fractal and fractional

Article

Discovery of Intrinsic Ferromagnetism Induced by Memory Effects in Low-Dimensional System

Shaolong Zeng [1], Xuejin Wan [1], Yangfan Hu [1,*], Shijing Tan [2] and Biao Wang [1,*]

[1] Research Institute of Interdisciplinary Science & School of Materials Science and Engineering, Dongguan University of Technology, Dongguan 523820, China; zengshlong@mail3.sysu.edu.cn (S.Z.); wanxj@dgut.edu.cn (X.W.)

[2] Hefei National Laboratory for Physical Sciences at the Microscale, Department of Chemical Physics, University of Science and Technology of China, Hefei 230026, China; tansj@ustc.edu.cn

* Correspondence: huyf@dgut.edu.cn (Y.H.); wangbiao@mail.sysu.edu.cn (B.W.)

Abstract: The impact of dynamic processes on equilibrium properties is a fundamental issue in condensed matter physics. This study investigates the intrinsic ferromagnetism generated by memory effects in the low-dimensional continuous symmetry Landau–Ginzburg model, demonstrating how memory effects can suppress fluctuations and stabilize long-range magnetic order. Our results provide compelling evidence that tuning dynamical processes can significantly alter the behavior of systems in equilibrium. We quantitatively evaluate how the emergence of the ferromagnetic phase depends on memory effects and confirm the presence of ferromagnetism through simulations of hysteresis loops, spontaneous magnetization, and magnetic domain structures in the 1D continuous symmetry Landau–Ginzburg model. This research offers both theoretical and numerical insights for identifying new phases of matter by dynamically modifying equilibrium properties.

Keywords: magnetic responses; fractional temporal derivatives; Landau–Ginzburg model; hysteresis loops; spontaneous magnetization low-dimensional system

Citation: Zeng, S.; Wan, X.; Hu, Y.; Tan, S.; Wang, B. Discovery of Intrinsic Ferromagnetism Induced by Memory Effects in Low-Dimensional System. *Fractal Fract.* **2024**, *8*, 668. https://doi.org/10.3390/fractalfract8110668

Academic Editor: Haci Mehmet Baskonus

Received: 7 October 2024
Revised: 12 November 2024
Accepted: 14 November 2024
Published: 16 November 2024

1. Introduction

Understanding the behavior of matter through dynamical processes and microscopic interactions has long been a central endeavor in statistical physics [1]. The study of dynamical processes has garnered widespread attention across various fields, including the evolution and transformation of matter [2], electron transport [3], diffusion [4], and particle formation and annihilation [5]. Furthermore, in the research on system evolution, dynamic phase transitions akin to equilibrium phase transitions have been discovered [6], providing an important theoretical foundation for regulating system properties through dynamical processes. Although in traditional thermodynamics, the equilibrium state of a system is considered time-independent, and these results have been widely verified in Markovian systems [1], whether memory effects can alter the equilibrium characteristics of non-Markovian systems and introduce a time-dependent characteristic variable in thermodynamic functions remains an open scientific question. Indeed, in spin glasses the observed memory effects not only lead to the existence of multiple metastable states and break down the fluctuation-dissipation theorem [7], but also influence the equilibrium state, resulting in different magnetization strengths, correlation lengths, and magnetic susceptibilities [8]. Additionally, studies on ferromagnetic phase transitions have shown that memory effects also lead to higher critical temperatures and exhibit critical characteristics of higher-dimensional systems near continuous phase transition points [9]. It is well known that memory effects also have been found in non-Ohmic dissipative quantum systems [10], viscous fluids [11], and aging and rejuvenation [12] and so on. In general, when the relaxation time scales of a system and its environment are comparable, the system's past information may be reintroduced through its interactions with the environment, giving rise to long-range temporal correlations [13]. Mathematically, the introduction of fractional temporal derivatives has significantly broadened the scope of dynamical research by offering a more general framework for describing systems with memory effects and nonlocal temporal behaviors [14]. This approach has been extensively applied to various equations in physics, including the Schrödinger equation [15], Fokker–Planck equation [16], Langevin equation [17] and so on.

Low-dimensional magnetic materials, characterized by their smooth surfaces and atomically thin layers, offer significant potential for the development of next-generation spintronic devices [18–21]. However, the Mermin–Wagner theorem states that fluctuations prevent stable long-range magnetic order and ferromagnetism in low-dimensional systems with continuous symmetry [22]. Even though the magnetic response of two-dimensional systems has been successfully explained due to anisotropy [23], and the temperature dependence of magnetization and resistivity in anisotropic structured crystals also provides insights for information transmission [24], the discovery of magnetic responses in one-dimensional chains at finite temperatures remains unexplained in experiments [25]. Additionally, in the last century, the introduction of spatial fractional derivatives in the Landau–Ginzburg model not only predicted new magnetic phenomena [26], but also established a theoretical basis for the magnetism induced by defect engineering and doping [27,28]. Nevertheless, relying solely on the introduction of spatial long-range interactions cannot fully and reasonably describe the dependence of magnetization strength, magnetic susceptibility, and specific heat on environmental variables near the critical point. In recent years, we have found that the introduction of fractional temporal derivatives leads to a redistribution of time and spatial dimensions, resulting in new critical exponents that can well explain the measurements observed in experiments [29,30].

In order to clarify the effect of memory on magnetic response, and to explain the magnetic responses observed in one-dimensional chains, we study the Landau–Ginzburg model with fractional temporal derivatives, as the fractional time derivative allows for tuning the memory effects in the dynamical process [14,31]. The Landau–Ginzburg model is an important framework for describing ferromagnetic phase transitions and understanding magnetic responses at low temperatures [32], successfully predicting the dependence of magnetic effects on environmental variables near the ferromagnetic transition point [33]. This work provides theoretical guidance for future applications aimed at regulating the magnetic response of systems through dynamical processes and establishes a coupling mechanism between space and time. We have investigated the intrinsic magnetism in low-dimensional systems induced by memory effects, including the following two aspects: (1) Quantitatively calculating the conditions under which ferromagnetism is induced by memory effects. (2) Simulating the continuous symmetric Landau–Ginzburg model with fractional temporal derivatives and validating the occurrence of ferromagnetism by comparing hysteresis effects, spontaneous magnetization, and magnetic domain structures.

2. Theory of Magnetic Respond Induced by Fractional Temporal Derivatives

We consider the extended Langevin equation with temporal fractional derivatives which is expressed as

$$\Gamma(N-\alpha)\,{}_{t_0}^{C}D_t^{\alpha}\phi(x,t) = -\lambda\frac{\delta H}{\delta\phi(x,t)} + \zeta, \tag{1}$$

where

$$H = \int d^d x\left[\frac{1}{2}\tau\phi(x,t)^2 + \frac{1}{2}(\bigtriangledown\phi(x,t))^2 - h\phi(x,t) + \frac{1}{4}u\phi(x,t)^4\right], \tag{2}$$

Equation (2) is a d-dimensional Landau–Ginzburg model in which the order parameter satisfies SU(2) group symmetry. It forms the theoretical foundation for describing phase transitions in the XY model and can predict the emergence of ferromagnetism and the associated rules of ferromagnetic phase transitions [32]. The $\phi(x,t)$ is the order parameter of two components at position x and time t; h is the external field; the kinetic coefficient λ

should be positive; τ is reduced temperature and defined as $\tau = (T - T_c)/T_c$ where T and T_c are temperature and critical temperature, respectively, u is coupling constant and must be positive. The term $\bigtriangledown\phi(x,t))^2$ represents the gradient term and spatial nearest-neighbor interactions, which describe the inhomogeneity of the system. In Equation (2), due to $\bigtriangledown\phi(x,t))^2$, the order parameter is arranged in parallel, resulting in lower internal energy. ${}_{t_0}^{C}D_t^{\alpha}$ is a Caputo derivative and can be expressed as [34,35]

$$ {}_{t_0}^{C}D_t^{\alpha}\phi(t) = \frac{1}{\Gamma(N-\alpha)}\int_{t_0}^{t} dt \frac{\phi^{(N)}(t')}{(t-t')^{\alpha} - N + 1} \tag{3} $$

with $\phi^{(N)}(t') = \partial^N\phi(t')/(\partial t')^N$, t_0 is the initial time. The introduced parameter $0 < \alpha < 1$ is dimensionless fractional order and represents memory effects [36]. The $\Gamma(N-\alpha)$ is the well-known Gamma function and N is the smallest positive integer greater than or equal to α, in our situation, $N = 1$ [37]. For $\alpha = 1$, Equation (1) recovers to the standard Langevin equation. The Gaussian noise is satisfied with

$$ \begin{aligned} &< \zeta(x,t) >= 0, \\ &< \zeta(x,t)\zeta(x',t') >= 2\lambda\delta(x-x')\delta(t-t'), \end{aligned} \tag{4} $$

where $<>$ represents average. Let the order parameter ϕ decompose the mean magnetization m, the value of fluctuation $\phi_{//}$ parallel to the direction of mean magnetization and the value of fluctuation $\phi_{\perp}$ perpendicular to the direction of the mean magnetization. It is expressed as

$$ \phi = m\vec{n} + \phi_{//}\vec{n} + \phi_{\perp}\vec{n}_{\perp}, \tag{5} $$

where $\vec{n}$ and $\vec{n}_{\perp}$ are unit vectors which are parallel and perpendicular to the direction of the mean magnetization, respectively. According to Equation (1), at equilibrium, with $h = 0$ and since $\bigtriangledown m = 0$, the equation for the mean magnetization m, derived from Equation (1), reduces to

$$ \tau m + u m^3 = 0. \tag{6} $$

By substituting Equations (5) and (6) into Equation (1), the part of $\phi_{//}$ in equilibrium is expressed as

$$ \Gamma(1-\alpha){}_{t_0}^{C}D_t^{\alpha}\phi_{//}(t) = -\lambda\tfrac{\delta H}{\delta\phi_{//}} + \zeta_{//}, \tag{7} $$

where

$$ \tfrac{\delta H}{\delta\phi_{//}} = -2\tau\phi_{//} + \bigtriangledown^2\phi_{//} + o\left(\phi_{//}^2, \phi_{\perp}^2\right), \tag{8} $$

o represents high-order small quantity and $\zeta_{//}$ is the white noise component parallel to the mean magnetization. Similarly, the dynamic equation for $m_{\perp}$ is given by

$$ \Gamma(1-\alpha){}_{t_0}^{C}D_t^{\alpha}\phi_{\perp}(t) = -\lambda\left[\bigtriangledown^2\phi_{\perp} + o\left(\phi_{\perp}^2, \phi_{//}^2\right)\right] + \zeta_{\perp}, \tag{9} $$

where $\zeta_{\perp}$ is the white noise perpendicular to the mean magnetization. After performing a Fourier transform on the order parameter $\phi(x,t) = \int d^d k \int dw \phi(k,w) e^{ikx - iwt}/(2\pi)^{d+1}$, the solutions for $\phi_{//}(k,w)$ and $\phi_{\perp}(k,w)$ from Equation (7) and Equation (9) are as follows:

$$ \phi_{//}(k,w) = \frac{\zeta_{//}}{\lambda(k^2 - 2\tau) - (iw)^{\alpha}} \tag{10} $$

and

$$ \phi_{\perp}(k,w) = \frac{\zeta_{\perp}}{\lambda k^2 - (iw)^{\alpha}} \tag{11} $$

According to the definition of responding functions $G_{//} = \phi_{//}/\zeta_{//}$ and $G_{\perp} = \phi_{\perp}/\zeta_{\perp}$. The responding functions can be given by

$$G_{//}(k,w) = \frac{1}{\lambda(k^2 - 2\tau) - (iw)^{\alpha}}. \tag{12}$$

and

$$G_{\perp}(k,w) = \frac{1}{\lambda k^2 - (iw)^{\alpha}}. \tag{13}$$

Then, the correlation function is defined as $(2\pi)^{d+1}\delta(w+w')\delta^d(k+k')C_{//}(k,k',w,w') =< \phi_{//}(k,w)\phi_{//}(k',w') >$. Due to the constraint of δ function, $C_{//}(k,w)$ is just contributed by $w' = -w$ and $k' = -k$ and expressed as

$$\begin{aligned} C_{//}(k,w) &=< \phi_{//}(k,w)\phi_{//}(-k,-w) > \\ &= 2\lambda G_{//}(k,w)G_{//}(-k,-w). \end{aligned} \tag{14}$$

Correspondingly, the correlation function $C_{\perp}(k,w) =< \phi_{\perp}(k,w)\phi_{\perp}(-k,-w) >$ in momentum-frequency space is expressed as

$$C_{\perp}(k,w) = 2\lambda G_{\perp}(k,w)G_{\perp}(-k,-w). \tag{15}$$

According to Equation (14), the correlation function $C_{//}$ in real space is expressed as

$$\begin{aligned} C_{//}(\Delta x, 0) &= \frac{1}{(2\pi)^{d+1}} \int d^d k \int dw e^{ik\Delta x} C_{//}(k,w) \\ &\propto \int d^d k \frac{e^{ik\Delta x}}{(k^2 - 2\tau)^{2-\frac{1}{\alpha}}}, \end{aligned} \tag{16}$$

where Δx represents the distance between two particles. Furthermore, the correlation function $G_{\perp}$ in real space is expressed as

$$\begin{aligned} C_{\perp}(\Delta x, 0) &= \frac{1}{(2\pi)^{d+1}} \int d^d k \int dw e^{ik\Delta x} C_{\perp}(k,w) \\ &\propto \int d^d k \frac{e^{ik\Delta x}}{k^{4-2\alpha}} = \Delta x^{4-\frac{2}{\alpha}-d}. \end{aligned} \tag{17}$$

For $\alpha = 1$, Equation (17) shows an infrared divergence when spatial dimension $d \leq 2$. It also implies that the long-range magnetic order can be disrupted by even the smallest fluctuations, preventing the emergence of a ferromagnetic phase. For models like the Ising model, the weak XY model, or the anisotropic Heisenberg model, where the spin energies in different directions are no longer degenerate, only the correlation function $G_{//}$ parallel to the average magnetization direction in real space needs to be considered. In these cases, due to suppression by temperature τ in Equation (16), it does not exhibit infrared divergence in $d = 2$, resulting in a stable ferromagnetic phase [23,38]. On the other hand, for $\alpha < 1$, the impact of fluctuations is suppressed by the history of the path, maintaining a stable ferromagnetic phase in low-dimensional spaces until $d < 4 - 2/\alpha$. As in Equations (13) and (15), we also should note the responding and correlation functions for fractional temporal derivatives do not satisfy the fluctuation-dissipation theorem.

3. Simulation of Continuous Symmetry Landau–Ginzburg Model

To verify the stable existence of the ferromagnetic phase induced by memory effects in low-dimensional spaces, we numerically simulated the 1D Landau–Ginzburg model with fractional temporal derivatives as described by Equations (1) and (2), 100 spins have been simulated. To ensure the convergence of the results, we assume that the memory is finite. During the simulation, we fixed this memory $\mathcal{T}_0 = 40$. Specifically, we examined the magnetic hysteresis loop, spontaneous magnetization, and magnetic domain structures. Periodic boundary conditions and an ordered initial state were applied throughout the simulation. We decomposed the order parameter into components along the x and z directions which are expressed as ϕ_x and ϕ_z, respectively. In order to fix the magnitude constraint, $\phi_x^2 + \phi_z^2 = 1$ should be satisfied. Let $\lambda = 1$ for convenience, the dynamic equations of ϕ_x and ϕ_z are expressed as

$$\Gamma(1-\alpha)_{t_0}^{C}D_t^{\alpha}\phi_x(x,t) = -\frac{\delta H}{\delta\phi_x(x,t)} + \cos\theta\zeta, \tag{18}$$

where

$$\frac{\delta H}{\delta\phi_x} = \tau\phi_x + \bigtriangledown^2\phi_x + u\phi_x^3 + u\phi_z^2\phi_x \tag{19}$$

and θ is the angle with the x-direction. Correspondingly, the dynamic equation of ϕ_z is expressed as

$$\Gamma(1-\alpha)_{t_0}^{C}D_t^{\alpha}\phi_z(x,t) = -\frac{\delta H}{\delta\phi_z(x,t)} + \sin\theta\zeta, \tag{20}$$

where

$$\frac{\delta H}{\delta\phi_z} = \tau\phi_z + \bigtriangledown^2\phi_z + u\phi_x^3 + u\phi_x^2\phi_z. \tag{21}$$

Figure 1 displays the magnetic hysteresis loops for the 1D continuous symmetric Landau model with fractional temporal derivatives at $\tau = -5$ and $\tau = -2$, respectively; 100 spins and 100 samples have been averaged. Magnetization is defined as $M =< m >$ where $m =< \phi >$, and the z-direction magnetization as $M_z =< m_z >$ with $m_z =< \phi_z >$. The initial state assumes all order parameters at $\phi_z = 1$ and $\phi_x = 0$. As the fractional order decreases, the hysteresis effect becomes more prominent. When fractional order $\alpha = 0.7 > 2/3$, the hysteresis effect nearly disappears. The comparison between Figure 1a,b suggests that this disappearance of hysteresis at $\alpha = 0.7$ is not temperature-driven, but rather due to the more stable ferromagnetic phase induced by the lower fractional order. The hysteresis arises from metastable states within the system, making the identification of these states crucial for validating the hysteresis loops. Figure 2 illustrates how the absolute value of the z-component of magnetization, $|M_z|$, evolves under an external z-directional field, $h_z = \pm 0.1$, applied parallel and antiparallel to the initial state. For $h_z = -0.1$, a plateau distinct from that for $h_z = 0.1$ emerges, and the spin structure at this plateau is depicted in Figure 3. Near the saturation field, the Landau–Ginzburg model with fractional temporal derivatives can also exhibit magnetic solitons. These persistent magnetic solitons lead to the formation of metastable states, resulting in a significant hysteresis loop.

Additionally, Figure 4 shows the evolution of $< m^2 >$ over time without an external field. As illustrated, after 100,000 steps, although the direction of magnetization remains uncertain, the magnitude of the squared magnetization stabilizes. Moreover, the smaller the fractional order, the larger the value of $< m^2 >$. This suggests that more pronounced fractional temporal derivatives make individual samples more likely to exhibit macroscopic magnetization. To more intuitively demonstrate the effect of fractional temporal derivatives on long-range ferromagnetic order, we present the magnetic domain structures of 10 randomly selected samples. As shown in Figure 5, a comparison between $\alpha = 0.7$ and $\alpha = 0.4$ reveals that the smaller the fractional order, the more stable the long-range ferromagnetic order and the larger the magnetic domains. This indicates that memory effects help suppress fluctuations and protect the stable existence of the long-range ferromagnetic order.

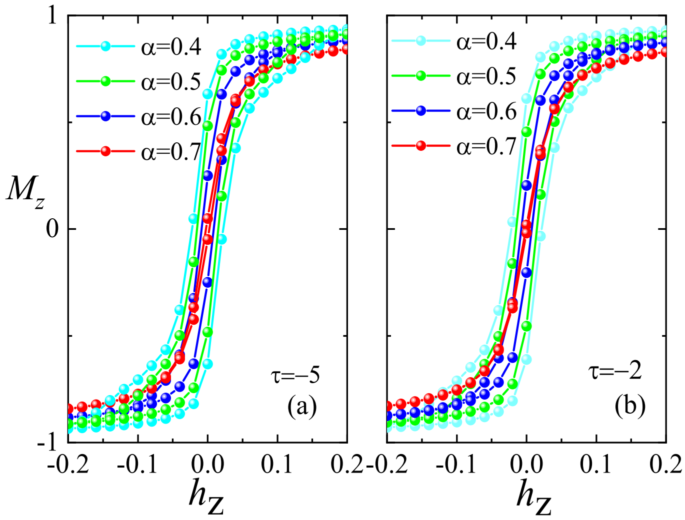

Figure 1. Magnetic hysteresis loop at $\tau = -5$ (**a**) and $\tau = -2$ (**b**) in the 1D continuous symmetry Landau–Ginzburg model for fractional temporal derivatives.

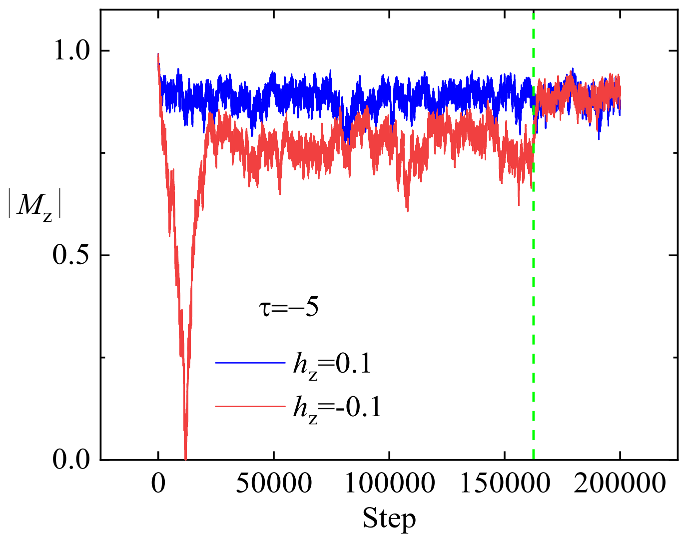

Figure 2. Absolute value of magnetization in the z-direction $|M_z|$ versus external field in the z-direction h_z for the 1D continuous symmetry Landau–Ginzburg model with fractional order $\alpha = 0.5$. The green dashed line corresponds to the position at 162,500 steps.

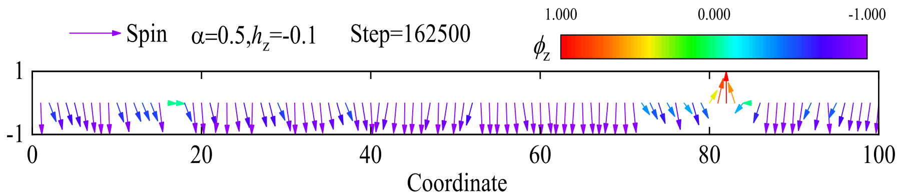

Figure 3. Magnetic structure of the 1D continuous symmetry Landau–Ginzburg model in Figure 2 under an external field $h_z = -0.1$ at step = 162,500. The color scale represents the projection of the order parameter in the z-direction.

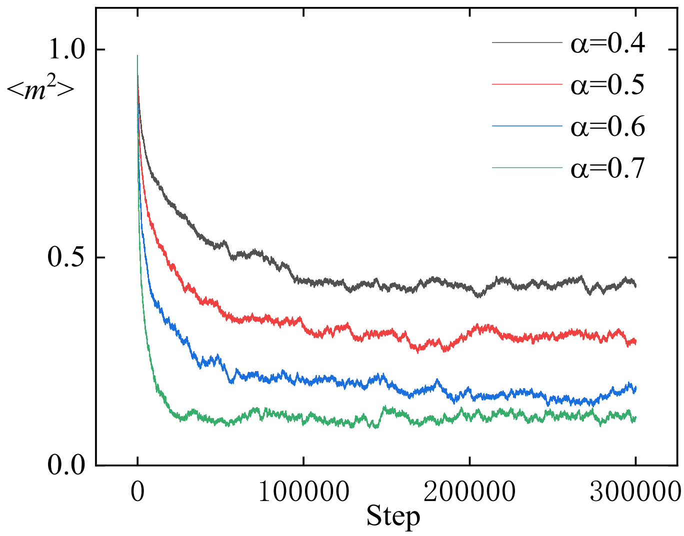

Figure 4. Spontaneous magnetization in the 1D continuous symmetry Landau–Ginzburg model with fractional temporal derivatives. Results averaged over 100 samples.

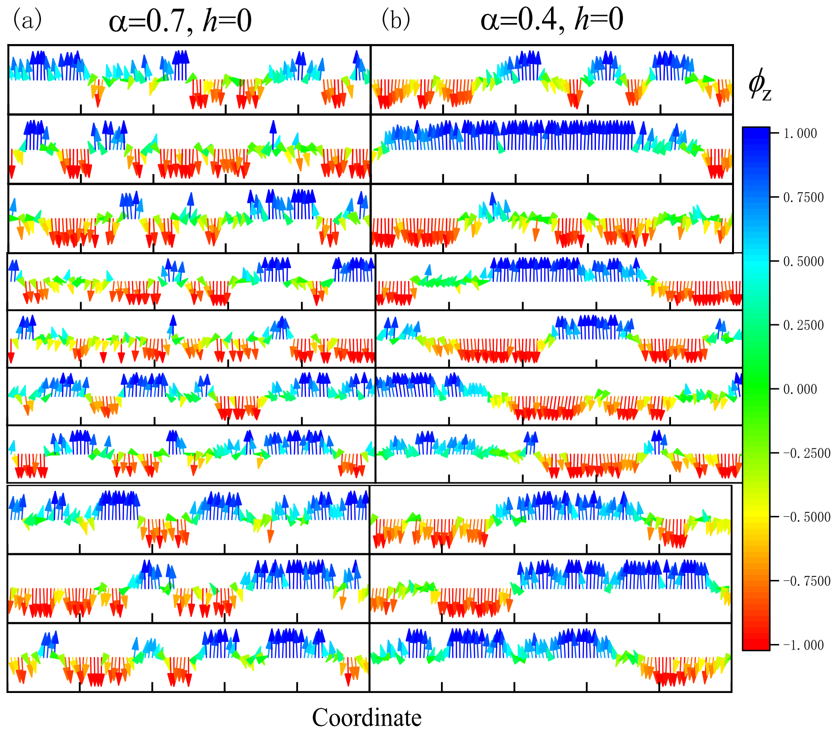

Figure 5. The magnetic structure with $\alpha = 0.4$ (**a**) and $\alpha = 0.7$ (**b**). The arrows represent the order parameter directions at different positions, and the color scale represents the projection of the order parameter in the z-direction.

Consequently, we validated the existence of a ferromagnetic phase induced by memory effects through the analysis of hysteresis loops, spontaneous magnetization, and magnetic domain structures. For $\alpha < 2/3$, we clearly identified the presence of metastable states, which are the primary cause of the hysteresis loops observed in the Landau–Ginzburg model with fractional temporal derivatives. Furthermore, although the magnetization direction in the continuous symmetric Landau model is not fixed, in the absence of an external field, a lower fractional order results in greater macroscopic magnetization in individual samples and larger magnetic domain structures. These results indicate that memory effects can induce the emergence of a ferromagnetic phase in low-dimensional systems. These discoveries also suggest that memory effects not only alter the dynamic behavior of the system but also modify its equilibrium properties, resulting in the formation of new phases.

4. Conclusions

In summary, we have demonstrated that intrinsic ferromagnetism can be induced in a low-dimensional continuous symmetric Landau–Ginzburg model through memory effects. We found that in low-dimensional systems, the relationship between the fractional order α and the spatial dimension d satisfies $\alpha < 2/(d-4)$, where the infrared divergence of the correlation function is suppressed by the fractional temporal derivatives, leading to the formation of a stable long-range magnetic order. We validated these theoretical results by simulating the hysteresis loops, spontaneous magnetization, and magnetic domain structures of the continuous symmetry Landau model with fractional temporal derivatives, confirming the positive role of memory effects in suppressing fluctuations and protecting the long-range ferromagnetic order.

These findings suggest that memory effects can alter the equilibrium properties of a system to achieve new phases, providing a potential method for modulating the equilibrium properties through dynamical processes. This research not only offers new perspectives and methods for understanding magnetic responses in low-dimensional systems, but more importantly, it theoretically and numerically validates the possibility of tuning the equilibrium properties of a system through dynamical processes. This provides an important reference for establishing a universal connection between dynamical processes, spatial interactions, and the macroscopic properties of systems.

Author Contributions: Conceptualization, S.Z.; Methodology, S.Z.; Software, X.W.; Formal analysis, Y.H.; Writing—original draft, S.Z.; Writing—review & editing, Y.H.; Funding acquisition, S.T. and B.W. All authors have read and agreed to the published version of the manuscript.

Funding: The work was supported by the NSFC (National Natural Science Foundation of China) through the funds with Grant Nos. 12172090, 11772360, 11832019, 12102091, the key projects of Guangdong-Dongguan Collaborative Fund (Grant No. 2023B1515120013).

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding authors.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Huang, K. *Introduction to Statistical Physics*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2009.
2. Zwanzig, R. *Nonequilibrium Statistical Mechanics*; Oxford University Press: Oxford, UK, 2001. [CrossRef]
3. Lemme, M.C.; Li, L.J.; Palacios, T.; Schwierz, F. Two-dimensional materials for electronic applications. *Mrs Bull.* **2014**, *39*, 711–718. [CrossRef]
4. Crank, J. *The Mathematics of Diffusion*; Oxford University Press: Oxford, UK, 1979.
5. Perkins, D.H. *Introduction to High Energy Physics*; Cambridge University Press: Cambridge, UK, 2000.
6. Hinrichsen, H. Non-equilibrium phase transitions. *Phys. A Stat. Mech. Its Appl.* **2006**, *369*, 1–28. [CrossRef]
7. Bouchaud, J.P.; Cugliandolo, L.F.; Kurchan, J.; Mézard, M. Out of equilibrium dynamics in spin-glasses and other glassy systems. *Spin Glas. Random Fields* **1998**, *12*, 161.
8. Bouchaud, J.P.; Dupuis, V.; Hammann, J.; Vincent, E. Separation of time and length scales in spin glasses: Temperature as a microscope. *Phys. Rev. B* **2001**, *65*, 024439. [CrossRef]
9. Zeng, S.; Zhong, F. Theory of critical phenomena with long-range temporal interaction. *Phys. Scr.* **2023**, *98*, 075017. [CrossRef]
10. Sperstad, I.B.; Stiansen, E.B.; Sudbø, A. Quantum criticality in spin chains with non-Ohmic dissipation. *Phys. Rev. B* **2012**, *85*, 214302. [CrossRef]
11. Harlow, F.H.; Welch, J.E. Numerical Calculation of Time-Dependent Viscous Incompressible Flow of Fluid with Free Surface. *Phys. Fluids* **1965**, *8*, 2182–2189. [CrossRef]
12. Keim, N.C.; Paulsen, J.D.; Zeravcic, Z.; Sastry, S.; Nagel, S.R. Memory formation in matter. *Rev. Mod. Phys.* **2019**, *91*, 035002. [CrossRef]
13. de Vega, I.; Alonso, D. Dynamics of non-Markovian open quantum systems. *Rev. Mod. Phys.* **2017**, *89*, 015001. [CrossRef]

14. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Preface. In *Theory and Applications of Fractional Differential Equations*; Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., Eds.; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204, pp. vii–x. [CrossRef]
15. Laskin, N. Fractional Schrödinger equation. *Phys. Rev. E* **2002**, *66*, 056108. [CrossRef]
16. Adelman, S.A. Fokker–Planck equations for simple non-Markovian systems. *J. Chem. Phys.* **1976**, *64*, 124–130. [CrossRef]
17. Jeon, J.H.; Metzler, R. Fractional Brownian motion and motion governed by the fractional Langevin equation in confined geometries. *Phys. Rev. E* **2010**, *81*, 021103. [CrossRef]
18. González-Herrero, H.; Gómez-Rodríguez, J.M.; Mallet, P.; Moaied, M.; Palacios, J.J.; Salgado, C.; Ugeda, M.M.; Veuillen, J.Y.; Yndurain, F.; Brihuega, I. Atomic-scale control of graphene magnetism by using hydrogen atoms. *Science* **2016**, *352*, 437–441. [CrossRef]
19. Telford, E.J.; Chica, D.G.; Ziebel, M.E.; Xie, K.; Manganaro, N.S.; Huang, C.Y.; Cox, J.; Dismukes, A.H.; Zhu, X.; Walsh, J.P.S.; et al. Designing Magnetic Properties in CrSBr through Hydrostatic Pressure and Ligand Substitution. *Adv. Phys. Res.* **2023**, *2*, 2300036. [CrossRef]
20. Zhang, G.; Yu, J.; Wu, H.; Yang, L.; Jin, W.; Zhang, W.; Chang, H. Field-free room-temperature modulation of magnetic bubble and stripe domains in 2D van der Waals ferromagnetic Fe3GaTe2. *Appl. Phys. Lett.* **2023**, *123*, 101901. [CrossRef]
21. Yao, Y.; Zhivulin, V.; Zykova, A.; Cherkasova, N.; Vinnik, D.; Trofimov, E.; Gudkova, S.; Zaitseva, O.; Taskaev, S.; Alyabyeva, L.; et al. High entropy BaFe12-x(Ti/Mn/Ga/In)xO19 (x = 1–7) oxides: Correlation of the composition, entropy state, magnetic characteristics, and terahertz properties. *Ceram. Int.* **2023**, *49*, 31549–31558. [CrossRef]
22. Mermin, N.D.; Wagner, H. Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models. *Phys. Rev. Lett.* **1966**, *17*, 1133–1136. [CrossRef]
23. Gong, C.; Li, L.; Li, Z.; Ji, H.; Stern, A.; Xia, Y.; Cao, T.; Bao, W.; Wang, C.; Wang, Y.; et al. Discovery of intrinsic ferromagnetism in two-dimensional van der Waals crystals. *Nature* **2017**, *546*, 265–269. [CrossRef]
24. Vinnik, D.A.; Starikov, A.Y.; Zhivulin, V.E.; Astapovich, K.A.; Turchenko, V.A.; Zubar, T.I.; Trukhanov, S.V.; Kohout, J.; Kmječ, T.; Yakovenko, O.; et al. Changes in the Structure, Magnetization, and Resistivity of BaFe12–xTixO19. *ACS Appl. Electron. Mater.* **2021**, *3*, 1583–1593. [CrossRef]
25. Gambardella, P.; Dallmeyer, A.; Maiti, K.; Malagoli, M.; Eberhardt, W.; Kern, K.; Carbone, C. Ferromagnetism in one-dimensional monatomic metal chains. *Nature* **2002**, *416*, 301–304. [CrossRef]
26. Fisher, M.E.; Ma, S.k.; Nickel, B.G. Critical Exponents for Long-Range Interactions. *Phys. Rev. Lett.* **1972**, *29*, 917–920. [CrossRef]
27. Nair, R.; Sepioni, M.; Tsai, I.L.; Lehtinen, O.; Keinonen, J.; Krasheninnikov, A.V.; Thomson, T.; Geim, A.; Grigorieva, I. Spin-half paramagnetism in graphene induced by point defects. *Nat. Phys.* **2012**, *8*, 199–202. [CrossRef]
28. Červenka, J.; Katsnelson, M.; Flipse, C. Room-temperature ferromagnetism in graphite driven by two-dimensional networks of point defects. *Nat. Phys.* **2009**, *5*, 840–844. [CrossRef]
29. Taroni, A.; Bramwell, S.T.; Holdsworth, P.C.W. Universal window for two-dimensional critical exponents. *J. Phys. Condens. Matter* **2008**, *20*, 275233. [CrossRef]
30. Wang, Q.H.; Bedoya-Pinto, A.; Blei, M.; Dismukes, A.H.; Hamo, A.; Jenkins, S.; Koperski, M.; Liu, Y.; Sun, Q.C.; Telford, E.J.; et al. The Magnetic Genome of Two-Dimensional van der Waals Materials. *ACS Nano* **2022**, *16*, 6960–7079. [CrossRef]
31. Jin, B. *Fractional Differential Equations*; Springer: Berlin/Heidelberg, Germany, 2021.
32. Hoffmann, K.; Tang, Q. *Ginzburg-Landau Phase Transition Theory and Superconductivity*; International Series of Numerical Mathematics; Birkhäuser Basel: Basel, Switzerland, 2012.
33. Cardy, J. *Scaling and Renormalization in Statistical Physics*; Cambridge University Press: Cambridge, UK, 1996; Volume 5.
34. Diethelm, K.; Ford, N. The analysis of fractional differential equations. In *Lecture Notes in Mathematics*; Springer Nature: Berlin, Germany, 2010; Volume 2004.
35. Samraiz, M.; Mehmood, A.; Iqbal, S.; Naheed, S.; Rahman, G.; Chu, Y.M. Generalized fractional operator with applications in mathematical physics. *Chaos Solitons Fractals* **2022**, *165*, 112830. [CrossRef]
36. Li, C.; Zeng, F. *Numerical Methods for Fractional Calculus*; CRC Press: Boca Raton, FL, USA, 2015.
37. Milici, C.; Drăgănescu, G.; Machado, J.T. *Introduction to Fractional Differential Equations*; Springer: Berlin/Heidelberg, Germany, 2018; Volume 25.
38. Thiel, L.; Wang, Z.; Tschudin, M.A.; Rohner, D.; Gutiérrez-Lezama, I.; Ubrig, N.; Gibertini, M.; Giannini, E.; Morpurgo, A.F.; Maletinsky, P. Probing magnetism in 2D materials at the nanoscale with single-spin microscopy. *Science* **2019**, *364*, 973–976. [CrossRef]