha p} = I_2 \otimes \begin{bmatrix} aX_p & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}. \quad (36)$$

Remark 7. *In order to provide greater clarity, the inequalities are not presented in full detail.* $\check{A}$, $\check{B}$, $\check{C}$, *and* $\check{X}_p$ *are shown in (28), and* p *is the number of iterations.* $\check{K}_l$ *is from Lemma 8.*

Remark 8. *The iterative process is similar to that in Algorithm 1. Due to the difficulty in solving*

$$\check{A}^T\check{X} + \check{X}^T\check{A} - \check{X}^T\check{B}\check{B}^T\check{X} + \tilde{Q} = 0,$$

$X_{\triangle,1}$ *is solved by (37), which is the sub-block in the first row and first column in the consolidation of the left matrices* $\check{A}^T\check{X} + \check{X}^T\check{A} - \check{X}^T\check{B}\check{B}^T\check{X}$*. In step 2,* S_p *is constructed and close to* $\check{X}_p$*.*

Remark 9. *After certain experiments, the initial value* $X_{\triangle,1}$ *is related to the convergence of the algorithm. Then, the selection of* $Q_{\alpha 01}$ *and* $Q_{\alpha 12}$ *in step 1 indirectly influences the convergence property. In some cases,* $Q_{\alpha 01}$ *and* $Q_{\alpha 12}$ *are set as* W^TW*, where* W *is nonsingular.* $Q_{\alpha 12} = I$ *is also a choice, which leads to a convergent result.*

Remark 10. *In most existing studies, the consensus problem of MASs relies on transformation towards to* $\mathcal{H}$*, such as* $\mathcal{H} = T^{-1}JT$*, where* J *is a diagonal matrix containing eigenvalues* λ_i *of* $\mathcal{H}$*. If each subsystem matrix* $A + \lambda_i BKC_y$ *satisfies the stability condition, i.e.,* $\left|\arg\left(\text{spec}\left(A + \lambda_i BKC_y\right)\right)\right| > \alpha\frac{\pi}{2}$*, the consensus of the MAS is achieved. However, when an MAS contains disturbances* $w(t)$ *or nonlinear terms, the analysis process becomes complex after multiplying the parameter matrix with the transformation matrix* T*. In contrast, ILMI algorithms contribute a holistic analysis of the MAS, which facilitates performance assessment under disturbances.*

Remark 11. *From Algorithms 1 and 2, it is evident that no constraint is imposed on the matrix variables containing* X *and* Y*.*

Algorithm 2 ILMI algorithm for $\alpha \in [1,2)$.

Step 1: Set $p = 1$ and select $Q_{\alpha 12} > 0$, then solve the Riccati equation:

$$a(I_{\mathcal{N}} \otimes A)^T X_{\Delta,p} + aX_{\Delta,p}(I_{\mathcal{N}} \otimes A) - a^2 X_{\Delta,p}(\mathcal{H} \otimes B)(\mathcal{H} \otimes B)^T X_{\Delta,p} + Q_{\alpha 12} = 0. \quad (37)$$

Step 2: Set $S_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} aX_{\Delta,p} & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} bX_{\Delta,p} & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}$.

Maximize τ_p subject to the following LMIs:

$$X_p > 0,$$

$$\begin{bmatrix} \check{A}^T\check{X}_p + \check{X}_p^T\check{A} - S_p^T\check{B}\check{B}^T\check{X}_p - \check{X}_p^T\check{B}\check{B}^T S_p + S_p^T\check{B}\check{B}^T S_p - \tau_p\check{P}_{\alpha p} & \left(\check{B}^T\check{X}_p + \check{K}_l\widetilde{C}\right)^T \\ \star & -I \end{bmatrix} < 0, \quad (38)$$

$$l = 1, \cdots, 2^{n_u}, \quad (39)$$

Step 3: Denote $\tau_{\max,p}$ as the maximum value of τ_p, if $\tau_{\max,p} \geqslant 0$, go to step 7.
Step 4: Minimize trace(X_p) subject to the LMIs (39) and (38) with $\tau_{\max,p}$ until the minimized trace$X_{\min,p}$ is obtained.
Step 5: Give a small tolerance $\delta > 0$, if $\| X_{\Delta,p} - X_{\min,p} \| < \delta$, then go to step 6; else, set $p = p + 1$ and $X_{\Delta,p} = X_{\min,p-1}$, and go back to step 2.
Step 6: The leader-following consensus of the FOMAS in (1) and (2) may not be achieved by static output feedback, stop.
Step 7: Maximize ε subject to (35), (23) with S_p and obtain K and F for achieving consensus, stop.

The existing main method involves SVD towards C_y, such as $C_y = U\begin{bmatrix}\tilde{C}_y & 0\end{bmatrix}V^T$, where U and V are the unitary matrices. Then, the structure of X is restricted as

$$X = V\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}V^T,$$

or an upper triangular matrix in special cases for satisfying the exchange condition $C_yX = X_eC_y$, where X_e is another matrix variable. Note that for satisfying the exchange condition, the stability criterion used is similar to that of integer-order systems, which is independent of the order α. Thus, the method is conservative.

Remark 12. *In Table 1, it is shown that this paper takes into account multiple scenarios in the study of FOMASs with $\alpha \in (0,1)$ or $[1,2)$. In the existing literature, there is relatively little research on the H_∞ consensus for FOMASs, e.g., [30]. With the further consideration of output feedback, the limitation to the matrix variables is a common feature of many articles and an essential condition. The ILMI algorithms presented in this article avoid this limitation and are used in cases where there are disturbances from the output information and the control input saturation. Thus, the ILMI algorithms have a broad scope of applicability.*

Table 1. Comparison of existing methods.

Ref.	Range of α	Output Feedback	H_∞	MAS	Input Saturation	No Limitation to Matrix Variables
[41]	$(0,1)$	✓	×	×	×	×
[25]	$(0,1)$	✓	×	✓	✓	×
[42]	$(0,1)$	✓	×	×	×	×
[43]	$(0,1)$	✓	×	×	×	×
[44]	$(0,1)$	✓	×	×	×	×
[30]	$(0,1)$	✓	✓	✓	×	×
[45]	1	×	✓	✓	×	-
[24]	$(0,2)$	✓	✓	×	×	×
[17]	$(0,2)$	×	×	✓	✓	-
Ours	$(0,1)$, $[1,2)$	✓	✓	✓	✓	✓

4. Numerical Examples

This section presents two numerical examples to verify the effectiveness of the proposed ILMI algorithms with $\alpha \in (0,1)$ and $[1,2)$, respectively.

Example 1. *Consider the FOMAS in (1) and (2) with $\alpha = 0.7$ and set $\gamma = 10$. The undirected graph is shown in Figure 2 and*

$$\mathcal{H} = \begin{bmatrix} 3.2 & -0.5 & 0 & -0.7 \\ -0.5 & 3.2 & -0.7 & 0 \\ 0 & -0.7 & 3.4 & -0.7 \\ -0.7 & 0 & -0.7 & 1.4 \end{bmatrix}.$$

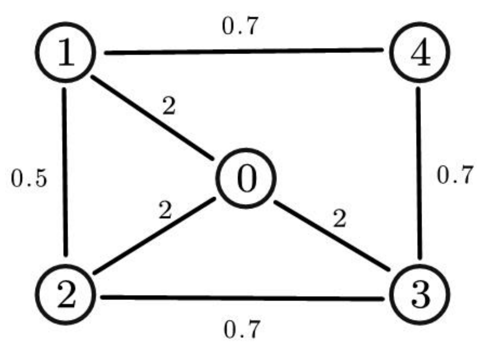

Figure 2. The weighted undirected graph in example 1.

The system matrices of each agent are

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 1 & -0.5 & -2 \\ 1 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0.5 \\ 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_y = \begin{bmatrix} 1 & 0.5 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Based on Algorithm 1, select $Q_{\alpha 01} = I$ in step 1. Solve Equation (25) and obtain

$$X_{\triangle,1} = \begin{bmatrix} 0.2453 & -0.0051 & 0.1220 & 0.0238 & 0.0142 & 0.0224 & 0.0215 & 0.0137 & 0.0194 & 0.0762 & 0.0474 & 0.0699 \\ -0.0051 & 0.2994 & -0.3851 & 0.0142 & 0.0555 & -0.0784 & 0.0137 & 0.0520 & -0.0725 & 0.0474 & 0.1822 & -0.2555 \\ 0.1220 & -0.3851 & 0.9754 & 0.0224 & -0.0784 & 0.2007 & 0.0194 & -0.0725 & 0.1838 & 0.0699 & -0.2555 & 0.6495 \\ 0.0238 & 0.0142 & 0.0224 & 0.2344 & -0.0122 & 0.1123 & 0.0287 & 0.0170 & 0.0272 & 0.0293 & 0.0187 & 0.0263 \\ 0.0142 & 0.0555 & -0.0784 & -0.0122 & 0.2724 & -0.3475 & 0.0170 & 0.0667 & -0.0943 & 0.0187 & 0.0712 & -0.0994 \\ 0.0224 & -0.0784 & 0.2007 & 0.1123 & -0.3475 & 0.8808 & 0.0272 & -0.0943 & 0.2418 & 0.0263 & -0.0994 & 0.2514 \\ 0.0215 & 0.0137 & 0.0194 & 0.0287 & 0.0170 & 0.0272 & 0.2401 & -0.0081 & 0.1170 & 0.0733 & 0.0455 & 0.0673 \\ 0.0137 & 0.0520 & -0.0725 & 0.0170 & 0.0667 & -0.0943 & -0.0081 & 0.2875 & -0.3681 & 0.0455 & 0.1752 & -0.2458 \\ 0.0194 & -0.0725 & 0.1838 & 0.0272 & -0.0943 & 0.2418 & 0.1170 & -0.3681 & 0.9318 & 0.0673 & -0.2458 & 0.6250 \\ 0.0762 & 0.0474 & 0.0699 & 0.0293 & 0.0187 & 0.0263 & 0.0733 & 0.0455 & 0.0673 & 0.4418 & 0.1170 & 0.3024 \\ 0.0474 & 0.1822 & -0.2555 & 0.0187 & 0.0712 & -0.0994 & 0.0455 & 0.1752 & -0.2458 & 0.1170 & 0.7689 & -1.0436 \\ 0.0699 & -0.2555 & 0.6495 & 0.0263 & -0.0994 & 0.2514 & 0.0673 & -0.2458 & 0.6250 & 0.3024 & -1.0436 & 2.6498 \end{bmatrix}.$$

Then, S_1 is obtained from step 2.

Maximize τ subject to (39) and (27) and the result is $\tau_{\max,1} = 6.6930 \times 10^{-4} > 0$. Go to step 7 and maximize ε subject to (8), (37), and (23) with S_p. The following feasible solutions are obtained:

$$\varepsilon = 9.8434,$$

$$K = \begin{bmatrix} 0.3339 & -0.3049 \\ -0.6050 & 0.3377 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.3350 & -0.3018 \\ -0.6050 & 0.3377 \end{bmatrix}.$$

Consider the case $\text{sat}(Ky(t)) = FC_y x(t)$. Calculating $\text{spec}(I_4 \otimes A + (\mathcal{H} \otimes B)(I_4 \otimes F)(I_4 \otimes C_y))$, one obtains $-1.3795 + 2.6138\text{j}$, $-1.3795 - 2.6138\text{j}$, $0.0422 + 1.6390\text{j}$, $0.0422 - 1.6390\text{j}$, $-0.9505 + 2.1520\text{j}$, $-0.9505 - 2.1520\text{j}$, $-0.6546 + 1.9703\text{j}$, $-0.6546 - 1.9703\text{j}$, -2.6729, -2.1906, -1.5200, -3.7509, which satisfies the stability condition in Lemma 2.

The state of each agent and the control input $u(t)$ are shown in Figure 3. At the beginning, the input is saturated. From Figures 4–6, it is shown that the FOMAS has achieved consensus. Figure 7 depicts that the error always tends to zero after the state change in leader.

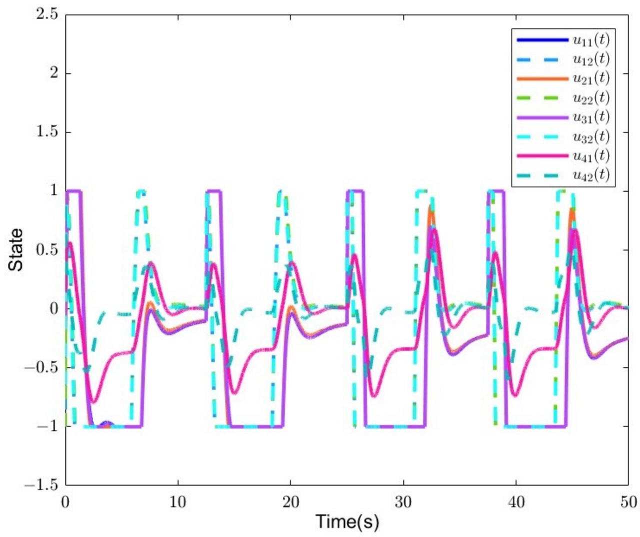

Figure 3. The control input of each agent in example 1.

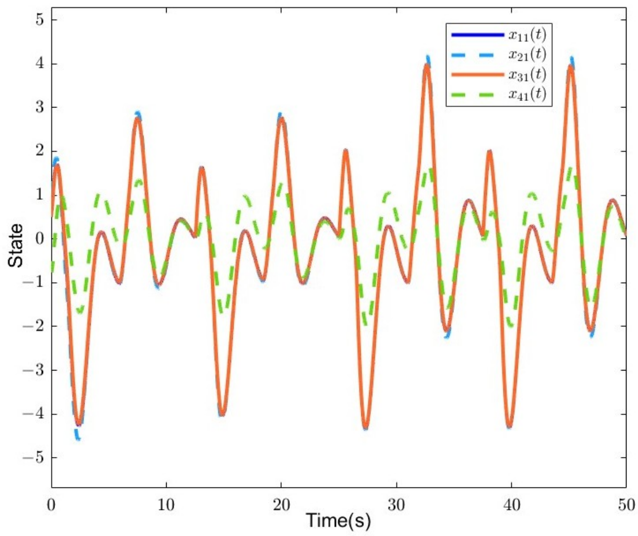

Figure 4. The state of each agent in example 1.

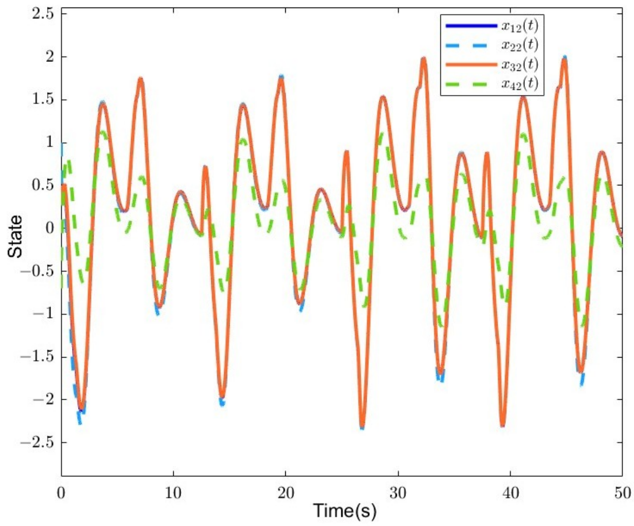

Figure 5. The state of each agent in example 1.

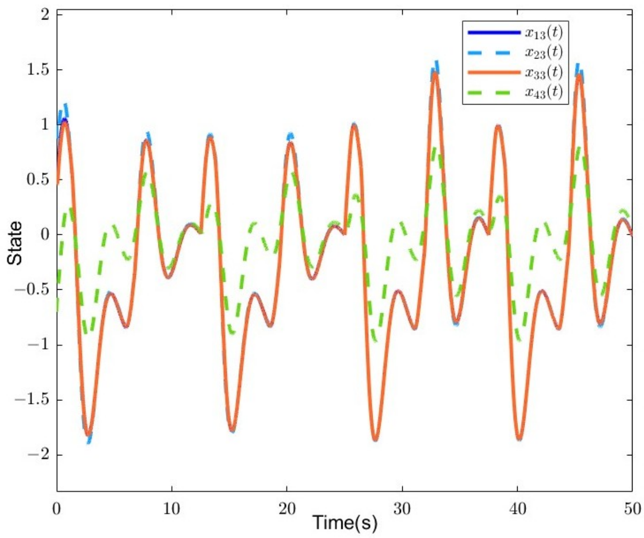

Figure 6. The state of each agent in example 1.

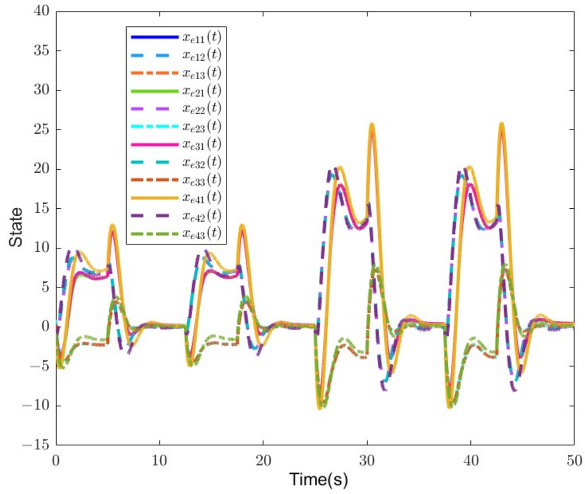

Figure 7. The state of error system in example 1.

Remark 13. *From the eigenvalues in the results, it is obvious that* $0.0422 \pm 1.6390\mathrm{j}$ *contain positive real parts. As mentioned in Remark 11, the stability criterion is not related to* α *for the exchange condition* $C_y X = X_e C_y$*. Thus, the eigenvalues of the matrix must have negative real parts. In other words, there exist gain matrices for the consensus of FOMASs, but they may not be obtained by using the SVD method. Thus, the ILMI algorithms in this paper are less conservative than the SVD method.*

Example 2. *Consider the FOMAS in (1) and (2) with* $\alpha = 1.3$ *and set* $\gamma = 20$*. The undirected graph is shown in Figure 8 and the parameter matrices are as follows:*

$$\mathcal{H} = \begin{bmatrix} 3.5 & -0.5 & 0 & -1 \\ -0.5 & 2 & -1 & 0 \\ 0 & -1 & 1.5 & -0.5 \\ -1 & 0 & -0.5 & 1.5 \end{bmatrix}.$$

$$A = \begin{bmatrix} -2 & 0.4 & 1 \\ 1 & -1 & -2 \\ 2 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},\ D_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix},\ D_3 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},\ C_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

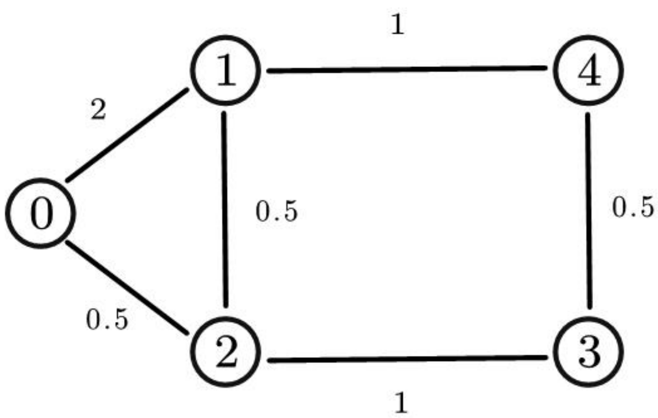

Figure 8. The weighted undirected graph in example 2.

Given $Q_{\alpha 12} = I$, the equation (37) has a feasible solution as

$$X_{\triangle,1} = \begin{bmatrix} 0.4009 & -0.2134 & 0.4429 & 0.1142 & 0.0233 & 0.1134 & 0.1326 & 0.0353 & 0.1167 & 0.1741 & 0.0283 & 0.1876 \\ -0.2134 & 0.3677 & -0.4388 & 0.0233 & 0.0205 & 0.0008 & 0.0353 & 0.0105 & 0.0287 & 0.0283 & 0.0365 & -0.0142 \\ 0.4429 & -0.4388 & 0.9906 & 0.1134 & 0.0008 & 0.1490 & 0.1167 & 0.0287 & 0.1075 & 0.1876 & -0.0142 & 0.2732 \\ 0.1142 & 0.0233 & 0.1134 & 0.6321 & -0.1767 & 0.6930 & 0.3452 & 0.0730 & 0.3401 & 0.1882 & 0.0520 & 0.1617 \\ 0.0233 & 0.0205 & 0.0008 & -0.1767 & 0.4202 & -0.4643 & 0.0730 & 0.0500 & 0.0220 & 0.0520 & 0.0149 & 0.0426 \\ 0.1134 & 0.0008 & 0.1490 & 0.6930 & -0.4643 & 1.3651 & 0.3401 & 0.0220 & 0.4164 & 0.1617 & 0.0426 & 0.1438 \\ 0.1326 & 0.0353 & 0.1167 & 0.3452 & 0.0730 & 0.3401 & 0.8325 & -0.1319 & 0.8855 & 0.2938 & 0.0716 & 0.2705 \\ 0.0353 & 0.0105 & 0.0287 & 0.0730 & 0.0500 & 0.0220 & -0.1319 & 0.4474 & -0.4464 & 0.0716 & 0.0347 & 0.0401 \\ 0.1167 & 0.0287 & 0.1075 & 0.3401 & 0.0220 & 0.4164 & 0.8855 & -0.4464 & 1.5914 & 0.2705 & 0.0401 & 0.2925 \\ 0.1741 & 0.0283 & 0.1876 & 0.1882 & 0.0520 & 0.1617 & 0.2938 & 0.0716 & 0.2705 & 0.7213 & -0.1652 & 0.7954 \\ 0.0283 & 0.0365 & -0.0142 & 0.0520 & 0.0149 & 0.0426 & 0.0716 & 0.0347 & 0.0401 & -0.1652 & 0.4385 & -0.4742 \\ 0.1876 & -0.0142 & 0.2732 & 0.1617 & 0.0426 & 0.1438 & 0.2705 & 0.0401 & 0.2925 & 0.7954 & -0.4742 & 1.5189 \end{bmatrix}.$$

From step 2, S_1 is set immediately.

Maximize τ subject to (39) and (38), then one obtains $\tau_{\max,1} = -0.0170 < 0$. Thus, go to step 4 and minimize $\mathrm{trace}(X_1)$ subject to the LMIs (39) and (38) with $\tau_{\max,1} = -0.0170 < 0$. The result is

$$X_{\min,1} = \begin{bmatrix} 1.0178 & -0.1140 & -0.0537 & 0.6247 & 0.0507 & -0.0730 & 0.8731 & 0.0244 & -0.1176 & 0.7418 & 0.0805 & -0.0479 \\ -0.1140 & 0.2079 & -0.1218 & 0.0507 & 0.0240 & 0.0649 & 0.0244 & 0.0682 & 0.0672 & 0.0805 & 0.0199 & 0.0873 \\ -0.0537 & -0.1218 & 0.7936 & -0.0730 & 0.0649 & 0.1732 & -0.1176 & 0.0672 & 0.2692 & -0.0479 & 0.0873 & 0.2169 \\ 0.6247 & 0.0507 & -0.0730 & 2.0204 & 0.0076 & -0.1279 & 1.8336 & 0.1095 & -0.1881 & 1.3089 & 0.0398 & -0.1901 \\ 0.0507 & 0.0240 & 0.0649 & 0.0076 & 0.2225 & 0.0026 & 0.1095 & 0.1078 & 0.1706 & 0.0398 & 0.0969 & 0.1023 \\ -0.0730 & 0.0649 & 0.1732 & -0.1279 & 0.0026 & 1.0694 & -0.1881 & 0.1706 & 0.5597 & -0.1901 & 0.1023 & 0.3911 \\ 0.8731 & 0.0244 & -0.1176 & 1.8336 & 0.1095 & -0.1881 & 3.1551 & 0.0700 & -0.2581 & 1.8251 & 0.0851 & -0.2382 \\ 0.0244 & 0.0682 & 0.0672 & 0.1095 & 0.1078 & 0.1706 & 0.0700 & 0.2907 & 0.1055 & 0.0851 & 0.1164 & 0.1577 \\ -0.1176 & 0.0672 & 0.2692 & -0.1881 & 0.1706 & 0.5597 & -0.2581 & 0.1055 & 1.4102 & -0.2382 & 0.1577 & 0.5402 \\ 0.7418 & 0.0805 & -0.0479 & 1.3089 & 0.0398 & -0.1901 & 1.8251 & 0.0851 & -0.2382 & 2.2835 & 0.0394 & -0.1132 \\ 0.0805 & 0.0199 & 0.0873 & 0.0398 & 0.0969 & 0.1023 & 0.0851 & 0.1164 & 0.1577 & 0.0394 & 0.2334 & 0.0353 \\ -0.0479 & 0.0873 & 0.2169 & -0.1901 & 0.1023 & 0.3911 & -0.2382 & 0.1577 & 0.5402 & -0.1132 & 0.0353 & 1.1664 \end{bmatrix}.$$

Set $X_{\triangle,2} = X_{\min,1}$ and obtain S_2. Then, maximize τ subject to (39) and (38), then one obtains $\tau_{\max,2} = 0.2062 > 0$. Jump to step 7, and maximize ε subject to the LMIs (23), $X > 0$, and (35) with S_2. The feasible solutions are

$$\varepsilon = 1.3548,$$

$$K = \begin{bmatrix} -1.6295 & -0.1643 \\ -0.9381 & -0.1125 \end{bmatrix},$$

$$F = \begin{bmatrix} -0.8538 & -0.0600 \\ -0.8470 & -0.0882 \end{bmatrix}.$$

The input saturation in the FOMAS is shown in Figure 9. The state of each agent and the error are shown in Figures 10–13. It is shown that the FOMAS has achieved leader-following consensus.

One can also choose another $Q_{\alpha 12}$ for different initial values $X_{\triangle,1}$. At the start of the algorithm, the positive definite matrix $Q_{\alpha 12}$ is selected as $I_4 \otimes W^T W$, where

$$W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Solve Equation (37) and obtain

$$X_{\triangle,1} = \begin{bmatrix} 0.5607 & -0.1210 & 0.8171 & 0.2470 & 0.1139 & 0.2463 & 0.2853 & 0.1571 & 0.2501 & 0.3677 & 0.1517 & 0.4034 \\ -0.1210 & 0.6278 & -0.3897 & 0.1139 & 0.1180 & 0.0723 & 0.1571 & 0.1001 & 0.1304 & 0.1517 & 0.2054 & 0.0793 \\ 0.8171 & -0.3897 & 1.3809 & 0.2463 & 0.0723 & 0.3039 & 0.2501 & 0.1304 & 0.2545 & 0.4034 & 0.0793 & 0.5480 \\ 0.2470 & 0.1139 & 0.2463 & 1.0534 & 0.0779 & 1.3582 & 0.7335 & 0.3517 & 0.7292 & 0.4094 & 0.2284 & 0.3490 \\ 0.1139 & 0.1180 & 0.0723 & 0.0779 & 0.9073 & -0.2933 & 0.3517 & 0.3369 & 0.2448 & 0.2284 & 0.1373 & 0.1906 \\ 0.2463 & 0.0723 & 0.3039 & 1.3582 & -0.2933 & 2.1199 & 0.7292 & 0.2448 & 0.8865 & 0.3490 & 0.1906 & 0.3408 \\ 0.2853 & 0.1571 & 0.2501 & 0.7335 & 0.3517 & 0.7292 & 1.4823 & 0.2894 & 1.7722 & 0.6347 & 0.3281 & 0.5836 \\ 0.1571 & 0.1001 & 0.1304 & 0.3517 & 0.3369 & 0.2448 & 0.2894 & 1.0944 & -0.1408 & 0.3281 & 0.2524 & 0.2503 \\ 0.2501 & 0.1304 & 0.2545 & 0.7292 & 0.2448 & 0.8865 & 1.7722 & -0.1408 & 2.6063 & 0.5836 & 0.2503 & 0.6436 \\ 0.3677 & 0.1517 & 0.4034 & 0.4094 & 0.2284 & 0.3490 & 0.6347 & 0.3281 & 0.5836 & 1.2340 & 0.1467 & 1.5745 \\ 0.1517 & 0.2054 & 0.0793 & 0.2284 & 0.1373 & 0.1906 & 0.3281 & 0.2524 & 0.2503 & 0.1467 & 1.0200 & -0.2612 \\ 0.4034 & 0.0793 & 0.5480 & 0.3490 & 0.1906 & 0.3408 & 0.5836 & 0.2503 & 0.6436 & 1.5745 & -0.2612 & 2.4336 \end{bmatrix}.$$

Then, maximize τ subject to (39) and (38), and the result is $\tau_{\max,1} = 0.0743 > 0$. Go to step 7. The subsequent solving process is omitted.

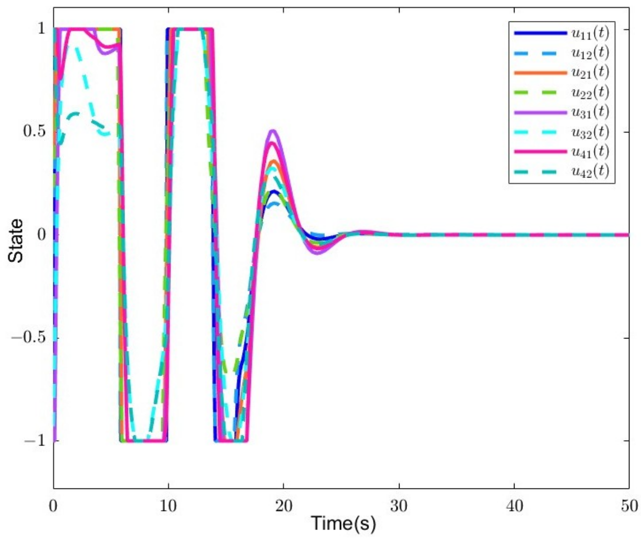

Figure 9. The control input of each agent in example 2.

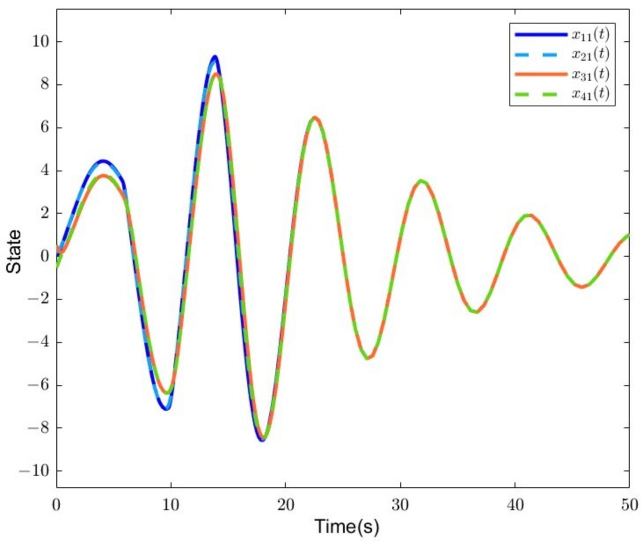

Figure 10. The state of each agent in example 2.

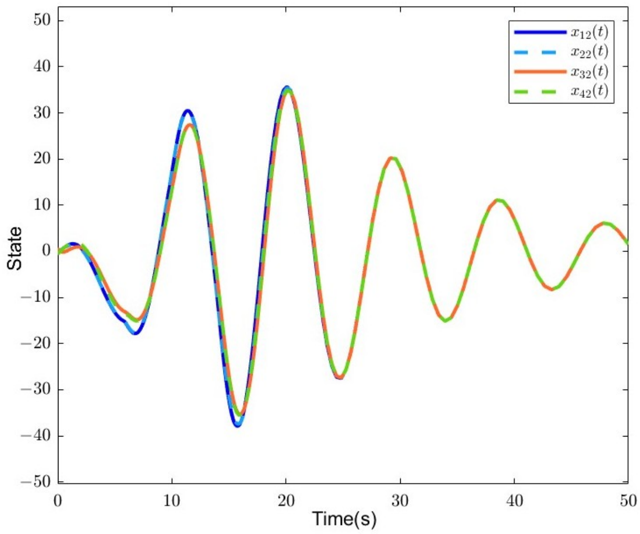

Figure 11. The state of each agent in example 2.

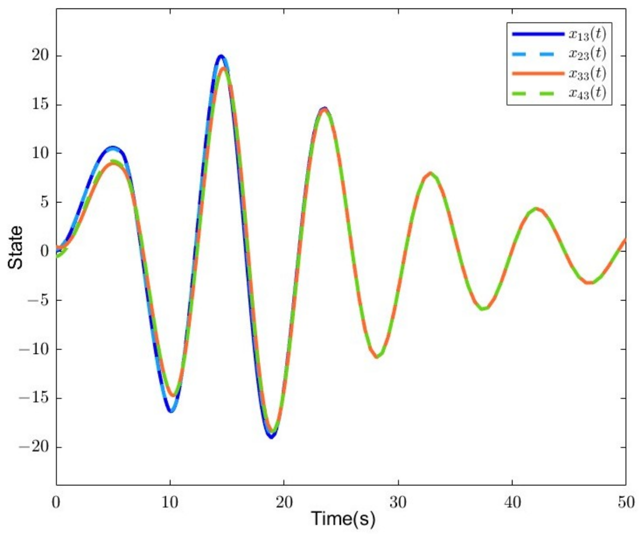

Figure 12. The state of each agent in example 2.

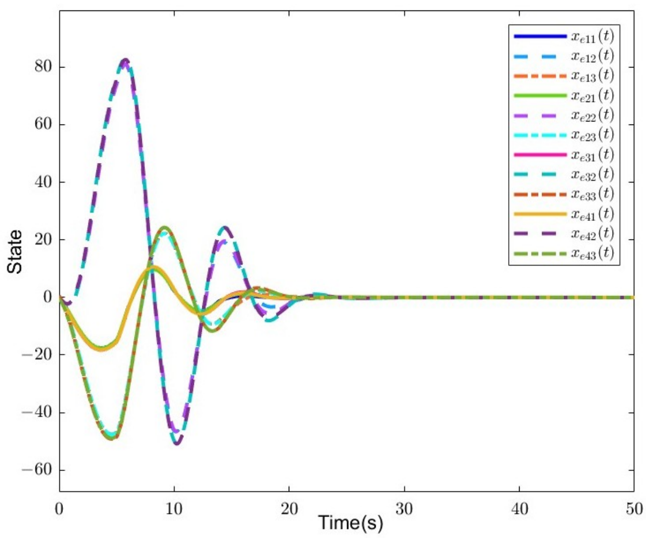

Figure 13. The state of error system in example 2.

Remark 14. *It is seen that the number of iterations in* $Q_{\alpha 12} = I_4 \otimes W^T W$ *is less than that in* $Q_{\alpha 12} = I$. $Q_{\alpha 12} = I$ *is convenient to set, but* I *is a diagonal matrix, which differs from the* $(a(\mathcal{H} \otimes B)^T X + \tilde{K}(I_4 \otimes C_y))^T (a(\mathcal{H} \otimes B)^T X + \tilde{K}(I_4 \otimes C_y))$ *under general conditions. Thus, the choice of* $Q_{\alpha 01}$ *and* $Q_{\alpha 12}$ *affects the number of iterations. When the algorithm cannot rapidly converge, replacement of the initial value may be considered. Choosing* $Q_{\alpha 12} > 0$ *and* $Q_{\alpha 01} > 0$ *is the key to ensuring fast convergence of the algorithm. Otherwise,* $\tau_{\max,1} = -2.2478$, $\tau_{\max,2} = -0.4005$, $\tau_{\max,3} = -0.0644$, *and* $\tau_{\max,4} = 0.1303$ *are obtained in step 2 after setting*

$$Q_{\alpha 12} = \begin{bmatrix} I_6 & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0,$$

which greatly increases the number of iterations.

5. Conclusions

The leader-following H_∞ consensus of FOMASs under input saturation via the output feedback was investigated. Lemmas 3 and 7 provided sufficient conditions for the H_∞ consensus for FOMASs in (1) and (2) with $\alpha \in (0, 1)$ and $[1, 2)$, respectively. Additionally, based on the ILMI approach, Algorithms 1 and 2 were presented to compute the gain matrices. In both ILMI algorithms, the methods to derive initial values were also given and ILMI algorithms solved the H_∞ consensus problem for FOMASs under actuator saturation. The algorithms are effective and convergent, avoiding matrix exchange condition in the SVD method and strong assumptions. The method also provides a holistic approach for calculating the gain matrices of MASs. Finally, numerical examples were conducted to validate the efficiency of the proposed approach.

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